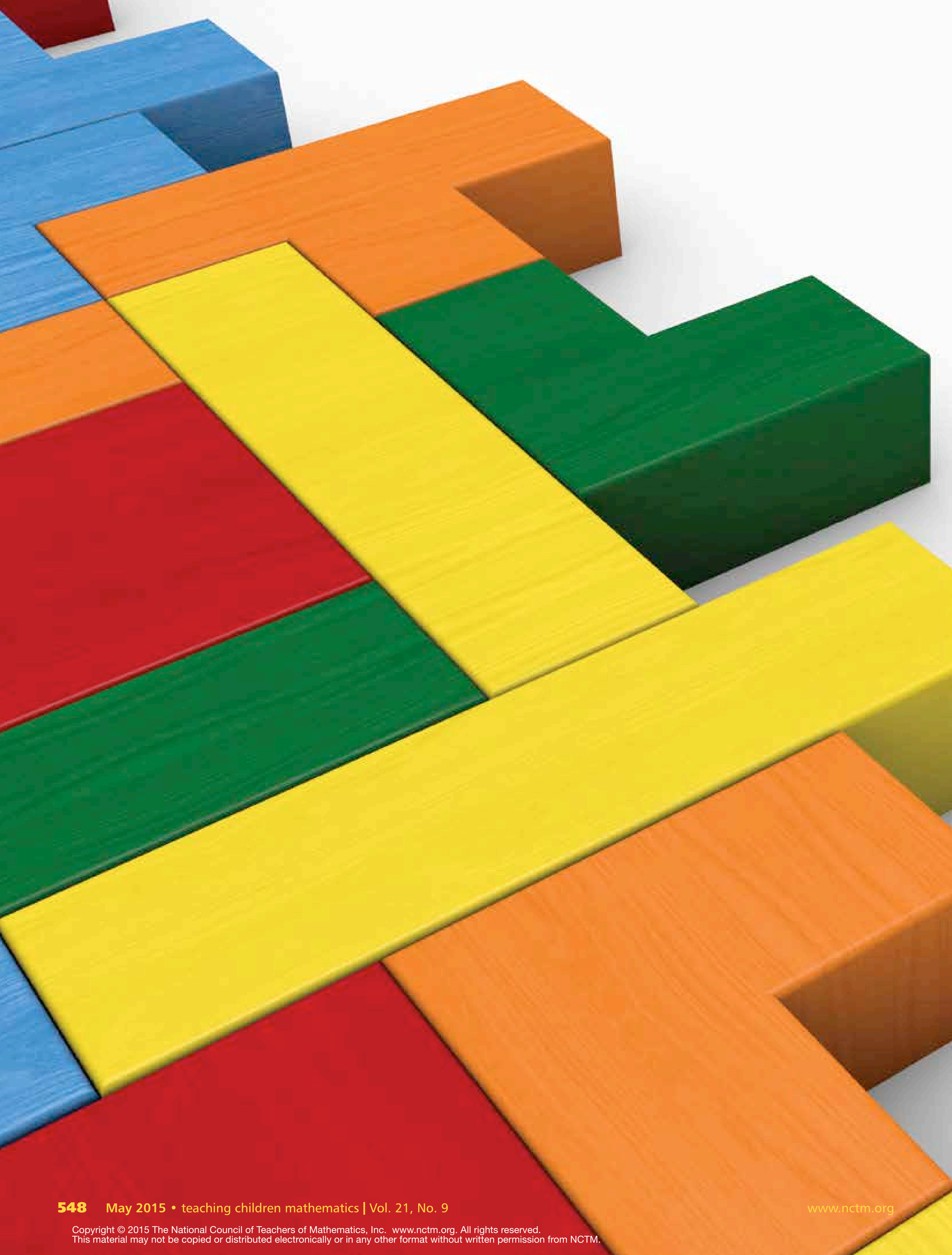


THREE STEPS TO MASTERING MULTIPLICATION FACTS

First understand what fluency is, then use these games and a sequence of strategies to help your students develop facility and confidence.

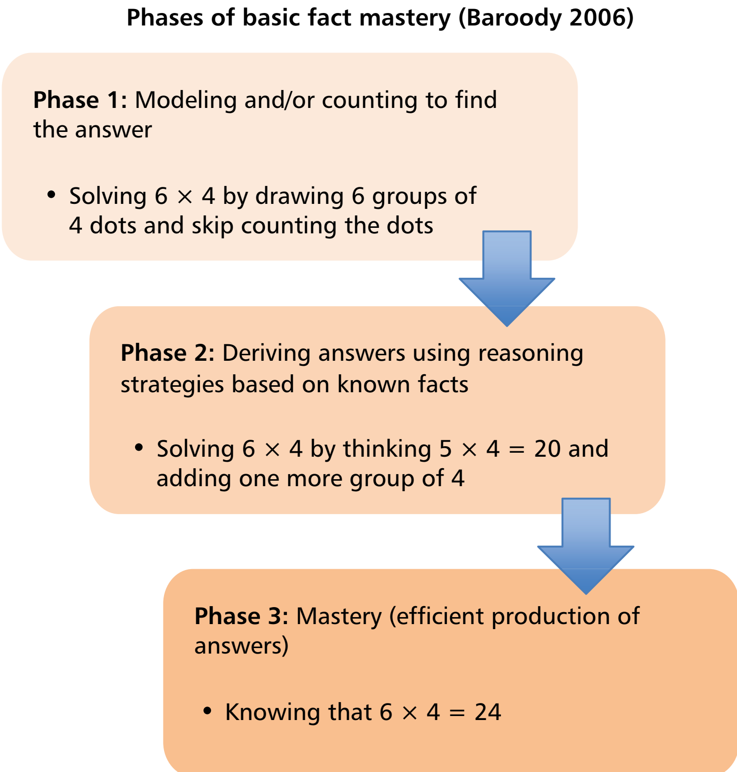
By Gina Kling and Jennifer M. Bay-Williams

"That was the day I decided I was bad at math."

Countless times, we have heard preservice and in-service teachers make statements such as this after sharing vivid memories of learning multiplication facts. Timed tests; public competitive games, such as Around the World; and visible displays of who has and has not mastered groups of facts still resonate as experiences that led them to doubt their mathematical abilities. Others who appeared to be successful with such activities have shared such statements as these: "We learned a song for every fact. I can find any fact quickly, but I still need to sing the song first" and "I use the nines finger trick but have no idea how or why it works." Are these people *truly* fluent with their multiplication facts?

LUCADP/THINKSTOCK

FIGURE 1 Students who learn multiplication facts through traditional approaches generally do not retain the facts because the method attempts to move students from phase 1 directly to phase 3 of Baroody's (2006) three developmental phases.



Helping students develop fluency with their multiplication facts is arguably one of the most important goals of teachers in grades 3–5. To achieve this goal, we must know the answers to these questions:

- What does fluency mean?
- What approaches to building fluency with multiplication facts help our students become confident and competent mathematical thinkers?
- What does meaningful practice look like?

We explore each of these questions in the sections that follow.

Understanding fluency

Teachers have many different opinions of what "fluency with multiplication facts" means. The Common Core State Standards for Mathematics (CCSSM) provide some guidance on understanding fluency:

> Fluently multiply and divide within 100, using strategies such as the relationship between multiplication and division (e.g., knowing that 8 × 5 = 40, one knows 40 ÷ 5 = 8) or properties of operations. By the end of Grade 3, know from memory all products of two one-digit numbers. (3.OA.C.7, CCSSI 2010, p. 23)

What *is* fluency?

CCSSM presents two significant aspects of fluency here. First, *fluently* means noticing

THINGLASS/THINKSTOCK

relationships and using strategies. According to CCSSM, fluency is "skill in carrying out procedures flexibly, accurately, efficiently and appropriately" (CCSSI 2010, p. 6). Thus, far from just being a measure of speed, fluency with multiplication facts involves flexibly and accurately using an appropriate strategy to find the answer efficiently.

Second, note that the phrase *know from memory* is used—*not* the term *memorization*. With repeated experiences working with number, students can come to "just know" that 2 × 6 = 12, without ever having had to memorize it. At this point, we say students have *mastered* their multiplication facts, as they have become so fluent at applying their strategies that they do so automatically, without hesitation.

How is fluency developed?

Students develop fluency as they progress through three developmental phases (Baroody 2006) (see **fig. 1**). Traditional approaches to learning multiplication facts (flash cards, drill, and timed testing) attempt to move students from phase 1 directly to phase 3. This approach is ineffective—many students do not retain what they memorized in the long term, moving to grade 4 and beyond still not knowing their facts. Even if students remember facts, they are unlikely to be fluent as defined above, as they will not have learned to flexibly apply strategies to find the answer to a multiplication fact (see **fig. 1**).

Research tells us that students must deliberately progress through these phases, with explicit development of reasoning strategies, which helps students master the facts and gives them a way to regenerate a fact if they have forgotten it. Students make more rapid gains in fact mastery when emphasis is placed on strategic thinking (National Research Council [NRC] 2001, Cook and Dossey 1982, Heege 1985, Thornton 1978). So, how do we help children progress through the three phases with respect to multiplication facts? Careful sequencing and explicit attention to strategy development is necessary.

Sequencing and developing strategies

We are familiar with the traditional sequence of learning multiplication facts: "master" the 0s,

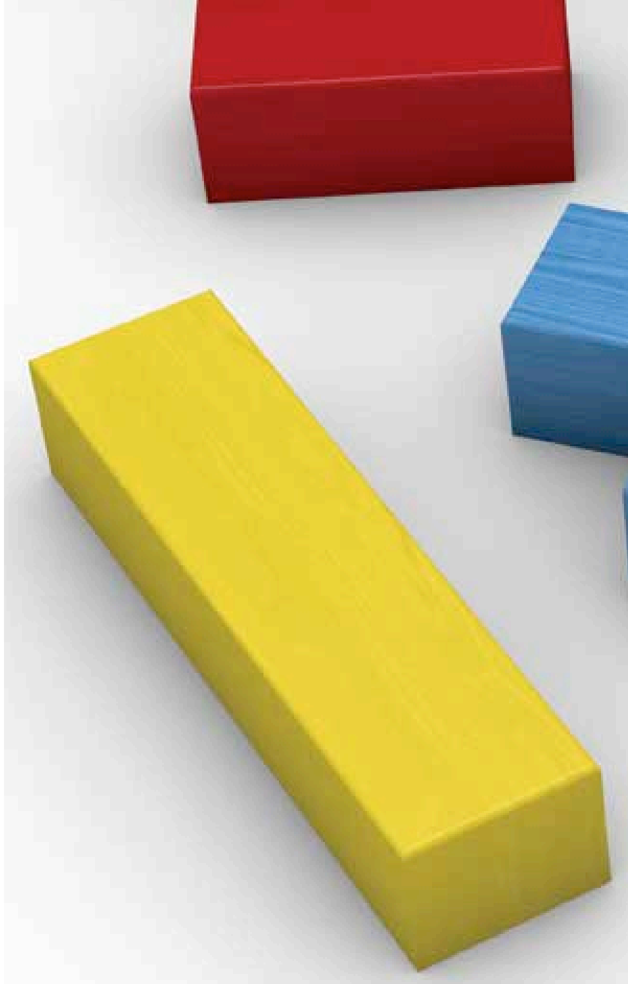
TABLE 1

The authors suggest this sequence and these strategies for teaching children fluency, flexibility, and reasoning strategies with multiplication facts.

Sequence and strategies for teaching multiplication facts

Foundational facts*	
1. 2s, 5s, and 10s (begin these late in second grade)	Use story problems, arrays, skip counting, and patterns on a hundred chart and a multiplication table to learn these facts.
2. 0s* 1's, multiplication squares (2 × 2, 3 × 3, etc.)	
Derived fact strategies	
3. Adding or subtracting a group	Start with a nearby 2s, 5s, or 10s fact, then subtact (or add) the group. *Example:* I don't know 9 × 6, so I think "**10** × 6 = 60" and subtract one group of 6 to get 54.
4. Halving and doubling	Look for an even factor. Find the fact for half of that factor, then double it. *Example:* I don't know 6 × 8, so I think "3 × 8 = 24" and double that to get 48.
5. Using a square product	Look for a nearby square. Find that fact and add on or subtract off the extra group. *Example:* I don't know 7 × 6. I use 6 × 6 = 36 and add one more 6 to get 42.
6. Decomposing a factor	Partition one of the factors into a convenient sum of known facts, find the two known facts, and combine the products. *Example:* I don't know 7 × 6. I break the 7 into 2 and 5, because I know 2 × 6 and 5 × 6. Then I add 12 and 30 to get 42.

* 0s are foundational but are not typically used for derived-fact strategies.



then 1's, then 2s, and so on. Yet, introducing multiplication facts in terms of their relative difficulty (starting with easiest facts) and clustering them around strategies is more effective (Thornton 1978; NRC 2001; Heege 1985; Van de Walle, Karp, and Bay-Williams 2012). On the basis of this research, as well as on classroom experience using these ideas, we suggest the sequence and strategies for fact instruction outlined in **table 1**.

Foundational facts

During the first few years of school, through the meaningful practices of skip counting, working with addition doubles, and representing multiplication and division situations and arrays, children begin to learn the first set of multiplication facts: the 2s, 5s, and 10s (Heege 1985, Kamii and Anderson 2003, Watanabe 2003). We recommend working with these foundational facts at the end of second grade, so that students entering third grade are fluent and ready to apply them to derive other facts.

The multiplication squares (e.g., 3 × 3), 0s, and 1's are the next set of foundational facts. Instead of teaching 0s and 1's with memorization "tricks," invite students to apply their understanding of multiplication (for example, that five groups of zero—or five "empty" groups—would give us zero objects, so 5 × 0 = 0). Exploring situations using these facts, drawing arrays, and looking for patterns in the multiplication table will help students learn these facts.

We describe the facts above as "foundational" because they lay the groundwork for derived-fact strategies. By definition, derived-fact strategies are based on facts students already *know*. Therefore, a lack of fluency with foundational facts can lead to frustration or inefficiency when students do not quickly "see" a known fact in the problem they are solving, preventing them from adopting important derived-fact strategies.

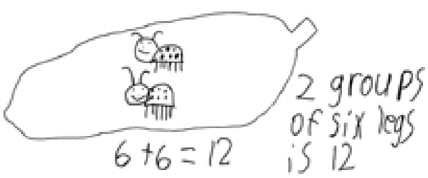
Derived facts

The foundational facts, learned and understood, can then be used for learning all other multiplication facts (phase 2). Carefully chosen contexts and sequencing can allow particular strategies to emerge. **Figure 2** presents an example for connecting 3s to 2s, where the context and structure of the stories facilitate students making this connection. Students can use 2s, 5s, and 10s facts to solve nearby facts, such as 3s, 4s, 6s, and 9s. For example, all 6 facts (6 × *n*) can be found by starting with five groups of the other factor, plus one more group of that factor (5 × *n* + *n*). The key is to

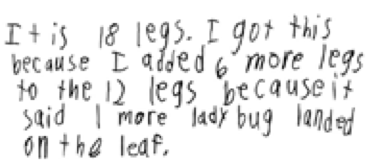
FIGURE 2

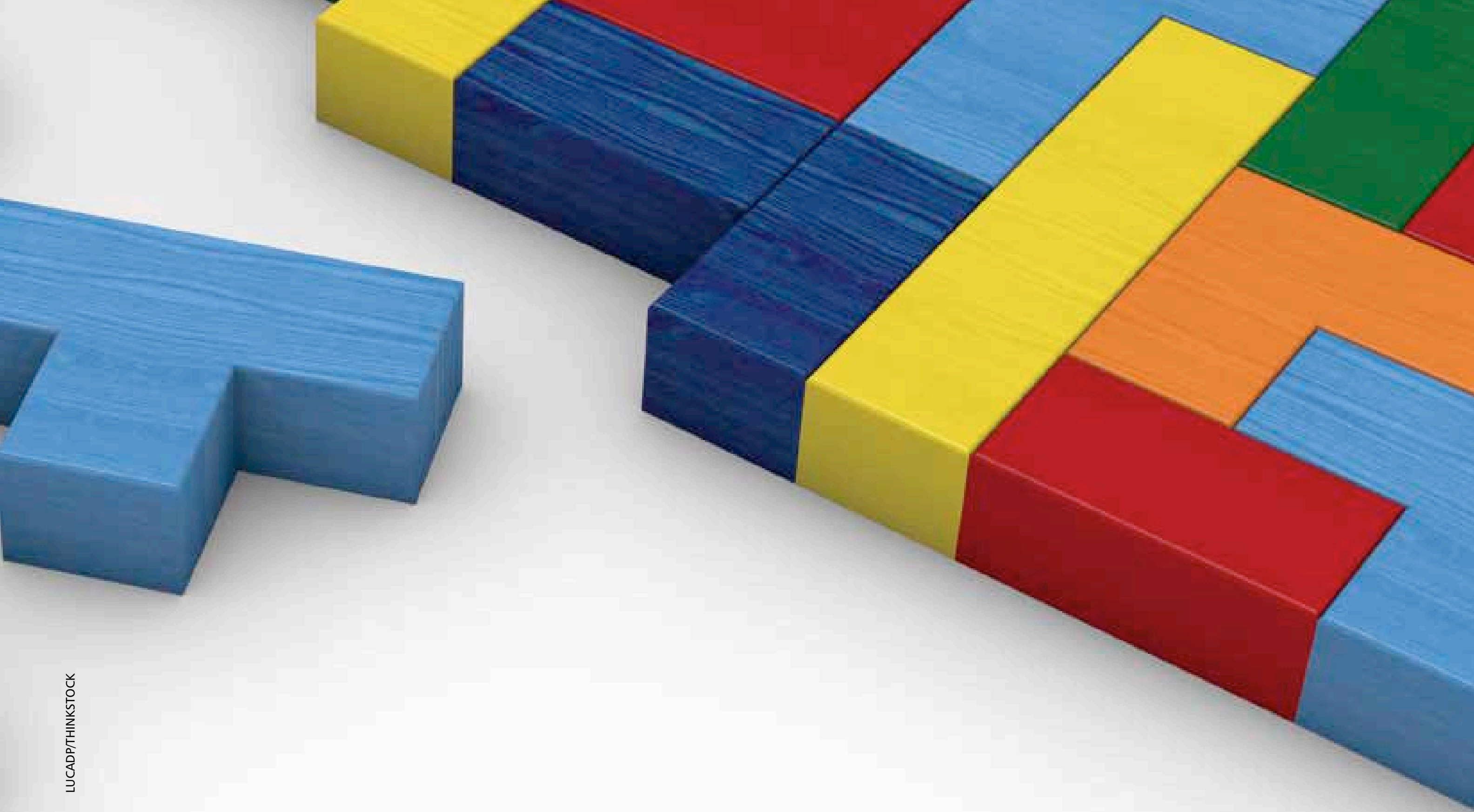
In this example for connecting 3s to 2s, the context and structure of the stories facilitate student learning. Story problems can connect foundational facts to other facts.

There are 2 ladybugs sitting on a leaf. Each ladybug has 6 legs. How many legs do they have altogether? Draw a picture to show your thinking.



A third ladybug lands on the leaf. How many legs are there altogether now? Explain how you can use your first picture to help you figure this out.





LUCADP/THINKSTOCK

help students think of how they can work from the first fact to derive the second, related fact, as opposed to starting over and drawing an entirely new picture (see **fig. 2**).

Facts that include even factors (e.g., 4s, 6s, and 8s) can be found through halving and then doubling. This has been shown to be a powerful strategy (Flowers and Rubenstein 2011, Heege 1985, Thornton 1978). A sequence of multiplication stories suggest using doubling to find the final product (see **fig. 3**). The area representation helps students visualize how doubling one of the factors leads to doubling the area, or product.

With sufficient experiences with halving/doubling relationships, students learn to work flexibly to apply this understanding to unknown facts. For example, for 6 × 7, students can think, "Half of 6 is 3, and I know that 3 × 7 = 21, so I double 21 and get 42."

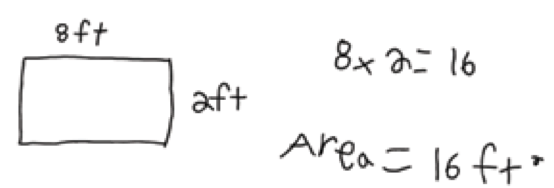
Multiplication squares can also be used to solve related facts, an effective approach for some of the most challenging facts, such as 7 × 8 or 6 × 7. Students can apply their understanding of adding or subtracting a group to a nearby square, such as by solving 7 × 8 by starting with 8 × 8 = 64 and subtracting one group of 8 to get 56.

Finally, any fact can be found by decomposing one of the factors to create known facts and then recomposing the entire product. This strategy is grounded in number sense and will serve students well as they look for efficient ways to solve multidigit multiplication problems. When

FIGURE 3

A sequence of multiplication stories suggests using doubling to find the final product.

Your class is building a sandbox for the 1st graders. The sandbox will be 2 feet wide and 8 feet long. What is the area of the sandbox? Draw a sketch of the sandbox and write a number model to show how you found the area.



You decide to make the sandbox 4 feet wide and 8 feet long instead. How can you use your work from the first problem to figure out the new area? Explain, using sketches and words to show your thinking.

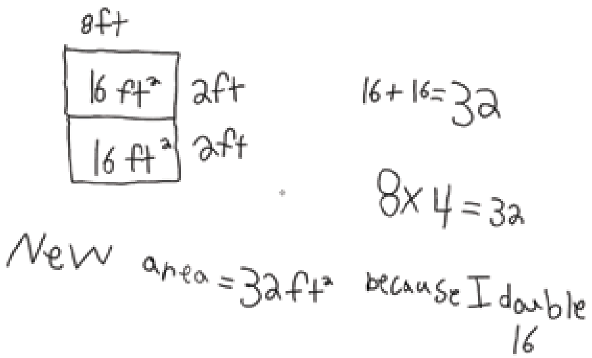


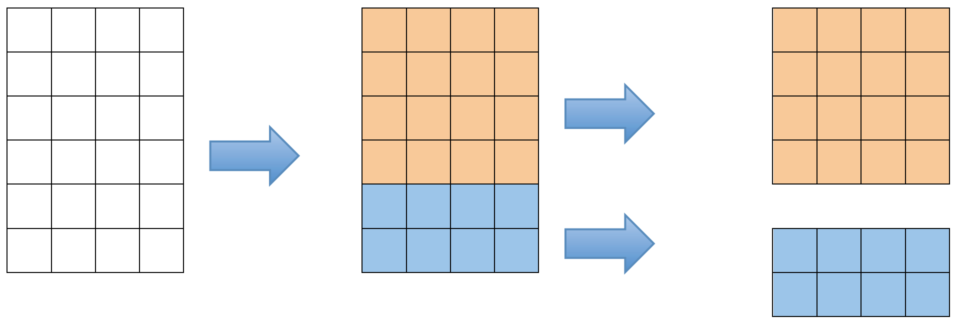
FIGURE 4

Array and area models and equal-groups interpretations work well for the early stages of learning the decomposing strategy, when using a representation is a crucial part of a student's process.

(a) Using an equal-groups interpretation to decompose the fact 7 × 8

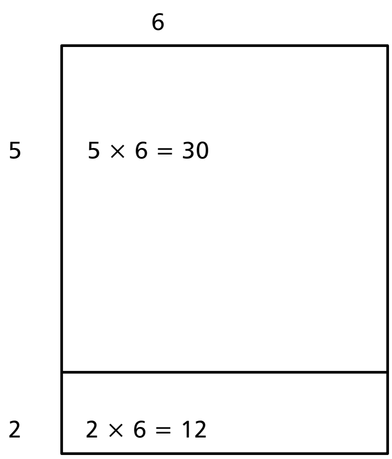
I think of 7 × 8 as 7 groups of 8 things. I don't know what that is, so I start with 5 groups of 8 things, which is 40.	5 × 8 = 40
I have to have 7 groups in the end, so I need to add 2 more groups of 8 things. I know that 2 groups of 8 things is 16.	2 × 8 = 16
So, to find 7 groups of 8 things, I add 40 + 16, which is 56.	7 × 8 = 5 × 8 + 2 × 8 = 40 + 16 = 56

(b) Using an array representation to decompose the fact 6 × 4



I can split my 6 × 4 array into two smaller arrays, one that is 4 × 4 and one that is 2 × 4. I know that 4 × 4 =16 and 2 × 4 = 8. I then add the smaller products of 16 and 8 and get 24 for my answer.

(c) Using an area representation to decompose the fact 7 × 6



$$\begin{aligned} 7 \times 6 &= 5 \times 6 + 2 \times 6 \\ &= 30 + 12 \\ &= 42 \end{aligned}$$

children first begin decomposing, using a representation is key to keeping track of their process; equal groups interpretations, array models, and area models all work well for this purpose (see **fig. 4**).

Properties of multiplication

Underlying all these strategies are the properties of multiplication, namely the Commutative, Associative, and Distributive Properties. In grade 3, the related Common Core standard does *not* say that students must be able to name the properties but to *apply* them. Students apply these properties intuitively as they attempt to make facts easier to solve (see **table 2**). Children will need frequent opportunities to explore, apply, and discuss multiplication strategies and properties throughout the year to move from fluency with strategies to mastery of all facts. This presents the need for meaningful practice.

Meaningful practice

There is no doubt that practicing multiplication facts is essential for mastering them (phase 3). To maximize precious class time spent practicing facts, embedding that practice in worthwhile mathematical activities is important. Drilling isolated facts may, over time, lead to memorization of those facts, but that is the *only* gain. In contrast, *meaningful practice* involves helping students learn their facts through rich, engaging mathematical activities that provide the additional benefits of promoting problem solving, reasoning, and communicating mathematical thinking. Meaningful practice of multiplication facts begins with the use of related problems like the examples given above. It can be sustained throughout the year by reminding students to think of strategies they know when solving an unknown fact and by expecting students to articulate those strategies verbally and in writing.

Multiplication fact games provide meaningful (and enjoyable) practice. Games involve many calculations in which efficiency is encouraged, without the stress of timed tests. Some games focus on particular fact strategies, whereas others provide general practice of all facts. Here we share three of our favorite multiplication games (see the more4u box at the end of this article for additional games). Several of the games refer to array cards. To make a set of array cards, use centimeter grid paper. Label each one with the facts written both ways (e.g., 3 × 8 and 8 × 3). Depending on the activity, you may also write the product on the back of each card (see **fig. 5**).

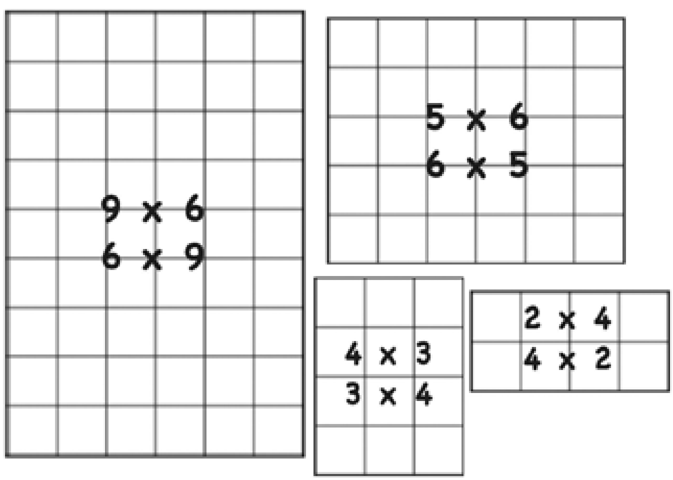
TABLE 2

Students need meaningful practice to move from fluency to mastery. CCSSM 3.OA.B.5: "Apply properties of operations as strategies to multiply and divide."

Applying the properties of multiplication	
Commutative property of multiplication	Important to all facts. If 6 × 4 = 24 is known, then 4 × 6 = 24 is also known. This cuts the learning of facts in half.
Associative property of multiplication	Used in derived facts, like doubling. A student sees 6 × 9 and thinks (2 × 3) × 9, which is the same as 2 × (3 × 9), which is 2 × 27, 54.
Distributive property of multiplication over addition	A student realizes that 8 × 7 = 8 × (5 + 2) and uses this to find the answer, thinking (8 × 5) + (8 × 2) = 40 + 16 = 56.

FIGURE 5

Game play can encourage mathematical efficiency without producing the anxiety of timed tests. These are examples of array cards cut from grid paper.



Strive to derive

This game mimics the thinking that students use in deriving facts, because students first see the actual fact and then visually partition it into two facts to find the derived facts and the answer. The first time students play this game, they could focus on using a particular

JENNIFER M. BAY-WILLIAMS

In the Cover It game, two players take turns drawing array cards to find a combination of two that will cover the original array. For example, if the 7×8 array were pulled and covered by 5×8 and 2×8 arrays, students would record $7 \times 8 = 5 \times 8 + 2 \times 8$.

FIGURE 6

These Strive to Derive game instructions focus on using the 2s, 5s, and 10s facts so that students begin thinking about which foundational facts can help them with the 3s, 6s, and 9s facts.

Strive to Derive game instructions

Materials

- Array cards (use arrays for 3s, 4s, 6s, and 9s)
- Uncooked spaghetti or thin sticks
- Two teacher-labeled dice, one with 3, 3, 6, 6, 9, 9; the other with 0, 1, 4, 6, 7, 8

Number of players

2–4

Instructions

1. Spread the array cards out so they can be seen.
2. Players alternate taking a turn. Player 1 rolls the dice, then—
   a. finds that array;
   b. partitions the array (using uncooked spaghetti) into two arrays, one of them being a 2, 5, or 10 fact;
   c. says or writes how to find the fact that he or she rolled.

Example

Lisa rolls a 6 and a 7. She pulls the 6×7 array card. She places spaghetti to show 5×7 and 1×7. She then says, "Six times seven is five times seven, thirty-five, and one more seven, forty-two."

Optional

Students can record their arrays: $6 \times 7 = 5 \times 7 + 7 = 42$.

3. If a player is able to illustrate and explain the fact using a derived fact, he or she scores a point.
4. The player returns the array card to the middle of the table. Play goes to the next player.
5. Play to ten points.

fact, for example, *Strive to Derive from 5*. The **figure 6** instructions focus on using 2s, 5s, and 10s so that students begin thinking about which foundational facts can help them with the 3s, 6s, and 9s. The game can decrease or increase in challenge by changing the values on the dice. The challenge of derived facts is to avoid getting bogged down in the step of choosing a known fact. The more this game is played, the more quickly students find a known fact and are able to use it to solve the given problem (see **fig. 6**).

Cover it

In this two-player matching game, students spread selected array cards so that all are visible (adapted from Russell and Economopoulos 2008). Player 1 pulls an array from the middle and gives it to player 2, who must find two arrays that exactly cover the array he or she received. If player 2 does this successfully, he or she keeps the three array cards. If player 2 cannot find a pair, player 1 gets a chance and can also win the cards. Players switch roles and continue. Students say or write the combinations that they have found to cover the original array.

Multiplication Tetris®

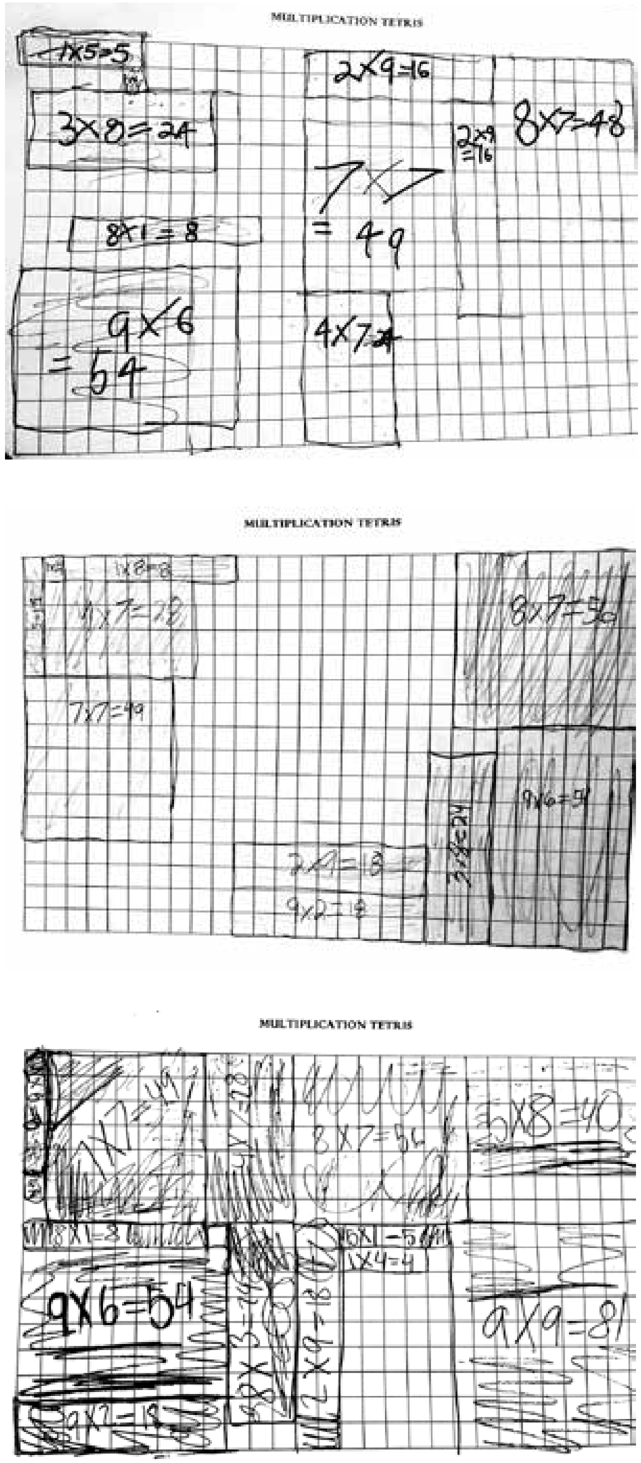
Tetris has entertained us for many years. Students love this mathematized version of Tetris. The goal is to stay in the game the longest by having room on your grid paper to fit a given rectangle. The teacher rolls two dice (regular dice, ten-sided dice, or teacher-labeled dice to emphasize particular facts such as 2s, 5s, and 10s). If the teacher rolls a 4 and a 6, each student decides where and in what orientation to best fit a 4×6 rectangle on the grid paper. Students trace either a 4×6 or a 6×4 array on their paper and write the multiplication fact. The teacher continues to roll, and students mark out the called rectangle somewhere on their grid (see **fig. 7**). When a student cannot fit a rectangle with the dimensions rolled, he or she is out of the game. The last students in the game are the winners. This game helps students see the facts as arrays while also reinforcing the commutativity of each fact (4 × 6 = 6 × 4 = 24).

You may have noticed that the grids in **figure 7** do not follow the conventional recording of rows × columns. These students had been focusing on the commutative property, and as they turned their rectangles to fit them on the page, they were thinking of 7 × 8 and 8 × 7 interchangeably. This is consistent with recommendations from CCSSM *Progressions* documents, which state the following:

> In the Array situations, the roles of the factors do not differ. One factor tells the number of rows in the array, and the other factor tells the number of columns in the situation. But rows and columns depend on the orientation of the array. If an array is rotated 90º, the rows

FIGURE 7

Multiplication Tetris helps students see multiplication facts as arrays. As the game boards show, students apply different strategies to placing their arrays.



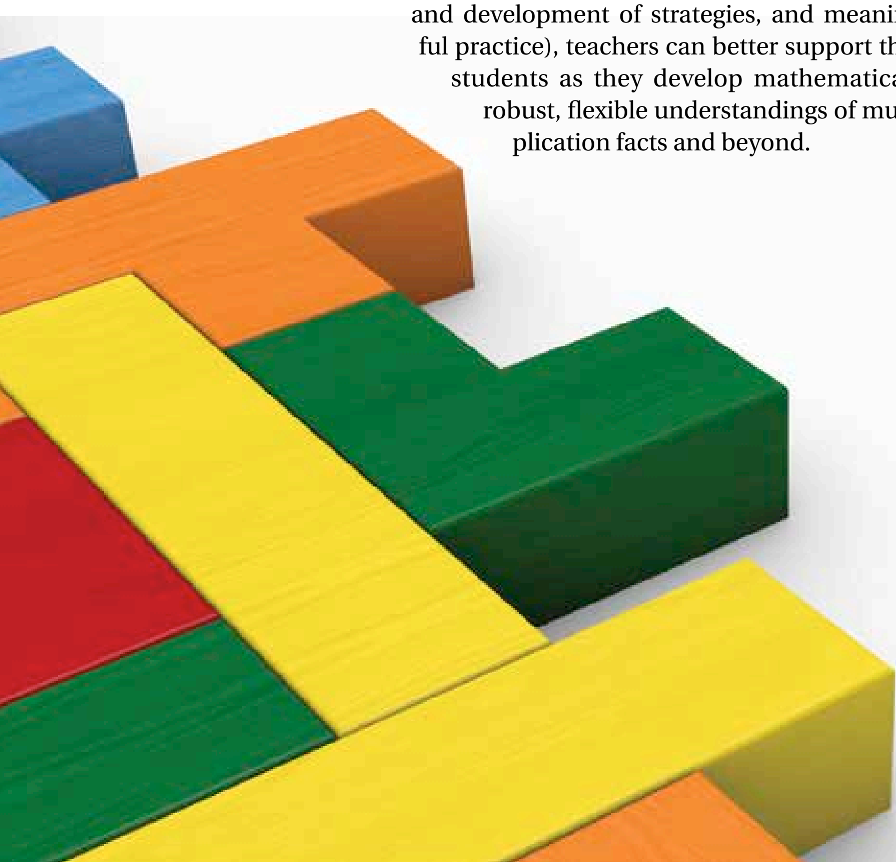
JENNIFER M. BAY-WILLIAMS

Multiplication Tetris can strengthen a student's concept of the commutativity property of each fact. Here, MacKenna decides where to place her array.

become columns and the columns become rows. This is useful for seeing the commutative property for multiplication. (Common Core Standards Writing Team 2011, p. 24)

Final thoughts

There is no question that the CCSSM expectation for mastery with all multiplication facts by the end of third grade is a daunting task. Decades of drill and timed testing have failed our students, often leading to a lack of fluency and a negative disposition toward mathematics. Even in cases where students are able to successfully complete tasks, such as timed tests, one might question the value of such assessments. Does a perfect score on a timed test really tell us anything about that student's *understanding*? Do we actually know if he or she is *fluent* as defined in this article? Couldn't we learn more by carefully observing and questioning students as they engage in meaningful practice playing games, in class discussions of strategies, or even through brief interviews with individual students (Kling and Bay-Williams 2014)? Such questions are worthy of careful consideration as one reflects on possible paths toward multiplication fact mastery. It is our hope that by following these three steps (understanding fluency, thoughtful sequencing and development of strategies, and meaningful practice), teachers can better support their students as they develop mathematically robust, flexible understandings of multiplication facts and beyond.



LUCADP/THINKSTOCK

BIBLIOGRAPHY

Baroody, Arthur J. 2006. "Why Children Have Difficulties Mastering the Basic Number Combinations and How to Help Them." *Teaching Children Mathematics* 13 (August): 22–31.

Bay-Williams, Jennifer M., and Gina Kling. 2014. "Enriching Addition and Subtraction Fact Mastery through Games." *Teaching Children Mathematics* 21 (November): 238–47.

Bell, Max, John Bretzlauf, Amy Dillard, Andy Isaacs, Kathleen Pitvorec, Jean Bell, Mary Ellen Dairyko, Robert Hartfield, James McBride, and Pater Saecker. 2012. *Everyday Mathematics: Common Core State Standards ed*. Chicago, IL: McGraw-Hill.

Common Core Standards Writing Team. 2011. *Progressions Documents for the Common Core Math Standards*: *Draft K–5 Progression on Counting and Cardinality and Operations and Algebraic Thinking*. http://ime.math.arizona.edu/progressions/

Common Core State Standards Initiative (CCSSI). 2010. Common Core State Standards for Mathematics (CCSSM). Washington, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers. http://www.corestandards.org/wp-content/uploads/Math_Standards.pdf

Cook, Cathy J., and John A. Dossey. 1982. "Basic Fact Thinking Strategies for Multiplication—Revisited." *Journal for Research in Mathematics Education* 13 (3): 163–71. doi:http://dx.doi.org/10.2307/748553

Flowers, Judith M., and Rheta N. Rubenstein. 2010/2011. "Multiplication Fact Fluency Using Doubles." *Mathematics Teaching in the Middle School* 16 (December/January): 296–303.

Heege, Hans Ter. 1985. "The Acquisition of Basic Multiplication Skills." *Educational Studies in Mathematics* 16 (4): 375–88. doi:http://dx.doi.org/10.1007/BF00417193

Kamii, Constance, and Catherine Anderson. 2003. "Multiplication Games: How We Made and Used Them." *Teaching Children Mathematics* 10 (November): 135–41.

Kling, Gina, and Jennifer M. Bay-Williams. 2014. "Assessing Basic Fact Fluency." *Teaching Children Mathematics* 20 (April): 488–97.

National Research Council (NRC). 2001. *Adding It Up: Helping Children Learn Mathematics,* edited by Jeremy Kilpatrick, Jane Swafford,

and Bradford Findell. Washington, DC: National Academies Press.
Russell, Susan Jo, and Karen Economopoulos. 2008. *Investigations in Number, Data, and Space.* 2nd ed. New York: Pearson.
Thornton, Carol A. 1978. "Emphasizing Thinking Strategies in Basic Fact Instruction." *Journal for Research in Mathematics Education* 9 (3): 214–27. doi:http://dx.doi.org/10.2307/748999
Van de Walle, John. A., Karen S. Karp, and Jennifer M. Bay-Williams. 2012. *Elementary and Middle School Mathematics: Teaching Developmentally: Professional Development Edition for Mathematics Coaches and Other Teacher Leaders*. New York: Pearson.
Watanabe, Tad. 2003. "Teaching Multiplication: An Analysis of Elementary School Mathematics Teachers' Manuals from Japan and the United States." *The Elementary School Journal* 104 (2): 111–25. http://dx.doi.org/10.1086/499745

The authors thank Mrs. George and her fourth graders at Crestwood Elementary School in Kentucky, and Ellen Dairyko and Amanda Ruch at the University of Chicago for their contributions to this article.


Gina Kling, of Western Michigan University, **gina.garza-kling@wmich.edu**, and **Jennifer M. Bay-Williams**, **j.baywilliams@louisville.edu**, of the University of Louisville, have a shared enthusiasm for helping children develop fluency with their basic facts.

more4U

Access an appendix of additional games by navigating to the article online.