Georgia
Standards of Excellence
Curriculum Frameworks

Mathematics

GSE Algebra I

Unit 4: Modeling and Analyzing Exponential Functions

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Unit 4
Modeling and Analyzing Exponential Functions
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OVERVIEW

In this unit students will:

- Analyze exponential functions only.
- Build on and informally extend understanding of integer exponents to consider exponential functions.
- Use function notation.
- Interpret expressions for functions in terms of the situation they model.
- Interpret exponential functions that arise in applications in terms of the context.
- Analyze exponential functions and model how different representations may be used based on the situation presented.
- Build a function to model a relationship between two quantities.
- Recognize geometric sequences as exponential functions.
- Create new functions from existing functions.
- Construct and compare exponential models and solve problems.
- Reinforce previous understanding of characteristics of graphs and investigate key features of exponential graphs.
- Investigate a multiplicative change in exponential functions.
- Create and solve exponential equations.
- Apply related linear equations solution techniques and the laws of exponents to the creation and solution of simple exponential equations.

Because we continually make theories about dependencies between quantities in nature and society, functions are important tools in the construction of mathematical models. A function can be described in various ways, such as by a graph (e.g., the trace of a seismograph); by a verbal rule, as in, "I'll give you a state, you give me the capital city;" by an algebraic expression like $f(x) = a + bx$; or by a recursive rule. The graph of a function is often a useful way of visualizing the relationship of the function models, and manipulating a mathematical expression for a function can throw light on the function's properties.

Although the units in this instructional framework emphasize key standards and big ideas at specific times of the year, routine topics such as estimation, mental computation, and basic computation facts should be addressed on an ongoing basis. Ideas related to the eight practice standards should be addressed constantly as well. This unit provides much needed content information and excellent learning activities. However, the intent of the framework is not to provide a comprehensive resource for the implementation of all standards in the unit. A variety of resources should be utilized to supplement this unit. The tasks in this unit framework illustrate the types of learning activities that should be utilized from a variety of sources. To assure that this unit is taught with the appropriate emphasis, depth, and rigor, it is important that the **"Strategies for Teaching and Learning"** in the Comprehensive Course Overview and the tasks under the **"Evidence of Learning"** be reviewed early in the planning process.

STANDARDS ADDRESSED IN THIS UNIT

Mathematical standards are interwoven and should be addressed throughout the year in as many different units and activities as possible in order to emphasize the natural connections that exist among mathematical topics.

KEY STANDARDS ADDRESSED

Create equations that describe numbers or relationships

MGSE9-12.A.CED.1 Create equations and inequalities in one variable and use them to solve problems. Include equations arising from ~~linear, quadratic,~~ ~~simple rational, and~~ exponential functions (integer inputs only).

MGSE9-12.A.CED.2 Create ~~linear, quadratic, and~~ exponential equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. (The phrase “in two or more variables” refers to formulas like the compound interest formula, in which $A = P(1 + r/n)^{nt}$ has multiple variables.)

Build a function that models a relationship between two quantities

MGSE9-12.F.BF.1 Write a function that describes a relationship between two quantities.

MGSE9-12.F.BF.1a Determine an explicit expression and the recursive process (steps for calculation) from context. *For example, if Jimmy starts out with $15 and earns $2 a day, the explicit expression “2x+15” can be described recursively (either in writing or verbally) as “to find out how much money Jimmy will have tomorrow, you add $2 to his total today.”*
$J_n = J_{n-1} + 2, J_0 = 15$

MGSE9-12.F.BF.2 Write ~~arithmetic and~~ geometric sequences recursively and explicitly, use them to model situations, and translate between the two forms. Connect ~~arithmetic sequences to linear functions and~~ geometric sequences to exponential functions.

Build new functions from existing functions

MGSE9-12.F.BF.3 Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k\,f(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. ~~Include recognizing even and odd functions from their graphs and algebraic expressions for them.~~ *(Focus on vertical translations of graphs of linear and exponential functions. Relate the vertical translation of a linear function to its y-intercept.)*

Understand the concept of a function and use function notation

MGSE9-12.F.IF.1 Understand that a function from one set (the input, called the domain) to another set (the output, called the range) assigns to each element of the domain exactly one element of the range, i.e. each input value maps to exactly one output value. If f is a function, x is the input (an element of the domain), and f(x) is the output (an element of the range). Graphically, the graph is y = f(x).

MGSE9-12.F.IF.2 Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context.

MGSE9-12.F.IF.3 Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers. (Generally, the scope of high school math defines this subset as the set of natural numbers 1,2,3,4...) By graphing or calculating terms, students should be able to show how the recursive sequence $a_1=7$, $a_n=a_{n-1}+2$; the sequence $s_n = 2(n-1) + 7$; and the function f(x) = 2x + 5 (when x is a natural number) all define the same sequence.

Interpret functions that arise in applications in terms of the context

MGSE9-12.F.IF.4 Using tables, graphs, and verbal descriptions, interpret the key characteristics of a function which models the relationship between two quantities. Sketch a graph showing key features including: intercepts; interval where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; ~~and periodicity.~~

MGSE9-12.F.IF.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*

MGSE9-12.F.IF.6 Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.

Analyze functions using different representations

MGSE9-12.F.IF.7 Graph functions expressed algebraically and show key features of the graph both by hand and by using technology.

MGSE9-12.F.IF.7e Graph exponential ~~and logarithmic~~ functions, showing intercepts and end behavior, ~~and trigonometric functions, showing period, midline, and amplitude.~~

MGSE9-12.F.IF.9 Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one function and an algebraic expression for another, say which has the larger maximum.*

STANDARDS FOR MATHEMATICAL PRACTICE

Refer to the Comprehensive Course Overview for more detailed information about the Standards for Mathematical Practice.

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

ENDURING UNDERSTANDINGS

- Graph exponential equations in two variables.
- Create exponential equations in one variable and use them in a contextual situation to solve problems.
- Create equations in two or more variables to represent relationships between quantities.
- Solve exponential equations in one variable.
- Graph equations in two variables on a coordinate plane and label the axes and scales.
- The concept of a function and function notation.
- Geometric sequences are functions.
- Interpret exponential functions that arise in applications in terms of the context.
- Different representations may be used based on the situation presented.
- A function may be built to model a relationship between two quantities.
- New functions can be created from existing functions.
- Construct and compare exponential models and solve problems.

- Interpret expressions for functions in terms of the situation they model.

- The effect on the graph of replacing f(x) by f(x) + k, k f(x), f(kx), and f(x + k) for specific values of k (both positive and negative); find the value of k given the graphs.

- Recognize even and odd functions from their graphs and algebraic expressions for them.

ESSENTIAL QUESTIONS

- How do I use graphs to represent and solve real-world equations and inequalities?

- Why is the concept of a function important and how do I use function notation to show a variety of situations modeled by functions?

- Why are geometric sequences functions?

- How do I interpret functions that arise in applications in terms of context?

- How do I use different representations to analyze exponential functions?

- How do I build an exponential function that models a relationship between two quantities?

- How do I build new functions from existing functions?

- How can we use real-world situations to construct and compare exponential models and solve problems?

- How do I interpret expressions for functions in terms of the situation they model?

- How do I solve an exponential equation in one variable?

- What are the effects on the graph of replacing f(x) by f(x) + k, k f(x), f(kx), and f(x + k) for specific values of k (both positive and negative)? How do I find the value of k given the graphs?

- How do I distinguish between even and odd functions from their graphs and algebraic expressions for them?

CONCEPTS AND SKILLS TO MAINTAIN

Students are expected to have prior knowledge/experience related to the concepts and skills identified below. It may be necessary to pre–assess to determine whether instructional time should be spent on conceptual activities that help students develop a deeper understanding of these ideas.

- Using the Pythagorean Theorem
- Understanding slope as a rate of change of one quantity in relation to another quantity
- Interpreting a graph
- Creating a table of values
- Working with functions
- Writing a linear equation
- Using inverse operations to isolate variables and solve equations
- Maintaining order of operations
- Understanding notation for inequalities
- Being able to read and write inequality symbols
- Graphing equations and inequalities on the coordinate plane
- Understanding and use properties of exponents
- Graphing points
- Choosing appropriate scales and label a graph

In order for students to be successful, the following skills and concepts need to be maintained:

- Know how to solve equations, using the distributive property, combining like terms and equations with variables on both sides.
- Understand and be able to explain what a function is.
- Determine if a table, graph or set of ordered pairs is a function.
- Distinguish between linear and non-linear functions.

Students may not realize the importance of unit conversion in conjunction with computation when solving problems involving measurement. Since today's calculating devices often display 8 to 10 decimal places, students frequently express answers to a much greater degree of precision than is required.

Measuring commonly used objects and choosing proper units for measurement are part of the mathematics curriculum prior to high school. In high school, students experience a broader variety of units through real–world situations and modeling, along with the exploration of the different levels of accuracy and precision of the answers.

An introduction to the use of variable expressions and their meaning, as well as the use of variables and expressions in real–life situations, is included in the Expressions and Equations Domain of Grade 7.
Working with expressions and equations, including formulas, is an integral part of the curriculum in Grades 7 and 8. In high school, students explore in more depth the use of equations and inequalities

to model real–world problems, including restricting domains and ranges to fit the problem's context, as well as rewriting formulas for a variable of interest.

SELECTED TERMS AND SYMBOLS

The following terms and symbols are often misunderstood. These concepts are not an inclusive list and should not be taught in isolation. However, due to evidence of frequent difficulty and misunderstanding associated with these concepts, instructors should pay particular attention to them and how their students are able to explain and apply them.

The definitions below are for teacher reference only and are not to be memorized by the students. Students should explore these concepts using models and real life examples. Students should understand the concepts involved and be able to recognize and/or demonstrate them with words, models, pictures, or numbers.

The websites below are interactive and include a math glossary suitable for high school children. **Note – At the high school level, different sources use different definitions. Please preview any website for alignment to the definitions given in the frameworks.**
http://www.amathsdictionaryforkids.com/
This web site has activities to help students more fully understand and retain new vocabulary.
http://intermath.coe.uga.edu/dictnary/homepg.asp

Definitions and activities for these and other terms can be found on the Intermath website. Intermath is geared towards middle and high school students.

- **Algebra:** The branch of mathematics that deals with relationships between numbers, utilizing letters and other symbols to represent specific sets of numbers, or to describe a pattern of relationships between numbers.
- **Average Rate of Change.** The change in the value of a quantity by the elapsed time. For a function, this is the change in the *y*-value divided by the change in the *x*-value for two distinct points on the graph.
- **Coefficient.** A number multiplied by a variable in an algebraic expression.
- **Continuous.** Describes a connected set of numbers, such as an interval.
- **Discrete.** A set with elements that are disconnected.
- **Domain.** The set of *x*-coordinates of the set of points on a graph; the set of *x*-coordinates of a given set of ordered pairs. The value that is the input in a function or relation.
- **End Behaviors.** The appearance of a graph as it is followed farther and farther in either direction.

- **Equation:** A number sentence that contains an equals symbol.
- **Explicit Expression.** A formula that allows direct computation of any term for a sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$.
- **Exponential Function.** A nonlinear function in which the independent value is an exponent in the function, as in $y = ab^x$.
- **Exponential Model.** An exponential function representing real-world phenomena. The model also represents patterns found in graphs and/or data.
- **Expression.** Any mathematical calculation or formula combining numbers and/or variables using sums, differences, products, quotients including fractions, exponents, roots, logarithms, functions, or other mathematical operations.
- **Factor.** For any number x, the numbers that can be evenly divided into x are called factors of x. For example, the number 20 has the factors 1, 2, 4, 5, 10, and 20.
- **Geometric Sequence.** A sequence of numbers in which the ratio between any two consecutive terms is the same. In other words, you multiply by the same number each time to get the next term in the sequence. This fixed number is called the common ratio for the sequence.
- **Horizontal Translation.** A shift in which a plane figure moves horizontally.
- **Interval Notation.** A notation representing an interval as a pair of numbers. The numbers are the endpoints of the interval. Parentheses and/or brackets are used to show whether the endpoints are excluded or included.
- **Irrational Number.** A number whose decimal form is nonterminating and nonrepeating. Irrational numbers cannot be written in the form a/b, where a and b are integers (b cannot be zero). So all numbers that are not rational are irrational.
- **Natural Numbers.** The set of numbers 1, 2, 3, 4, ... Also called counting numbers.
- **Ordered Pair**: A pair of numbers, (x, y), that indicate the position of a point on a Cartesian plane.
- **Parameter.** The independent variable or variables in a system of equations with more than one dependent variable.
- **Range.** The set of all possible outputs of a function.

- **Rational Number**. A number that can be written as a/b where a and b are integers, but b is not equal to 0.

- **Real Numbers.** All the rational and irrational numbers; that is, all of the numbers that can be expressed as decimals.

- **Recursive Formula.** A formula that requires the computation of all previous terms to find the value of a_n.

- **Reflection.** A transformation that "flips" a figure over a mirror or reflection line.

- **Term.** A value in a sequence--the first value in a sequence is the 1st term, the second value is the 2nd term, and so on; a term is also any of the monomials that make up a polynomial.

- **Variable:** A letter or symbol used to represent a number.

- **Vertical Translation.** A shift in which a plane figure moves vertically.

- **Whole Numbers.** **The set of numbers 0, 1, 2, 3, 4,….**

- ***X*-intercept.** The point where a line meets or crosses the *x*-axis.

- ***Y*-intercept.** The point where a line meets or crosses the *y*-axis.

The Properties of Operations

Here *a*, *b* and *c* stand for arbitrary numbers in a given number system. The properties of operations apply to the rational number system, the real number system, and the complex number system.

Associative property of addition	$(a + b) + c = a + (b + c)$
Commutative property of addition	$a + b = b + a$
Additive identity property of 0	$a + 0 = 0 + a = a$
Existence of additive inverses	For every a there exists $–a$ so that $a + (–a) = (–a) + a = 0$.
Associative property of multiplication	$(a \times b) \times c = a \times (b \times c)$
Commutative property of multiplication	$a \times b = b \times a$
Multiplicative identity property of 1	$a \times 1 = 1 \times a = a$
Existence of multiplicative inverses	For every $a \neq 0$ there exists $1/a$ so that $a \times 1/a = 1/a$

	$\times a = 1$.
Distributive property of multiplication over addition	$a \times (b + c) = a \times b + a \times c$

The Properties of Equality

Here a, b and c stand for arbitrary numbers in the rational, real, or complex number systems.

Reflexive property of equality	$a = a$
Symmetric property of equality	If $a = b$, then $b = a$.
Transitive property of equality	If $a = b$ and $b = c$, then $a = c$.
Addition property of equality	If $a = b$, then $a + c = b + c$.
Subtraction property of equality	If $a = b$, then $a - c = b - c$.
Multiplication property of equality	If $a = b$, then $a \times c = b \times c$.
Division property of equality	If $a = b$ and $c \neq 0$, then $a \div c = b \div c$.
Substitution property of equality	If $a = b$, then b may be substituted for a in any expression containing a.

EVIDENCE OF LEARNING

By the conclusion of this unit, students should be able to demonstrate the following competencies:

- Define and use function notation, evaluate functions at any point in the domain, give general statements about how *f(x)* behaves at different regions in the domain (as *x* gets very large or very negative, close to 0 etc.), and interpret statements that use function notation.
- Explain the difference and relationship between domain and range and find the domain and range of a function from a contextual situation, function equation, table, or graph.
- Examine data (from a table, graph, or set of points) and determine if the data represent a function and explain any conclusions that can be drawn.
- Write a function from a sequence or a sequence from a function.
- Explain how a geometric sequence is related to its algebraic function notation.
- Interpret *x* and *y* intercepts, where the function is increasing or decreasing, where it is positive or negative, its end behaviors, given the graph, table, or algebraic representation of a linear or exponential function in terms of the context of the function.
- Find and/or interpret appropriate domains and ranges for authentic exponential functions.
- Calculate and interpret the average rate of change over a given interval of a function from a function equation, graph or table, and explain what that means in terms of the context of the function.
- Estimate the rate of change of a function from its graph at any point in its domain.
- Explain the relationship between the domain of a function and its graph in general and/or to the context of the function.
- Graph an exponential function using technology.
- Sketch the graph of an exponential function accurately identifying *x*- and *y*-intercepts and asymptotes.
- Describe the end behavior of an exponential function (what happens as *x* approaches positive or negative infinity).
- Discuss and compare two different functions represented in different ways (tables, graphs or equations). Discussion and comparisons should include: identifying differences in rates of change, intercepts, and/or where each function is greater or less than the other.

- Identify the effect on the graph of replacing f(x) by f(x) + k, k f(x), f(kx), and f(x + k) for specific values of k (both positive and negative); find the value of k given the graphs.

- Experiment with cases and illustrate an explanation of the effects on the graph using technology.

- Write recursive and explicit formulas for geometric sequences.

- Construct and compare exponential models and solve problems. Recognize situations in which a quantity either grows or decays by a constant percent rate.

- Recognize even and odd functions from their graphs and algebraic expressions for them.

- Apply related linear equations solution techniques and the laws of exponents to the creation and solution of simple exponential equations.

TEACHER RESOURCES

The following pages include teacher resources that teachers may wish to use to supplement instruction.

- Web Resources
- Compare / Contrast: Exponential Functions
- Graphic Organizer: Graphing Transformations

Web Resources

The following list is provided as a sample of available resources and is for informational purposes only. It is your responsibility to investigate them to determine their value and appropriateness for your district. GADOE does not endorse or recommend the purchase of or use of any particular resource.

The following pages include teacher resources that teachers may wish to use for the purpose of instruction and assessment:

- The Georgia Online Formative Assessment Resource (GOFAR) accessible through SLDS contains test items related to content areas assessed by the Georgia Milestones Assessment System and NAEP. Teachers and administrators can utilize the GOFAR to develop formative and summative assessments, aligned to the state-adopted content standards, to assist in informing daily instruction.

Students, staff, and classes are prepopulated and maintained through the State Longitudinal Data System (SLDS). Teachers and Administrators may view Exemplars and Rubrics in Item Preview. A scoring code may be distributed at a local level to help score constructed response items.

For GOFAR user guides and overview, please visit:
https://www.gadoe.org/Curriculum-Instruction-and-Assessment/Assessment/Pages/Georgia-Online-Formative-Assessment-Resource.aspx

- Illustrative Mathematics
  https://www.illustrativemathematics.org/
  Standards are illustrated with instructional and assessment tasks, lesson plans, and other curriculum resources.

- Illuminations
  http://illuminations.nctm.org
  NCTM resources for teaching mathematics.

Compare / Contrast: Exponential Functions

Show similarities and differences between the two exponent functions:
What things are being compared? How are they similar? How are they different?

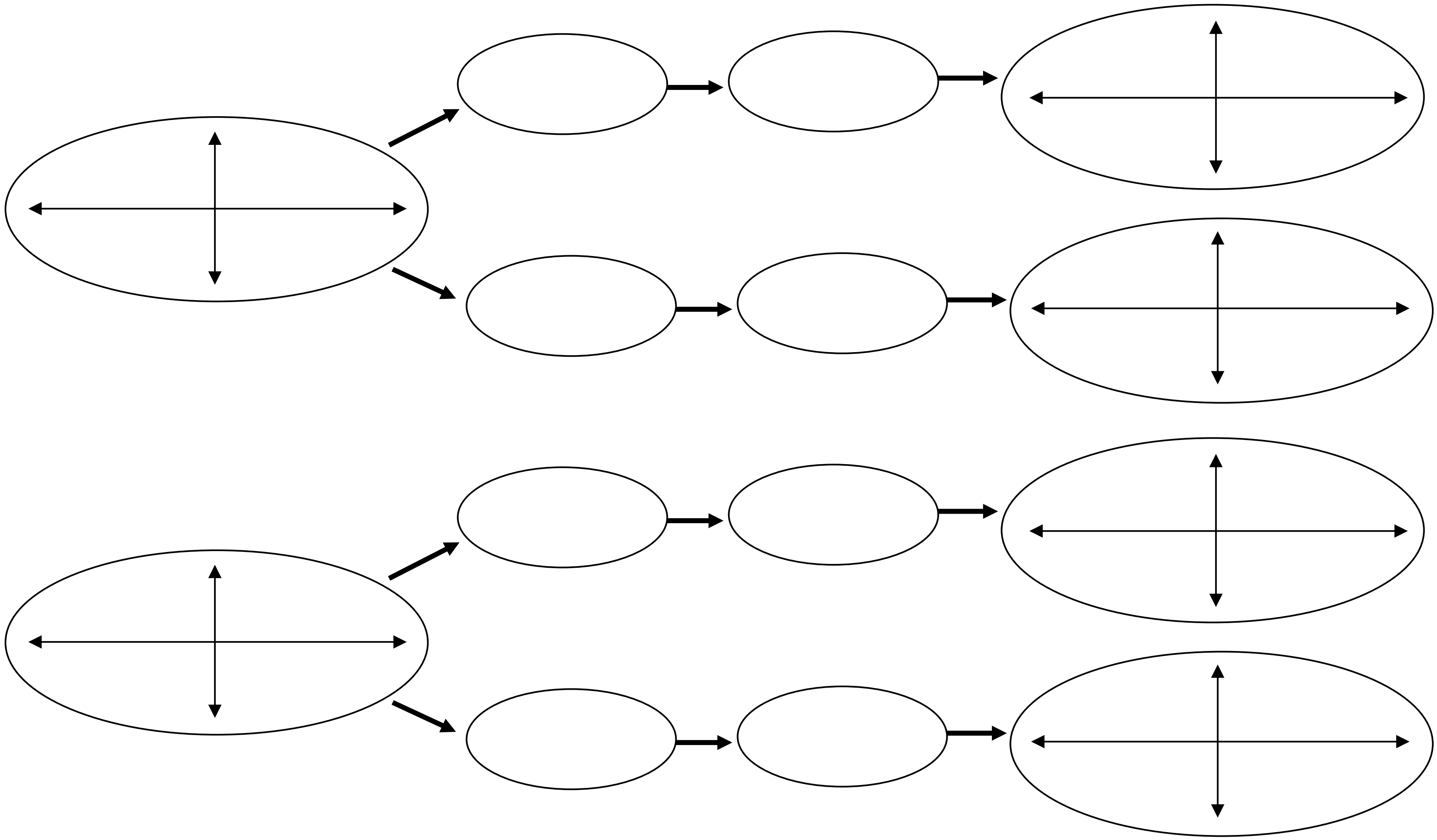
Attribute	Function 1	Function 2
Rate of change		
Domain & Range		
Intercepts		
Asymptotes		
End Behavior		

Example Functions to Graph and Discuss:

$$f(x) = (1/2)^x + 3 \qquad f(x) = 2^x + 3$$

Graphic Organizer: Graphing Transformations

Parent Function and Graph | **Equation** | **Transformation** | **Graph**



Graphic Organizer by Dale Graham and Linda Meyer Thomas County Central High School; Thomasville GA

FORMATIVE ASSESSMENT LESSONS (FAL)

Formative Assessment Lessons are intended to support teachers in formative assessment. They reveal and develop students' understanding of key mathematical ideas and applications. These lessons enable teachers and students to monitor in more detail their progress towards the targets of the standards. They assess students' understanding of important concepts and problem solving performance, and help teachers and their students to work effectively together to move each student's mathematical reasoning forward.

More information on Formative Assessment Lessons may be found in the Comprehensive Course Overview.

SPOTLIGHT TASKS

A Spotlight Task has been added to each CCGPS mathematics unit in the Georgia resources for middle and high school. The Spotlight Tasks serve as exemplars for the use of the Standards for Mathematical Practice, appropriate unit-level Georgia Standards of Excellence, and research-based pedagogical strategies for instruction and engagement. Each task includes teacher commentary and support for classroom implementation. Some of the Spotlight Tasks are revisions of existing Georgia tasks and some are newly created. Additionally, some of the Spotlight Tasks are 3-Act Tasks based on 3-Act Problems from Dan Meyer and Problem-Based Learning from Robert Kaplinsky.

3-ACT TASKS

A Three-Act Task is a whole group mathematics task consisting of 3 distinct parts: an engaging and perplexing Act One, an information and solution seeking Act Two, and a solution discussion and solution revealing Act Three.

More information along with guidelines for 3-Act Tasks may be found in the Comprehensive Course Overview.

TASKS

Task Name *Suggested Time*	**Task Type** *Grouping Strategy*	**Content Addressed**	**Standards**
Paper Folding Task *60-90 min*	Constructing Task *Partners/Groups*	• Modeling real-life situations with exponential functions • Interpret key features of exponential functions in context	A.CED.1, 2 F.IF.1, 2, 4, 5, 6, 7, 7e F.BF.1, 3 A.SSE.1, 1a, 1b
Multiplying Cells *30-45 minutes* PDF	Short Cycle Task *Individual/Partners*	• Use exponential functions to model real-life situations	A.CED.1, 2 F.BF.1, 1a, 2 N.Q.1
The Marvel of Medicine *30-45 minutes*	Constructing Task *Individual/Partners*	• Apply exponential functions to real-life situations • Analyze exponential functions in context	A.CED.1, 2 F.IF.1, 2, 4, 5, 9 F.BF.1, 3 N.Q.1, 2, 3
High Functioning *90-120 minutes*	Practice Task *Individual/Partners*	• Find the value of k given the graphs • Identify even and odd function • Relate vertical translations of a linear function to its y intercept	F.BF.3
How Long Does It Take? *60 min*	Constructing Task *Partners/Small Groups*	• Exploring Exponential Phenomena • Analyzing exponential equations	A.CED.1, 2 F.IF.1, 2, 4, 5, 7, 7e F.BF.1 A.SSE.1, 1a, 1b

Task Name ***Suggested Time***	**Task Type** ***Grouping Strategy***	**Content Addressed**	**Standards**
Growing by Leaps and Bounds Part I and II *60-90 minutes* Part III *90 minutes*	Culminating Task *Individual/Partners*	• Graphing equations on coordinate axes with labels and scales • Interpreting expressions that represent a quantity in terms of its context • Determining constraints • Modeling with exponential functions • Interpreting functions and its key features • Analyzing sequences as functions	A.CED.1, 2 F.BF.1, 1a, 2, 3 F.IF.1, 2, 3, 4, 5, 7, 7e, 9

Paper Folding (Constructing Task)

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Adapted from PBS Mathline: http://www.pbs.org/teachers/mathline/lessonplans/pdf/hsmp/rhinos.pdf

Introduction

Students will use paper folding to model exponential functions. Students will collect data, create scatterplots, and determine algebraic models that represent their functions. Students begin this lesson by collecting data within their groups. They fold a sheet of paper and determine the area of the smallest region after each fold. Next they draw a scatterplot of their data and determine by hand an algebraic model. This investigation allows students to explore the patterns of exponential models in tables, graphs, and symbolic form.

Mathematical Goals

- Write and graph an equation to represent an exponential relationship.
- Model a data set using an equation.
- Choose the best form of an equation to model exponential functions.
- Graph equations on coordinate axes with labels and scales.
- Interpret key features of graphs in context.

Essential Questions

- How do I create equations in two variables to represent relationships between quantities?
- How can I represent patterns algebraically? What types of patterns exist in exponential relationships?
- How do I interpret key features of graphs in context?

GEORGIA STANDARDS OF EXCELLENCE

MGSE9-12.A.CED.1 Create equations and inequalities in one variable and use them to solve problems. Include equations arising from ~~linear, quadratic,~~ ~~simple rational, and~~ exponential functions (integer inputs only).

MGSE9-12.A.CED.2 Create ~~linear, quadratic, and~~ exponential equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. (The phrase "in two or more variables" refers to formulas like the compound interest formula, in which $A = P(1 + r/n)^{nt}$ has multiple variables.)

MGSE9-12.F.IF.1 Understand that a function from one set (the input, called the domain) to another set (the output, called the range) assigns to each element of the domain exactly one element of the range, i.e. each input value maps to exactly one output value. If f is a function, x is the input (an element of the domain), and f(x) is the output (an element of the range). Graphically, the graph is y = f(x).

MGSE9-12.F.IF.2 Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context.

MGSE9-12.F.IF.4 Using tables, graphs, and verbal descriptions, interpret the key characteristics of a function which models the relationship between two quantities. Sketch a graph showing key features including: intercepts; interval where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; ~~and periodicity.~~

MGSE9-12.F.IF.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*

MGSE9-12.F.IF.6 Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.

MGSE9-12.F.IF.7 Graph functions expressed algebraically and show key features of the graph both by hand and by using technology.

MGSE9-12.F.IF.7e Graph exponential ~~and logarithmic~~ functions, showing intercepts and end behavior, ~~and trigonometric functions, showing period, midline, and amplitude.~~

MGSE9-12.F.BF.1 Write a function that describes a relationship between two quantities.

MGSE9-12.F.BF.3 Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k\,f(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. ~~Include recognizing even and odd functions from their graphs and algebraic expressions for them.~~ *(Focus on vertical translations of graphs of linear and exponential functions. Relate the vertical translation of a linear function to its y-intercept.)*

RELATED STANDARDS

MGSE9-12.A.SSE.1 Interpret expressions that represent a quantity in terms of its context.

MGSE9-12.A.SSE.1a Interpret parts of an expression, such as terms, factors, and coefficients, in context.

MGSE9-12.A.SSE.1b Given situations which utilize formulas or expressions with multiple terms and/or factors, interpret the meaning (in context) of individual terms or factors.

STANDARDS FOR MATHEMATICAL PRACTICE

1. Make sense of problems and persevere in solving them.
   Students will have to make several attempts to fold the paper to get the appropriate folds.
4. Model with mathematics.
   Students will create a table, graph, and equation representing the data collected.
8. Look for and express regularity in repeated reasoning.
   Students will recognize a pattern in the number of sections created by each additional fold.

Background Knowledge

- Students can make a table, graph, and equation
- Students can choose an appropriate scale for a graph.
- Students have some knowledge of exponents (e.g., $x^0 = 1$ if $x \neq 0$)

Common Misconceptions

- Students may not understand how to show a fractional "area" of a piece of paper.

Materials

- Graph paper
- Paper for folding

Grouping

- Partners/Groups

Differentiation

Extension:

- Try with different kinds of paper.
- Discuss discrete versus continuous functions.
- Part III of the task.

Intervention:

- Pair with resident experts.
- Do a group demonstration.

Formative Assessment Questions

- How might this problem change with a different kind of paper?
- What are the limiting factors to the activity?
- How might you fold the paper to form a linear sequence?

Paper Folding – Teacher Notes

Part I: Number of Sections

Comments
Students will need guidance in determining a mathematical model that represents the data. This is their first exposure to modeling exponential functions.

# of Folds	# of Sections
0	*1*
1	*2*
2	*4*
3	*8*
4	*16*
5	*32*
6	*64*

1. **1.** Fold an 8.5” × 11” sheet of paper in half and determine the number of sections the paper has after you have made the fold.

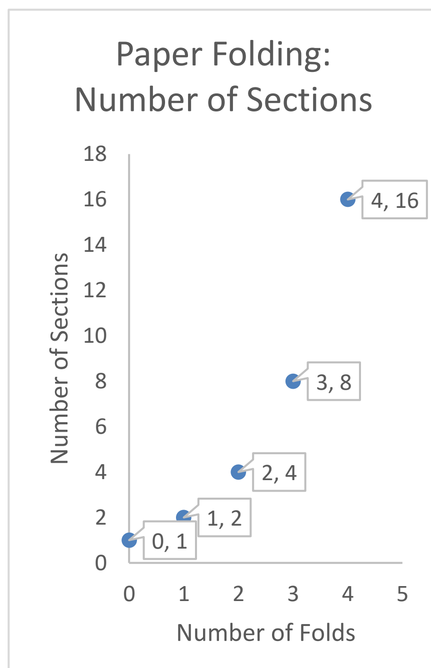
2. **2.** Record this data in the table and continue in the same manner until it becomes too hard to fold the paper.

3. **3.** What might be different if you tried this experiment with an 8.5 x 11” sheet of wax paper or tissue paper?

   Solution
   The results would be exactly the same, but you would be able to make more folds and collect more data because the paper would be thinner.

4. **4.** Identify the independent and dependent variables in the situation. Make a scatter plot of your data in a separate sheet of graph paper. Why do we not connect the dots? What do you notice about the shape of the scatter plot?

   Solution
   Students may provide answers in a variety of ways.
   The number of folds is an independent variable, and the number of sections is a dependent variable. The scatter plot is shown on the right. It does not make sense to connect the dots as you can only make a whole number of folds.
   The graph is shaped as a curve; as x-values increase, the y-values increase at a faster rate.



5. **5. Describe the y-intercept, domain, and range of the function. Note any restrictions, if applicable.**

Solution
The y-intercept is (0, 1), which represents the fact that we start off with one sheet of paper before we begin folding it.
The domain could be described to a set of nonnegative integers (whole numbers): $\{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}, \dots\}$.
It does not make sense to include fractional parts as the number of folds cannot be a fraction. Students will benefit from a discussion on restricting the domain further and answering such questions as, "Could we make infinitely many folds in real life?" "How many folds will it be possible to make in real life?" "Could we restrict the domain to 10, 9, 8, or 7 folds?"
Please see the following Myth Busters video:
https://www.youtube.com/watch?v=kRAEBbotuIE
The range could be described as a set of positive integers (natural numbers): $\{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}, \dots\}$.
It does not make sense to include fractional parts as we start off with one piece of paper and continue counting the number of sections, which cannot be a fractional part in this context. Students will benefit from a discussion on restricting the range further and answering such questions as, "Could we make infinitely many sections of paper in real life?" "How would our range restrictions depend on the domain restrictions above?"

6. Determine a mathematical model that represents this data by examining the patterns in the table.
Solution

$y = 2^x$

7. Calculate and compare the rates of change on the intervals [1, 3] and [4, 6]. What do you notice?
Solution
The rate of change on the interval [1, 3] is $\frac{\mathbf{8-2}}{\mathbf{3-1}} = \mathbf{3}$ ***sections per fold.***

The rate of change on the interval [4, 6] is $\frac{\mathbf{64-16}}{\mathbf{6-4}} = \mathbf{24}$ ***sections per fold.***

As the number of folds increases, the number of sections of paper increases at a faster rate, which explains the shape of the graph – the exponential model.

Part II: Area of Smallest Section

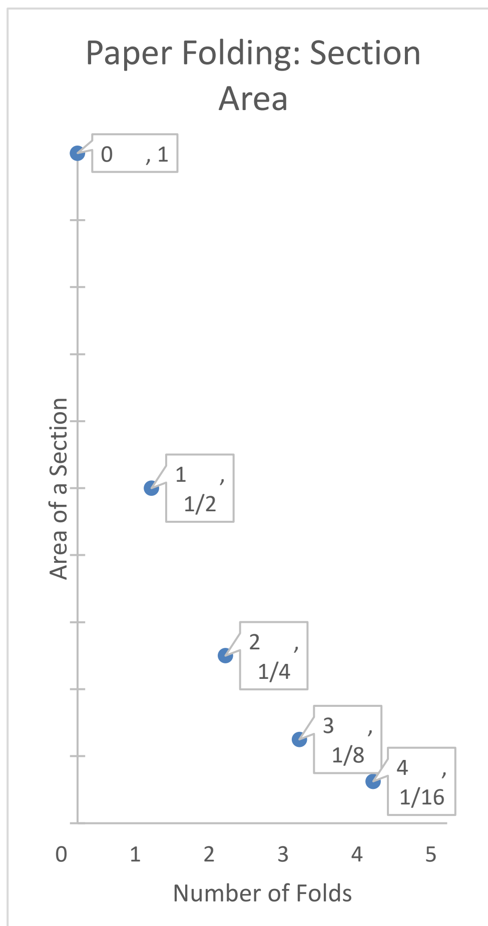
Comments

Students will need guidance in determining a mathematical model that represents the data. This is their first exposure to modeling exponential functions.

8. **Fold an 8.5” × 11” sheet of paper in half and determine the area of *one* of the two sections after you have made the fold.**

9. **Record this data in the table and continue in the same manner until it becomes too hard to fold the paper.**
Solution

# of Folds	Area of Smallest Section
0	**1**
1	***1/2***
2	***1/4***
3	***1/8***
4	***1/16***
5	***1/32***
6	***1/64***



10. **Make a scatter plot of your data on a separate sheet of graph paper. Describe the shape of the scatter plot.**

Solution:* *The scatter plot is shaped like a curve; decreasing. As the x-values increase,

the y-values decrease at a slower rate.

11. Describe the y-intercept, domain, and range of the function. Note any restrictions, if applicable to the context.

Solution

The y-intercept is (0, 1), same as in Part I of the task as we start off with one sheet of paper before we begin folding the paper and counting the area of the sections.

As in part I of the task, the domain of the function can be restricted to a set of nonnegative integers (whole numbers): $\{0, 1, 2, 3, 4, 5, 6, \ldots\}$.

It does not make sense to include fractional parts as the number of folds cannot be a fractional part. Students will benefit from a discussion on restricting the domain further and answering such questions as, "Could we make infinitely many folds in real life?" "How many folds will it be possible to make in real life?" "Could we restrict the domain to 10, 9, 8, or 7 folds?"

The range could be described as a set of positive rational numbers, powers of 1/2: $\left\{\ldots, \frac{1}{64}, \frac{1}{32} . \frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1\right\}$. ***The maximum y-value is 1 as we start folding 1 sheet of paper; the area of a section of paper cannot be negative. Unlike in part I of the task, it does make sense to include fractional parts as we start off with one piece of paper and continue finding the area of each smaller section in relationship to the area of the original one whole piece of paper.***

Students would benefit from a discussion on just how small the area of the smallest section would be in real life and how close the area could approach zero. Relate this discussion to Part I of the task.

12. Determine a mathematical model that represents this data by examining the patterns in the table.

Solution

$y = (1/2)^x$

Part III: Compare and Contrast – Extension

13. How do the scatter plots in Parts I and II of the task compare?

Solution

Both scatter plots are shaped like curves and model exponential situations. The scatter plot in Part I is increasing, which models exponential growth, and the scatter plot in Part II is decreasing, which models exponential decay.
If plotted on the same coordinate plane, it will be easy to see that the scatter plots

are symmetrical about the y-axis (reflected across the y-axis).

14. How do the equations in parts I and II of the task compare?

Solution

Both represent exponential functions. The "doubling" in part I is represented by the ratio (base) of "2" in the equation. This situation represents exponential growth.

In part II, the area of each consequent section is half of its previous area, which is represented by the fraction of "1/2", which is the base of the exponent in the equation.

In science, the exponential decay by a factor of ½ is referred to as "half-life".

If we compare the equations further, we will notice that ½ could be written as 2^{-1}, ***which would change the equation*** $y = 1/2^x$ ***to*** $y = 2^{-x}$.

Upon comparing the equations $y = 2^x$ ***(part I of the task) and*** $y = 2^{-x}$ ***(part II of the task), students may further conclude that the change from a positive exponent "x" to a negative exponent "-x" resulted in a reflection of the graph across the y – axis.***

Paper Folding (Constructing Task)

Name_______________________________ **Date**_________________

Adapted from PBS Mathline: http://www.pbs.org/teachers/mathline/lessonplans/pdf/hsmp/rhinos.pdf

Mathematical Goals

- Write and graph an equation to represent an exponential relationship.
- Model a data set using an equation.
- Choose the best form of an equation to model exponential functions.
- Graph equations on coordinate axes with labels and scales.
- Interpret key features of graphs in context.

Essential Questions

- How do I create equations in two variables to represent relationships between quantities?
- How can I represent patterns algebraically? What types of patterns exist in exponential relationships?
- How do I interpret key features of graphs in context?

GEORGIA STANDARDS OF EXCELLENCE

MGSE9-12.A.CED.1 Create equations and inequalities in one variable and use them to solve problems. Include equations arising from ~~linear, quadratic,~~ ~~simple rational, and~~ exponential functions (integer inputs only).

MGSE9-12.A.CED.2 Create ~~linear, quadratic, and~~ exponential equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. (The phrase "in two or more variables" refers to formulas like the compound interest formula, in which $A = P(1 + r/n)^{nt}$ has multiple variables.)

MGSE9-12.F.IF.1 Understand that a function from one set (the input, called the domain) to another set (the output, called the range) assigns to each element of the domain exactly one element of the range, i.e. each input value maps to exactly one output value. If f is a function, x is the input (an element of the domain), and f(x) is the output (an element of the range). Graphically, the graph is y = f(x).

MGSE9-12.F.IF.2 Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context.

MGSE9-12.F.IF.4 Using tables, graphs, and verbal descriptions, interpret the key characteristics of a function which models the relationship between two quantities. Sketch a graph showing key features including: intercepts; interval where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; ~~and periodicity.~~

MGSE9-12.F.IF.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*

MGSE9-12.F.IF.6 Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.

MGSE9-12.F.IF.7 Graph functions expressed algebraically and show key features of the graph both by hand and by using technology.

> **MGSE9-12.F.IF.7e** Graph exponential ~~and logarithmic~~ functions, showing intercepts and end behavior, ~~and trigonometric functions, showing period, midline, and amplitude.~~

MGSE9-12.F.BF.1 Write a function that describes a relationship between two quantities.

MGSE9-12.F.BF.3 Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k\,f(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. ~~Include recognizing even and odd functions from their graphs and algebraic expressions for them.~~ *(Focus on vertical translations of graphs of linear and exponential functions. Relate the vertical translation of a linear function to its y-intercept.)*

RELATED STANDARDS

MGSE9-12.A.SSE.1 Interpret expressions that represent a quantity in terms of its context.

> **MGSE9-12.A.SSE.1a** Interpret parts of an expression, such as terms, factors, and coefficients, in context.
>
> **MGSE9-12.A.SSE.1b** Given situations which utilize formulas or expressions with multiple terms and/or factors, interpret the meaning (in context) of individual terms or factors.

STANDARDS FOR MATHEMATICAL PRACTICE

1. Make sense of problems and persevere in solving them.
4. Model with mathematics.
8. Look for and express regularity in repeated reasoning.

Paper Folding (Constructing Task)

Name__________________________________ **Date**___________________

Part I: Number of Sections

1. Fold an 8.5” × 11” sheet of paper in half and determine the number of sections the paper has after you have made the fold.

2. Record this data in the table and continue in the same manner until it becomes too hard to fold the paper.

# of Folds	# of Sections
0	
1	
2	
3	
4	
5	
6	

3. What might be different if you tried this experiment with an 8.5 x 11” sheet of wax paper or tissue paper?

4. Identify the independent and dependent variables in the situation. Make a scatter plot of your data in a separate sheet of graph paper. Why do we not connect the dots? What do you notice about the shape of the scatter plot?

5. Describe the y-intercept, domain, and range of the function. Note any restrictions, if applicable.

6. Determine a mathematical model that represents this data by examining the patterns in the table.

7. Calculate and compare the rates of change on the intervals [1, 3] and [4, 6]. What do you notice?

Part II: Area of Smallest Section

8. Fold an 8.5” × 11” sheet of paper in half and determine the area of *one* of the two sections after you have made the fold.

9. Record this data in the table and continue in the same manner until it becomes too hard to fold the paper.

10. Make a scatter plot of your data on a separate sheet graph paper. Describe the shape of the scatter plot.

# of Folds	Area of Smallest Section
0	1
1	
2	
3	
4	
5	
6	

11. Describe the y-intercept, domain, and range of the function. Note any restrictions, if applicable to the context.

12. Determine a mathematical model that represents this data by examining the patterns in the table.

Part III: Compare and Contrast – Extension

13. How do the scatter plots in Parts I and II of the task compare?

14. How do the equations in parts I and II of the task compare?

Multiplying Cells (Short Cycle Task)

Back to Task Table

Source: Formative Assessment Lesson Materials from Mathematics Assessment Project
http://www.map.mathshell.org/materials/download.php?fileid=788

Task Comments and Introduction

Tasks and lessons from the Mathematics Assessment Project are specifically designed to help teachers effectively formatively assess their students. The way the tasks and lessons are designed gives the teacher a clear understanding of what the students are able to do and not do. Within the lesson, teachers will find suggestions and question prompts that will help guide students towards understanding. For more information access the MAP website:
http://www.map.mathshell.org/materials/background.php?subpage=summative

The task, *Multiplying Cells*, is a Mathematics Assessment Project Assessment Task that can be found at the website:
http://www.map.mathshell.org/materials/tasks.php?taskid=268&subpage=apprentice
The PDF version of the task can be found at the link below:
http://www.map.mathshell.org/materials/download.php?fileid=788
The scoring rubric can be found at the following link:
http://www.map.mathshell.org/materials/download.php?fileid=789

Mathematical Goals

- Use exponential functions to model real-world situations.

Essential Questions

- How can I use exponential functions to model real-world situations?

GEORGIA STANDARDS OF EXCELLENCE

MGSE9-12.A.CED.1 Create equations and inequalities in one variable and use them to solve problems. Include equations arising from ~~linear, quadratic, simple rational, and~~ exponential functions (integer inputs only).

MGSE9-12.A.CED.2 Create ~~linear, quadratic, and~~ exponential equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. (The phrase "in two or more variables" refers to formulas like the compound interest formula, in which $A = P(1 + r/n)^{nt}$ has multiple variables.)

MGSE9-12.F.BF.1 Write a function that describes a relationship between two quantities.

MGSE9-12.F.BF.1a Determine an explicit expression and the recursive process (steps for calculation) from context. *For example, if Jimmy starts out with $15 and earns $2 a day, the explicit expression "2x+15" can be described recursively (either in writing or verbally) as "to find out how much money Jimmy will have tomorrow, you add $2 to his total today."* $J_n = J_{n-1} + 2, J_0 = 15$

MGSE9-12.F.BF.2 Write ~~arithmetic and~~ geometric sequences recursively and explicitly, use them to model situations, and translate between the two forms. Connect ~~arithmetic sequences to linear functions and~~ geometric sequences to exponential functions.

<u>RELATED STANDARDS</u>

MGSE9-12.N.Q.1 Use units of measure (linear, area, capacity, rates, and time) as a way to understand problems:

a. Identify, use, and record appropriate units of measure within context, within data displays, and on graphs;
b. Convert units and rates using dimensional analysis (English-to-English and Metric-to-Metric without conversion factor provided and between English and Metric with conversion factor);
c. Use units within multi-step problems and formulas; interpret units of input and resulting units of output.

<u>STANDARDS FOR MATHEMATICAL PRACTICE</u>

3. Construct viable arguments and critique the reasoning of others.
7. Look for and make use of structure.

Background Knowledge

- Students can work with exponents.
- Students recognize exponential relationships.

Common Misconceptions

- Students may think about sequences recursively but incorrectly write their pattern as explicit formulas.
- Students may interpret 2^3 as $2 \cdot 3$, or they may believe the growth is linear.

Materials

- see FAL website

Grouping

- Individual/Partners

The Marvel of Medicine (Constructing Task)

Back to Task Table

Introduction

The source of the task is the Charles A. Dana Center at The University of Texas at Austin (2011). Algebra Assessments through the Common Core (grades 6-12).

Comments

Though the graphing calculator is not required from this task, students' prior knowledge of how to use a graphing calculator would be beneficial.

Scaffolding Questions

- How much is the initial dosage?
- What percentage of the medicine is left in the body after 1 hour? Express this percentage as a decimal.
- How is the amount of medicine in the body at 2 hours related to the amount at 1 hour?
- What are the variables in this situation?
- How do the variables change in relation to each other?
- How would a graph help you understand the problem situation?

Mathematical Goals

- Apply exponential functions to real-life situations
- Analyze exponential functions in context

Essential Questions

- How do we interpret exponential functions in context?
- How do we use exponential functions to represent real-life situations?

GEORGIA STANDARDS OF EXCELLENCE

MGSE9-12.A.CED.1 Create equations and inequalities in one variable and use them to solve problems. Include equations arising from ~~linear, quadratic, simple rational, and~~ exponential functions (integer inputs only).

MGSE9-12.A.CED.2 Create ~~linear, quadratic, and~~ exponential equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. (The phrase "in two or more variables" refers to formulas like the compound interest formula, in which $A = P(1 + r/n)^{nt}$ has multiple variables.)

MGSE9-12.F.IF.1 Understand that a function from one set (the input, called the domain) to another set (the output, called the range) assigns to each element of the domain exactly one element of the range, i.e. each input value maps to exactly one output value. If f is a function, x is the input (an element of the domain), and f(x) is the output (an element of the range). Graphically, the graph is y = f(x).

MGSE9-12.F.IF.2 Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context.

MGSE9-12.F.IF.4 Using tables, graphs, and verbal descriptions, interpret the key characteristics of a function which models the relationship between two quantities. Sketch a graph showing key features including: intercepts; interval where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; ~~and periodicity.~~

MGSE9-12.F.IF.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*

MGSE9-12.F.IF.9 Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one function and an algebraic expression for another, say which has the larger maximum.*

MGSE9-12.F.BF.1 Write a function that describes a relationship between two quantities.

MGSE9-12.F.BF.3 Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k\,f(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. ~~Include recognizing even and odd functions from their graphs and algebraic expressions for them.~~ *(Focus on vertical translations of graphs of linear and exponential functions. Relate the vertical translation of a linear function to its y-intercept.)*

RELATED STANDARDS

MGSE9-12.N.Q.1 Use units of measure (linear, area, capacity, rates, and time) as a way to understand problems:

a. Identify, use, and record appropriate units of measure within context, within data displays, and on graphs;
b. Convert units and rates using dimensional analysis (English-to-English and Metric-to-Metric without conversion factor provided and between English and Metric with conversion factor);
c. Use units within multi-step problems and formulas; interpret units of input and resulting units of output.

MGSE9-12.N.Q.2 Define appropriate quantities for the purpose of descriptive modeling. Given a situation, context, or problem, students will determine, identify, and use appropriate quantities for representing the situation.

MGSE9-12.N.Q.3 Choose a level of accuracy appropriate to limitations on measurement when reporting quantities. *For example, money situations are generally reported to the nearest cent (hundredth). Also, an answers' precision is limited to the precision of the data given.*

STANDARDS FOR MATHEMATICAL PRACTICE

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Materials Needed:

- Graphing or scientific calculator

Grouping

- Individual/Partners

The Marvel of Medicine (Constructing Task) - Teacher Notes

Part I. A doctor prescribes 400 milligrams of medicine to treat an infection. Each hour following the initial dose, 85% of the concentration remains in the body from the preceding hour.

1. Complete the table showing the amount of medicine remaining after each hour.

Number of Hours	Process	Number of Milligrams Remaining in the Body
0	400	400
1	400 (0.85)	340
2	400 (0.85)(0.85)	
3		
4		
5		

Solutions
Repeated multiplication by 0.85 was used to complete the table.

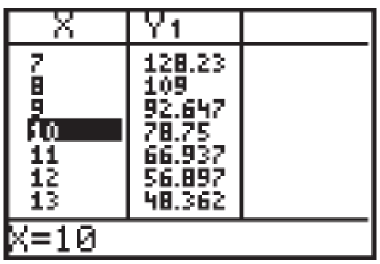
Number of Hours	Process	Number of Milligrams Remaining in the Body
0	400	400
1	400 (0.85) $400(0.85)^1$	340
2	400 (0.85)(0.85) $400(0.85)^2$	***289***
3	***400 (0.85)(0.85)(0.85)*** $400(0.85)^3$	***245.65***
4	***400 (0.85)(0.85)(0.85)(0.85)*** $400(0.85)^4$	***208.8***
5	***400 (0.85)(0.85)(0.85)(0.85)(0.85)*** $400(0.85)^5$	***177.48***

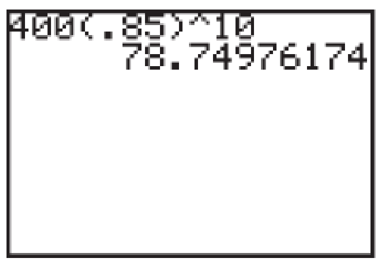
2. Using symbols and words, describe the functional relationship in this situation. Discuss the domain and range of both the function rule and the problem situation.

Solutions
The function that models this situation is $y = 400(0.85)^x$. The x-values represent the number of hours the medicine is in the patient's system. The y-values represent the amount of medicine (in milligrams) that remains in the patient's system. Each hour, 15% of the medicine filters through the body, leaving 85% of the medicine in the body to fight the infection. A scatterplot can be used to analyze the data.

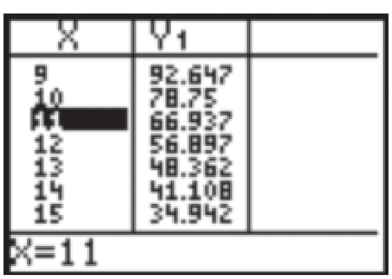
The domain of the function is the set of all real numbers, but the domain of the problem situation is the set of nonnegative numbers because x represents time in this situation. The range of the function is the set of all real numbers greater than 0, but the range for the problem situation is the set of all numbers less than or equal to 400 but greater than 0.

3. Determine the amount of medicine left in the body after 10 hours. Justify your answer in two ways.
Solutions
After 10 hours, there will be 78.75 milligrams left in the body. The calculator can be used in two ways to explore the amount of medicine left in the patient's system after different numbers of hours. Students could input the function into y = and look at the table, or they could use the calculator to evaluate the algebraic expression at 10 hours.

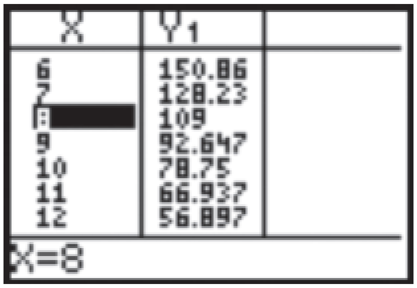




4. When does the amount of medicine still in the body reach 60 milligrams? Explain how you know.
Use a table to find when the dose of medicine is at 60 milligrams. The patient will have 60 milligrams left in his system between 11 and 12 hours after the initial dosage.



5. Suppose that the level of medicine in the patient's body must maintain a level greater than 100 milligrams. How often does the patient need to take the medicine?
Solutions
The patient needs to take the medicine every 8 hours.



X	Y1
6	150.86
7	128.23
8	109
9	92.647
10	78.75
11	66.937
12	56.897

X=8

Part II. What if?

6. If the rule had been $y = 500(0.85)^x$ instead of $y = 400(0.85)^x$, how would this situation be different from the given situation?
   Solutions
   The initial dose of the medicine changes from 400 to 500 milligrams.

7. What would the equation be if the concentration of medicine decreased by 30% each hour?
   Solutions
   If the concentration was reduced by 30% each hour, then 70% of the original dose would be left in the patient's body after 1 hour. If the original dose was 400 milligrams, the equation would be $y = 400(0.70)^x$

8. If the patient took a second 400-milligram dose at the twelfth hour, how much medicine would the patient have in his system at the fifteenth hour?
   Solutions
   The amount of medicine in the patient's system at the fifteenth hour is approximately 280.59 milligrams.

The Marvel of Medicine (Constructing Task)

Name__________________________ Date____________________

Mathematical Goals

- Apply exponential functions to real-life situations
- Analyze exponential functions in context

Essential Questions

- How do we interpret exponential functions in context?
- How do we use exponential functions to represent real-life situations?

GEORGIA STANDARDS OF EXCELLENCE

MGSE9-12.A.CED.1 Create equations and inequalities in one variable and use them to solve problems. Include equations arising from ~~linear, quadratic,~~ ~~simple rational, and~~ exponential functions (integer inputs only).

MGSE9-12.A.CED.2 Create ~~linear, quadratic, and~~ exponential equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. (The phrase "in two or more variables" refers to formulas like the compound interest formula, in which $A = P(1 + r/n)^{nt}$ has multiple variables.)

MGSE9-12.F.IF.1 Understand that a function from one set (the input, called the domain) to another set (the output, called the range) assigns to each element of the domain exactly one element of the range, i.e. each input value maps to exactly one output value. If f is a function, x is the input (an element of the domain), and f(x) is the output (an element of the range). Graphically, the graph is y = f(x).

MGSE9-12.F.IF.2 Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context.

MGSE9-12.F.IF.4 Using tables, graphs, and verbal descriptions, interpret the key characteristics of a function which models the relationship between two quantities. Sketch a graph showing key features including: intercepts; interval where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; ~~and periodicity.~~

MGSE9-12.F.IF.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*

MGSE9-12.F.IF.9 Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one function and an algebraic expression for another, say which has the larger maximum.*

MGSE9-12.F.BF.1 Write a function that describes a relationship between two quantities.

MGSE9-12.F.BF.3 Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k\,f(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. ~~Include recognizing even and odd functions from their graphs and algebraic expressions for them.~~ *(Focus on vertical translations of graphs of linear and exponential functions. Relate the vertical translation of a linear function to its y-intercept.)*

RELATED STANDARDS

MGSE9-12.N.Q.1 Use units of measure (linear, area, capacity, rates, and time) as a way to understand problems:

a. Identify, use, and record appropriate units of measure within context, within data displays, and on graphs;
b. Convert units and rates using dimensional analysis (English-to-English and Metric-to-Metric without conversion factor provided and between English and Metric with conversion factor);
c. Use units within multi-step problems and formulas; interpret units of input and resulting units of output.

MGSE9-12.N.Q.2 Define appropriate quantities for the purpose of descriptive modeling. Given a situation, context, or problem, students will determine, identify, and use appropriate quantities for representing the situation.

MGSE9-12.N.Q.3 Choose a level of accuracy appropriate to limitations on measurement when reporting quantities. *For example, money situations are generally reported to the nearest cent (hundredth). Also, an answers' precision is limited to the precision of the data given.*

STANDARDS FOR MATHEMATICAL PRACTICE

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

<u>The Marvel of Medicine (Constructing Task)</u>

Name__________________________ Date_____________________

Part I. A doctor prescribes 400 milligrams of medicine to treat an infection. Each hour following the initial dose, 85% of the concentration remains in the body from the **preceding hour.**

1. Complete the table showing the amount of medicine remaining after each hour.

Number of Hours	Process	Number of Milligrams Remaining in the Body
0	400	400
1	400 (0.85)	340
2	400 (0.85)(0.85)	
3		
4		
5		

2. Using symbols and words, describe the functional relationship in this situation. Discuss the domain and range of both the function rule and the problem situation.

3. Determine the amount of medicine left in the body after 10 hours. Justify your answer in two ways.

4. When does the amount of medicine still in the body reach 60 milligrams? Explain how you know.

5. Suppose that the level of medicine in the patient's body must maintain a level greater than 100 milligrams. How often does the patient need to take the medicine?

Part II. What if?

6. If the rule had been $y = 500(0.85)^x$ instead of $y = 400(0.85)^x$, how would this situation be different from the given situation?

7. What would the equation be if the concentration of medicine decreased by 30% each hour?

8. If the patient took a second 400-milligram dose at the twelfth hour, how much medicine would the patient have in his system at the fifteenth hour?

High Functioning! (Practice Task)

Back to Task Table

Introduction

Students are introduced to transformations of functions. In this task, students will focus on vertical translations of graphs of linear and make connections to the *y*-intercept. Transformations are approached from the perspective of using a constant, *k*, to make changes to the function. They will be able to answer questions such as: What happens when you add *k* to the input? What happens when *k* is a negative number?

Mathematical Goals

- Use graphs of vertical translations to determine function rules.
- Relate vertical translations of a linear function to its *y*-intercept.
- Recognize even and odd functions from their graphs and algebraic expressions for them.

Essential Questions

- How are functions affected by adding or subtracting a constant to the function?
- How does the vertical translation of a linear function model translations for other functions?

GEORGIA STANDARDS OF EXCELLENCE

MGSE9-12.F.BF.3 Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k\,f(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. ~~Include recognizing even and odd functions from their graphs and algebraic expressions for them.~~ *(Focus on vertical translations of graphs of linear and exponential functions. Relate the vertical translation of a linear function to its y-intercept.)*

STANDARDS FOR MATHEMATICAL PRACTICE

3. Construct viable arguments and critique the reasoning of others.
Students will make predictions based on observations of shifts and defend their reasoning.

6. Attend to precision.
Students will need to graph equations precisely and use specific points to show the shifts.

7. Look for and express regularity in repeated reasoning.
Students will look for patterns and use them to determine general rules.

Background Knowledge

- Students can graph linear equations.
- Students can read, write, and interpret function notation.
- Students need to have a basic understanding of translations of geometric figures.
- Students need to have a basic understanding of rotations of geometric figures.

Common Misconceptions

- Students may make very general/tangential observations and need to be directed.
- Students may not understand function notation well enough to write rules.

Materials

- Graphing Calculator (optional)

Grouping

- Partner / Individual

Differentiation

Extension:

- Students can use a graphing calculator to model more examples of linear and exponential functions and see the transformations that occur.

Intervention:

- Use strategic grouping to pair struggling students with resident experts.

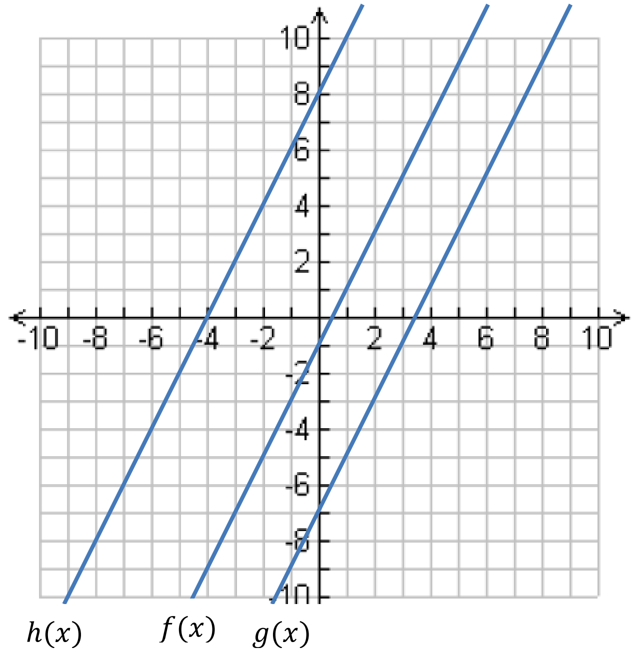
Formative Assessment Questions

- How are translations of linear functions related to the y-intercept?
- How can we determine the relative location of a function given a translation in function notation?

High Functioning – Teacher Notes

1. Graph and label the following functions.

   $f(x) = 2x - 1$
   $g(x) = 2x - 7$
   $h(x) = 2x + 8$



2. What observations can you make about the three functions? Be sure to include observations about the characteristics and the location of the functions.

   Solution:
   Answers may vary. Students should not that the slopes of the functions are the same and that the lines are parallel. They should see that the y-intercept has changed in each function causing the function to move up and down.

3. Analyze specifically what happens to the y-intercepts $f(0)$, $g(0)$, and $h(0)$ in the three functions.
   How does the *y*-intercept change from…
   a. $f \rightarrow g$?
      Solution:
      from -1 to -7; down 6

   b. $g \rightarrow h$?
      Solution:
      from -7 to +8; up 15

   c. $f \rightarrow h$?
      Solution:
      from -1 to +8; up 9

   d. $h \rightarrow f$?
      Solution:
      from +8 to -1; down 9

4. Find…

a. $f(1)$
Solution:
2(1) – 1 = 1

b. $g(1)$
Solution:
2(1) – 7 = -5

c. $h(1)$
Solution:
2(1) + 8 = 10

5. What changes in the output as you go from…
   a. $f(1) \rightarrow g(1)$?
   Solution:
   from 1 to -5; down 6

   b. $g(1) \rightarrow h(1)$?
   Solution:
   from -5 to 10; up 15

   c. $f(1) \rightarrow h(1)$?
   Solution:
   from 1 to 10; up 9

   d. $h(1) \rightarrow f(1)$?
   Solution:
   from 10 to 1; down 9

6. Comparing your answers to 3 and 5, what predictions can you make about other inputs?

   Solution:
   The shifts in #3 (input x = 0) and #5 (input x = 1) are the same. We can predict that the shifts will be the same regardless of input.

7. Write an algebraic rule for the following shifts.
   a. $f(x) \rightarrow g(x)$
      Solution:
      $g(x) = f(x) - 6$

   b. $g(x) \rightarrow h(x)$
      Solution:
      $h(x) = g(x) + 15$

   c. $f(x) \rightarrow h(x)$
      Solution:
      $h(x) = f(x) + 9$

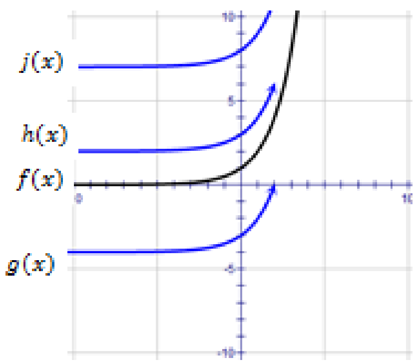
   d. $h(x) \rightarrow f(x)$
      Solution:
      $f(x) = h(x) - 9$

8. Write a general rule for a vertical translation.

   Solution:
   Answers may vary. A vertical translation is a shift up or down on the coordinate plane caused by the addition (or subtraction) of a constant to the function.

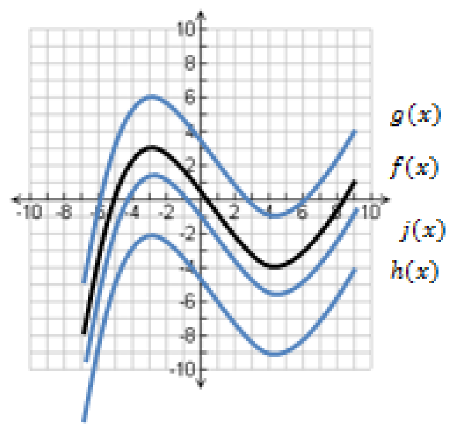
Using the functions below, draw and label the given translations.

9.



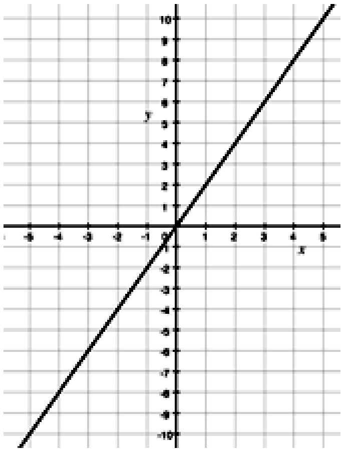
   a. $g(x) = f(x) - 4$
   b. $h(x) = f(x) + 2$
   c. $j(x) = f(x) + 7$

10.



   a. $g(x) = f(x) + 3$
   b. $h(x) = f(x) - 5$
   c. $j(x) = f(x) - 2$

11. The graph of the **odd function** $f(x) = 2x$ is shown below. Fill in the table.



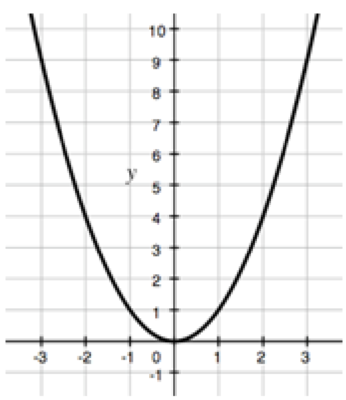
f (-1) = ***- 2***	***f (1) =*** ***2***
f (-4) = ***- 8***	***f (4) =*** ***8***
f (-8) = ***- 16***	***f (8) = 16***
f (-25) = ***- 50***	***f (25) =*** ***50***

12. What characteristics do you notice about odd functions based on the points in the table?
Solution: ***Opposite inputs give opposite outputs.***

13. Now rotate your paper 180˚ so that the graph is upside down. What further observation can you make about characteristics of odd functions?

Solution: ***Odd functions have rotational symmetry about the origin. Since f(x) is a linear function passing through the origin, its image looks the same after the graph is rotated so the graph is upside down.***

14. The graph of the **even function** $g(x) = x^2$ is shown below. Fill in the table.



f (-1) = ***1***	***f (1) =*** ***1***
f (-2) = ***4***	***f (2) =*** ***4***
f (-3) = ***9***	***f (3) =*** ***9***
f (-4) = ***16***	***f (4) =*** ***16***

15. What characteristics do you notice about even functions based on the table?

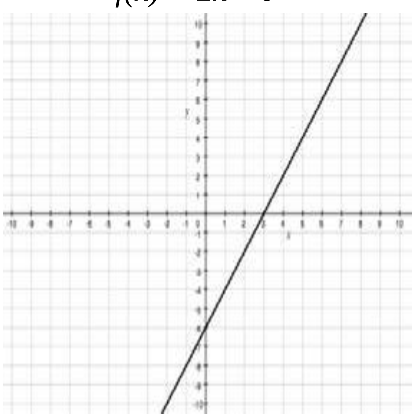
Solution:
Opposite inputs give the same output.

16. Fold the graph of $g(x)$ along the y-axis. What further observations can you make about the characteristics of even functions?

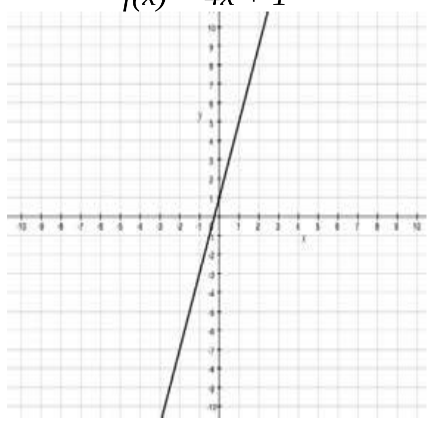
Solution:
Even functions have symmetry over the y-axis. If an even function is reflected over the y-axis it will land back on itself.

17. Graph the three functions below and explain why they are neither even nor odd.

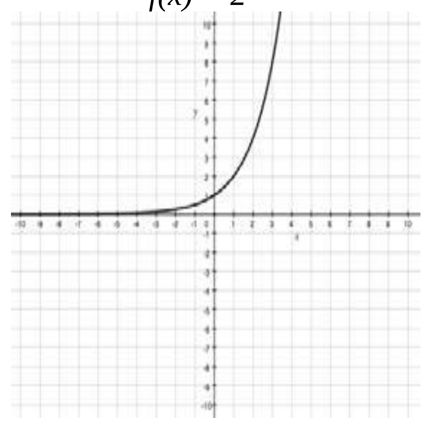
$f(x) = 2x – 6$



$f(x) = 4x + 1$



$f(x) = 2^x$



18. Now demonstrate algebraically that the three functions are neither even nor odd by using the inputs 2 and -2.

Solution

f (2) = 2(2) – 6 = -2
f(-2) = 2(-2) – 6 = -10
These are neither the same nor opposite.

g(2) = 4(2) + 1 = 9
g(-2) = 4(-2) + 1 = -7
These are neither the same nor opposite.

h(2) = 2^2 = 4
h(-2) = 2^{-2} = ¼
These are neither the same nor opposite.

High Functioning (Practice Task)

Name________________________________ **Date**_________________

Mathematical Goals

- Use graphs of vertical translations to determine function rules.
- Relate vertical translations of a linear function to its *y*-intercept.
- Recognize even and odd functions from their graphs and algebraic expressions for them.

Essential Questions

- How are functions affected by adding or subtracting a constant to the function?
- How does the vertical translation of a linear function model translations for other functions?

GEORGIA STANDARDS OF EXCELLENCE

MGSE9-12.F.BF.3 Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k\,f(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. ~~Include recognizing even and odd functions from their graphs and algebraic expressions for them.~~ *(Focus on vertical translations of graphs of linear and exponential functions. Relate the vertical translation of a linear function to its y-intercept.)*

STANDARDS FOR MATHEMATICAL PRACTICE

3. Construct viable arguments and critique the reasoning of others.
6. Attend to precision.
7. Look for and express regularity in repeated reasoning.

High Functioning (Practice Task)

Name__________________________________ **Date**__________________

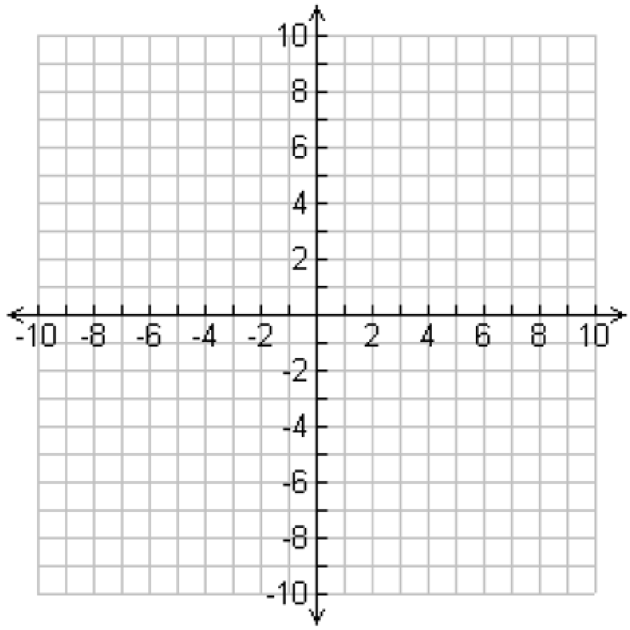
1. Graph and label the following functions.

   $f(x) = 2x - 1$

   $g(x) = 2x - 7$

   $h(x) = 2x + 8$

2. What observations can you make about the three functions? Be sure to include observations about the characteristics and the location of the functions.



3. Analyze specifically what happens to the y-intercepts $f(0)$, $g(0)$, and $h(0)$ in the three functions.
   How does the *y*-intercept change from…
   a. $f \rightarrow g$?

   b. $g \rightarrow h$?

   c. $f \rightarrow h$?

   d. $h \rightarrow f$?

4. Find…
   a. $f(1)$

   b. $g(1)$

   c. $h(1)$

5. What changes in the output as you go from…
   a. $f(1) \rightarrow g(1)$?

   b. $g(1) \rightarrow h(1)$?

   c. $f(1) \rightarrow h(1)$?

   d. $h(1) \rightarrow f(1)$?

6. Comparing your answers to 3 and 5, what predictions can you make about other inputs?

7. Write an algebraic rule for the following shifts.
   a. $f(x) \rightarrow g(x)$

   b. $g(x) \rightarrow h(x)$

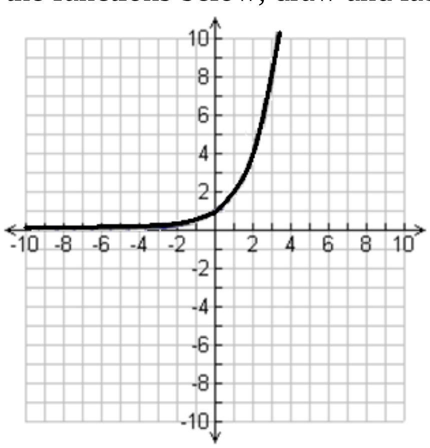
   c. $f(x) \rightarrow h(x)$

   d. $h(x) \rightarrow f(x)$

8. Write a general rule for a vertical translation.

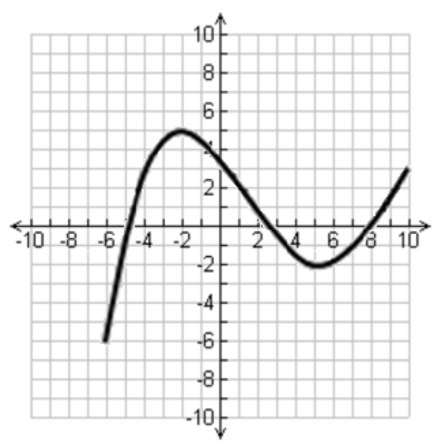
Using the functions below, draw and label the given translations.

9.



a. $g(x) = f(x) - 4$
b. $h(x) = f(x) + 2$
c. $j(x) = f(x) + 7$

10.



a. $g(x) = f(x) + 3$
b. $h(x) = f(x) - 5$
c. $j(x) = f(x) - 2$

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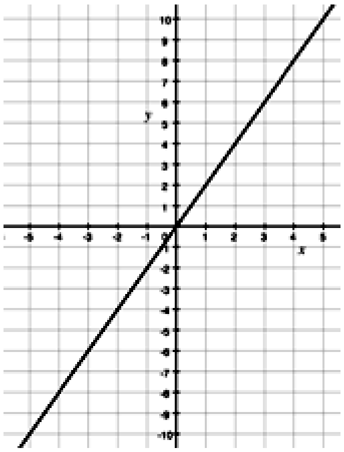
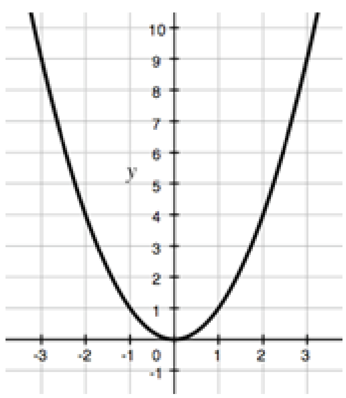


f (-1) =	***f (1) =***
f (-4) =	***f (4) =***
f (-8) =	***f (8) =***
f (-25) =	***f (25) =***

12. What characteristics do you notice about odd functions based on the points in the table?

13. Now rotate your paper 180˚ so that the graph is upside down. What further observation can you make about characteristics of odd functions?

14. The graph of the **even function** $g(x) = x^2$ is shown below. Fill in the table.



f (-1) =	***f (1) =***
f (-2) =	***f (2) =***
f (-3) =	***f (3) =***
f (-4) =	***f (4) =***

15. What characteristics do you notice about even functions based on the table?

16. Fold the graph of $g(x)$ along the y-axis. What further observations can you make about the characteristics of even functions?

17. Graph the three functions below and explain why they are neither even nor odd.

$f(x) = 2x - 6$ $\qquad$ $f(x) = 4x + 1$ $\qquad$ $f(x) = 2^x$

18. Now demonstrate algebraically that the three functions are neither even nor odd by using the inputs 2 and -2.

How Long Does It Take? (Constructing Task)

Back to Task Table

Mathematical Goals

- Explore exponential phenomena
- Analyze exponential equations

Essential Questions:

- How do we use exponential functions to represent real-life phenomena?
- How do we analyze exponential equations?

GEORGIA STANDARDS OF EXCELLENCE

MGSE9-12.A.CED.1 Create equations and inequalities in one variable and use them to solve problems. Include equations arising from ~~linear, quadratic,~~ ~~simple rational, and~~ exponential functions (integer inputs only).

MGSE9-12.A.CED.2 Create ~~linear, quadratic, and~~ exponential equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. (The phrase "in two or more variables" refers to formulas like the compound interest formula, in which $A = P(1 + r/n)^{nt}$ has multiple variables.)

MGSE9-12.F.IF.1 Understand that a function from one set (the input, called the domain) to another set (the output, called the range) assigns to each element of the domain exactly one element of the range, i.e. each input value maps to exactly one output value. If f is a function, x is the input (an element of the domain), and f(x) is the output (an element of the range). Graphically, the graph is y = f(x).

MGSE9-12.F.IF.2 Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context.

MGSE9-12.F.IF.4 Using tables, graphs, and verbal descriptions, interpret the key characteristics of a function which models the relationship between two quantities. Sketch a graph showing key features including: intercepts; interval where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; ~~and periodicity.~~

MGSE9-12.F.IF.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*

MGSE9-12.F.IF.7 Graph functions expressed algebraically and show key features of the graph both by hand and by using technology.

MGSE9-12.F.IF.7e Graph exponential ~~and logarithmic~~ functions, showing intercepts and end behavior, ~~and trigonometric functions, showing period, midline, and amplitude.~~

MGSE9-12.F.BF.1 Write a function that describes a relationship between two quantities.

RELATED STANDARDS

MGSE9-12.A.SSE.1 Interpret expressions that represent a quantity in terms of its context.

> **MGSE9-12.A.SSE.1a I**nterpret parts of an expression, such as terms, factors, and coefficients, in context.
>
> **MGSE9-12.A.SSE.1b** Given situations which utilize formulas or expressions with multiple terms and/or factors, interpret the meaning (in context) of individual terms or factors.

STANDARDS FOR MATHEMATICAL PRACTICE

2. Reason abstractly and quantitatively by requiring students to make sense of quantities and their relationships to one another in problem situations.
3. Construct viable arguments and critique the reasoning of others by engaging students on discussion of why they agree or disagree with responses, decide whether they make sense, and ask useful questions to clarify or improve the arguments.
7. Look for and make use of structure by expecting students to apply rules, look for patterns and analyze structure.

Materials Needed:

- Graphing Calculator
- Graph paper

How Long Does It Take? (Constructing Task) – Teacher Notes

Before sending astronauts to investigate the new planet of Exponentia, NASA decided to run a number of tests on the astronauts.

1. A specific multi-vitamin is eliminated from an adult male's bloodstream at a rate of about 20% per hour. The vitamin reaches peak level in the bloodstream of 300 milligrams.

    a. How much of the vitamin remains 2 hours after the peak level? 5 hours after the peak level? Make a table of values to record your answers. Write expressions for how you obtain your answers.

Solution

Time (hours) since peak	**0**	**1**	**2**	**3**	**4**	**5**
Vitamin concentration in bloodstream (mg)	**300**	**240**	**192**	**153.6**	**122.88**	**98.304**

300*(1 -.2) = 240; 240 * .8 = 192; 192 * .8 = 153.6; 153.6 * .8 = 122.88; 122.88 * .8 = 98.304

    b. Using your work from (a), write expressions for each computed value using the original numbers in the problem (300 and 20%). For example, after 2 hours, the amount of vitamin left is [300 * (1 - .2)]* (1 - .2).

Solution:

After 1 hour: 300 * (1 - .2) = 240; after 2 hours: $300 * (1 - .2)^2 = 192$;

After 3 hours: $300 * (1 - .2)^3 = 153.6$; after 4 hours: $300 * (1 - .2)^4 = 122.88$;

After 5 hours: $300 * (1 - .2)^5 = 98.304$

    c. Using part (b), write a function for the vitamin level with respect to the number of hours after the peak level, *x*.

Solution

$f(x) = 300(1 - .2)^x$ or $f(x) = 300(.8)^x$

d. How would use the function you wrote in (c) to answer the questions in (a)? Use your function and check to see that you get the same answers as you did in part (a).

Solution

$f(0) = 300(.8)^0 = 300$

$f(1) = 300(.8)^1 = 240$

$f(2) = 300(.8)^2 = 192$

$f(3) = 300(.8)^3 = 153.6$

$f(4) = 300(.8)^4 = 122.88$

$f(5) = 300(.8)^5 = 98.304$

e. After how many hours will there be less than 10 mg of the vitamin remaining in the bloodstream? Explain how you would determine this answer using both a table feature and a graph.

Solution

Between 15 and 16 hours after peak, the concentration dips below 10 mg. Using the table feature, at 15 hours, the concentration is 10.555 mg whereas at 16 hours, it is 8.4442. Using a graph, we can trace the graph to determine where the vitamin concentration dips below 10 mg. This occurs around 15.3 hours.

f. Write an equation that you could solve to determine when the vitamin concentration is exactly 10 mg. Could you use the table feature to solve this equation? Could you use the graph feature? How could you use the intersection feature of your calculator? Solve the problem using one of these methods.

Solution

$300(.8)^x = 10$***; You could use the table feature to approximate the solution by making smaller and smaller table steps. The trace feature could also be used to approximate the solution. If the original equation was graphed as Y1 and you let Y2 = 10, we could look for the intersection. Using this last method, the solution is 15.242194 hours or approximately 15 hours, 14 minutes, 32 seconds.***

g. How would you solve the equation you wrote in (f) algebraically? What is the first step**?**

Solution

The first step would be to divide by 300. → $.8^x = 1/30$.

To finish solving the problem algebraically, we must know how to find inverses of exponentials. This topic will be explored later in Algebra II.

2. **Extension** - A can of Instant Energy, a 16-ounce energy drink, contains 80 mg of caffeine. Suppose the caffeine in the bloodstream peaks at 80 mg. If ½ of the caffeine has been eliminated from the bloodstream after 5 hours (the half-life of caffeine in the bloodstream is 5 hours), complete the following:

   a. How much caffeine will remain in the bloodstream after 5 hours? 10 hours? 1 hour? 2 hours? Make a table to organize your answers. Explain how you came up with your answers. (Make a conjecture. You can return to your answers later to make any corrections.)

Solution

Time (hours) since peak	*0*	*1*	*2*	*5*	*10*
Caffeine in bloodstream (mg)	*80*	*69.644*	*60.629*	*40*	*20*

(Explanations will vary. At this point, students may not have the answers for 1 hour and 2 hours correct. They will be able to go back and correct it.)

$f(t) = 80\ (1/2)^{(t/5)}$, t – the number of hours.

How Long Does It Take? (Constructing Task)

Name__________________________________ Date____________________

Mathematical Goals
- Explore exponential phenomena
- Analyze exponential equations

Essential Questions:

- How do we use exponential functions to represent real-life phenomena?
- How do we analyze exponential equations?

GEORGIA STANDARDS OF EXCELLENCE

MGSE9-12.A.CED.1 Create equations and inequalities in one variable and use them to solve problems. Include equations arising from ~~linear, quadratic,~~ ~~simple rational, and~~ exponential functions (integer inputs only).

MGSE9-12.A.CED.2 Create ~~linear, quadratic, and~~ exponential equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. (The phrase "in two or more variables" refers to formulas like the compound interest formula, in which $A = P(1 + r/n)^{nt}$ has multiple variables.)

MGSE9-12.F.IF.1 Understand that a function from one set (the input, called the domain) to another set (the output, called the range) assigns to each element of the domain exactly one element of the range, i.e. each input value maps to exactly one output value. If f is a function, x is the input (an element of the domain), and f(x) is the output (an element of the range). Graphically, the graph is y = f(x).

MGSE9-12.F.IF.2 Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context.

MGSE9-12.F.IF.4 Using tables, graphs, and verbal descriptions, interpret the key characteristics of a function which models the relationship between two quantities. Sketch a graph showing key features including: intercepts; interval where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; ~~and periodicity.~~

MGSE9-12.F.IF.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*

MGSE9-12.F.IF.7 Graph functions expressed algebraically and show key features of the graph both by hand and by using technology.

MGSE9-12.F.IF.7e Graph exponential ~~and logarithmic~~ functions, showing intercepts and end behavior, ~~and trigonometric functions, showing period, midline, and amplitude.~~

MGSE9-12.F.BF.1 Write a function that describes a relationship between two quantities.

RELATED STANDARDS

MGSE9-12.A.SSE.1 Interpret expressions that represent a quantity in terms of its context.

MGSE9-12.A.SSE.1a Interpret parts of an expression, such as terms, factors, and coefficients, in context.

MGSE9-12.A.SSE.1b Given situations which utilize formulas or expressions with multiple terms and/or factors, interpret the meaning (in context) of individual terms or factors.

STANDARDS FOR MATHEMATICAL PRACTICE

2. Reason abstractly and quantitatively by requiring students to make sense of quantities and their relationships to one another in problem situations.
3. Construct viable arguments and critique the reasoning of others by engaging students on discussion of why they agree or disagree with responses, decide whether they make sense, and ask useful questions to clarify or improve the arguments.
7. Look for and make use of structure by expecting students to apply rules, look for patterns and analyze structure.

Materials Needed:

- Graphing Calculator
- Graph paper

How Long Does It Take? (Constructing Task)

Name_________________________________ Date___________________

Before sending astronauts to investigate the new planet of Exponentia, NASA decided to run a number of tests on the astronauts.

1. A specific multi-vitamin is eliminated from an adult male's bloodstream at a rate of about 20% per hour. The vitamin reaches peak level in the bloodstream of 300 milligrams.

- How much of the vitamin remains 2 hours after the peak level? 5 hours after the peak level? Make a table of values to record your answers. Write expressions for how you obtain your answers.

b. Using your work from (a), write expressions for each computed value using the original numbers in the problem (300 and 20%). For example, after 2 hours, the amount of vitamin left is [300 * (1 - .2)]* (1 - .2).

c. Using part (b), write a function for the vitamin level with respect to the number of hours after the peak level, *x*.

d. How would use the function you wrote in (c) to answer the questions in (a)? Use your function and check to see that you get the same answers as you did in part (a).

e. After how many hours will there be less than 10 mg of the vitamin remaining in the bloodstream? Explain how you would determine this answer using both a table feature and a graph.

f. Write an equation that you could solve to determine when the vitamin concentration is exactly 10 mg. Could you use the table feature to solve this equation? Could you use the graph feature? How could you use the intersection feature of your calculator? Solve the problem using one of these methods.

g. How would you solve the equation you wrote in (f) algebraically? What is the first step?

2. **Extension** - A can of Instant Energy, a 16-ounce energy drink, contains 80 mg of caffeine. Suppose the caffeine in the bloodstream peaks at 80 mg. If ½ of the caffeine has been eliminated from the bloodstream after 5 hours (the half-life of caffeine in the bloodstream is 5 hours), complete the following:

a. How much caffeine will remain in the bloodstream after 5 hours? 10 hours? 1 hour? 2 hours? Make a table to organize your answers. Explain how you came up with your answers. (Make a conjecture. You can return to your answers later to make any corrections.)

Growing by Leaps and Bounds (Culminating Task)

Back to Task Table

Comments on the Task:

The following task covers the majority of the material from the unit. When using this task you should be aware that there are some new material portions. Part I is very similar to the paper folding task creating the growth function $y = 2^x$. Given the length of the task, Parts I and II should be done separately from Part III. Additionally, it should be noted in Part III that interest can be modeled by a linear or exponential model based on simple or compound interest. If available, this task is a good opportunity to teach students about Excel programming.

Introduction

At this point in their study students have had experience investigating exponential functions. Students have also extended their understanding of exponents to include all integer values but have not yet discussed rational exponents. In Part I, students investigate a mathematical model of spreading a rumor in which the domain of the function is limited to a finite set of nonnegative integers. In Part II, students compare two exponential functions. In Part III, students work with the compound interest formula.

Mathematical Goals

- Create exponential equations from contextual situations.
- Write and graph an equation to represent an exponential relationship.
- Graph equations on coordinate axes with labels and scales.
- Use technology to explore exponential graphs.
- Interpret key features of graphs in context.

Essential Questions

- How do I interpret exponential functions in context?

GEORGIA STANDARDS OF EXCELLENCE

MGSE9-12.A.CED.1 Create equations and inequalities in one variable and use them to solve problems. Include equations arising from ~~linear, quadratic,~~ ~~simple rational, and~~ exponential functions (integer inputs only).

MGSE9-12.A.CED.2 Create ~~linear, quadratic, and~~ exponential equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. (The phrase "in two or more variables" refers to formulas like the compound interest formula, in which $A = P(1 + r/n)^{nt}$ has multiple variables.)

MGSE9-12.F.BF.1 Write a function that describes a relationship between two quantities.

MGSE9-12.F.BF.1a Determine an explicit expression and the recursive process (steps for calculation) from context. *For example, if Jimmy starts out with $15 and earns $2 a day, the explicit expression "2x+15" can be described recursively (either in writing or verbally) as "to find out how much money Jimmy will have tomorrow, you add $2 to his total today."* $J_n = J_{n-1} + 2, J_0 = 15$

MGSE9-12.F.BF.2 Write ~~arithmetic and~~ geometric sequences recursively and explicitly, use them to model situations, and translate between the two forms. Connect ~~arithmetic sequences to linear functions and~~ geometric sequences to exponential functions.

MGSE9-12.F.BF.3 Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k\,f(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. ~~Include recognizing even and odd functions from their graphs and algebraic expressions for them.~~ *(Focus on vertical translations of graphs of linear and exponential functions. Relate the vertical translation of a linear function to its y-intercept.)*

MGSE9-12.F.IF.1 Understand that a function from one set (the input, called the domain) to another set (the output, called the range) assigns to each element of the domain exactly one element of the range, i.e. each input value maps to exactly one output value. If f is a function, x is the input (an element of the domain), and f(x) is the output (an element of the range). Graphically, the graph is y = f(x).

MGSE9-12.F.IF.2 Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context.

MGSE9-12.F.IF.3 Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers. (Generally, the scope of high school math defines this subset as the set of natural numbers 1,2,3,4...) By graphing or calculating terms, students should be able to show how the recursive sequence $a_1=7$, $a_n=a_{n-1}+2$; the sequence $s_n = 2(n-1) + 7$; and the function f(x) = 2x + 5 (when x is a natural number) all define the same sequence.

MGSE9-12.F.IF.4 Using tables, graphs, and verbal descriptions, interpret the key characteristics of a function which models the relationship between two quantities. Sketch a graph showing key features including: intercepts; interval where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; ~~and periodicity.~~

MGSE9-12.F.IF.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*

MGSE9-12.F.IF.7 Graph functions expressed algebraically and show key features of the graph both by hand and by using technology.

MGSE9-12.F.IF.7e Graph exponential ~~and logarithmic~~ functions, showing intercepts and end behavior, ~~and trigonometric functions, showing period, midline, and amplitude.~~

MGSE9-12.F.IF.9 Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one function and an algebraic expression for another, say which has the larger maximum.*

STANDARDS FOR MATHEMATICAL PRACTICE

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Students should use all eight SMPs when exploring this task.

Background Knowledge

- Students will apply everything they have learned in this unit.

Common Misconceptions

- Address misconceptions brought to light during the rest of the unit.

Materials

- Graph paper
- Graphing utility
- Optional: spreadsheet software (e.g., Excel)

Grouping

- Individual / Partners

Growing by Leaps and Bounds (Culminating Task) – Teacher Notes

Part I: Meet Linda

Linda's lifelong dream has been to open her own business. After working, sacrificing, and saving, she finally has enough money to open up an ice cream business. The grand opening of her business is scheduled for the Friday of Memorial Day weekend. She would like to have a soft opening for her business on the Tuesday before. The soft opening should give her a good idea of any supply or personnel issues and give her time to correct them before the big official opening.

A soft opening means that the opening of the business is not officially announced; news of its opening is just spread by word of mouth (see, not all rumors are bad!). Linda needs a good idea of when she should begin the rumor in order for it to spread reasonably well before her soft opening. She has been told that about 10% of the people who know about an event will actually attend it. Based on this assumption, if she wants to have about 50 people visit her store on the Tuesday of the soft opening, she will need 500 people to know about it.

1. Linda plans to tell one person each day and will ask that person to tell one other person each day through the day of the opening, and so on. Assume that each new person who hears about the soft opening is also asked to tell one other person each day through the day of the opening and that each one starts the process of telling their friends on the day after he or she first hears. When should Linda begin telling others about the soft opening in order to have at least 500 people know about it by the day it occurs?
   Comments

   With the table in item 2 below, it is likely that many students will organize their work in a similar way. Whether or not they use such a table, they will need to count up from the first day that Linda begins to spread the news to find out how many days it will take for the number of people who know to reach 500 and then count backwards to determine the day Linda should start. The Memorial Day reference in the problem gives a convenient way to express the answer.

   Solution

   Linda should tell her first person about the soft opening on Monday two weeks before Memorial Day because. See →

   1st day: Linda tells one other person, so 2 people know.

   2nd day: Each of the two people who know tell another person, so 4 people know.

   3rd day: Each of the four people who know tell another person, so 8 people know.

   4th day: 16 people know

   5th day: 32 people know

   The 9th day corresponds to the Tuesday

of the soft opening. So, the 2nd day is the Tuesday one week before, and the 1st day is the Monday that is two weeks before Memorial Day.

6th day: 64 people know

7th day: 128 people know

8th day: 256 people know

9th day: 512 people know

2. Let x represent the day number and let y be the number of people who know about the soft opening on day x. Consider the day before Linda told anyone to be Day 0, so that Linda is the only person who knows about the opening on Day 0. Day 1 is the first day that Linda told someone else about the opening.

   a. Complete the following table.

Day	0	1	2	3	4	5
Number of people who know	1	2	**4**	**8**	**16**	**32**

Comments

The table of values is limited to fewer than that needed to answer the question in item 1 so that the graph in part b below will show the y–intercept and the shape typical of an exponential function.

Solutions
The completed table is shown above.

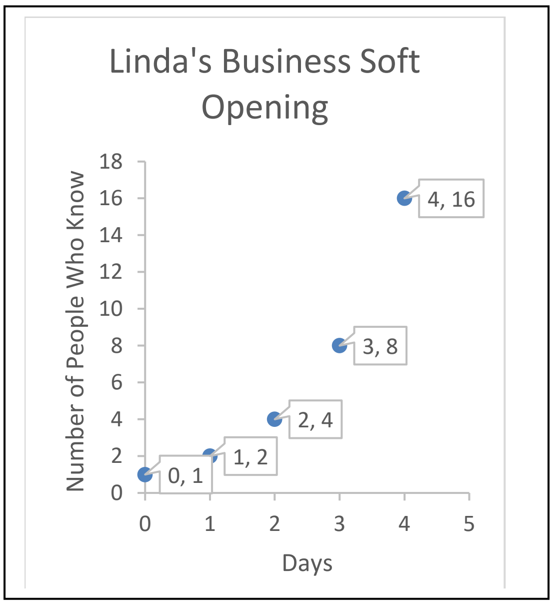
3. Graph the points from the table in part 2a.

Comments

It is recommended that students draw this graph by hand on graph paper.

Solutions

The graph is shown below:



a. Does it make sense to connect the dots on the graph? Why or why not?

Comments

The question of whether to connect the dots was prominent in students' early formal s reminds students to think of the meaning of points on the graph and to consider what meaningful in the situation. The description here is that the output is total number of Fractional parts of a day are not meaningful. Note: It would be impossible to draw a a continuous time domain since we do not know when during the day each person wh

Solutions

No, it does not make sense to connect the dots. Connecting the dots would imply time passing continuously. We do not know when during the day people hear about the soft opening. We just have a count of the total number of people who know on each day.

b. What does point (5, 32) represent in this situation? Describe the point in a form of a function notation.

Solutions: Point (5, 32) represents the value of the function at x =5. In the context, on day 5, 32 people knew about the soft opening of Linda's business: f (5) = 32

c. Describe the domain of the function. What does the domain mean in this situation? What are the restrictions of the domain that arise from the context?

Comments: Students may express the correct answer in a variety of ways.

Solutions: The domain in this function could be described as follows: The set of all nonnegative integers less than or equal to 9 $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. ***The domain of the function represents the days on which the rumor about the soft opening of Linda's business spreads. The domain is restricted to nonnegative integers (whole numbers) as we consider the initial day, day 0, the day before the rumor started spreading; it does not make sense to include fractional values of the domain as it describes the number of whole days.***

d. Describe the range of the function. What does the range mean in context of the situation? What are the restrictions of the range that arise from the context?

Solutions: The range in this function could be described as follows: The set of all positive integers (natural numbers) less than or equal to 512 or : $\{1, 2, 4, 8, 16, 32, 64, 128, 256, 512\}$. ***The range of the function represents the number of people who know about the soft opening of Linda's business. The range is restricted to positive integers (natural numbers) as one person knows about the business opening on the initial day, day 0; it does not make sense to include fractional values of the range as it describes the number of people who know about the business soft opening.***

e. What is the y-intercept in this problem? What does the y-intercept represent in this situation?

Solutions

The y – intercept is (0, 1). The y – intercept represents the number of people who know about the business opening on day 0 (the initial day; the day before the rumor about the soft opening starts spreading.)

4. The spread of a rumor or the spread of a disease can be modeled by a type of function known as exponential function; in particular, an exponential ***growth*** function. An **exponential function** has the form

   $f\left(x\right)=a\cdot b^{x}$,

   where *a* is a non–zero real number and *b* is a positive real number other than 1. An exponential growth function has a value of b that is greater than 1. Values "a" and "b" are the parameters of the function.

   Write explicit and recursive equations that describe the relationship between *x* (day) and *y* (number of people who know) for the situation of spreading the news about the soft opening of Linda's ice cream store. What type of a sequence does the function represent? What do the parts of the equations represent in terms of the context?

   Comments
   Students should easily see that the outputs of the function are powers of 2 and then note that the day number and the power of 2 are the same.

   Solutions
   Explicit equation 1:
   $y = 1\ (2)^{x}$ ***or*** $y = 2^{x}$
   "1" represents the fact that only one person, Linda, knew about the soft opening of her business on day 0, the day before the news started spreading.
   "2" represents the fact that the number of people who know about the business sofr opening increases by the ratio of "2".

   Explicit equation 2:
   $y = 2\ (2)^{x-1}$
   "2" (the first factor in the equation above) represents the fact that two people (Linda plus one more person) knew about the soft opening of Linda's business on day 1.
   "2" (the base) represents the fact that the number of people who know about the business increases by the ratio of "2".

   Recursive Equation:
   $a_n = a_{n-1}\ (2) \quad a_1 = 2$

The "2" in the recursive equation above represents the fact that the number of people who know about the business increases by the ratio of "2".
$a_1 = 2$ as two people know on day one; "2" is the first term of the sequence.
The function represents a geometric sequence as the output of the function increases by an equal ratio (factor).

5. Does your equation describe the relationship between day and number who know about Linda's ice cream store soft opening completely? Why or why not?

Comments
The point of this question is that students realize that the domain is restricted in ways not implied by the equation. Since students have not yet studied a definition for non–integer exponents, they may believe that the equation makes sense only for integer exponents. However, they know about negative integer exponents and thus need to explicitly excluded these from the domain. They also need to exclude integers greater than 9 from the domain since Linda's method of spreading the news of the soft opening stops on the day of the opening. If students state the correct inequalities but do not explicitly state that the exponents should be integers, teachers need to explain that this restriction must be included since other numbers can be exponents, although they will not study other exponents explicitly until a later course.
Solutions
No, the equation does not describe the relationship completely because the domain needs to be restricted to the integers 0, 1, 2, . . . , 9, and this information is not included in the equation.

Part II: What if?

6. Predict how the graph would change if Linda had told two people each day rather than one and had asked that each person also tell two other people each day?

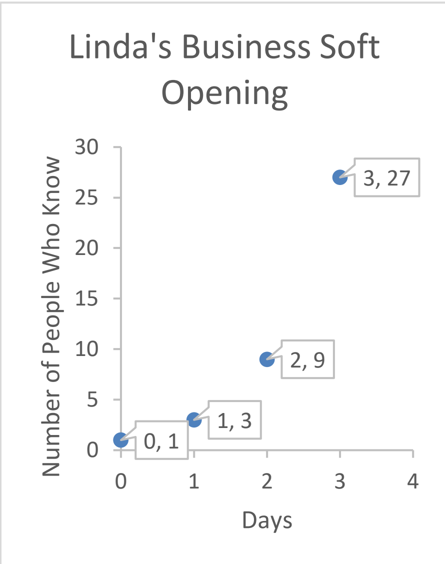
Solutions
The point (0, 1) would stay the same since on Day 0 Linda would still be the only person who knows about the opening. But for the other days, more people would know so the points for the other days would be higher.

7. Graph the hypothetical situation described in item 6. How did the graph change as compared to part I of the task?

Comments

This question asks students to think in terms of function values or points on the graph, but they will have to think through the situation in a similar manner to the original. If students answer more generally here without being specific about new function values, then they will have more work to do in item 8 to find the new equation.

Solutions



The point (0, 1) stayed the same since on Day 0 Linda would still be the only person who knows about the opening. But for the other days, more people would know so the points for the other days are higher.
In particular, on the first day in part I of the task, 2 people (Linda and one more person) knew about the business opening, which corresponded to point (1, 2) on the graph; on the first day in part II of the task, 3 people (Linda and the two people she tells) would know giving the point (1, 3). On the second day, in part I of the task, 4 people knew, which corresponded to point (2, 4) on the graph; in part II of the task, because each of the 3 who know will tell 2 others giving a total of 3 + 2(3) =9. So the point for Day 2 is (2, 9). We can continue in this way for the other points. As the number of days increases, the number of people who know about the business in parts I and II of the task differs more and more significantly.

8. How would the equation change if Linda had told two people each day rather than one and had asked that each person also tell two other people each day? What would be the values of *a* and *b* in this case? How does the change in parameter b affect the output values of the function?

Comments
In addition to giving the formula, students must specify the values of a and b to reinforce the definition of exponential function.

Solutions
The equation would have a 3 as base for the exponent instead of a 2, that is, the equation would be $y = 3^x$. ***In this case, a = 1 and b = 3. The change in parameter b from 2 to 3 accounts for the faster growth in output values. For example, on day 5 in part I of the task,***

32 people knew about the business opening. On day 5 in part II of the task, 243 people would know about the business opening. The conclusion made based on the analysis of the equation in item 8 corresponds to the conclusion made based on the analysis of the graph in item 7: as the number of days increases, the number of people who know about the business in parts I and II of the task differs more and more significantly

9. How long would it take for at least 500 people to find out about the opening if the rumor spread at this new rate?

Comments

Students can use the graph or a table of values to determine the answer. If they draw a graph using a graphing utility, they should realize that the points of this function are only the integer valued points on the continuous graph shown. It is likely that most students will just count up to find the first power of 3 that is greater than 500.

Solutions

It would take 6 days for at least 500 people to find out.

Day number, x	**0**	**1**	**2**	**3**	**4**	**5**	**6**
No. people who know, $y = 3^x$	**1**	**3**	**9**	**27**	**81**	**243**	**729**

10. What if on day *0*, two people instead of one knew about the business opening? What would the new equation be? What effect would this change in parameter *a* have on the graph?

Solutions

The new equation would be $y = 2(3)^x$ ***The graph's y-intercept would change to (0, 2), which would mean that two people instead of one would know initially, before the news started spreading. The growth rate would not change as the base is the same and equals to 3. The output values in item 10 would always be twice the output values in items 7, 8, and 9 (please see the graph and table above).***

Part III: The Beginning of a Business

How in the world did Linda ever save enough to buy the franchise to an ice cream store? Her mom used to say, "That Linda, why she could squeeze a quarter out of a nickel!" The truth is that Linda learned early in life that patience with money is a good thing. When she was just about 9 years old, she asked her dad if she could put her money in the bank. He took her to the bank and she opened her very first savings account.

Each year until Linda was 16, she deposited her birthday money into her savings account. Her grandparents (both sets) and her parents each gave her money for her birthday that was equal to twice her age; so on her ninth birthday, she deposited \$54 (\$18 from each couple).

Linda's bank paid her 3% interest, compounded quarterly. The bank calculated her interest using the following standard formula:

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

where A = final amount, P = principal amount, r = interest rate, n = number of times per year the interest is compounded, and t is the number of years the money is left in the account.

1. Verify the first entry in the following chart, and then complete the chart to calculate how much money Linda had on her 16th birthday. Do not round answers until the end of the computation, then give the final amount rounded to the nearest cent.

Age	Birthday \$	Amount from previous year plus Birthday	Total at year end
9	54	0	55.63831630
10			
11			
12			
.	.	.	.

• •	• •	• •	• •

Comments

Students will need exploration time to understand the compound interest formula. They may need to look up the term "principal amount" to understand that it refers to the amount deposited into the account. They will also need to realize that the interest rate r must be expressed in decimal form. If they enter numbers in their calculators following the formula exactly, they may need to be reminded about order of operations and that the calculator will not make the correct calculation unless the expression, nt, in the exponent is put in parentheses during calculation.

Some students may benefit from verifying the meaning of the compound interest formula by stepping through the compound interest calculation as four applications of simple interest using a rate of $\frac{0.03}{4} = 0.0075$ ***for each quarter for four quarters of one year as shown in the table below.***

Quarter number	*Amount invested at beginning of quarter*	*Amount of interest paid*	*Amount at end of quarter*
1	**54**	**54(.0075) = 0.405**	**54.405**
2	**54.405**	**0.4080375**	**54.8130375**
3	**54.8130375**	**0.4110977813**	**55.22413528**
4	**55.22413528**	**0.4141810146**	**55.63831630**

Advanced students may benefit from seeing how the compound interest formula is developed using calculations similar to the above but using P for the amount of money, as shown below. One quarter is one–fourth of a year, so the number of quarters is always 4 times the number of years.

#. of	*# of*	*Amt. invested at beginning of quarte*	*Amount of*	*Amount at end of quarter*

yrs	*qtrs*	*r*	*interest paid*	
1/4	**1**	P	$P(.0075)$	$P(1 + 0.0075)$
1/2	**2**	$P(1.0075)$	$[P(1.0075)](.0075)$	$P(1 + 0.0075)(1 + .0075) = P(1.0075)^2$
3/4	**3**	$P(1.0075)^2$	$[P(1.0075)^2](.0075)$	$[P(1.0075)^2](1 + .0075) = P(1.0075)^3$
1	**4**	$P(1.0075)^3$	$[P(1.0075)^3](.0075)$	$[P(1.0075)^3](1 + .0075) = P(1.0075)^4$
5/4	**5**	$P(1.0075)^4$	$[P(1.0075)^4](.0075)$	$[P(1.0075)^4](1 + .0075) = P(1.0075)^5$
3/2	**6**	$P(1.0075)^5$	$[P(1.0075)^5](.0075)$	$[P(1.0075)^5](1 + .0075) = P(1.0075)^6$

Solutions

For her deposit at age 9, P = 54, r = 0.03, n = 4, t = 4.

$$A = 54\left(1 + \frac{0.03}{4}\right)^{(4\cdot 1)} = 54(1.0075)^4$$

$$A \approx 54(1.030339191) = 54.6383163 \approx 54.63832, \textit{ as in the chart}$$

Age	$ received on this Birthday	Amt from previous year plus Birthday	Total at year end
8	N/A	N/A	0
9	54	54	55.63832
10	**60**	**115.63832**	**115.63832(1.0075)^4 =119.14669**
11	**66**	**185.14669**	**185.14669(1.0075)^4 = 190.76389**
12	**72**	**262.76389**	**262.76389(1.0075)^4 = 270.73593**
13	**78**	**348.73593**	**348.73593(1.0075)^4 = 359.31630**
14	**84**	**443.31630**	**443.31630(1.0075)^4 = 456.76616**
15	**90**	**546.76616**	**546.76616(1.0075)^4 = 563.35460**

On the day before her 16th birthday, a year after her 15th, Linda had $563.35.

2. On her 16th birthday, the budding entrepreneur asked her parents if she could invest in the stock market. She studied the newspaper, talked to her economics teacher, researched a few companies and finally settled on the stock she wanted. She invested all of her money in the stock and promptly forgot about it. When she graduated from college on her 22nd birthday, she received a statement from her stocks and realized that her stock had appreciated an average of 10% per year. How much was her stock worth on her 22nd birthday?

Comments

The challenge for students here is realizing that the information that Linda's stock had appreciated an average of 10% per year means that the money grew as if it were invested at 10% compounded annually for the 6 years.

Solution

Linda's stock was worth \$998.01 by application of the compound interest formula with $P = 563.35$, $r = 0.10$, $n = 1$, ***and*** $t = 6$:

$$A = 563.35\left(1+\frac{0.10}{1}\right)^{(1\cdot 6)} = 563.35(1.1)^6 = 563.35(1.771561) \approx 998.01$$

3. When Linda graduated from college, she received an academic award that carried a \$500 cash award. On her 22nd birthday, she used the money to purchase additional stock. She started her first job immediately after graduation and decided to save \$50 each month. On her 23rd birthday she used the \$600 (total of her monthly amount) savings to purchase new stock. Each year thereafter she increased the total of her savings by \$100 and, on her birthday each year, used her savings to purchase additional stock. Linda continued to learn about stocks and managed her accounts carefully. On her 35th birthday she looked back and saw that her stock had appreciated at 11% during the first year after college and that the rate of appreciation increased by 0.25% each year thereafter. At age 34, she cashed in enough stock to make a down payment on a bank loan to purchase her business. What was her stock worth on her 34th birthday? Use a table like the one below to organize your calculations.

Age	Amt from previous year	Amt Linda added from savings that year	Amount invested for the year	Interest rate for the year	Amt at year end
22	998.01	500	1498.01	11.00%	1662.79
23	1662.79	600		11.25%	
24		700		11.50%	
25		800		11.75%	
. .	. .	. .	. .	. .	. .

•	•	•	•	•	•

Comments

This item brings closure to this part of the learning task. The calculations are simple applications of the compound interest formula. If students have access to a spreadsheet program, having them set up the spreadsheet formulas is a possible extension of this activity.

Solutions

***At age 34, Linda's stock was worth* $30,133.63.**

The completed table is given on the next page as an Excel file.

Age	*Amt from previous year*	*Amt Linda added*	*Amt invested for the year*	*Interest rate for the year*	*Amt at year end*
22	998.01	500	1498.01	0.1100	1662.79
23	1662.79	600	2262.79	0.1125	2517.36
24	2517.36	700	3217.36	0.1150	3587.35
25	3587.35	800	4387.35	0.1175	4902.86
26	4902.86	900	5802.86	0.1200	6499.21
27	6499.21	1000	7499.21	0.1225	8417.86
28	8417.86	1100	9517.86	0.1250	10707.59
29	10707.59	1200	11907.59	0.1275	13425.81
30	13425.81	1300	14725.81	0.1300	16640.17
31	16640.17	1400	18040.17	0.1325	20430.49
32	20430.49	1500	21930.49	0.1350	24891.11
33	24891.11	1600	26491.11	0.1375	30133.63

The same spreadsheet with formulas turned on is pasted in below.

Age	*Amt from previous year*	*Amt Linda added*	*Amt invested for the year*	*Interest rate for the year*	*Amt at year end*
22	998.01	500	=B2+C2	0.11	=D2*(1+E2)
23	=F2	600	=B3+C3	0.1125	=D3*(1+E3)
24	=F3	700	=B4+C4	0.115	=D4*(1+E4)
25	=F4	800	=B5+C5	0.1175	=D5*(1+E5)
26	=F5	900	=B6+C6	0.12	=D6*(1+E6)
27	=F6	1000	=B7+C7	0.1225	=D7*(1+E7)
28	=F7	1100	=B8+C8	0.125	=D8*(1+E8)
29	=F8	1200	=B9+C9	0.1275	=D9*(1+E9)
30	=F9	1300	=B10+C10	0.13	=D10*(1+E10)
31	=F10	1400	=B11+C11	0.1325	=D11*(1+E11)
32	=F11	1500	=B12+C12	0.135	=D12*(1+E12)
33	=F12	1600	=B13+C13	0.1375	=D13*(1+E13)

Growing by Leaps and Bounds (Culminating Task)

Name________________________________ **Date**_________________

Mathematical Goals

- Create exponential equations from contextual situations.
- Write and graph an equation to represent an exponential relationship.
- Graph equations on coordinate axes with labels and scales.
- Use technology to explore exponential graphs.
- Interpret key features of graphs in context.

Essential Questions

- How do I interpret exponential functions in context?

GEORGIA STANDARDS OF EXCELLENCE

MGSE9-12.A.CED.1 Create equations and inequalities in one variable and use them to solve problems. Include equations arising from ~~linear, quadratic,~~ ~~simple rational, and~~ exponential functions (integer inputs only).

MGSE9-12.A.CED.2 Create ~~linear, quadratic, and~~ exponential equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. (The phrase "in two or more variables" refers to formulas like the compound interest formula, in which $A = P(1 + r/n)^{nt}$ has multiple variables.)

MGSE9-12.F.BF.1 Write a function that describes a relationship between two quantities.

> **MGSE9-12.F.BF.1a** Determine an explicit expression and the recursive process (steps for calculation) from context. *For example, if Jimmy starts out with $15 and earns $2 a day, the explicit expression "2x+15" can be described recursively (either in writing or verbally) as "to find out how much money Jimmy will have tomorrow, you add $2 to his total today."* $J_n = J_{n-1} + 2, J_0 = 15$

MGSE9-12.F.BF.2 Write ~~arithmetic and~~ geometric sequences recursively and explicitly, use them to model situations, and translate between the two forms. Connect ~~arithmetic sequences to linear functions and~~ geometric sequences to exponential functions.

MGSE9-12.F.BF.3 Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k\,f(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. ~~Include recognizing even and odd functions from their graphs and algebraic expressions for them.~~ *(Focus on vertical translations of graphs of linear and exponential functions. Relate the vertical translation of a linear function to its y-intercept.)*

MGSE9-12.F.IF.1 Understand that a function from one set (the input, called the domain) to another set (the output, called the range) assigns to each element of the domain exactly one element of the range, i.e. each input value maps to exactly one output value. If f is a function, x is the input (an element of the domain), and f(x) is the output (an element of the range). Graphically, the graph is y = f(x).

MGSE9-12.F.IF.2 Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context.

MGSE9-12.F.IF.3 Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers. (Generally, the scope of high school math defines this subset as the set of natural numbers 1,2,3,4...) By graphing or calculating terms, students should be able to show how the recursive sequence $a_1=7$, $a_n=a_{n-1}+2$; the sequence $s_n = 2(n-1) + 7$; and the function f(x) = 2x + 5 (when x is a natural number) all define the same sequence.

MGSE9-12.F.IF.4 Using tables, graphs, and verbal descriptions, interpret the key characteristics of a function which models the relationship between two quantities. Sketch a graph showing key features including: intercepts; interval where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; ~~and periodicity.~~

MGSE9-12.F.IF.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*

MGSE9-12.F.IF.7 Graph functions expressed algebraically and show key features of the graph both by hand and by using technology.

> **MGSE9-12.F.IF.7e** Graph exponential ~~and logarithmic~~ functions, showing intercepts and end behavior, ~~and trigonometric functions, showing period, midline, and amplitude.~~

MGSE9-12.F.IF.9 Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one function and an algebraic expression for another, say which has the larger maximum.*

STANDARDS FOR MATHEMATICAL PRACTICE

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.

8. Look for and express regularity in repeated reasoning.

Growing by Leaps and Bounds (Culminating Task)

Name________________________________ **Date**_________________

Part I: Meet Linda

Linda's lifelong dream has been to open her own business. After working, sacrificing, and saving, she finally has enough money to open up an ice cream business. The grand opening of her business is scheduled for the Friday of Memorial Day weekend. She would like to have a soft opening for her business on the Tuesday before. The soft opening should give her a good idea of any supply or personnel issues and give her time to correct them before the big official opening.

A soft opening means that the opening of the business is not officially announced; news of its opening is just spread by word of mouth (see, not all rumors are bad!). Linda needs a good idea of when she should begin the rumor in order for it to spread reasonably well before her soft opening. She has been told that about 10% of the people who know about an event will actually attend it. Based on this assumption, if she wants to have about 50 people visit her store on the Tuesday of the soft opening, she will need 500 people to know about it.

1. Linda plans to tell one person each day and will ask that person to tell one other person each day through the day of the opening, and so on. Assume that each new person who hears about the soft opening is also asked to tell one other person each day through the day of the opening and that each one starts the process of telling their friends on the day after he or she first hears. When should Linda begin telling others about the soft opening in order to have at least 500 people know about it by the day it occurs?

2. Let x represent the day number and let y be the number of people who know about the soft opening on day x. Consider the day before Linda told anyone to be Day 0, so that Linda is the only person who knows about the opening on Day 0. Day 1 is the first day that Linda told someone else about the opening.

   a. Complete the following table.

Day	0	1	2	3	4	5
Number of people who know	1	2				

3. Graph the points from the table in part 2a.

a. Does it make sense to connect the dots on the graph? Why or why not?

b. What does point (5, 32) represent in this situation? Describe the point in a form of a function notation.

c. Describe the domain of the function. What does the domain mean in this situation? What are the restrictions of the domain that arise from the context?

d. Describe the range of the function. What does the range mean in context of the situation? What are the restrictions of the range that arise from the context?

e. What is the y-intercept in this problem? What does the y-intercept represent in this situation?

4. The spread of a rumor or the spread of a disease can be modeled by a type of function known as exponential function; in particular, an exponential ***growth*** function. An **exponential function** has the form

$$f(x) = a \cdot b^x,$$

where a is a non–zero real number and b is a positive real number other than 1. An exponential growth function has a value of b that is greater than 1. Values “a” and “b” are the parameters of the function.

Write explicit and recursive equations that describe the relationship between x (day) and y (number of people who know) for the situation of spreading the news about the soft opening of Linda's ice cream store. What type of a sequence does the function represent? What do the parts of the equations represent in terms of the context?

5. Does your equation describe the relationship between day and number who know about Linda's ice cream store soft opening completely? Why or why not?

Part II: What if?

6. Predict how the graph would change if Linda had told two people each day rather than one and had asked that each person also tell two other people each day?

7. Graph the hypothetical situation described in item 6. How did the graph change as compared to part I of the task?

8. How would the equation change if Linda had told two people each day rather than one and had asked that each person also tell two other people each day? What would be the values of *a* and *b* in this case? How does the change in parameter b affect the output values of the function?

9. How long would it take for at least 500 people to find out about the opening if the rumor spread at this new rate?

10. What if on day 0, two people instead of one knew about the business opening? What would the new equation be? What effect would this change in parameter *a* have on the graph?

Part III: The Beginning of a Business

How in the world did Linda ever save enough to buy the franchise to an ice cream store? Her mom used to say, "That Linda, why she could squeeze a quarter out of a nickel!" The truth is that Linda learned early in life that patience with money is a good thing. When she was just about 9 years old, she asked her dad if she could put her money in the bank. He took her to the bank and she opened her very first savings account.

Each year until Linda was 16, she deposited her birthday money into her savings account. Her grandparents (both sets) and her parents each gave her money for her birthday that was equal to twice her age; so on her ninth birthday, she deposited $54 ($18 from each couple).

Linda's bank paid her 3% interest, compounded quarterly. The bank calculated her interest using the following standard formula:

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

where A = final amount, P = principal amount, r = interest rate, n = number of times per year the interest is compounded, and t is the number of years the money is left in the account.

1. Verify the first two rows of the following chart, and then complete the chart to calculate how much money Linda had on her 16th birthday. Do not round answers until the end of the computation, then give the final amount rounded to the nearest cent.

Age	Birthday Money	Amount from previous year plus birthday money	Total at year end
8	N/A	N/A	0
9	54	0 + 54 = 54	55.63831630
10			
11			
12			
N	N	N	N

2. On her 16th birthday, the budding entrepreneur asked her parents if she could invest in the stock market. She studied the newspaper, talked to her economics teacher, researched a few companies and finally settled on the stock she wanted. She invested all of her money in the stock and promptly forgot about it. When she graduated from college on her 22nd birthday, she received a statement from her stocks and realized that her stock had appreciated an average of 10% per year. How much was her stock worth on her 22nd birthday?

3. When Linda graduated from college, she received an academic award that carried a $500 cash award. On her 22nd birthday, she used the money to purchase additional stock. She started her first job immediately after graduation and decided to save $50 each month. On her 23rd birthday she used the $600 (total of her monthly amount) savings to purchase new stock. Each year thereafter she increased her total of her savings by $100 and, on her birthday each year, used her savings to purchase additional stock. Linda continued to learn about stocks and managed her accounts carefully. On her 35th birthday she looked back and saw that her stock had appreciated at 11% during the first year after college and that the rate of appreciation increased by 0.25% each year thereafter. At age 34, she cashed in enough stock to make a down payment on a bank loan to purchase her business. What was her stock worth on her 34th birthday? Use a table like the one below to organize your calculations.

Age	Amount from previous year	Amount Linda added from savings that year	Amount invested for the year	Interest rate for the year	Amount at year end
22	998.01	500	1498.01	11.00%	1662.79
23	1662.79	600		11.25%	
24		700		11.50%	
25		800		11.75%	
⋮	⋮	⋮	⋮	⋮	⋮

Additional Tasks

Back to Task Table

The tasks featured in this table provide additional resources and supplemental tasks to be incorporated into unit 4 instruction as deemed appropriate by the instructor.

UNIT4: Modeling and Analyzing Exponential Functions	Standards Addressed in the Task
One Grain of Rice NCTM Illuminations Task http://illuminations.nctm.org/Lesson.aspx?id=2497	F-BF, F-IF
More Graphing Stories **http://graphingstories.com** Site Developed by Dan Meyer (1-3 Stories)	F-BF, F-IF
Incredible Shrinking Dollar http://mrmeyer.com/threeacts/shrinkingdollar/ Task Developed by Dan Meyer	F – BF, F-IF
Domino Skyscraper http://mrmeyer.com/threeacts/dominoskyscraper/ Dan Meyer Three Act Task	F-BF, F-LE, F-IF