CCGPS Frameworks

Mathematics

CCGPS Pre-Calculus

Unit 1: Conics

Dr. John D. Barge, State School Superintendent
"Making Education Work for All Georgians"

Unit 1
Conics

TABLE OF CONTENTS

OVERVIEW

In this unit students will:

- Build upon the understanding of the algebraic representations of circles and parabolas.
- Develop the understanding of the geometric description and equations for the conic sections, ellipses and hyperbolas.

Although the units in this instructional framework emphasize key standards and big ideas at specific times of the year, routine topics such as estimation, mental computation, and basic computation facts should be addressed on an ongoing basis. Ideas related to the eight practice standards should be addressed constantly as well. To assure that this unit is taught with the appropriate emphasis, depth, and rigor, it is important that the tasks listed under "Evidence of Learning" be reviewed early in the planning process. A variety of resources should be utilized to supplement this unit. This unit provides much needed content information, but excellent learning activities as well. The tasks in this unit illustrate the types of learning activities that should be utilized from a variety of sources.

STANDARDS ADDRESSED IN THIS UNIT

Mathematical standards are interwoven and should be addressed throughout the year in as many different units and activities as possible in order to emphasize the natural connections that exist among mathematical topics.

Translate between the geometric description and the equation for a conic section.
MCC9-12.G.GPE.3 (+) Derive the equations of ellipses and hyperbolas given the foci, using the fact that the sum or difference of distances from the foci is constant.

RELATED STANDARDS

Translate between the geometric description and the equation for a conic section.
MCC9-12.G.GPE.1 Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.

MCC9-12.G.GPE.2 Derive the equation of a parabola given a focus and directrix.

STANDARDS FOR MATHEMATICAL PRACTICE

The Standards for Mathematical Practice describe varieties of expertise that mathematics educators at all levels should seek to develop in their students. These practices rest on important "processes and proficiencies" with longstanding importance in mathematics education. The first of these are the NCTM process standards of problem solving, reasoning and proof, communication, representation, and connections. The second are the strands of mathematical proficiency specified in the National Research Council's report *Adding It Up*: adaptive reasoning, strategic competence, conceptual understanding (comprehension of mathematical concepts, operations and relations), procedural fluency (skill in carrying out procedures flexibly, accurately, efficiently and appropriately), and productive disposition (habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy).

1. **Make sense of problems and persevere in solving them.** High school students start to examine problems by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. By high school, students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. They check their answers to problems using different methods and continually ask themselves, "Does this make sense?" They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

2. **Reason abstractly and quantitatively.** High school students seek to make sense of quantities and their relationships in problem situations. They abstract a given situation and represent it symbolically, manipulate the representing symbols, and pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Students use quantitative reasoning to create coherent representations of the problem at hand; consider the units involved; attend to the meaning of quantities, not just how to compute them; and know and flexibly use different properties of operations and objects.

3. **Construct viable arguments and critique the reasoning of others.** High school students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. High school students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. High school students learn to determine domains to which an argument applies, listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

4. **Model with mathematics.** High school students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. High school students making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

5. **Use appropriate tools strategically.** High school students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. High school students should be sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. They are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.

6. **Attend to precision.** High school students try to communicate precisely to others by using clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context. By the time they reach high school they have learned to examine claims and make explicit use of definitions.

7. **Look for and make use of structure.** By high school, students look closely to discern a pattern or structure. In the expression $x^2 + 9x + 14$, older students can see the 14 as 2×7 and the 9 as $2 + 7$. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 – 3(x – y)^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real

numbers *x* and *y*. High school students use these patterns to create equivalent expressions, factor and solve equations, and compose functions, and transform figures.

8. **Look for and express regularity in repeated reasoning.** High school students notice if calculations are repeated, and look both for general methods and for shortcuts. Noticing the regularity in the way terms cancel when expanding $(x - 1)(x + 1)$, $(x - 1)(x^2 + x + 1)$, and $(x - 1)(x^3 + x^2 + x + 1)$ might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, derive formulas or make generalizations, high school students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

Connecting the Standards for Mathematical Practice to the Standards for Mathematical Content

The Standards for Mathematical Practice describe ways in which developing student practitioners of the discipline of mathematics should engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle and high school years. Designers of curricula, assessments, and professional development should all attend to the need to connect the mathematical practices to mathematical content in mathematics instruction.

The Standards for Mathematical Content are a balanced combination of procedure and understanding. Expectations that begin with the word "understand" are often especially good opportunities to connect the practices to the content. Students who do not have an understanding of a topic may rely on procedures too heavily. Without a flexible base from which to work, they may be less likely to consider analogous problems, represent problems coherently, justify conclusions, apply the mathematics to practical situations, use technology mindfully to work with the mathematics, explain the mathematics accurately to other students, step back for an overview, or deviate from a known procedure to find a shortcut. In short, a missing mathematical knowledge effectively prevents a student from engaging in the mathematical practices.

In this respect, those content standards which set an expectation of understanding are potential "points of intersection" between the Standards for Mathematical Content and the Standards for Mathematical Practice. These points of intersection are intended to be weighted toward central and generative concepts in the school mathematics curriculum that most merit the time, resources, innovative energies, and focus necessary to qualitatively improve the curriculum, instruction, assessment, professional development, and student achievement in mathematics.

ENDURING UNDERSTANDINGS

- Conic sections are quadratic relations that can be expressed generally by the form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ and the comparison of the coefficients A and C reveal the specific type of conic.

- All conic sections are defined by the relationship of their locus of points to fixed points known as foci.
- Ellipses arise from a locus of points that represent a constant sum of distances from two fixed points (foci).
- Hyperbolas arise from a locus of points that represent a constant absolute value of difference of distances from two fixed points (foci).

ESSENTIAL QUESTIONS

- What role do foci play in the definition of conic quadratic relations?
- How can ellipses be defined in relation to their foci?
- How can hyperbolas be defined in relation to their foci?
- How can conic sections be identified by the A and C coefficients from the general form of quadratic relations?
- How can we solve real-world problems using what we know about conics?

CONCEPTS/SKILLS TO MAINTAIN

It is expected that students will have prior knowledge/experience related to the concepts and skills identified below. It may be necessary to pre-assess in order to determine if time needs to be spent on conceptual activities that help students develop a deeper understanding of these ideas.

- quantitative reasoning
- seeing the generalizability of relationships in building quadratic relations (and geometric concepts in general)
- using algebraic methods, such as completing the square, to change forms of equations
- see relationships between algebraic manipulation of equations and characteristics of corresponding graphs

SELECTED TERMS AND SYMBOLS

The following terms and symbols are often misunderstood. These concepts are not an inclusive list and should not be taught in isolation. However, due to evidence of frequent difficulty and misunderstanding associated with these concepts, instructors should pay particular attention to them and how their students are able to explain and apply them.

The definitions below are for teacher reference only and are not to be memorized by the students. Students should explore these concepts using models and real life examples. Students should understand the concepts involved and be able to recognize and/or demonstrate them with words, models, pictures, or numbers.

http://intermath.coe.uga.edu/dictnary/homepg.asp
Definitions and activities for these and other terms can be found on the Intermath website. Because Intermath is geared towards middle and high school, grade 3-5 students should be directed to specific information and activities.

- **Cone**: A three dimensional figure with a circular or elliptical base and one vertex.
- **Coplanar**: Set of points, lines, rays, line segments, etc., that lie in the same plane.
- **Ellipse:** A curved line forming a closed loop, where the sum of the distances from two points (foci) to every point on the line is constant.
- **Focus:** one of the fixed points from which the distances to any point of a given curve, such as an ellipse or parabola, are connected by a linear relation.
- **Hyperbola:** A plane curve having two branches, formed by the intersection of a plane with both halves of a right circular cone at an angle parallel to the axis of the cone. It is the locus of points for which the difference of the distances from two given points is a constant.
- **Locus of Points:** A group of points that share a property.
- **Plane**: One of the basic undefined terms of geometry. A plane goes on forever in all directions (in two-dimensions) and is "flat" (i.e., it has no thickness).

CLASSROOM ROUTINES

The importance of continuing the established classroom routines cannot be overstated. Daily routines must include quantitative reasoning and critical thinking. Students should work collaboratively to develop an authentic understanding of the mathematics in a course, and application-oriented activities using data analysis and model-building should be a regular occurrence. Mathematics should be presented in its many contexts so that students can see the true inter-disciplinary nature of the field, and it's absolutely essential role in providing the structure for so many seemingly non-related disciplines.

STRATEGIES FOR TEACHING AND LEARNING

- Students should be actively engaged by developing their own understanding.
- Mathematics should be represented in as many ways as possible by using graphs, tables, pictures, symbols and words.
- Interdisciplinary and cross curricular strategies should be used to reinforce and extend the learning activities.
- Appropriate manipulatives and technology should be used to enhance student learning.
- Students should be given opportunities to revise their work based on teacher feedback, peer feedback, and metacognition which includes self-assessment and reflection.
- Students should write about the mathematical ideas and concepts they are learning.
- Consideration of all students should be made during the planning and instruction of this unit. Teachers need to consider the following:
    - What level of support do my struggling students need in order to be successful with this unit?
    - In what way can I deepen the understanding of those students who are competent in this unit?
    - What real life connections can I make that will help my students utilize the skills practiced in this unit?

EVIDENCE OF LEARNING

By the conclusion of this unit, students should be able to demonstrate the following competencies:

- Identify the general and standard forms of the four types of conic sections.
- Use the method of completing the square to convert from general form to standard form of a conic equation.
- Understand conic sections pictorially as the intersection of a plane and a double-napped cone, algebraically as the result of specific quadratic equations, and geometrically as the relationship of the locus of points to foci.
- Use basic conic understanding to apply to realistic phenomena.

TASKS

The following tasks represent the level of depth, rigor, and complexity expected of all Pre-Calculus students in the culminating unit of the course. These tasks or tasks of similar depth and rigor should be used to demonstrate evidence of learning. It is important that all elements of a task be addressed throughout the learning process so that students understand what is expected of them. While some tasks are identified as a performance task, they may also be used for teaching and learning (learning task).

Scaffolding Task	Tasks that build up to the learning task.
Learning Task	Constructing understanding through deep/rich contextualized problem solving tasks.
Practice Task	Tasks that provide students opportunities to practice skills and concepts.
Performance Task	Tasks which may be a formative or summative assessment that checks for student understanding/misunderstanding and or progress toward the standard/learning goals at different points during a unit of instruction.
Culminating Task	Designed to require students to use several concepts learned during the unit to answer a new or unique situation. Allows students to give evidence of their own understanding toward the mastery of the standard and requires them to extend their chain of mathematical reasoning.
Formative Assessment Lesson (FAL)	Lessons that support teachers in formative assessment which both reveal and develop students' understanding of key mathematical ideas and applications. These lessons enable teachers and students to monitor in more detail their progress towards the targets of the standards.

Task Name	Task Type *Grouping Strategy*	Content Addressed
Our Only Focus: Circles & Parabolas Review	Learning/Review *Individual/Partner*	Graphs and equations of circles and parabolas
The Focus is the Foci: Ellipses and Hyperbolas	Learning *Individual/Partner*	Derive the equations of ellipses and hyperbolas given the foci, using the fact that the sum or difference of distances from the foci is constant.
A Conic Application	Performance *Individual/Partner*	Apply knowledge of ellipses to a realistic scenario.

Our Only Focus: Circles & Parabolas Review

Since this is a review circles and parabolas from Analytic Geometry, current Pre-Calculus standards do not apply, but the related standards from Analytic Geometry obviously do; they are given below.

Translate between the geometric description and the equation for a conic section.
MCC9-12.G.GPE.1 Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.

MCC9-12.G.GPE.2 Derive the equation of a parabola given a focus and directrix.

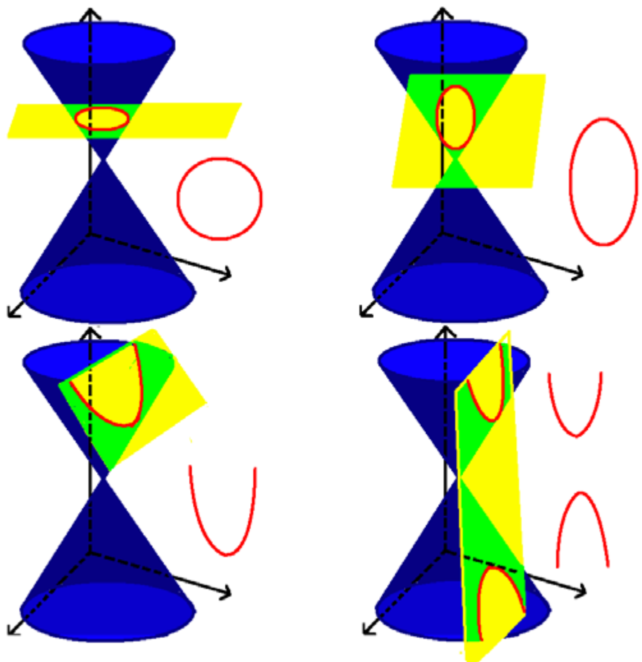
Introduction

It would be pedagogically awkward to begin a study of the ellipses and hyperbolas without a thorough review of their more basic cousins – circles and parabolas. First, for most students, it will have been two years since they last saw these two conic sections in Analytic Geometry, and no doubt many of the details have been forgotten. Those details, however, are important. For example, the general form of quadratic relations obviously applies to all conic sections. Even more importantly, the idea of a locus of points situated about a focus (or foci) is generalizable to all four of these planar graphs.

It is important here also to make a statement here about the notion of a focus for a circle. It is somewhat mathematically controversial to call the center of a circle its focus, but it is indeed where the two foci in this special case of an ellipse come together to define the relationship with the locus – and therefore, the center is presented as a focus in this activity. It is also up to the teacher to make things as generalizable as possible – for example, every conic can be defined by its relationship to two foci (including the circle scenario above or the fact that one of the foci of a parabola is a point at infinity), or, every conic can be defined by the relationship between a single focus and a single directrix. Also, the related idea of eccentricity is left out of our discussion. A teacher can pursue these ideas based on their own preferences and classroom context.

Our Only Focus: Circles & Parabolas Review

For most students, you last learned about conic sections in Analytic Geometry, which was a while ago. Before we begin looking back over the first two types of conic sections that you have already discovered, let's take a look at the geometric meaning of a conic section. First, why "conic"? Conic sections can be defined several ways, and what we'll focus on in this unit is deriving the formulas for the last two types of conic sections from special points called foci. But for the purpose of (re)introduction, the geometric meaning of "conic" comes into focus.



The reason we call these graphs conic sections is that they represent different slices of a double-napped cone. The diagram above shows how the different graphs can be "sliced off" of the figure. You should notice right away that the vast majority of these graphs are relations, not functions – in fact, only one case of parabolas (those featured in your past learning about quadratic functions) are actually functions. That doesn't limit the usefulness of these special planar graphs, however.

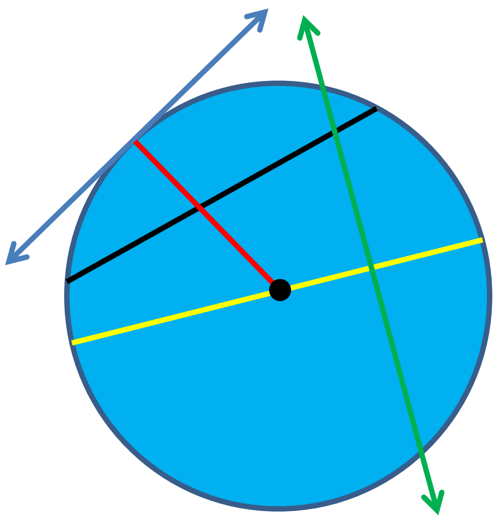
The General Form of a Quadratic Relation:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \text{ where } A, B, C \text{ cannot all be zero}$$

The graphs of all conic sections follow this same equation. It should be noted that for our more basic purposes, the B coefficient will always be zero.

Consider…the Circle

Circles should be old news to you, but just as a quick review let's see if you can remember their important parts.



For the circle to the left, label the following features on the diagram:

Center	***Black circle***
Diameter	***Yellow line segment***
Radius	***Red line segment***
Chord	***Black line segment***
Secant Line	***Green line***
Tangent Line	***Blue line***
Point of Tangency	***Where the red line segment meets the blue line***

Now let's try some very quick circle review using a few of these terms.

(a) What relationship do the points making up the graph of a circle have to the center?

All the points on the circle are equidistant from the center.

(b) What relationship do the radius and the diameter of a circle have?

The diameter has twice the length of the radius. Or "The diameter is twice the radius."

(c) What is a tangent line and what relationship does it have to the radius that it meets at the point of tangency?

A tangent line intersects the circle at exactly one place, known as the point of tangency. The radius that meets the tangent line at this point forms a right angle with the tangent line.

(d) If a circle has a diameter with endpoints $(-2,-5)$ and $(3,4)$, what is...

(i) the diameter of the circle?

Using the distance formula,

$$\sqrt{\left(3-(-2)\right)^2+\left(4-(-5)\right)^2}=\sqrt{106}$$

(ii) the center of the circle?

Using the midpoint formula,

$$\left(\frac{-2+3}{2},\frac{-5+4}{2}\right)=\left(\frac{1}{2},-\frac{1}{2}\right)$$

(iii) the radius of the circle?

$$\frac{\sqrt{106}}{2}$$

(iv) the slope of the radius from the center to $(3,4)$?

Using the slope formula,

$$\frac{-\frac{1}{2}-4}{\frac{1}{2}-3}=\frac{-\frac{9}{2}}{-\frac{5}{2}}=\frac{9}{5}$$

(v) the equation of the tangent line that intersects the circle at the point $(-2, -5)$?

The slope of the radius connecting the center to this point is the same as that found in part (iv) since both radii are two halves of the same diameter. And since the tangent line will meet this radius at a right angle, we can use a perpendicular slope to $\frac{9}{5}$, which is $-\frac{5}{9}$, and by substituting into the point-slope form of a line, we have

$$(y - (-5)) = -\frac{5}{9}(x - (-2))$$

$$(y + 5) = -\frac{5}{9}(x + 2)$$

$$y + 5 = -\frac{5}{9}x - \frac{10}{9}$$

$$y = -\frac{5}{9}x - \frac{55}{9}$$

If you answered (a) correctly, you know that the locus of points making up a circle are equidistant from the circle's center. This leads to an important idea about the center – it serves as the focus of the circle. The points making up the circle are all entirely dependent upon the location of that important focal point.

Now let's review the standard form of the equation describing a circle.

Standard Form of a Circle:

$$(x - h)^2 + (y - k)^2 = r^2 \text{ with center at } (h, k) \text{ and radius } r$$

This is the most useful form of a circle in terms of recognizing important pieces and for graphing and was the emphasis of your previous work with circles.

Let's try writing a few equations in standard form.

1. Write the equation for the circle with a diameter containing the endpoints $(-3, 0)$ and $(3, 0)$.

$$\boldsymbol{x^2 + y^2 = 9}$$

2. Write the equation for the entire set of points that are 4 units away from $(1, -5)$.

$$\boldsymbol{(x-1)^2 + (y+5)^2 = 16}$$

3. Write the equation of the circle with a radius from the center at $(2, 7)$ to an endpoint at $(6, 5)$.

$$\boldsymbol{(x-2)^2 + (y-7)^2 = 20}$$

And now let's review how to take a circle in a different form and change it to the more useable standard form. For example, let's look at the following:

$$x^2 + y^2 + 6x - 2y + 1 = 0$$

Notice that this circle is presented in the **general form** $Ax^2 + Cy^2 + Dx + Ey + F = 0$ where $A = 1, C = 1, D = 6, E = -2,$ and $F = 1$. As you work through the next set of problems, see if you recognize any patterns in the coefficients for general form, and then see if you can find other patterns using the general form equations for other conic sections. In any case, this general form is not useful in terms of graphing, or picking out the radius, diameter, or center. So we need to put the equation into standard form. To do this by completing the square, first group like variables together and move the constant to the other side of the equation.

$$x^2 + y^2 + 6x - 2y + 1 = 0$$

$$x^2 + 6x + y^2 - 2y = -1$$

Once we've gotten like variables together and sent the constant to the other side, we have to complete the square by taking the coefficient of the linear term for both variables, dividing it by 2, and squaring the quotients. Add both of these squares to both sides of your equation.

$$\left(x^2 + 6x + \left(\frac{6}{2}\right)^2\right) + \left(y^2 - 2y + \left(\frac{-2}{2}\right)^2\right) = -1 + \left(\frac{6}{2}\right)^2 + \left(\frac{-2}{2}\right)^2$$

$$(x^2 + 6x + 9) + (y^2 - 2y + 1) = 9$$

Now, all we must do is factor our two perfect square trinomials and we'll have standard form.

$$(x + 3)^2 + (y - 1)^2 = 9$$

Now we know that the circle has a center of $(-3, 1)$ and a radius of 3, facts not obvious from the original general form.

Put the following equations into standard form.

1. $x^2 + y^2 - 4x + 12y - 6 = 0$

$\boldsymbol{(x - 2)^2 + (y + 6)^2 = 46}$

2. $x^2 - 6x = y - y^2 + 7$

$\boldsymbol{(x - 3)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{65}{4}}$

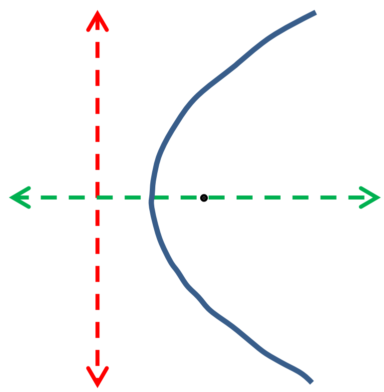
3. $\frac{7x^2}{3} + \frac{7y^2}{3} = 1$

$\boldsymbol{x^2 + y^2 = \frac{3}{7}}$

(Re)Presenting the…Parabola

The conic parabolas you learned about in Analytic Geometry were either functions (opened either up or down) or relations (opened either left or right). Let's do a little parabola review and see what you remember about these.

Label the following features on the sketch to the left.



Vertex: ***where axis of symmetry meets the parabola***

Focus: ***black dot***

Axis of Symmetry: ***green dotted line***

Directrix: ***red dotted line***

The directed distance p (label 2 different places): ***the distance from the vertex to the focus and from directrix to the vertex.***

Now let's see if you can answer some basic questions.

(a) What relationship does the locus of points forming a parabola have with the focus and directrix?

Each point on the parabola is equidistant from the focus and directrix.

(b) What relationship does the vertex have with the focus and the directrix?

The vertex the midpoint of the segment representing the (shortest distance from the focus to the directrix.

(c) What relationship does the directed distance p have with the focus and the directrix?

p is the distance from the vertex to the focus and from the directrix to the vertex.

Just as with circles, the most useable form for parabolas is standard form. Therefore, we need to know the following:

Standard Form of a Parabola and Related Information
With vertex (h, k) and directed distance from the vertex to the focus p:

$$Vertical\ Axis\ of\ Symmetry: (x-h)^2 = 4p(y-k)$$

If p is positive, the parabola opens up; if p is negative, the parabola opens down.

$$Horizontal\ Axis\ of\ Symmetry: (y-k)^2 = 4p(x-h)$$

If p is positive, the parabola opens to the right; if p is negative, the parabola opens to the left.

Let's try writing a few equations in standard form.

1. Write the equation of the parabola with a vertex at the origin and a focus at $(5, 0)$.

$$\boldsymbol{y^2 = 20x}$$

2. Write the equation of the parabola with focus at $(-3, 3)$ and directrix at $y = 9$.

$$\boldsymbol{(x+3)^2 = -12(y-6)}$$

3. Write the equation of the parabola that opens to the left, contains a distance of 5 between the focus and the directrix, and contains a vertex at $(9, 6)$.

$$\boldsymbol{(y-6)^2 = -10(x-9)}$$

Just as with circles, often you will be given either an equation for a parabola that is not in standard form for a parabola and you'll need to convert the equation to standard form. Consider the following equation of a parabola:

$$5y^2 - 6x + 10y - 7 = 0$$

This parabola has been written in general form. Using what we know about the coefficients from general form, we have $C = 5, D = -6, E = 10$, and $F = -7$. It's easy to see that the y term is squared, so either the parabola will open left or right, but beyond this, it's difficult to tell anything else about the relation. Therefore, once again, we will have to convert to standard form by manipulating terms and completing the square:

$$5y^2 - 6x + 10y - 7 = 0$$

$$5y^2 + 10y = 6x + 7$$

$$5\left(y^2 + 2y + \left(\frac{2}{2}\right)^2\right) = 6x + 7 + 5\left(\frac{2}{2}\right)^2$$

$$5(y^2 + 2y + 1) = 6x + 12$$

$$5(y + 1)^2 = 6(x + 2)$$

$$(y + 1)^2 = \frac{6}{5}(x + 2)$$

So what do we now know? Well, we know the vertex of the parabola is at $(-2, -1)$. We know the parabola opens to the right because p is positive. How do we know it's positive? Let's see…

Standard Form: $(y - k)^2 = 4p(x - h)$

so

$$4p = \frac{6}{5} \ so\ p = \frac{6}{20} = \frac{3}{10}$$

Therefore the focus is at $\left(-2 + \frac{3}{10}, -1\right) = \left(\frac{-17}{10}, -1\right)$ and the directrix would be at $x = -2 - \frac{3}{10}$ which simplifies to. $x = -\frac{23}{10}$

Convert the following equations of parabolas into standard form.

1. $x^2 + x - y = 5$

$\left(x + \frac{1}{2}\right)^2 = \left(y + \frac{21}{4}\right)$

2. $2y^2 + 16y = -x - 27$

$(y + 4)^2 = -\frac{1}{2}(x - 5)$

3. $x = -y^2 + 6y - 5$

$(\boldsymbol{y} - \mathbf{3})^{\mathbf{2}} = -(\boldsymbol{x} - \mathbf{4})$

Two more things before we go…

Circles and parabolas are from the past – they're not our focus now. But the next two conic sections are built upon your knowledge of these simplest of conics. Therefore, think about (and answer!) these two questions.

1. We know that the general form of a quadratic relation is

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

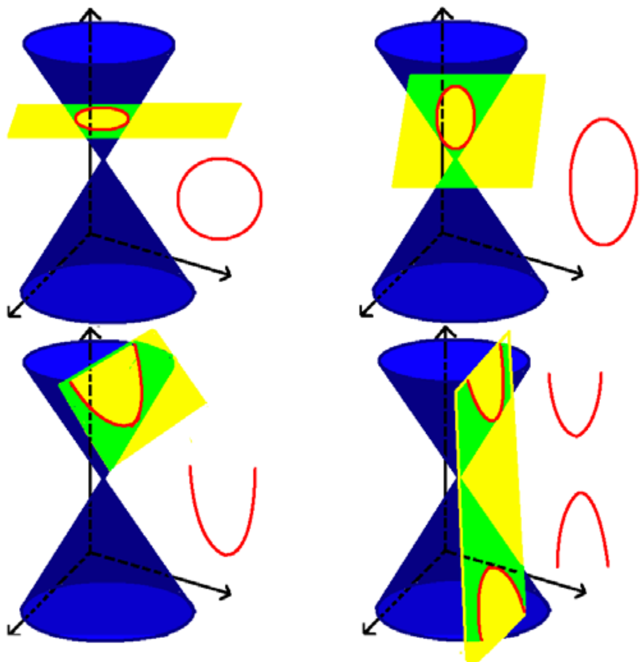
What relationship do the coefficients A and C have for a circle? For a parabola?

A and C are equal in equations of circles; parabolas have either A or C equal 0, but not both.

2. Why was this activity named "our only focus"?
Both of these conics have only one focus (the way that they are presented here). Teachers should see the introduction to this activity for further information.

Our Only Focus: Circles & Parabolas Review

For most students, you last learned about conic sections in Analytic Geometry, which was a while ago. Before we begin looking back over the first two types of conic sections that you have already discovered, let's take a look at the geometric meaning of a conic section. First, why "conic"? Conic sections can be defined several ways, and what we'll focus on in this unit is deriving the formulas for the last two types of conic sections from special points called foci. But for the purpose of (re)introduction, the geometric meaning of "conic" comes into focus.



The reason we call these graphs conic sections is that they represent different slices of a double-napped cone. The diagram above shows how the different graphs can be "sliced off" of the figure. You should notice right away that the vast majority of these graphs are relations, not functions – in fact, only one case of parabolas (those featured in your past learning about quadratic functions) are actually functions. That doesn't limit the usefulness of these special planar graphs, however.

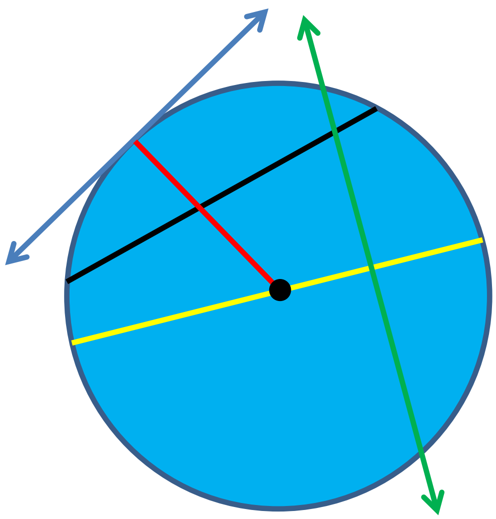
The General Form of a Quadratic Relation:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \ \ where \ \ A, B, C \ cannot \ all \ be \ zero$$

The graphs of all conic sections follow this same equation. It should be noted that for our more basic purposes, the B coefficient will always be zero.

Consider…the Circle

Circles should be old news to you, but just as a quick review let's see if you can remember their important parts.



For the circle to the left, label the following features on the diagram:

Center
Diameter
Radius
Chord
Secant Line
Tangent Line
Point of Tangency

Now let's try some very quick circle review using a few of these terms.

(a) What relationship do the points making up the graph of a circle have to the center?

(b) What relationship do the radius and the diameter of a circle have?

(c) What is a tangent line and what relationship does it have to the radius that it meets at the point of tangency?

(d) If a circle has a diameter with endpoints $(-2,-5)$ and $(3,4)$, what is...

(i) the diameter of the circle?

(ii) the center of the circle?

(iii) the radius of the circle?

(iv) the slope of the radius from the center to $(3,4)$?

(v) the equation of the tangent line that intersects the circle at the point $(-2,-5)$?

If you answered (a) correctly, you know that the locus of points making up a circle are equidistant from the circle's center. This leads to an important idea about the center – it serves as the <u>focus</u> of the circle. The points making up the circle are all entirely dependent upon the location of that important focal point.

Now let's review the standard form of the equation describing a circle.

Standard Form of a Circle:

$$(x-h)^2+(y-k)^2=r^2 \text{ with center at } (h,k) \text{ and radius } r$$

This is the most useful form of a circle in terms of recognizing important pieces and for graphing and was the emphasis of your previous work with circles.

Let's try writing a few equations in standard form.

1. Write the equation for the circle with a diameter containing the endpoints $(-3,0)$ and $(3,0)$.

2. Write the equation for the entire set of points that are 4 units away from $(1,-5)$.

3. Write the equation of the circle with a radius from the center at $(2,7)$ to an endpoint at $(6,5)$.

And now let's review how to take a circle in a different form and change it to the more useable standard form. For example, let's look at the following:

$$x^2+y^2+6x-2y+1=0$$

Notice that this circle is presented in the **general form** $Ax^2+Cy^2+Dx+Ey+F=0$ where $A=1, C=1, D=6, E=-2$, and $F=1$. As you work through the next set of problems, see if you recognize any patterns in the coefficients for general form, and then see if you can find other patterns using the general form equations for other conic sections. In any case, this general form is not useful in terms of graphing, or picking out the radius, diameter, or center. So we need to put the equation into standard form. To do this by completing the square, first group like variables together and move the constant to the other side of the equation.

$$x^2+y^2+6x-2y+1=0$$

$$x^2 + 6x + y^2 - 2y = -1$$

Once we've gotten like variables together and sent the constant to the other side, we have to complete the square by taking the coefficient of the linear term for both variables, dividing it by 2, and squaring the quotients. Add both of these squares to both sides of your equation.

$$\left(x^2 + 6x + \left(\frac{6}{2}\right)^2\right) + \left(y^2 - 2y + \left(\frac{-2}{2}\right)^2\right) = -1 + \left(\frac{6}{2}\right)^2 + \left(\frac{-2}{2}\right)^2$$

$$(x^2 + 6x + 9) + (y^2 - 2y + 1) = 9$$

Now, all we must do is factor our two perfect square trinomials and we'll have standard form.

$$(x + 3)^2 + (y - 1)^2 = 9$$

Now we know that the circle has a center of $(-3, 1)$ and a radius of 3, facts not obvious from the original general form.

Put the following equations into standard form.

1. $x^2 + y^2 - 4x + 12y - 6 = 0$

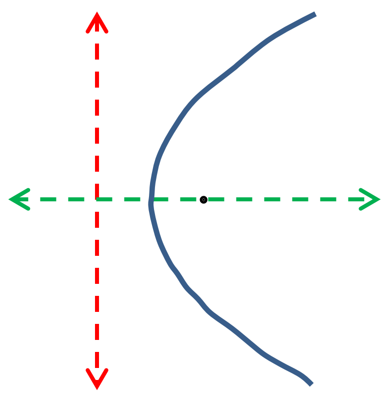
2. $x^2 - 6x = y - y^2 + 7$

3. $\frac{7x^2}{3} + \frac{7y^2}{3} = 1$

(Re)Presenting the…Parabola

The conic parabolas you learned about in Analytic Geometry were either functions (opened either up or down) or relations (opened either left or right). Let's do a little parabola review and see what you remember about these.

The conic parabolas you learned about in Analytic Geometry were either functions (opened either up or down) or relations (opened either left or right). Let's do a little parabola review and see what you remember about these.



Label the following features on the sketch to the left.

Vertex
Focus
Axis of Symmetry
Directrix
The directed distance p (label 2 different places)

Now let's see if you can answer some basic questions.

(a) What relationship does the locus of points forming a parabola have with the focus and directrix?

(b) What relationship does the vertex have with the focus and the directrix?

(c) What relationship does the directed distance p have with the focus and the directrix?

Just as with circles, the most useable form for parabolas is standard form. Therefore, we need to know the following:

Standard Form of a Parabola and Related Information
With vertex (h, k) and directed distance from the vertex to the focus p:

$$Vertical\ Axis\ of\ Symmetry: (x-h)^2 = 4p(y-k)$$

If p is positive, the parabola opens up; if p is negative, the parabola opens down.

$$Horizontal\ Axis\ of\ Symmetry: (y-k)^2 = 4p(x-h)$$

If p is positive, the parabola opens to the right; if p is negative, the parabola opens to the left.

Let's try writing a few equations in standard form.

1. Write the equation of the parabola with a vertex at the origin and a focus at $(5, 0)$.

2. Write the equation of the parabola with focus at $(-3, 3)$ and directrix at $y = 9$.

3. Write the equation of the parabola that opens to the left, contains a distance of 5 between the focus and the directrix, and contains a vertex at $(9, 6)$.

Just as with circles, often you will be given either an equation for a parabola that is not in standard form and you'll need to convert the equation to standard form. Consider the following equation of a parabola:

$$5y^2 - 6x + 10y - 7 = 0$$

This parabola has been written in general form. Using what we know about the coefficients from general form, we have $C = 5, D = -6, E = 10$, and $F = -7$. It's easy to see that the y term is squared, so either the parabola will open left or right, but beyond this, it's difficult to tell anything else about the relation. Therefore, once again, we will have to convert to standard form by manipulating terms and completing the square:

$$5y^2 - 6x + 10y - 7 = 0$$

$$5y^2 + 10y = 6x + 7$$

$$5\left(y^2 + 2y + \left(\frac{2}{2}\right)^2\right) = 6x + 7 + 5\left(\frac{2}{2}\right)^2$$

$$5(y^2 + 2y + 1) = 6x + 12$$

$$5(y + 1)^2 = 6(x + 2)$$

$$(y + 1)^2 = \frac{6}{5}(x + 2)$$

So what do we now know? Well, we know the vertex of the parabola is at. $(-2, -1$. We know the parabola opens to the right because p is positive. How do we know it's positive? Let's see…

Standard Form: $(y - k)^2 = 4p(x - h)$

so

$$4p = \frac{6}{5} \; so \; p = \frac{6}{20} = \frac{3}{10}$$

Therefore the focus is at $\left(-2 + \frac{3}{10}, -1\right) = \left(\frac{-17}{10}, -1\right)$ and the directrix would be at $x = -2 - \frac{3}{10}$ which simplifies to. $x = -\frac{23}{10}$

Convert the following equations of parabolas into standard form.

1. $x^2 + x - y = 5$

2. $2y^2 + 16y = -x - 27$

3. $x = -y^2 + 6y - 5$

Two more things before we go…

Circles and parabolas are from the past – they're not our focus now. But the next two conic sections are built upon your knowledge of these simplest of conics. Therefore, think about (and answer!) these two questions.

1. We know that the general form of a quadratic relation is

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

What relationship do the coefficients A and C have for a circle? For a parabola?

2. Why was this activity named "our only focus"?

The Focus is the Foci: Ellipses and Hyperbolas

Common Core Standards

Translate between the geometric description and the equation for a conic section.
MCC9-12.G.GPE.3 (+) Derive the equations of ellipses and hyperbolas given the foci, using the fact that the sum or difference of distances from the foci is constant.

Introduction

This task is designed to be an introductory guide to ellipses and hyperbolas, focusing on conics sections pictorially, algebraically, and geometrically. There aren't a large number of problems given, and the ones that are presented ask students to look at conic equations algebraically and graphically. This task is not meant as a substitute for other instruction or assignments that would be given in your classroom in support of this unit.

The focus of the standard for this unit is not to present conics as a vast memorization exercise, but emphasizes that students understand how the locus of points that make up ellipses and hyperbolas relate to the fixed points known as foci. The task begins with an often-used construction activity for ellipses using string, paper, and a pencil. While this is certainly not a novel activity, it is a simple, quick, and efficient way of getting students to think of conic sections beyond formulas that they may never understand and often only memorize or use their algebraic intuition to muddle through. The rest of the task is developed on this simple foundation as an introduction to the importance of foci.

Of course, as always with conic sections, algebra is needed, especially to convert between forms and to find some specific important details of conic graphs. Some of these problems reinforce the algebraic manipulation and logic that many students are relatively weak at.

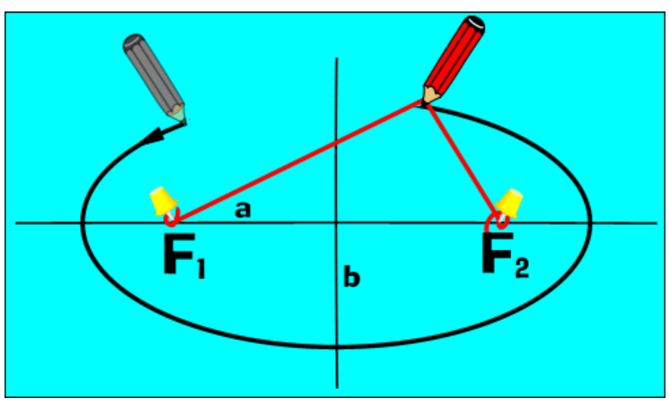
It is left up to the teacher here to encourage students to look at the connections between the A and C coefficients from the general form of quadratic relations, and to mix the four types of conics together to allow students to show their ability to distinguish between types, both in standard and general form. Also, the teacher needs to make sure that students understand that a circle is not really its own category of conic, but is a special case of ellipses. Here, a discussion of eccentricity and/or the idea of the two foci of an ellipse coming together at the center of a circle due to the equal lengths of its axes of symmetry may be helpful.

The Focus is the Foci: Ellipses and Hyperbolas

Ellipses and their Foci

The first type of quadratic relation we want to discuss is an ellipse. In terms of its "conic" definition, you can see how a plane would intersect with a cone, making the ellipse below.

We're going to start our study of ellipses by doing a very basic drawing activity. You should have two thumb tacks, a piece of string, a piece of cardboard, and a pencil. Attach the string via the two tacks to the piece of cardboard. Make sure to leave some slack in the string when you pin the ends down so that you can actually draw your outline! Trace out the ellipse by moving the pencil around as far as it will go with the string, making sure that the string is held tight against the pencil.



Of course, you should notice that the shape that results from this construction is an oval. (The term oval is not precise and includes many closed rounded shapes. This particular oval is an ellipse.)

(a) What do the two thumbtacks represent in this activity?

The two thumbtacks represent foci of the ellipse.

(b) A "locus" of points is a set of points that share a property. Thinking about the simple activity that you just completed, what is the property shared by the entire set of points that make up the ellipse?

An ellipse is the entire set of points on a plane where the sum of the distances from two fixed foci is a constant.

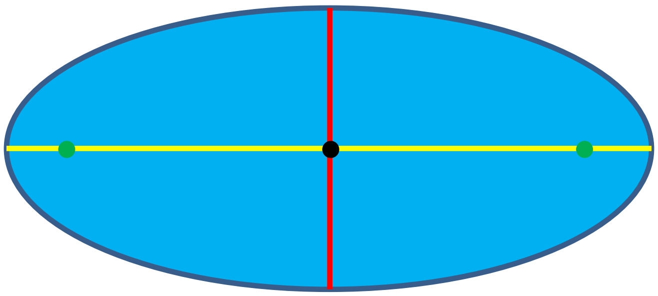
c) Move the string and consider other ellipses created in this manner with different foci. How does the placement of the foci affect the size of the ellipse? How do you know?

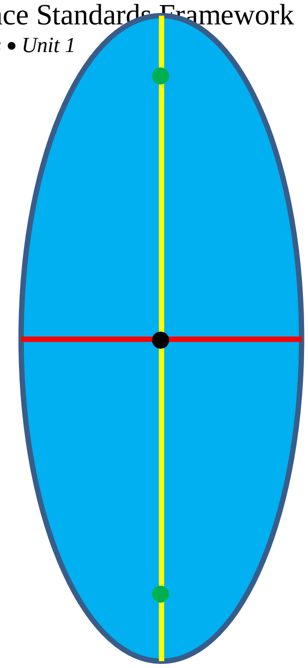
When the foci are farther apart, the ellipse is more elongated. When the foci are closer together, the ellipse more closely resembles a circle.

d) What is the length of the string in relation to these ellipses?

The length of the string is the same as the length of the major axis of the ellipse.

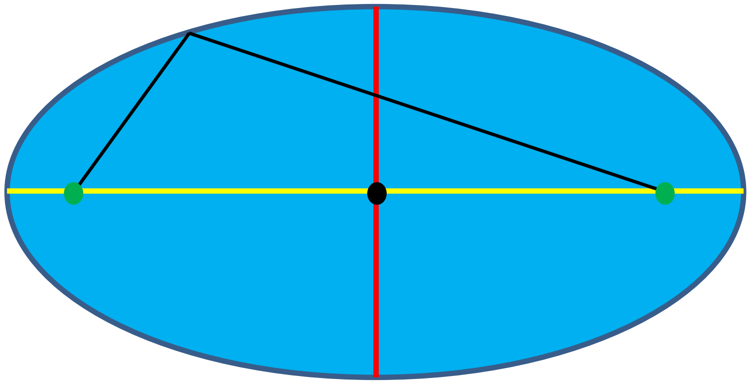
Now let's look at an ellipse.



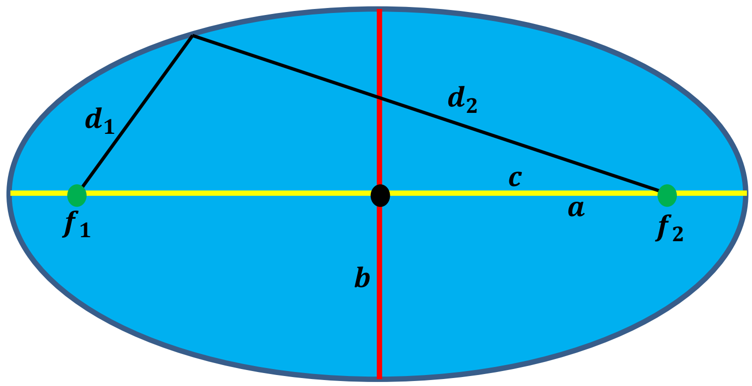


There are two types of ellipses that we'll be interested in during this unit – a horizontally oriented ellipse (left) and a vertically oriented ellipse (right). (Like all conic sections, these relations can be rotated diagonally if they contain the Bxy term from the general form of a quadratic relation, $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, but we won't be dealing with these right now – they'll come up in Calculus later on.)

What is the primary difference here? It's a matter of the major axis. Since an ellipse contains two axes of symmetry, the major axis is the longer, and the minor axis is the shorter. The major axis contains the two foci of the ellipse and has vertices as its endpoints. (The endpoints of the minor axis are called the co-vertices.) And speaking of foci, the green dots that you see represent the foci (the thumbtacks you just used), so let's go back and look at how an ellipse can be constructed (or defined) by using its two foci.



What you will hopefully recognize here is that the black lines represent where the string would have been as you were tracing with your pencil. And if you were able to answer (b) above correctly, you already know the relationship that binds the points of an ellipse together – the sum of the distances from each focus to any point on the ellipse remains constant. Another important piece of information that you may have noticed is that when you took your pencil and traced to one of the vertices (endpoints of the major axis), the entire length of string was being used going in a single direction. Therefore, the length of the string ended up being the length of your major axis! So let's fill this ellipse in with some important information.



Let's define what we see in the diagram:

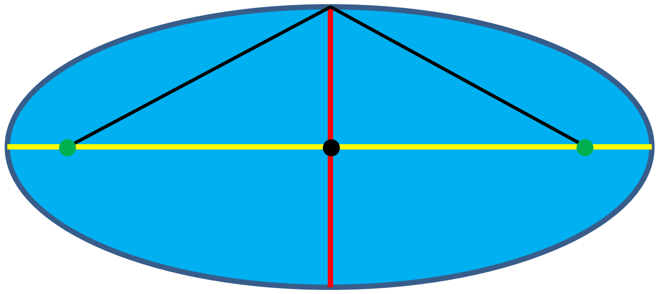
- $\boldsymbol{a}$ is half the length of the major axis
- $\boldsymbol{b}$ is half the length of the minor axis
- $\boldsymbol{c}$ is the length from the center to a focus
- $\boldsymbol{d_1}$ is the distance between the first focus (f_1) and the point of interest
- $\boldsymbol{d_2}$ is the distance between the second focus (f_2) and the point of interest

Using these pieces of an ellipse, we can write out some important facts about this planar curve.

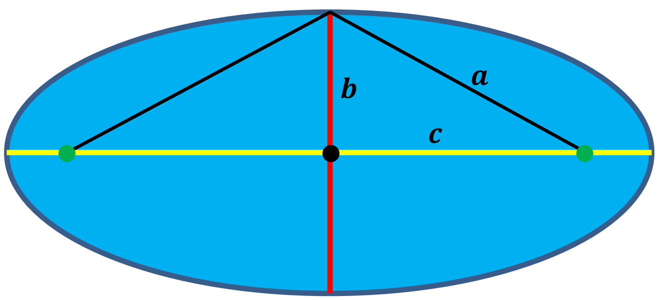
- The length of the major axis is $2a$.
- The length of the minor axis is $2b$.
- The distance between f_1 and f_2 is $2c$.
- $d_1 + d_2 = C$ where C is a constant. This is true regardless of the individual values of d_1 and d_2. What is the value of this constant? *2a*

There's one other important relationship among these variables that we need to explore. Remember that we concluded that the length of your string was the length of the major axis?

That has some important implications. Let's look at the diagram below.



Here's what we can deduce from the information we have thus far. When the string was in this position, then d_1 and d_2 were of equal length, forming two right triangles with the axes of the ellipse. Since we already know that the entire string length is equal to the major axis length, $2a$, that leads to an important conclusion, namely that $\boldsymbol{d_1 + d_2 = 2a}$. Since d_1 and d_2 are of equal length at this position, that means they both equal a, so we can see the following…



Since we're dealing with a right triangle, the Pythagorean Theorem applies, so

$$\boldsymbol{a^2 = b^2 + c^2}$$

or, to rearrange…

$$\boldsymbol{c^2 = a^2 - b^2}$$

Geometric Definition of an Ellipse:

The set of all points in a plane such that the sum of the distances from two fixed points (foci) is constant.

Standard Form of an Ellipse with Center $(\boldsymbol{h}, \boldsymbol{k})$:

$$Horizontal\ Major\ Axis: \frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

$$Vertical\ Major\ Axis: \frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$$

where $a > b$ and $c^2 = a^2 - b^2$

Teacher Note: There is quite a jump from the definition of the ellipse to this formula. To help make the connections first either A) derive the formula for the case with center at the origin using the definition and then ask students to translate remembering their translation skills with circles or else B) look at the equation of a circle, divide it by the square of the radius and then ask how we might alter this to allow for different radii in the x and y directions.

Just as with circles and parabolas, we often have to write the equation of an ellipse in standard form (as always, a more useful form) when it is given in another form. And once again, we'll be using the method of completing the square to convert to standard form. For example, if given the equation

$$25x^2 + 9y^2 - 200x + 18y + 184 = 0$$

we should recognize this equation as being in general form. We will now convert to standard form by completing the square.

$$25x^2 - 200x + 9y^2 + 18y = -184$$

$$25(x^2 - 8x) + 9(y^2 + 2y) = -184$$

$$25\left(x^2 - 8x + \left(\frac{8}{2}\right)^2\right) + 9\left(y^2 + 2y + \left(\frac{2}{2}\right)^2\right) = -184 + 25\left(\frac{8}{2}\right)^2 + 9\left(\frac{2}{2}\right)^2$$

$$25(x-4)^2 + 9(y+1)^2 = 225$$

$$\frac{25(x-4)^2}{225} + \frac{9(y+1)^2}{225} = 1$$

$$\frac{(x-4)^2}{9} + \frac{(y+1)^2}{25} = 1$$

So from our standard form, we know that the center of the ellipse is $(4, -1)$. We know the ellipse has a vertical major axis (since the denominator is larger under the y-term) and we know that $a = 5$ and $b = 3$. Therefore, the vertices would be at $(4, -1 + 5)$ and $(4, -1 - 5)$, simplifying to $(4, 4)$ and $(4, -6)$ and the co-vertices would be at $(4 + 3, -1)$ and $(4 - 3, -1)$ which simplifies to (7,-1) and (1,-1). To find the coordinates of the foci, we'd do the following:

$$c^2 = a^2 - b^2$$

$$c^2 = 25 - 9 = 16$$

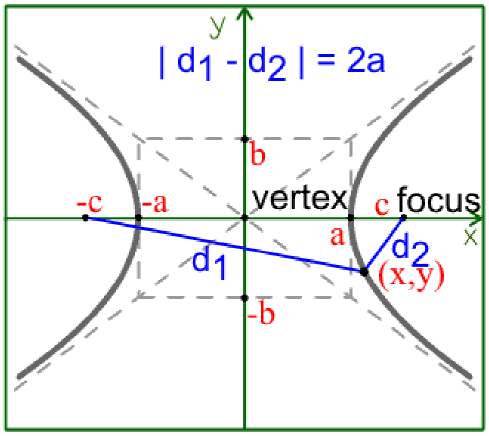
$$c = 4$$

So the foci are at $(4, -1 + 4)$ and $(4, -1 - 4)$, simplifying to $(4, 3)$ and $(4, -5)$.

Hyperbolas and their Foci

Although hyperbolas and ellipses are quite different, their formulas and foci relationships are similar, making it is easy to confuse their characteristics. Be careful when working with both of these conics.

We spent quite a bit of time deriving the definition of an ellipse from the relationship of its locus to its foci, and we're going to introduce hyperbolas with the same type of thinking, but we'll be brief.



Although on a smaller scale, the graph of the hyperbola looks like an inverted ellipse, on a larger scale, the hyperbola extends forever and approaches 2 intersecting lines called asymptotes. Remember that we define an ellipse as the set of all points on a plane where the sum of the distances from two fixed points called foci is a constant. The relationship of a hyperbola to its foci is slightly different. Notice the blue line segments representing d_1 and d_2. These are no longer being added to obtain a constant – they are now being subtracted!

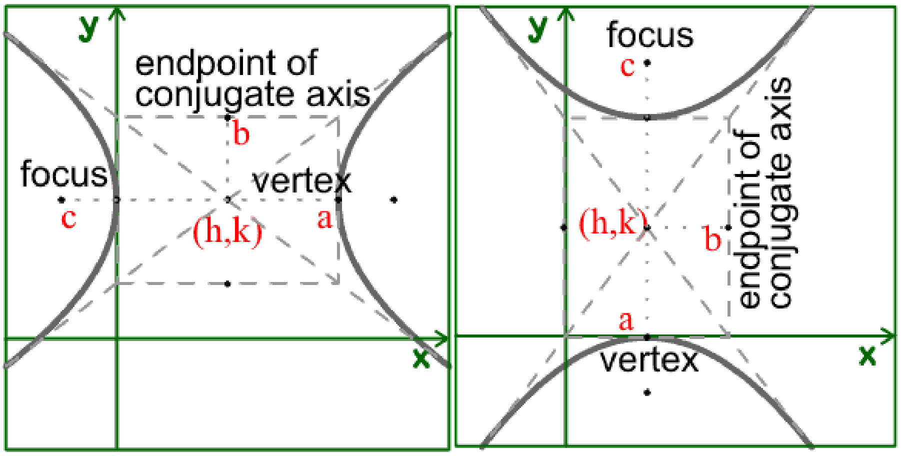
Geometric Definition of a Hyperbola:

The set of all points in a plane such that the absolute value of the difference of the distances from two fixed points (foci) is constant.

Note that $|d_1 - d_2| = 2a$, and a is still the distance from the center to a vertex, but we no longer call the axis containing the vertices and foci the major axis. We now call the segment joining the vertices the transversal axis, and it no longer must be the longer of the two axes. The other axis of symmetry for a hyperbola contains the conjugate axis, and these 2 axes bisect each other. Here are some other important characteristics of the graphs of hyperbolas:

- The center is the starting point at (h, k).
- The transverse axis contains the foci and the vertices.
- Transverse axis length = $2a$. This is also the constant that the difference of the distances must be.
- Conjugate axis length = $2b$.
- Distance between foci = $2c$.
- The foci are within the curve.
- Since the foci are the farthest away from the center, c is the largest of the three lengths, and the Pythagorean relationship is: $c^2 = a^2 + b^2$.

The two types of hyperbolas that we will study are: horizontally (left) and vertically (right) oriented.



The Standard Form of a Hyperbola with Center (h,k):

$$Horizontal\ Transverse\ Axis: \frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$

Since the change in y is b and the change in x is a, the slope of the asymptotes will be $\pm\frac{b}{a}$. The equations of the asymptotes will be $(y-k)=\pm\frac{b}{a}(x-h)$.

$$Vertical\ Transverse\ Axis: \frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$$

Since the change in y is a and the change in x is b, the slope of the asymptotes will be $\pm\frac{a}{b}$. The equations of the asymptotes will be $(y-k)=\pm\frac{a}{b}(x-h)$.

It's probably not surprising that, given the two focal distances are subtracted for hyperbolas instead of added, the standard form of a hyperbola involved subtraction instead of addition (like circles and ellipses).

And just as with circles, parabolas, and ellipses, we sometimes have to take a hyperbola written in another form and convert it to standard form in order to pick out the necessary information to graph the relation accurately. For example, consider the general form of this hyperbola:

$$9x^2 - 4y^2 + 90x + 32y + 197 = 0$$

$$9x^2 + 90x - 4y^2 + 32y = -197$$

As always, we need to complete the square in order to convert forms.

$$9(x^2 + 10x) - 4(y^2 - 8y) = -197$$

$$9\left(x^2 + 10x + \left(\frac{10}{2}\right)^2\right) - 4\left(y^2 - 8y + \left(\frac{8}{2}\right)^2\right) = -197 + 9\left(\frac{10}{2}\right)^2 - 4\left(\frac{8}{2}\right)^2$$

$$9(x+5)^2 - 4(y-4)^2 = -36$$

$$\frac{9(x+5)^2}{-36} - \frac{4(y-4)^2}{-36} = \frac{-36}{-36}$$

$$-\frac{(x+5)^2}{4} + \frac{(y-4)^2}{9} = 1$$

or

$$\frac{(y-4)^2}{9} - \frac{(x+5)^2}{4} = 1$$

So now we know that $a = 3$ and $b = 2$ and that the center of this hyperbola is $(-5, 4)$ and, since it has a vertical transverse axis, the vertices of the hyperbola are at $(-5, 4+3)$ and $(-5, 4-3)$, which become $(-5, 7)$ and $(-5, 1)$. To find the foci, we use

$$c^2 = a^2 + b^2$$

$$c^2 = 9 + 4 = 13$$

$$c = \sqrt{13}$$

So the foci are at $\left(-5, 4 + \sqrt{13}\right)$ and $\left(-5, 4 - \sqrt{13}\right)$.

So now all we need to do is find the asymptotes. Again, we know that this hyperbola has a vertical transverse axis, and therefore the slope of the asymptotes will be $\pm\frac{\Delta y}{\Delta x} = \pm\frac{3}{2}$. Therefore,

$$(y-4) = \pm\frac{3}{2}(x-(-5))$$

$$y-4 = \frac{3}{2}x + \frac{15}{2} \text{ and } y-4 = -\frac{3}{2}x - \frac{15}{2}$$

Therefore, the equations of the asymptotes are

$$y = \frac{3}{2}x + \frac{23}{2} \text{ and } y = -\frac{3}{2}x - \frac{7}{2}$$

And now it's your turn…

For the following, put the equation in standard form, label the important pieces, and sketch the graph of the relation.

1. $4x^2 + 9y^2 - 16x + 90y + 205 = 0$

$\frac{(x-2)^2}{9} + \frac{(y+5)^2}{4} = 1$

2. $100x^2 + 36y^2 > 3600$

$\frac{x^2}{36} + \frac{y^2}{100} > 1$

3. $9x^2 - 4y^2 - 54x - 16y - 79 = 0$

$\frac{(x-3)^2}{16} - \frac{(y+2)^2}{36} = 1$

4. $25x^2 - 4y^2 + 200x - 8y + 796 = 0$

$-\frac{(x+4)^2}{16} + \frac{(y+1)^2}{100} = 1$

or $\frac{(y+1)^2}{100} - \frac{(x+4)^2}{16} = 1$

5. Write the equation of a hyperbola whose center is at the origin, has a horizontal transverse axis and has asymptotes of $y = \pm\frac{5}{7}x$.

$$\frac{x^2}{49} - \frac{y^2}{25} = 1$$

6. Write the equation of the ellipse with major axis of length 12 and foci (3, 0) and (-3, 0).

$$c^2 = a^2 - b^2$$

$$9 = 36 - b^2$$

$$27 = b^2$$

$$\frac{x^2}{36} + \frac{y^2}{27} = 1$$

7. Write the equation of a hyperbola with asymptote $y - 2 = \frac{1}{3}(x + 4)$ and vertical transverse axis.

$$\frac{(y-2)^2}{1} - \frac{(x+4)^2}{9} = 1$$

Can you write the equation of another such hyperbola?

Any hyperbola of the form:

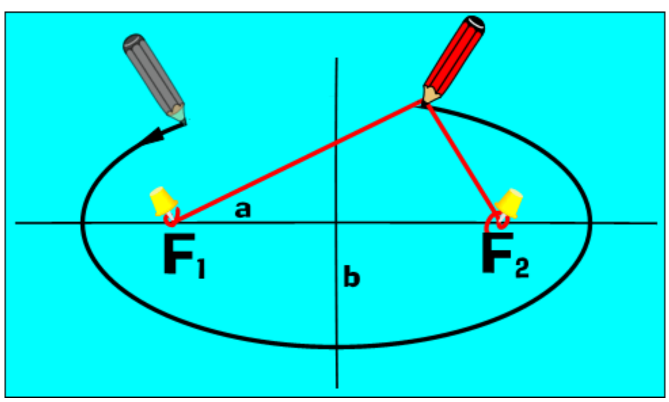
$$\frac{(y-2)^2}{1k} - \frac{(x+4)^2}{9k} = 1$$

The Focus is the Foci: Ellipses and Hyperbolas

Ellipses and their Foci

The first type of quadratic relation we want to discuss is an ellipse. In terms of its "conic" definition, you can see how a plane would intersect with a cone, making the ellipse below.

We're going to start our study of ellipses by doing a very basic drawing activity. You should have two thumb tacks, a piece of string, a piece of cardboard, and a pencil. Attach the string via the two tacks to the piece of cardboard. Make sure to leave some slack in the string when you pin the ends down so that you can actually draw your outline! Trace out the ellipse by moving the pencil around as far as it will go with the string, making sure that the string is held tight against the pencil.



Of course, you should notice that the shape that results from this construction is an oval.

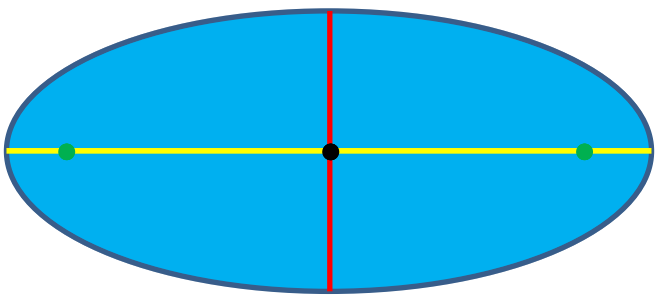
(a) What do the two thumbtacks represent in this activity?

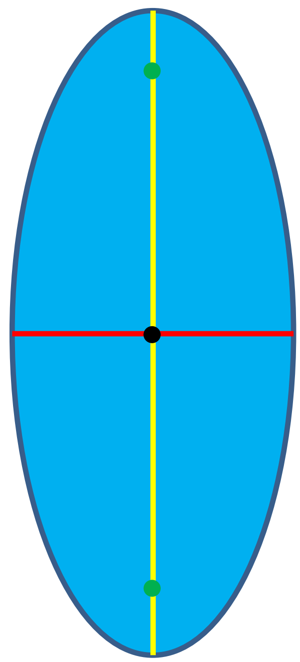
(b) A "locus" of points is a set of points that share a property. Thinking about the simple activity that you just completed, what is the property shared by the entire set of points that make up the ellipse?

c) Move the string and consider other ellipses created in this manner with different foci. How does the placement of the foci affect the size of the ellipse? How do you know?

d) What is the length of the string in relation to these ellipses?

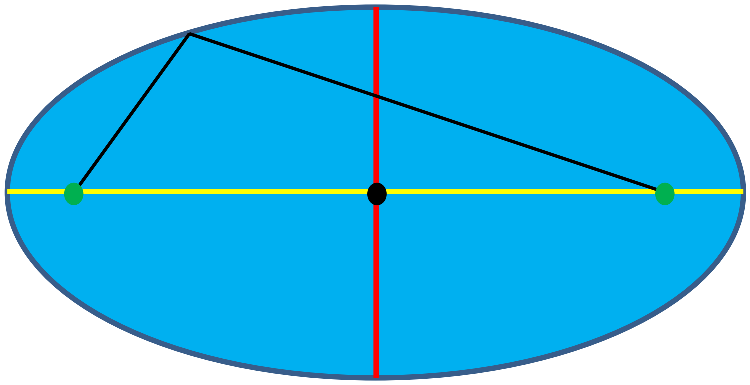
Now let's look at an ellipse.



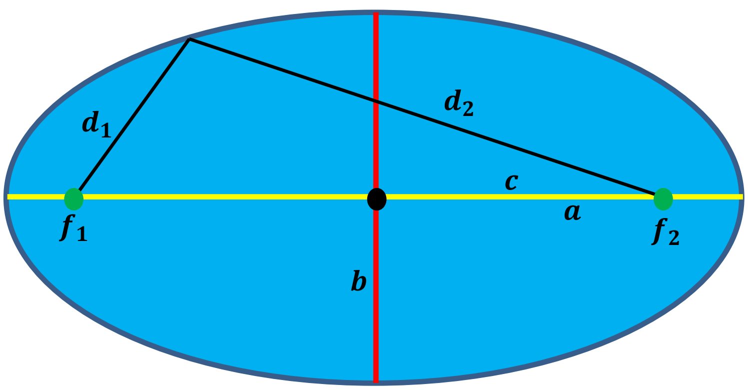


There are two types of ellipses that we'll be interested in during this unit – a horizontally oriented ellipse (left) and a vertically oriented ellipse (right). (Like all conic sections, these relations can be rotated diagonally if they contain the Bxy term from the general form of a quadratic relation, $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, but we won't be dealing with these right now – they'll come up in Calculus later on.)

What is the primary difference here? It's a matter of the major axis. Since an ellipse contains two axes of symmetry, the major axis is the longer, and the minor axis is the shorter. The major axis contains the two foci of the ellipse and has vertices as its endpoints. The endpoints of the minor axis are called the co-vertices. And speaking of foci, the green dots that you see represent the foci (the thumbtacks you just used), so let's go back and look at how an ellipse can be constructed (or defined) by using its two foci.



What you will hopefully recognize here is that the black lines represent where the string would have been as you were tracing with your pencil. And if you were able to answer (b) above correctly, you already know the relationship that binds the points of an ellipse together – the sum of the distances from each focus to any point on the ellipse remains constant. Another important piece of information that you may have noticed is that when you took your pencil and traced to one of the vertices (endpoints of the major axis), the entire length of string was being used going in a single direction. Therefore, the length of the string ended up being the length of your major axis! So let's fill this ellipse in with some important information.



Let's define what we see in the diagram:

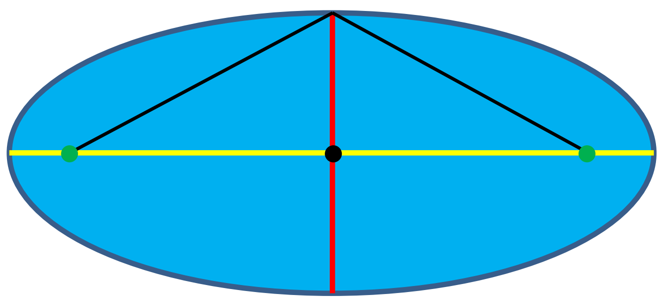
- a is half the length of the major axis
- b is half the length of the minor axis

- $\boldsymbol{c}$ is the length from the center to a focus
- $\boldsymbol{d_1}$ is the distance between the first focus (f_1) and the point of interest
- $\boldsymbol{d_2}$ is the distance between the second focus (f_2) and the point of interest

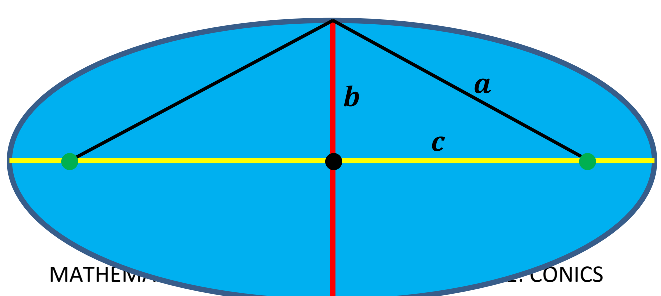
Using these pieces of an ellipse, we can write out some important facts about this planar curve.

- The length of the major axis is $2a$.
- The length of the minor axis is $2b$.
- The distance between f_1 and f_2 is $2c$.
- $d_1 + d_2 = C$ where $C$is a constant. This is true regardless of the individual values of d_1 and d_2. What is the value of this constant?

There's one other important relationship among these variables that we need to explore. Remember that we concluded that the length of your string was the length of the major axis? That has some important implications. Let's look at the diagram below.



Here's what we can deduce from the information we have thus far. When the string was in this position, then d_1 and d_2 were of equal length, forming two right triangles with the axes of the ellipse. Since we already know that the entire string length is equal to the major axis length, $2a$, that leads to an important conclusion, namely that $\boldsymbol{d_1 + d_2 = 2a}$. Since d_1 and d_2 are of equal length at this position, that means they both equal a, so we can see the following…



Since we're dealing with a right triangle, the Pythagorean Theorem applies, so

$$\boldsymbol{a^2 = b^2 + c^2}$$

or, to rearrange…

$$\boldsymbol{c^2 = a^2 - b^2}$$

Geometric Definition of an Ellipse:

The set of all points in a plane such that the sum of the distances from two fixed points (foci) is constant.

Standard Form of an Ellipse with Center $(\boldsymbol{h}, \boldsymbol{k})$:

$Horizontal\ Major\ Axis$: $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$

$Vertical\ Major\ Axis$: $\dfrac{(x-h)^2}{b^2} + \dfrac{(y-k)^2}{a^2} = 1$

where $a > b$ and $c^2 = a^2 - b^2$

Just as with circles and parabolas, we often have to write the equation of an ellipse in standard form (as always, a more useful form) when it is given in another form. And once again, we'll be using the method of completing the square to convert to standard form. For example, if given the equation

$$25x^2 + 9y^2 - 200x + 18y + 184 = 0$$

we should recognize this equation as being in general form. We will now convert to standard form by completing the square.

$$25x^2 - 200x + 9y^2 + 18y = -184$$

$$25(x^2 - 8x) + 9(y^2 + 2y) = -184$$

$$25\left(x^2 - 8x + \left(\frac{8}{2}\right)^2\right) + 9\left(y^2 + 2y + \left(\frac{2}{2}\right)^2\right) = -184 + 25\left(\frac{8}{2}\right)^2 + 9\left(\frac{2}{2}\right)^2$$

$$25(x-4)^2 + 9(y+1)^2 = 225$$

$$\frac{25(x-4)^2}{225} + \frac{9(y+1)^2}{225} = 1$$

$$\frac{(x-4)^2}{9} + \frac{(y+1)^2}{25} = 1$$

So from our standard form, we know that the center of the ellipse is $(4, -1)$. We know the ellipse has a vertical major axis (since the denominator is larger under the y-term) and we know that $a = 5$ and $b = 3$. Therefore, the vertices would be at $(4, -1 + 5)$ and $(4, -1 - 5)$, simplifying to $(4, 4)$ and $(4, -6)$ and the $\text{co} - \text{vertices}$ would be at $(4 + 3, -1)$ and $(4 - 3, -1)$ which simplifies to $(7, -1)$ and $(1, -1)$. To find the coordinates of the foci, we'd do the following:

$$c^2 = a^2 - b^2$$

$$c^2 = 25 - 9 = 16$$

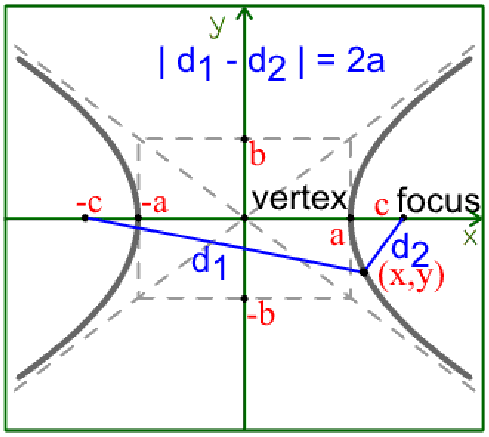
$$c = 4$$

So the foci are at $(4, -1 + 4)$ and $(4, -1 - 4)$, simplifying to $(4, 3)$ and $(4, -5)$.

Hyperbolas and their Foci

Although hyperbolas and ellipses are quite different, their formulas and foci relationships are similar, making it is easy to confuse their characteristics. Be careful when working with both of these conics.

We spent quite a bit of time deriving the definition of an ellipse from the relationship of its locus to its foci, and we're going to introduce hyperbolas with the same type of thinking, but we'll be brief.



Although on a smaller scale, the graph of the hyperbola looks like an inverted ellipse, on a larger scale, the hyperbola extends forever and approaches 2 intersecting lines called asymptotes. Remember that we define an ellipse as the set of all points on a plane where the sum of the distances from two fixed points called foci is a constant. The relationship of a hyperbola to its foci is slightly different. Notice the blue line segments representing d_1 and d_2. These are no longer being added to obtain a constant – they are now being subtracted!

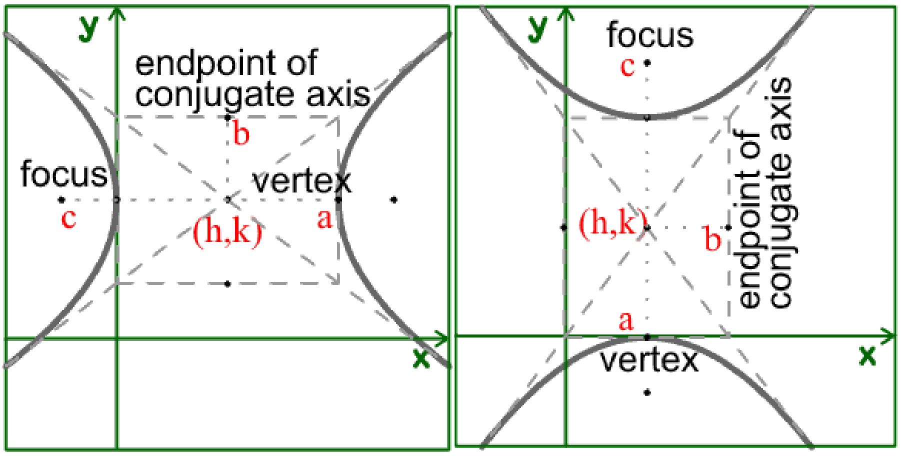
Geometric Definition of a Hyperbola:

The set of all points in a plane such that the absolute value of the difference of the distances from two fixed points (foci) is constant.

Note that $|d_1 - d_2| = 2a$, and a is still the distance from the center to a vertex, but we no longer call the axis containing the vertices and foci the major axis. We now call the segment joining the vertices the transversal axis, and it no longer must be the longer of the two axes. The other axis of symmetry for a hyperbola contains the conjugate axis, and these 2 axes bisect each other. Here are some other important characteristics of the graphs of hyperbolas:

- The center is the starting point at (h, k).
- The transverse axis contains the foci and the vertices.
- Transverse axis length = $2a$. This is also the constant that the difference of the distances must be.
- Conjugate axis length = $2b$.
- Distance between foci = $2c$.
- The foci are within the curve.
- Since the foci are the farthest away from the center, c is the largest of the three lengths, and the Pythagorean relationship is: $c^2 = a^2 + b^2$.

The two types of hyperbolas that we will study are: horizontally (left) and vertically (right) oriented.



The Standard Form of a Hyperbola with Center (h, k):

$Horizontal\ Transverse\ Axis$: $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$

Since the change in y is b and the change in x is a, the slope of the asymptotes will be $\pm\frac{b}{a}$. The equations of the asymptotes will be $(y-k) = \pm\frac{b}{a}(x-h)$.

$Vertical\ Transverse\ Axis$: $\dfrac{(y-k)^2}{a^2} - \dfrac{(x-h)^2}{b^2} = 1$

Since the change in y is a and the change in x is b, the slope of the asymptotes will be $\pm\frac{a}{b}$. The equations of the asymptotes will be$(y-k) = \pm\frac{a}{b}(x-h)$.

It's probably not surprising that, given that the two focal distances are subtracted for hyperbolas instead of added, the standard form of a hyperbola involved subtraction instead of addition (like circles and ellipses).

And just as with circles, parabolas, and ellipses, we sometimes have to take a hyperbola written in another form and convert it to standard form in order to pick out the necessary information to graph the relation accurately. For example, consider the general form of this hyperbola:

$$9x^2 - 4y^2 + 90x + 32y + 197 = 0$$

$$9x^2 + 90x - 4y^2 + 32y = -197$$

As always, we need to complete the square in order to convert forms.

$$9(x^2 + 10x) - 4(y^2 - 8y) = -197$$

$$9\left(x^2 + 10x + \left(\frac{10}{2}\right)^2\right) - 4\left(y^2 - 8y + \left(\frac{8}{2}\right)^2\right) = -197 + 9\left(\frac{10}{2}\right)^2 - 4\left(\frac{8}{2}\right)^2$$

$$9(x+5)^2 - 4(y-4)^2 = -36$$

$$\frac{9(x+5)^2}{-36} - \frac{4(y-4)^2}{-36} = \frac{-36}{-36}$$

$$-\frac{(x+5)^2}{4} + \frac{(y-4)^2}{9} = 1$$

$$\frac{(y-4)^2}{9} - \frac{(x+5)^2}{4} = 1$$

So now we know that $a = 3$ and $b = 2$ and that the center of this hyperbola is $(-5, 4)$ and, since it has a vertical transverse axis, the vertices of the hyperbola are at $(-5, 4+3)$ and $(-5, 4-3)$, which become $(-5, 7)$ and $(-5, 1)$. To find the foci, we use

$$c^2 = a^2 + b^2$$

$$c^2 = 9 + 4 = 13$$

$$c = \sqrt{13}$$

So the foci are at $\left(-5, 4+\sqrt{13}\right)$ and $\left(-5, 4-\sqrt{13}\right)$.

So now all we need to do is find the asymptotes. Again, we know that this hyperbola has a vertical transverse axis, and therefore the slope of the asymptotes will be $\pm\frac{\Delta y}{\Delta x} = \pm\frac{3}{2}$. . Therefore,

$$(y-4) = \pm\frac{3}{2}(x-(-5))$$

$$y-4 = \frac{3}{2}x + \frac{15}{2} \text{ and } y-4 = -\frac{3}{2}x - \frac{15}{2}$$

Therefore, the equations of the asymptotes are

$$y = \frac{3}{2}x + \frac{23}{2} \text{ and } y = -\frac{3}{2}x - \frac{7}{2}$$

And now it's your turn…

For the following, put the equation in standard form, label the important pieces, and sketch the graph of the relation.

1. $4x^2 + 9y^2 - 16x + 90y + 205 = 0$

2. $100x^2 + 36y^2 > 3600$

3. $9x^2 - 4y^2 - 54x - 16y - 79 = 0$

4. $25x^2 - 4y^2 + 200x - 8y + 796 = 0$

5. Write the equation of a hyperbola whose center is at the origin, has a horizontal transverse axis and has asymptotes of $y = \pm\frac{5}{7}x$.

6. Write the equation of the ellipse with major axis of length 12 and foci (3, 0) and (-3, 0).

7. Write the equation of a hyperbola with asymptote $y - 2 = \frac{1}{3}(x + 4)$ and vertical transverse axis.

Can you write the equation of another such hyperbola?

A Conic Application

Translate between the geometric description and the equation for a conic section.
MCC9-12.G.GPE.3 (+) Derive the equations of ellipses and hyperbolas given the foci, using the fact that the sum or difference of distances from the foci is constant.

Introduction

This short performance task is designed to have students apply what they know about ellipses to a realistic situation. Usually, conic sections are not presented as a system of inequalities serving as geometric constrains, nor is area often applied to these planar graphs. This problem uses both of those ideas in the problem-solving process.

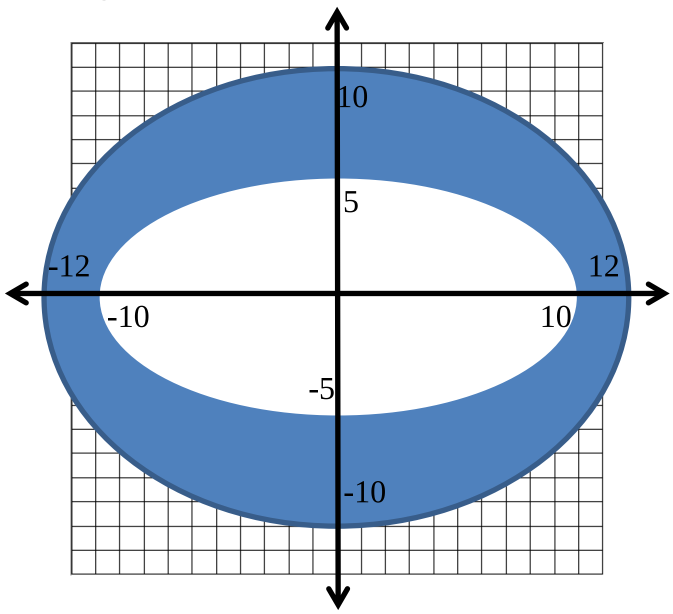
A Conic Application

Suppose that you are Chief Mathematician for the Swindle & Rob Construction Company. Your company has a contract to build a football stadium in the form of two concentric ellipses, with the field inside the inner ellipse, and seats between the two ellipses. The seats are in the intersection of the graphs of

$$x^2 + 4y^2 \geq 100 \text{ and } 25x^2 + 36y^2 \leq 3600$$

where each unit on the graph represents 10 meters.

(a) Draw a graph of the seating area.



(b) In your research, you find that the area of an elliptical region is πab, where a and b are half the lengths of the major and minor axes, respectively. The Engineering Department estimates that each seat occupies 0.8 square meter. What is the seating capacity of the stadium?

27,489 seats

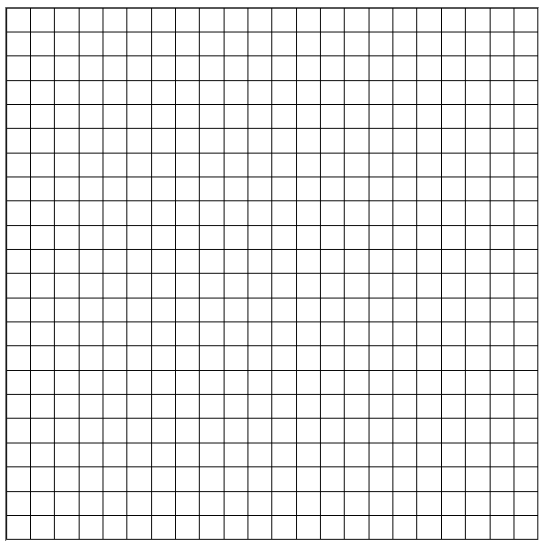
A Conic Application

Suppose that you are Chief Mathematician for the Swindle & Rob Construction Company. Your company has a contract to build a football stadium in the form of two concentric ellipses, with the field inside the inner ellipse, and seats between the two ellipses. The seats are in the intersection of the graphs of

$$x^2 + 4y^2 \geq 100 \text{ and } 25x^2 + 36y^2 \leq 3600$$

where each unit on the graph represents 10 meters.

(a) Draw a graph of the seating area.



(b) In your research, you find that the area of an elliptical region is πab, where a and b are half the lengths of the major and minor axes, respectively. The Engineering Department estimates that each seat occupies 0.8 square meter. What is the seating capacity of the stadium?