CCGPS Frameworks

Mathematics

CCGPS Pre-Calculus
Unit 3: Trigonometry of General Triangles

Unit 3

Trigonometry of General Triangles

TABLE OF CONTENTS

OVERVIEW

In this unit students will:
- Expand the use of trigonometric functions beyond right triangles into more general triangles.
- Develop the trigonometric formula for area of triangle.
- Use the Laws of Sines and Cosines to solve problems.

Although the units in this instructional framework emphasize key standards and big ideas at specific times of the year, routine topics such as estimation, mental computation, and basic computation facts should be addressed on an ongoing basis. Ideas related to the eight practice standards should be addressed constantly as well. To assure that this unit is taught with the appropriate emphasis, depth, and rigor, it is important that the tasks listed under "Evidence of Learning" be reviewed early in the planning process. A variety of resources should be utilized to supplement this unit. This unit provides much needed content information, but excellent learning activities as well. The tasks in this unit illustrate the types of learning activities that should be utilized from a variety of sources.

STANDARDS ADDRESSED IN THIS UNIT

Mathematical standards are interwoven and should be addressed throughout the year in as many different units and activities as possible in order to emphasize the natural connections that exist among mathematical topics. Students will build on their previous experience with trigonometric functions to include applications of the Laws of Sines and Cosines.

KEY STANDARDS

Apply trigonometry to general triangles

MCC9-12.G.SRT.9 (+) Derive the formula A= (1/2)ab sin(C) for the area of a triangle by drawing an auxiliary line from a vertex perpendicular to the opposite side.

MCC9-12.G.SRT.10 (+) Prove the Laws of Sines and Cosines and use them to solve problems.

MCC9-12.G.SRT.11 (+) Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles (e.g., surveying problems, resultant forces).

RELATED STANDARDS

Define trigonometric ratios and solve problems involving right triangles

MCC9-12.G.SRT.6 Understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles.

MCC9-12.G.SRT.7 Explain and use the relationship between the sine and cosine of complementary angles.

MCC9-12.G.SRT.8 Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems.

STANDARDS FOR MATHEMATICAL PRACTICE

The Standards for Mathematical Practice describe varieties of expertise that mathematics educators at all levels should seek to develop in their students. These practices rest on important "processes and proficiencies" with longstanding importance in mathematics education. The first of these are the NCTM process standards of problem solving, reasoning and proof, communication, representation, and connections. The second are the strands of mathematical proficiency specified in the National Research Council's report *Adding It Up*: adaptive reasoning, strategic competence, conceptual understanding (comprehension of mathematical concepts, operations and relations), procedural fluency (skill in carrying out procedures flexibly, accurately, efficiently and appropriately), and productive disposition (habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy).

1. **Make sense of problems and persevere in solving them.** High school students start to examine problems by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. By high school, students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. They check their answers to problems using different methods and continually ask themselves, "Does this make sense?" They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

2. **Reason abstractly and quantitatively.** High school students seek to make sense of quantities and their relationships in problem situations. They abstract a given situation and represent it symbolically, manipulate the representing symbols, and pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Students use quantitative reasoning to create coherent representations of the problem at hand; consider the units involved; attend to the meaning of quantities, not just how to compute them; and know and flexibly use different properties of operations and objects.

3. **Construct viable arguments and critique the reasoning of others.** High school students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. High school students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. High school students learn to determine domains to which an argument applies, listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

4. **Model with mathematics.** High school students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. High school students making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

5. **Use appropriate tools strategically.** High school students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. High school students should be sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. They are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.

6. **Attend to precision.** High school students try to communicate precisely to others by using clear definitions in discussion with others and in their own reasoning. They state

the meaning of the symbols they choose, specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context. By the time they reach high school they have learned to examine claims and make explicit use of definitions.

7. **Look for and make use of structure.** By high school, students look closely to discern a pattern or structure. In the expression $x^2 + 9x + 14$, older students can see the 14 as 2×7 and the 9 as $2 + 7$. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 - 3(x - y)^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers x and y. High school students use these patterns to create equivalent expressions, factor and solve equations, and compose functions, and transform figures.

8. **Look for and express regularity in repeated reasoning.** High school students notice if calculations are repeated, and look both for general methods and for shortcuts. Noticing the regularity in the way terms cancel when expanding $(x - 1)(x + 1)$, $(x - 1)(x^2 + x + 1)$, and $(x - 1)(x^3 + x^2 + x + 1)$ might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, derive formulas or make generalizations, high school students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

Connecting the Standards for Mathematical Practice to the Standards for Mathematical Content

The Standards for Mathematical Practice describe ways in which developing student practitioners of the discipline of mathematics should engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle and high school years. Designers of curricula, assessments, and professional development should all attend to the need to connect the mathematical practices to mathematical content in mathematics instruction.

The Standards for Mathematical Content are a balanced combination of procedure and understanding. Expectations that begin with the word "understand" are often especially good opportunities to connect the practices to the content. Students who do not have an understanding of a topic may rely on procedures too heavily. Without a flexible base from which to work, they may be less likely to consider analogous problems, represent problems coherently, justify conclusions, apply the mathematics to practical situations, use technology mindfully to work with the mathematics, explain the mathematics accurately to other students, step back for an overview, or deviate from a known procedure to find a shortcut. In short, a missing mathematical knowledge effectively prevents a student from engaging in the mathematical practices.

In this respect, those content standards which set an expectation of understanding are potential "points of intersection" between the Standards for Mathematical Content and the Standards for Mathematical Practice. These points of intersection are intended to be weighted toward central and generative concepts in the school mathematics curriculum that most merit the time, resources, innovative energies, and focus necessary to qualitatively improve the curriculum, instruction, assessment, professional development, and student achievement in mathematics.

ENDURING UNDERSTANDINGS

- Derive a formula for the area of a triangle using two sides and a non-included angle
- Calculate the area of a general triangle using $A = \dfrac{1}{2} ab(Sin\,C)$
- Verify the Law of Sines and the Law of Cosines for a general triangle.
- Apply the Law of Sines and the Law of Cosines to solve problems.

ESSENTIAL QUESTIONS

- How can I calculate the area of any triangle given only two sides and a non-included angle?
- How can I apply trigonometric relationships to non-right triangles?
- What is the least amount of information that is sufficient to find all six parts of a triangle?

CONCEPTS/SKILLS TO MAINTAIN

It is expected that students will have prior knowledge/experience related to the concepts and skills identified below. It may be necessary to pre-assess in order to determine if time needs to be spent on conceptual activities that help students develop a deeper understanding of these ideas.
- number sense
- computation with whole numbers and decimals, including application of order of operations
- calculating the area of a triangle
- solving trigonometric equations
- using inverse trigonometric functions to solve problems
- constructing altitudes in a triangle
- performing operations with trigonometric functions

SELECTED TERMS AND SYMBOLS

The following terms and symbols are often misunderstood. These concepts are not an inclusive list and should not be taught in isolation. However, due to evidence of frequent

difficulty and misunderstanding associated with these concepts, instructors should pay particular attention to them and how their students are able to explain and apply them.

The definitions below are for teacher reference only and are not to be memorized by the students. Students should explore these concepts using models and real life examples. Students should understand the concepts involved and be able to recognize and/or demonstrate them with words, models, pictures, or numbers.

The website below is interactive and includes a math glossary.

http://intermath.coe.uga.edu/dictnary/homepg.asp
Definitions and activities for these and other terms can be found on the Intermath website. Links to external sites are particularly useful.

- **Altitude of a Triangle**: The perpendicular distance between a vertex of a triangle and the side opposite that vertex. Sometimes called the height of a triangle. Also, sometimes the line segment itself is referred to as the altitude.

- **Hinge Theorem**: If two sides of one triangle are congruent to two sides of another triangle, and the included angle of the first is larger than the included angle of the second, then the third side of the first triangle is longer than the third side of the second triangle. (Wikipedia)

- **Included Angle**: The angle between two given sides of a triangle

- **Law of Cosines**: The square of the length of any side of a triangle equals the sum of the squares of the lengths of the other two sides minus twice the product of the lengths of the other two sides and the cosine of the angle between them. (Swokowski, Cole)

- **Law of Sines**: In any triangle, the ratio of the sine of an angle to the side opposite that angle is equal to the ratio of the sine of another angle to the side opposite that angle (Swokowski, Cole)

- **Oblique Triangle**: A triangle that is not a right triangle

- **Vertex of a Triangle**: The common endpoint of the two legs that serve as the sides of a triangle

CLASSROOM ROUTINES

The importance of continuing the established classroom routines cannot be overstated. Daily routines must include such obvious activities as estimating, analyzing data, describing patterns, and answering daily questions. They should also include less obvious routines, such as how to select materials, how to use materials in a productive manner, how to put materials away, how to access classroom technology such as computers and calculators. An additional routine is to allow plenty of time for children to explore new materials before attempting any directed activity

with these new materials. The regular use of routines is important to the development of students' number sense, flexibility, fluency, collaborative skills and communication. These routines contribute to a rich, hands-on standards based classroom and will support students' performances on the tasks in this unit and throughout the school year.

STRATEGIES FOR TEACHING AND LEARNING

- Students should be actively engaged by developing their own understanding.

- Mathematics should be represented in as many ways as possible by using graphs, tables, pictures, symbols and words.

- Interdisciplinary and cross curricular strategies should be used to reinforce and extend the learning activities.

- Appropriate manipulatives and technology should be used to enhance student learning.

- Students should be given opportunities to revise their work based on teacher feedback, peer feedback, and metacognition which includes self-assessment and reflection.

- Students should write about the mathematical ideas and concepts they are learning.

- Consideration of all students should be made during the planning and instruction of this unit. Teachers need to consider the following:

  - What level of support do my struggling students need in order to be successful with this unit?

  - In what way can I deepen the understanding of those students who are competent in this unit?

  - What real life connections can I make that will help my students utilize the skills practiced in this unit?

- A discussion of precision before this unit will help students with the standard for mathematical practice that states, "They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context." An excessive number of decimal places may not be appropriate for real world problems. Significant digits may be discussed or the class may agree that when measurements are uncertain, 3 or 4 significant digits should be used.

EVIDENCE OF LEARNING

By the conclusion of this unit, students should be able to demonstrate the following competencies:
- Demonstrate the process of deriving the general formula for the area of a generic triangle
- Verify the Law of Sines
- Verify the Law of Cosines
- Apply the Law of Sines to solve for missing pieces of a generic triangle
- Apply the Law of Cosines to solve for missing pieces of a generic triangle
- Understand and apply the Law of Sines and the Law of Cosines to solve problems in the context of a real-world situation.

TASKS

The following tasks represent the level of depth, rigor, and complexity expected of all Pre-Calculus students. These tasks or tasks of similar depth and rigor should be used to demonstrate evidence of learning. It is important that all elements of a task be addressed throughout the learning process so that students understand what is expected of them. While some tasks are identified as a performance task, they may also be used for teaching and learning (learning task).

Scaffolding Task	Tasks that build up to the learning task.
Learning Task	Constructing understanding through deep/rich contextualized problem solving tasks.
Practice Task	Tasks that provide students opportunities to practice skills and concepts.
Performance Task	Tasks which may be a formative or summative assessment that checks for student understanding/misunderstanding and or progress toward the standard/learning goals at different points during a unit of instruction.
Culminating Task	Designed to require students to use several concepts learned during the unit to answer a new or unique situation. Allows students to give evidence of their own understanding toward the mastery of the standard and requires them to extend their chain of mathematical reasoning.
Formative Assessment Lesson (FAL)	Lessons that support teachers in formative assessment which both reveal and develop students' understanding of key mathematical ideas and applications. These lessons enable teachers and students to monitor in more detail their progress towards the targets of the standards.

Task Name	Task Type *Grouping Strategy*	Content Addressed
Finding a New Area Formula for Triangles	Learning Task *Individual/Partner Task*	Determining the area of triangles with the aid of trigonometry.
Proving the Law of Cosines	Learning Task *Individual/Partner Task*	Derivation of the Law of Cosines
Proving the Law of Sines	Learning Task *Individual/Partner Task*	Derivation of the Law of Sines
The Hinge Theorem	Extension Task *Partner/Small Group Task*	Investigating the Hinge Theorem using trigonometry.
Culminating Task: Combining Lots	Performance Task *Individual/Partner Task*	Utilize the standards in order to determine missing measurements.

FINDING A NEW AREA FORMULA FOR TRIANGLES LEARNING TASK

Common Core Standards:

MCC9-12.G.SRT.9 -Derive the formula A= (1/2)ab sin(C) for the area of a triangle by drawing an auxiliary line from a vertex perpendicular to the opposite side

Introduction:

This task is designed to help a student create a new formula for calculating the area of a triangle by combining the familiar formula for area with the trigonometric ratios that they have learned previously.

FINDING A NEW AREA FORMULA FOR TRIANGLES

A developer needs to find the area of some plots of land he is interested in buying. Each plot is owned by a different person and neither owner knows the actual area of the land. The diagram below illustrates the plots he wants to buy but he wants to know the area before buying it.

Your task is to calculate the total areas of the plots.

1. Recall that the formula for the area of a triangle is $A = \frac{1}{2}bh$

    where *b* is the length of the base and *h* is the height, perpendicular to the base. Can the formula be used in this situation? Why or why not?

    The area formula cannot be used with these triangles with the given information. The height must be measured perpendicular from the base, and such a measure is not given.
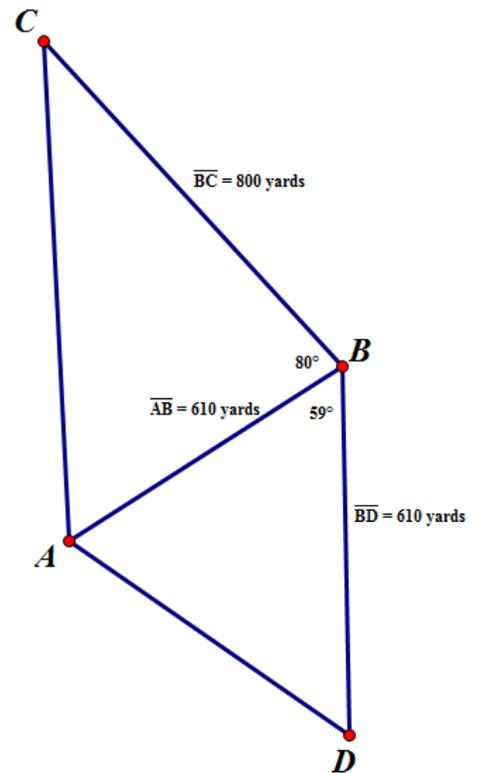
$\overline{BC} = 800$ yards

80°
59°
$\overline{AB} = 610$ yards
$\overline{BD} = 610$ yards

2. What information would be helpful in determining the area of the triangles? (Hint: Can you draw another line that would help?)

Students should recognize that they can draw a perpendicular through point C and point D. This is a good place to revisit constructions, particularly constructing a perpendicular through a given point.
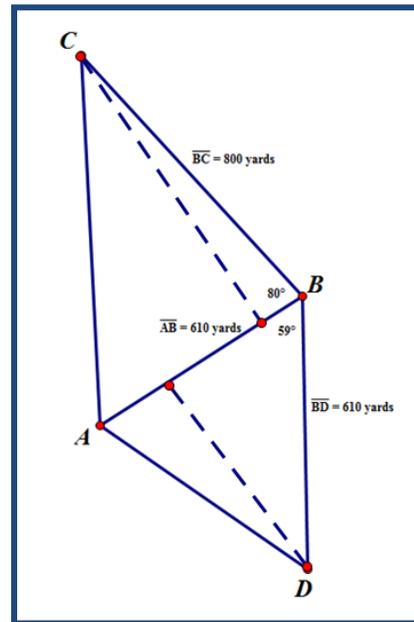
3. Calculate the height of ΔABC . Find the area. Use your strategy to calculate the area of ΔABD . What is the total area of the two properties?

Since students should be in a trigonometry frame of mind, they should think of using trig ratios to find the height of the triangle. Although the task is designed to lead this way, you should encourage other methods of solution and see where that takes your class. Remember, the ultimate goal is to encourage student's mathematical thinking and problem solving.

$$\sin \angle ABC = \frac{h}{\overline{BC}}$$

To find the height of ABC: $\sin 80° = \dfrac{h}{800}$

$$h = 800\sin 80°$$
$$h = 787.85\, yds$$

Thus the area of the triangle is: $\dfrac{1}{2}\overline{AB} \times h = \dfrac{1}{2}(610)(787.85) = 240293.09\, yds^2$

$$\sin \angle ABD = \frac{h}{BD}$$

To find the height of ABD: $\sin 59° = \dfrac{h}{610}$

$$h = 610\sin 59°$$

$$h = 522.87\, yds$$

Thus the area of triangle ABC is: $\dfrac{1}{2}\overline{AB} \times h = \dfrac{1}{2}(610)(522.87) = 159475.35\, yds^2$

The total area of the two plots is: $240293.09 + 159475.35 = 399768.44\, yds^2$

Or more realistically about 399800 yds²

4. After working on this problem, would it be possible to generalize this method for use on any triangle?

   Construct any triangle. Label the angles A, B, and C. Label the side opposite from A with *a*, the side opposite B with *b*, and the side opposite C with *c*.

   Construct any altitude from a vertex.

   Now calculate the height of your triangle.

This is perhaps the most important part of the task. Here the student will take a concrete idea (area) and generalize it to fit any oblique triangle. Do not emphasize memorizing the formula, as that is a low functioning task. Instead, challenge students to understand where the formula comes from and be able to construct the formula from scratch.

   Now that you know the height of the triangle, you can write a general formula for the area of any triangle. Use $A = \dfrac{1}{2}bh$ as a starting point.

Students should have any one of the following 3 formulas:

$$A = \frac{1}{2}ac(\sin B)$$

$$A = \frac{1}{2}ab(\sin C)$$

$$A = \frac{1}{2}bc(\sin A)$$

In addition, students should be able to generalize all three of these formulas by understanding that the area of a triangle equals one-half the product of the lengths of any two sides and the sine of the included angle.

5. Compare your formula with another student. Did you get the same thing? Could you both be right?

This is where students justify their answers and explain to other students their methods and findings. Students should arrive at the conclusion that any of the three formulas above are correct.

6. Would it be possible to develop a third formula for the area? If so, find it. If not, explain why not.

The pairs should then attempt to find the third iteration of the above formulas, depending on the formulas they found already.

FINDING A NEW AREA FORMULA FOR TRIANGLES LEARNING TASK

A developer needs to find the area of some plots of land he is interested in buying. Each plot is owned by a different person and neither owner knows the actual area of the land. The diagram below illustrates the plots he wants to buy but he wants to know the area before buying it.

Your task is to calculate the total areas of the plots.

1. Recall that the formula for the area of a triangle is $A = \frac{1}{2}bh$

    where b is the length of the base and h is the height, perpendicular to the base. Can the formula be used in this situation? Why or why not?

C

$\overline{BC} = 800$ yards

B

80°

$\overline{AB} = 610$ yards   59°

$\overline{BD} = 610$ yards

A

D

2. What information would be helpful in determining the area of the triangles? (Hint: Can you draw another line that would help?)

3. Calculate the height of ΔABC. Find the area. Use your strategy to calculate the area of ΔABD. What is the total area of the two properties?

4. After working on this problem, would be possible to generalize this method for use on any triangle?

   Construct any triangle. Label the angles A, B, and C. Label the side opposite from A with a, the side opposite B with b, and the side opposite C with c.

   Construct any altitude from a vertex.

   Now calculate the height of your triangle.

   Now that you know the height of the triangle, you can write a general formula for the area of any triangle. Use $A = \frac{1}{2}bh$ as a starting point.

5. Compare your formula with another student. Did you get the same thing? Could you both be right?

6. Would it be possible to develop a third formula for the area? If so, find it. If not, explain why not.

PROVING THE LAW OF COSINES

Common Core Standards

MCC9-12.G.SRT.10 Prove the Laws of Sines and Cosines and use them to solve problems.

MCC9-12.G.SRT.11 Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles

Introduction

The purpose of this task is to guide students through the derivation of the Law of Cosines. The teacher should spend the extra time and effort in helping students understand the conceptual foundation for the Law of Cosines, and not just memorizing the formula. A few problems have been provided at the end of the task, but the classroom teacher should provide ample opportunities to practice and improve fluency.

PROVING THE LAW OF COSINES

During a baseball game an outfielder caught a ball hit to dead center field, 400 feet from home plate. If the distance from home plate to first base is 90 feet, how far does the outfielder have to throw the ball to get it to first base?

1. Model the problem with a picture. Be sure to label information that you know.

2. Do you have enough information to solve the problem? If not, what is missing?

We know two sides of the triangle. We also know the
angle at home base is 45° (The 'diamond' is a square.
The diagonal of a square bisects the angle at the vertex.)

Outfielder

x

400 feet

First Base

90 feet

Home plate

Typically, you have solved triangles that are right triangles. This is a case where we do not have a right triangle to solve. We know two sides and one included angle. In this task, you will develop a method for solving triangles like this using trigonometry. We will come back to the baseball example later. For now, consider the triangle below. Follow these steps to derive a way to solve for c knowing just that much information. For this example, assume we know measurements for segments a, b, and angle C.

A

c

b

h

a-x

x

B

C

a

3. What does segment *h* represent? What are its properties and what does it do to the large triangle?

    h is the altitude of the triangle. It is perpendicular to the base and divides the larger triangle into two right triangles.

    Comment:
    Get students to realize that the altitude does not necessarily bisect the angle!

4. Write an equation that represents c^2 . Explain the method you used.

    According to the Pythagorean Theorem:
    $$c^2 = h^2 + (a-x)^2 = h^2 + a^2 - 2ax + x^2$$

5. Now write an equation that represents h^2 in terms of *b* and *x*. Substitute this expression into the expression you wrote in #4. Expand and simplify.

    According to the Pythagorean Theorem:

    $$h^2 = b^2 - x^2$$

    When you substitute this into the c^2 equation you get:

    $$c^2 = b^2 + a^2 - 2ax$$

6. Now write an expression that represents *x* in terms of the angle *C*. Substitute this expression into the equation you wrote in #5. Simplify completely.

Students should use the cosine ratio here to arrive at: $\cos C = \dfrac{x}{b}$
$$x = b(\cos C)$$

Substituting this in gives: $c^2 = b^2 + a^2 - 2ab(\cos C)$, *one of the versions of the Law of Cosines.*

Your answer to #6 is one of three formulas that make up the **Law of Cosines**. Each of the formulas can be derived in the same way you derived this one by working with each vertex and the other heights of the triangle.

Law of Cosines
Let a, b, and c be the lengths of the legs of a triangle opposite angles A, B, and C. Then,
$$a^2 = b^2 + c^2 - 2bc(cosA)$$
$$b^2 = a^2 + c^2 - 2ac(cosB)$$
$$c^2 = a^2 + b^2 - 2ab(cosC)$$

These formulas can be used to solve for unknown lengths and angles in a triangle.

Students should also be able to verbally articulate the generalization of these Laws. It should go something like this: The square of the length of any side of a triangle is equal to the sum of the squares of the other two sides minus two times the product of the lengths of the other two sides and the cosine of their included angle.

7. Solve the baseball problem at the beginning of this task using the Law of Cosines.

In this problem a = 90, b= 400 and C=45°.
Using the Law of Cosines:
$c^2 = a^2 + b^2 - 2abcosC$
$c^2 = 90^2 + 400^2 - 2(90)(400)cos45°$
$c \approx 342$
The distance is about 342 feet from dead centerfield to 1st base.

Here are a few problems to help you apply the Law of Cosines.

8. Two airplanes leave an airport, and the angle between their flight paths is 40°. An hour later, one plane has traveled 300 miles while the other has traveled 200 miles. How far apart are the planes at this time?
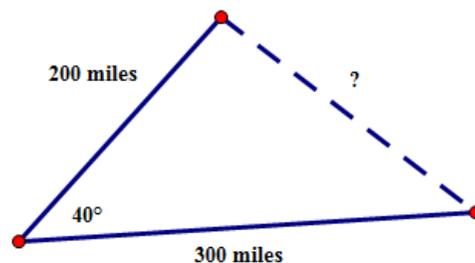
$a^2 = b^2 + c^2 - 2bccosA$
Let a be the missing side, b = 300, c = 200 and A = 40.
$a^2 = 300^2 + 200^2 - 2(300)(200)cos(40)$
$a^2 = 38074.66683$
$a \approx 195.13\ miles$
The planes are approximately 195 miles apart.

9. A triangle has sides of 8 and 7 and the angle between these sides is 35°. Solve the triangle. (Find all missing angles and sides.)

$a^2 = b^2 + c^2 - 2bc\cos A$
b = 7, c = 8 and A = 35.
$a^2 = 7^2 + 8^2 - 2(7)(8)\cos(35)$
$a^2 \approx 21.255$
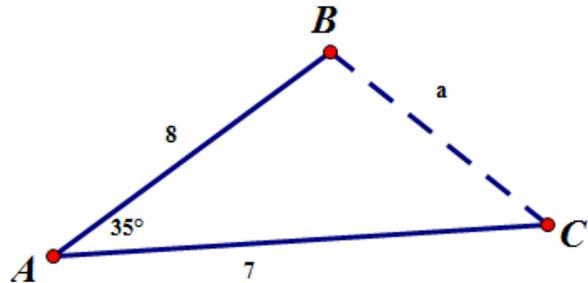$a \approx 4.61$

To find angle C
$c^2 = a^2 + b^2 - 2ab\cos C$
$8^2 = 4.61^2 + 7^2 - 2(4.61)(7)(\cos C)$
$64 = 70.2521 - 64.54(\cos C)$
0.0969 = cos C
84.44 =C

B = 180 – 84.44 – 35 = 60.56

10. Three soccer players are practicing on a field. The triangle they create has side lengths of 18, 14, and 15 feet. At what angles are they standing from each other?

To find angle C
$c^2 = a^2 + b^2 - 2ab\cos C$
$15^2 = 14^2 + 18^2 - 2(14)(18)(\cos C)$
$225 = 520 - 504(\cos C)$
0.5853 = cos C
54.17°=C

To find angle B
$b^2 = a^2 + c^2 - 2ac\cos B$
$18^2 = 14^2 + 15^2 - 2(14)(15)(\cos B)$
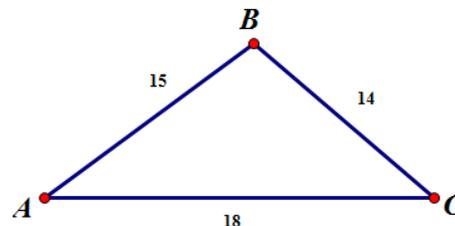$324 = 421 - 420(\cos C)$
0.231 = cos C
76.65° =C

To find angle A
A = 180 – 54.17 – 76.65 = 49.18°

11. Is it possible to know two sides of a triangle and the included angle and not be able to solve for the third side?

Using the Law of Cosines, you will always be able to find the third side of the triangle.
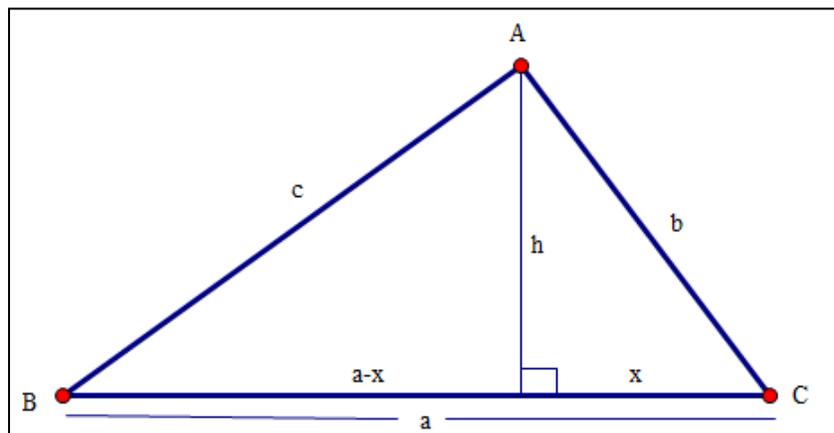
PROVING THE LAW OF COSINES

During a baseball game an outfielder caught a ball hit to dead center field, 400 feet from home plate. If the distance from home plate to first base is 90 feet, how far does the outfielder have to throw the ball to get it to first base?

1. Model the problem with a picture. Be sure to label information that you know.

2. Do you have enough information to solve the problem? If not, what is missing?

Typically, you have solved triangles that are right triangles. This is a case where we do not have a right triangle to solve. We know two sides and one included angle. In this task, you will develop a method for solving triangles like this using trigonometry. We will come back to the baseball example later. For now, consider the triangle below. Follow these steps to derive a way to solve for c knowing just that much information. For this example, assume we know measurements for segments *a*, *b*, and angle *C*.

3. What does segment *h* represent? What are its properties and what does it do to the large triangle?

4. Write an equation that represents c^2 . Explain the method you used.

5. Now write an equation that represents h^2 in terms of b and x. Substitute this expression into the expression you wrote in #4. Expand and simplify.

6. Now write an expression that represents x in terms of the angle C. Substitute this expression into the equation you wrote in #5. Simplify completely.

Your answer to #6 is one of three formulas that make up the **Law of Cosines**. Each of the formulas can be derived in the same way you derived this one by working with each vertex and the other heights of the triangle.

Law of Cosines
Let a, b, and c be the lengths of the legs of a triangle opposite angles A, B, and C. Then,
$$a^2 = b^2 + c^2 - 2bc(cosA)$$
$$b^2 = a^2 + c^2 - 2ac(cosB)$$
$$c^2 = a^2 + b^2 - 2ab(cosC)$$

These formulas can be used to solve for unknown lengths and angles in a triangle.

7. Solve the baseball problem at the beginning of this task using the Law of Cosines.

Here are a few problems to help you apply the Law of Cosines.

8. Two airplanes leave an airport, and the angle between their flight paths is 40°. An hour later, one plane has traveled 300 miles while the other has traveled 200 miles. How far apart are the planes at this time?

9. A triangle has sides of 8 and 7 and the angle between these sides is 35°. Solve the triangle. (Find all missing angles and sides.)

10. Three soccer players are practicing on a field. The triangle they create has side lengths of 18, 14, and 15 feet. At what angles are they standing from each other?

11. Is it possible to know two sides of a triangle and the included angle and not be able to solve for the third side?

PROVING THE LAW OF SINES

Common Core Standards:

MCC9-12.G.SRT.10 Prove the Laws of Sines and Cosines and use them to solve problems.

MCC9-12.G.SRT.11 Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles

Introduction

The purpose of this task is to guide students through the process of deriving the Law of Sines. Emphasis should be placed on conceptual understanding rather than memorizing a formula.

PROVING THE LAW OF SINES

When using the Law of Cosines it is necessary to know the angle included between two sides. However, there are times the angle that is known is not the angle included between two known sides. And there are other times where we might know two angles and only one side. Consider the following triangle: The angle labels should be closer to the same size as the side labels.
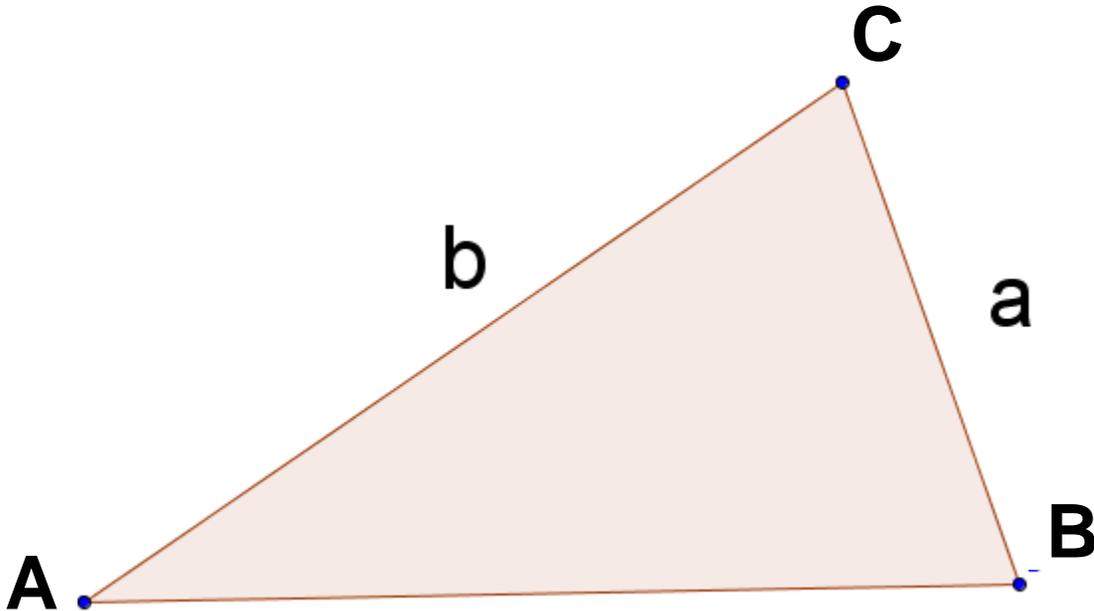
1. Construct an altitude through *C*. Label it *h*.

2. Write an equation for h in terms of angle B. Also write an equation for h in terms of angle A.

   Students should be familiar with altitudes from the last task.

   $\sin B = \dfrac{opp}{hyp} = \dfrac{h}{a}$ $\sin A = \dfrac{opp}{hyp} = \dfrac{h}{b}$

   $h = a\sin B$ $h = b\sin A$

3. Write a new equation relating the two equations together.

   By noticing that h=h and then substituting in, we get:

   $a\sin B = b\sin A$

   $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$

4. Now construct another altitude through B. Label it j.

5. Write an equation for j in terms of angle A. Also write an equation for j in terms of angle C.

   $\sin A = \dfrac{opp}{hyp} = \dfrac{j}{c}$ $\sin C = \dfrac{opp}{hyp} = \dfrac{j}{a}$

   $j = c\sin A$ $j = a\sin C$

6. Write a new equation relating the two equations together.

   By noticing that j=j and then substituting in, we get:

   $c\sin A = a\sin C$

   $\dfrac{a}{\sin A} = \dfrac{c}{\sin C}$

7. Using the Transitive Property of Equality, write an equation relating the equations in #3 and #6.

   Thus by the transitive property: $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

What you have arrived at is known as the **Law of Sines**.

Law of Sines
Let a, b, and c be the lengths of the legs of a triangle opposite angles A, B, and C. Then,

$$\frac{a}{sinA} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Here is an example of applying the Law of Sines. In the triangle to the right the measurements for angle B and A are missing and the length of AC. Follow the steps below to solve for the measure using the Law of Sines.

B

m $\overline{BC}$ = 5.4 cm

m $\overline{AB}$ = 6.7 cm

m∠BCA = 33.0° • *C*

A
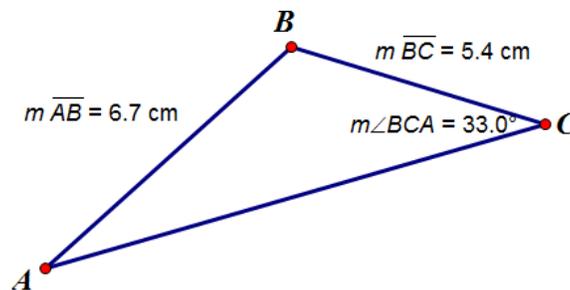
Students should focus on understanding and being able to explain the definition of the law of sines. The ratio of the sine of an angle to the side opposite the angle is equal to the ratio of the sine of another angle to the side opposite that angle.

This is the information that we know so far.

$$\frac{6.7}{\sin 33} = \frac{5.4}{\sin A} = \frac{b}{\sin B}$$

- Solve the first two ratios to find A.

$$5.4\sin 33 = 6.7\sin A$$

$$\sin A = \frac{5.4\sin 33}{6.7} = .4389$$

$$A = 26.04$$

- Find angle B. (Remember that the sum of the interior angles in a triangle is always 180 degrees)

$$B = 180 - 26 - 33 = 121°$$

Comment:
If students use the Law of Sines to solve this, they will get the wrong answer. This is a good opportunity to discuss the range of the $\sin^{-1}$ function and why it isn't a good idea to use the law of sines to find the largest angle in a triangle.
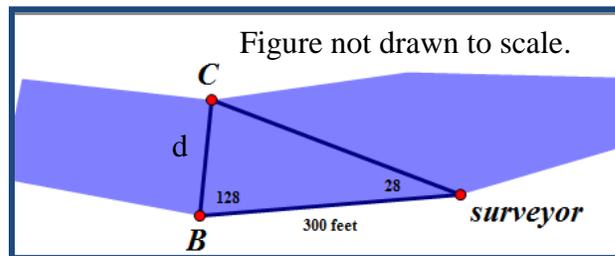
- Use either the first or second ratio with the last ratio to solve for side b.

$$\frac{6.7}{sin\ 33} = \frac{b}{sin\ 121}$$

$$\frac{6.7}{sin\ 33} * sin\ 121 = b$$

$$b = 10.55$$

Here are some problems that will help you practice applying the Law of Sines. Begin each problem by drawing a picture to model the situation. Then solve the problem.

8. A surveyor is near a river and wants to calculate the distance across the river. He measures the angle between his observations of two points on the shore, one on his side and one on the other side, to be 28°. The distance between him and the point on his side of the river can be measured and is 300 feet. The angle formed by him, the point on his side of the river, and the point on the opposite side of the river is 128°. What is the distance across the river? Remember that the distance across the river should be the shortest distance.

Angle c = 24°

$$\frac{300}{sin\ 24°} = \frac{d}{sin\ 28°}$$

$$\frac{300 sin\ 28°}{sin\ 24°} = d$$

d = 346 feet .

Figure not drawn to scale.

C

d

128   28

B   300 feet   *surveyor*

9. Two people are observing a hot air balloon from different places on level ground. The angles of elevation are $20°$ and $50°$. The two people are 7 miles apart and the balloon is between them. How high is the balloon off the ground?

First find the distance from the balloon to the person with the 50 degree line of sight.

$$\frac{7}{\sin 110°} = \frac{a}{\sin 20°}$$

$a = 2.548$
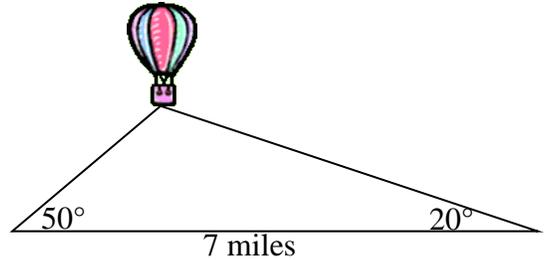
Now use side a to find the height.

$$\sin 50° = \frac{h}{2.548}$$

$h = 1.95$
The balloon is about 1.95 miles off the ground

PROVING THE LAW OF SINES

When using the Law of Cosines it is necessary to know the angle included between two sides. However, there are times the angle that is known is not the angle included between two known sides. And there are other times where we might know two angles and only one side. Consider the following triangle:
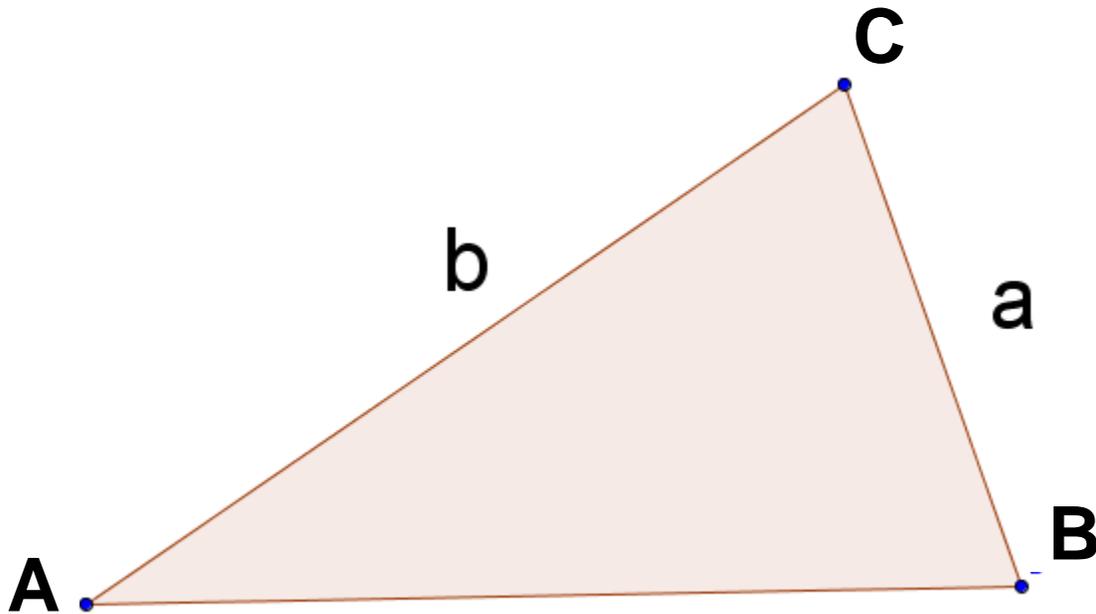
1. Construct an altitude through *C*. Label it *h*.

2. Write an equation for *h* in terms of angle *B*. Also write an equation for *h* in terms of angle *A*.

3. Write a new equation relating the two equations together.

4. Now construct another altitude through *B*. Label it *j*.

5. Write an equation for *j* in terms of angle *A*. Also write an equation for *j* in terms of angle *C*.

6. Write a new equation relating the two equations together.

7. Using the Transitive Property of Equality, write an equation relating the equations in #3 and #6.

What you have arrived at is known as the **Law of Sines**.

Law of Sines
Let a, b, and c be the lengths of the legs of a triangle opposite angles A, B, and C. Then,
$$\frac{a}{sinA} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Here is an example of applying the Law of Sines. In the triangle to the right the measurements for angle B and A are missing and the length of AC. Follow the steps below to solve for the measure using the Law of Sines.

$m\,\overline{BC} = 5.4$ cm

$m\,\overline{AB} = 6.7$ cm

$m\angle BCA = 33.0°$ C
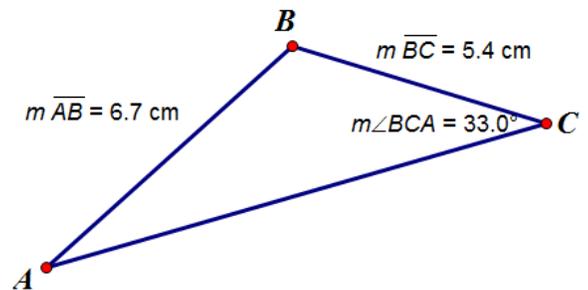
This is the information that we know so far.
$$\frac{6.7}{\sin 33} = \frac{5.4}{\sin A} = \frac{b}{\sin B}$$

- Solve the first two ratios to find A.

- Find angle B. (Remember that the sum of the interior angles in a triangle is always 180 degrees)

- Use either the first or second ratio with the last ratio to solve for side b.

Here are some problems that will help you practice applying the Law of Sines. Begin each problem by drawing a picture to model the situation. Then solve the problem.

8. A surveyor is near a river and wants to calculate the distance across the river. He measures the angle between his observations of two points on the shore, one on his side and one on the other side, to be 28º. The distance between him and the point on his side of the river can be measured and is 300 feet. The angle formed by him, the point on his side of the river, and the point on the opposite side of the river is 128º. What is the distance across the river? Remember that the distance across the river should be the shortest distance.

9. Two people are observing a hot air balloon from different places on level ground. The angles of elevation are $20°$ and $50°$. The two people are 7 miles apart and the balloon is between them. How high is the balloon off the ground?

THE HINGE THEOREM

Common Core Standards

MCC9-12.G.SRT.10 Prove the Laws of Sines and Cosines and use them to solve problems.

MCC9-12.G.SRT.11 Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles

Introduction:

Although the Hinge Theorem is not part of the standards, this task makes a connection to concepts already known by the student and the Law of Sines. This exploration works well with construction software if it is available.

THE HINGE THEOREM

From your previous math experience you know that the measures of two sides and a non-included angle will not necessarily work together to create a triangle. The Hinge Theorem is a geometric theorem that focuses on this idea. You may have explored this idea when you studied congruent triangles.

Consider the two triangles below. Given sides of 7 cm and 4.2 cm with a non-included angle of 30°, there are two triangles that can be created. This is why angle-side-side is not a congruency theorem for triangles.

In trigonometry we consider this to be the ambiguous case for solving triangles.

Looking at the two triangles above, why do you think this theorem might be called the Hinge Theorem?

BC seems to swing back and forth (like a hinge) to create the 2 different triangles.

What is the relationship between the measure of ∠ABC in the first figure and m∠ABC in the second figure?

They are supplementary.

Consider triangle ABC to the right. Note that
m∠A = 30°, AC = 7 cm and BC = 3.5 cm.
Are there still 2 possible triangles? Explain your answer.
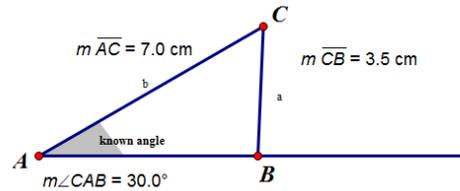
No, since the angle opposite the 30° angle is half the adjacent side, this must be a 30°-60°-90° triangle. BC is too short to "swing".

Redraw triangle ABC so that m∠A = 30°, AC = 7 cm and BC = 9 cm. Are there still 2 triangles? Explain your answer.

No, BC is too long to swing to the other side of AC and still have a 30° angle.

Redraw triangle ABC so that m∠A = 30°, AC = 7 cm and BC = 2 cm. Are there still 2 triangles? Explain your answer.

No, since BC is shorter than 3.5, it is too short to reach line AB, so no triangle can be created.

1. Order the sides of the triangle to the right from longest to shortest. (The figure is NOT drawn to scale.)

a>c>b
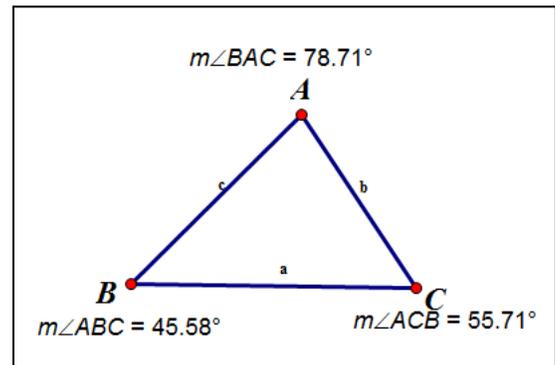
2. How did you decide the order of the sides?

The larger the angle the larger the side opposite the angle.
The longest side is opposite the largest angle.
The shortest side is always opposite the smallest angle.

3. Sketch the information given and decide if it is possible to create a triangle with the given information. Explain your answer.

Triangle 1: A = 40°, a = 8 cm and b = 5 cm.

> *To make a right triangle the side would need to be 5.14 cm. The side of 5 is too short so this will not make a triangle.*

Triangle 2: A = 150°, a = 5 cm and b = 8cm

> *The side opposite the obtuse angle should be the longest side in the triangle. But it's shorter than the other given side. Therefore, you cannot make a triangle with these measures.*

Be sure to take time to analyze the data you are given when you are solving a triangle to determine if you might have a set of data that will not yield a triangle. This could save you a lot of work!

Remember that the calculator only yields inverse sine values between -90° and 90°, so it will never let you know if there are obtuse angles in your triangles.

4. Solve the following triangle. Determine if there is no solution, one solution or two solutions.

Measure of angle A = 35°, a = 10 cm and b = 16 cm

If this were going to make a right triangle, a would have to be equal to 16sin35°.
16 sin35° = 9.18
Since a is greater than that but still less than 16, there will be two solutions.

$$\frac{10}{\sin 35} = \frac{16}{\sin B}$$

B = 66.6° or 113.4°

C= 180 – 66.6 – 35 = 78.4°   or C = 180 – 113.4 – 35 = 31.6

$$\frac{10}{\sin 35} = \frac{c}{\sin 78.4} \qquad\qquad \frac{10}{\sin 35} = \frac{c}{\sin 31.6}$$

c = 17.1   *c = 9.13*

5. A ship traveled 60 miles due east and then adjusted its course 15 degrees northward. After traveling for a while the ship turned back towards port. The ship arrived in port 139 miles later. How far did the ship travel on the second leg of the journey? What angle did the ship turn through when it headed back to port?

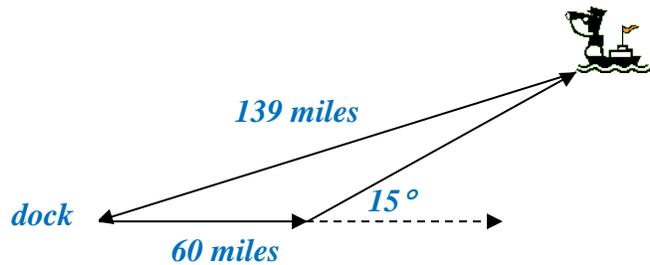$$\frac{139}{\sin 165} = \frac{60}{\sin C}$$

C = 6.4 degrees
A = 180 – 165 – 6.4 = 8.6
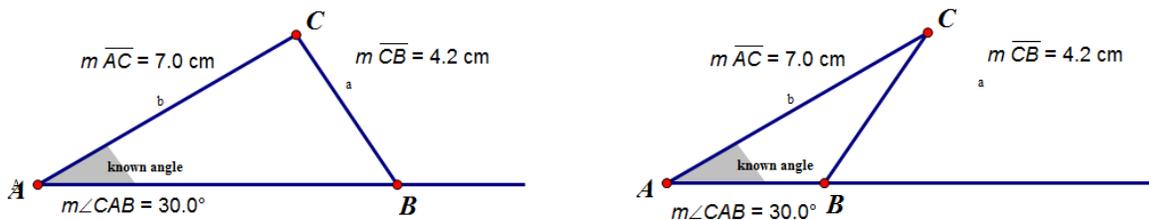
$$\frac{139}{\sin 165} = \frac{a}{\sin 8.6}$$

A
a = 80.2 miles

139 miles

dock

15°

60 miles

THE HINGE THEOREM

From your previous math experience you know that the measures of two sides and a non-included angle will not necessarily work together to create a triangle. The Hinge Theorem is a geometric theorem that focuses on this idea. You may have explored this idea when you studied congruent triangles.

Consider the two triangles below. Given sides of 7 cm and 4.2 cm with a non-included angle of 30°, there are two triangles that can be created. This is why angle-side-side is not a congruency theorem for triangles.

In trigonometry we consider this to be the ambiguous case for solving triangles.

Looking at the two triangles above, why do you think this theorem might be called the Hinge Theorem?

What is the relationship between the measure of ∠ABC in the first figure and m∠ABC in the second figure?

Consider triangle ABC to the right. Note that m∠A = 30°, AC = 7 cm and BC = 3.5 cm.
Are there still 2 possible triangles? Explain your answer.

Redraw triangle ABC so that m∠A = 30°, AC = 7 cm and BC = 9 cm. Are there still 2 triangles? Explain your answer.

Redraw triangle ABC so that m∠A = 30°, AC = 7 cm and BC = 2 cm. Are there still 2 triangles? Explain your answer.

1. Order the sides of the triangle to the right from longest to shortest. (The figure is NOT drawn to scale.

$m\angle BAC = 78.71°$

$m\angle ABC = 45.58°$ $m\angle ACB = 55.71°$
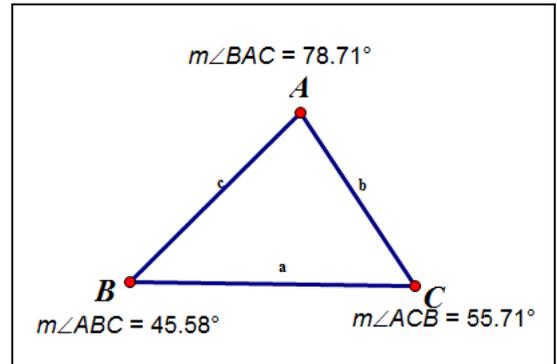
2. How did you decide the order of the sides?

3. Sketch the information given and decide if it is possible to create a triangle with the given information. Explain your answer.

Triangle 1: A = 40°, a = 8 cm and b = 5 cm.

Triangle 2: A = 150°, a = 5 cm and b = 8cm

Be sure to take time to analyze the data you are given when you are solving a triangle to determine if you might have a set of data that will not yield a triangle. This could save you a lot of work!

Remember that the calculator only yields inverse sine values between -90° and 90°, so it will never let you know if there are obtuse angles in your triangles.

4. Solve the following triangle. Determine if there is no solution, one solution or two solutions.

    Measure of angle A = 35°, a = 10 cm and b = 16 cm

5. A ship traveled 60 miles due east and then adjusted its course 15 degrees northward. After traveling for a while the ship turned back towards port. The ship arrived in port 139 miles later. How far did the ship travel on the second leg of the journey? What angle did the ship turn through when it headed back to port?

CULMINATING TASK: COMBINING LOTS

An antique map was found in the attic of a local courthouse. It shows some measurements from a local farm that was divided into 5 parts. Some of the measurements have faded with age, so you must find the remaining measurements, as well as calculate the total area and perimeter of the outside of the property. All lengths are measured in miles.
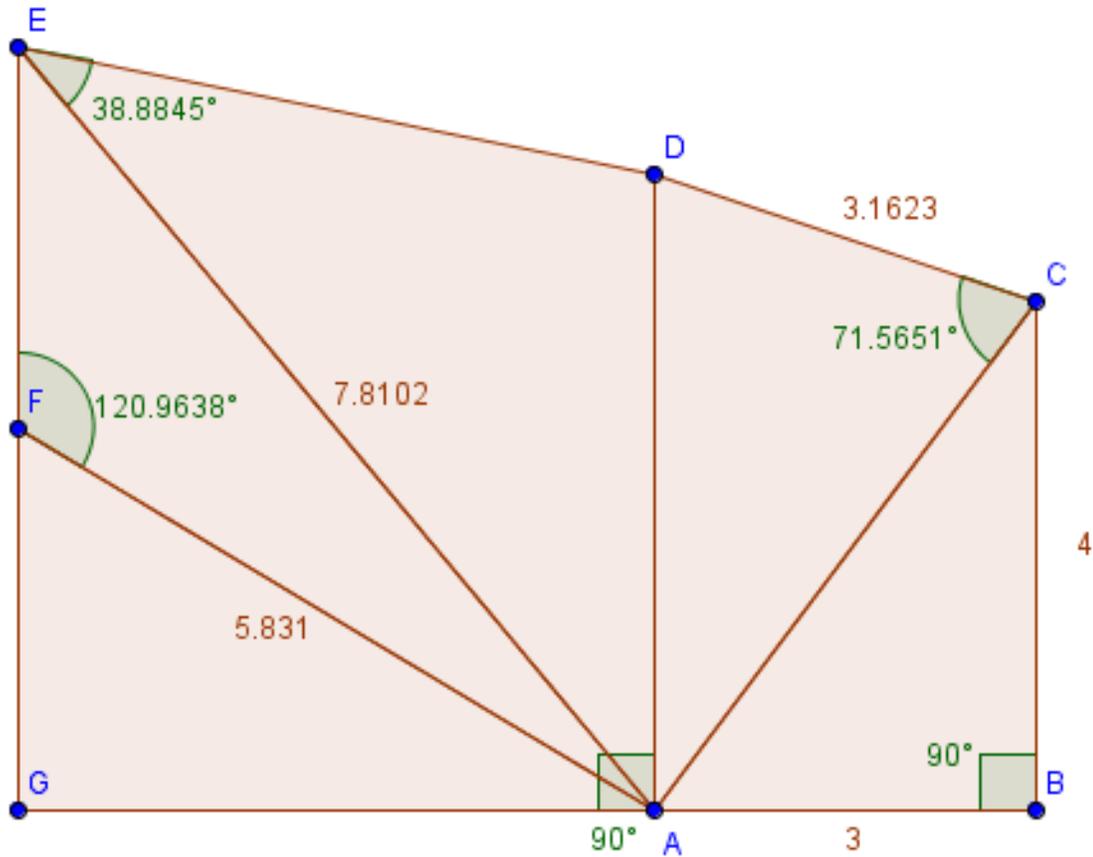
From the Pythagorean Theorem, $\overline{AC} = 5$

From the Law of Cosines, $\overline{AD}^2 = 5^2 + 3.1623^2 - 2(5)(3.1623)\cos 71.5651° = 25$
$\overline{AD} = 5$

From the Law of Sines: $\dfrac{7.8102}{\sin \angle EDA} = \dfrac{5}{\sin 38.8845°}$
$\angle EDA = 101.3117°$ *(The obtuse solution)*

From subtraction: $\angle EAD = 180° - (38.8845° + 101.3117°) = 39.8038°$

From Law of Sines: $\dfrac{\overline{ED}}{\sin 39.8038°} = \dfrac{5}{\sin 38.8845°}$

$\overline{ED} = 5.0988$

From Law of Sines: $\dfrac{7.8102}{\sin 120.9638°} = \dfrac{5.831}{\sin \angle AEF}$

$\angle AEF = 39.8062°$

By Subtraction: $\angle EAC = 180° - (39.8062° + 120.9638°) = 19.23°$

From Law of Sines: $\dfrac{7.8102}{\sin 120.9638°} = \dfrac{\overline{EF}}{\sin 19.23°}$

$\overline{EF} = 3$

From Subtraction: $\angle GAF = 90° - (39.8038° + 19.23°) = 30.9662°$

From Subtraction: $\angle AFG = 180° - 120.9638° = 59.0362°$

From Law of Sines: $\dfrac{5.831}{\sin 90°} = \dfrac{\overline{AG}}{\sin 59.0362°}$

$\overline{AG} = 5$

From Law of Sines: $\dfrac{5.831}{\sin 90°} = \dfrac{\overline{FG}}{\sin 30.9638°}$

$\overline{FG} = 3$

It should be noted that the methods here are only one possible way to find the missing parts. Encourage multiple solutions to this problem

Perimeter = 3+4+3.1623+5.0988+3+3+5=26.2611 km

Area: $\Delta ABC = \frac{1}{2}(3)(4) = 6$

$\Delta ADC = \frac{1}{2}(5)(3.1623)\sin 71.5651 = 7.5$

$\Delta ADE = \frac{1}{2}(7.8102)(5.0988)\sin 38.8845 = 12.5$

$\Delta AEF = \frac{1}{2}(3)(5.831)\sin 120.9638 = 7.5$

$\Delta AGF = \frac{1}{2}(5)(3) = 7.5$

Total Area: $6 + 7.5 + 12.5 + 7.5 + \&.5 = 41\ km^2$

CULMINATING TASK: COMBINING LOTS

An antique map was found in the attic of a local courthouse. It shows some measurements from a local farm that was divided into 5 parts. Some of the measurements have faded with age, so you must find the remaining measurements, as well as calculate the total area and perimeter of the outside of the property. All lengths are measured in miles.