# CCGPS Frameworks

# Mathematics

CCGPS Pre-Calculus
Unit 4: Trigonometric Identities

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*"Making Education Work for All Georgians"*

# Unit 4

# Trigonometric Identities

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## OVERVIEW

In this unit, students will:

- build upon their work with trigonometric identities with addition and subtraction formulas
- will look at addition and subtraction formulas geometrically
- prove addition and subtraction formulas
- use addition and subtraction formulas to solve problems.

Although the units in this instructional framework emphasize key standards and big ideas at specific times of the year, routine topics such as estimation, mental computation, and basic computation facts should be addressed on an ongoing basis. Ideas related to the eight practice standards should be addressed constantly as well. To assure that this unit is taught with the appropriate emphasis, depth, and rigor, it is important that the tasks listed under "Evidence of Learning" be reviewed early in the planning process. A variety of resources should be utilized to supplement this unit. This unit provides much needed content information, but excellent learning activities as well. The tasks in this unit illustrate the types of learning activities that should be utilized from a variety of sources.

## STANDARDS ADDRESSED IN THIS UNIT

Mathematical standards are interwoven and should be addressed throughout the year in as many different units and activities as possible in order to emphasize the natural connections that exist among mathematical topics. The standards call for a student to demonstrate mastery of the content by proving these addition and subtraction formulas using various methods.

## KEY STANDARDS

### Prove and apply trigonometric identities

**MCC9-12.F.TF.9 (+)** Prove the addition and subtraction formulas for sine, cosine, and tangent and use them to solve problems.

## STANDARDS FOR MATHEMATICAL PRACTICE

The Standards for Mathematical Practice describe varieties of expertise that mathematics educators at all levels should seek to develop in their students. These practices rest on important "processes and proficiencies" with longstanding importance in mathematics education. The first of these are the NCTM process standards of problem solving, reasoning and proof, communication, representation, and connections. The second are the strands of mathematical proficiency specified in the National Research Council's report *Adding It Up*: adaptive reasoning, strategic competence, conceptual understanding (comprehension of mathematical concepts, operations and relations), procedural fluency (skill in carrying out procedures flexibly, accurately, efficiently and appropriately), and productive disposition (habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy).

1. **Make sense of problems and persevere in solving them.** High school students start to examine problems by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. By high school, students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. They check their answers to problems using different methods and continually ask themselves, "Does this make sense?" They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

2. **Reason abstractly and quantitatively.** High school students seek to make sense of quantities and their relationships in problem situations. They abstract a given situation and represent it symbolically, manipulate the representing symbols, and pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Students use quantitative reasoning to create coherent representations of the problem at hand; consider the units involved; attend to the meaning of quantities, not just how to compute them; and know and flexibly use different properties of operations and objects.

3. **Construct viable arguments and critique the reasoning of others.** High school students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. High school students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. High school students learn to determine domains to which an argument applies, listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

4. **Model with mathematics.** High school students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. By high school, a

student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. High school students making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

5. **Use appropriate tools strategically.** High school students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. High school students should be sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. They are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.

6. **Attend to precision.** High school students try to communicate precisely to others by using clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context. By the time they reach high school they have learned to examine claims and make explicit use of definitions.

7. **Look for and make use of structure.** By high school, students look closely to discern a pattern or structure. In the expression $x^2 + 9x + 14$, older students can see the 14 as $2 \times 7$ and the 9 as $2 + 7$. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 – 3(x – y)^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers $x$ and $y$. High school students use these patterns to create equivalent expressions, factor and solve equations, and compose functions, and transform figures.

8. **Look for and express regularity in repeated reasoning.** High school students notice if calculations are repeated, and look both for general methods and for shortcuts. Noticing the regularity in the way terms cancel when expanding $(x - 1)(x + 1)$, $(x - 1)(x^2 + x + 1)$, and $(x - 1)(x^3 + x^2 + x + 1)$ might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, derive formulas or make generalizations, high school students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

**Connecting the Standards for Mathematical Practice to the Standards for Mathematical Content**

The Standards for Mathematical Practice describe ways in which developing student practitioners of the discipline of mathematics should engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle and high school years. Designers of curricula, assessments, and professional development should all attend to the need to connect the mathematical practices to mathematical content in mathematics instruction.

The Standards for Mathematical Content are a balanced combination of procedure and understanding. Expectations that begin with the word “understand” are often especially good opportunities to connect the practices to the content. Students who do not have an understanding of a topic may rely on procedures too heavily. Without a flexible base from which to work, they may be less likely to consider analogous problems, represent problems coherently, justify conclusions, apply the mathematics to practical situations, use technology mindfully to work with the mathematics, explain the mathematics accurately to other students, step back for an overview, or deviate from a known procedure to find a shortcut. In short, a missing mathematical knowledge effectively prevents a student from engaging in the mathematical practices.

In this respect, those content standards which set an expectation of understanding are potential “points of intersection” between the Standards for Mathematical Content and the Standards for Mathematical Practice. These points of intersection are intended to be weighted toward central and generative concepts in the school mathematics curriculum that most merit the time, resources, innovative energies, and focus necessary to qualitatively improve the curriculum, instruction, assessment, professional development, and student achievement in mathematics.

## ENDURING UNDERSTANDINGS

- Understand the concept of identity
- Prove the addition formula for sine, cosine and tangent
- Prove the subtraction formula for sine, cosine and tangent
- Use trigonometric functions to prove formulas

## ESSENTIAL QUESTIONS

- How can I add trigonometric functions?
- How can I subtract trigonometric functions?
- How can I prove the addition formula for trigonometric functions?
- How can I prove the subtraction formula for trigonometric functions?
- What is an identity?

## CONCEPTS/SKILLS TO MAINTAIN

It is expected that students will have prior knowledge/experience related to the concepts and skills identified below. It may be necessary to pre-assess in order to determine if time needs to be spent on conceptual activities that help students develop a deeper understanding of these ideas.

- number sense
- computation with whole numbers and decimals, including application of order of operations
- applications of the Pythagorean Theorem
- operations with trigonometric ratios
- operations with radians and degrees
- even and odd functions
- geometric constructions
- algebraic proofs
- geometric proofs
- methods of proof

## SELECTED TERMS AND SYMBOLS

The following terms and symbols are often misunderstood. These concepts are not an inclusive list and should not be taught in isolation. However, due to evidence of frequent difficulty and misunderstanding associated with these concepts, instructors should pay particular attention to them and how their students are able to explain and apply them.

**The definitions below are for teacher reference only and are not to be memorized by the students.** Students should explore these concepts using models and real life examples. Students should understand the concepts involved and be able to recognize and/or demonstrate them with words, models, pictures, or numbers.

The website below is interactive and includes a math glossary.

http://intermath.coe.uga.edu/dictnary/homepg.asp

Definitions and activities for these and other terms can be found on the Intermath website. Links to external sites are particularly useful.

- **Addition Identity for Cosine**: $\cos(x+y)=\cos x\cos y-\sin x\sin y$
- **Addition Identity for Sine**: $\sin(x+y)=\sin x\cos y+\cos x\sin y$
- **Addition Identity for Tangent**: $\tan(x+y)=\dfrac{\tan x+\tan y}{1-\tan x\tan y}$
- **Even Function**: a function with symmetry about the y-axis that satisfies the relationship $f(x)=f(-x)$
- **Identity**: an identity is a relation that is always true, no matter the value of the variable.
- **Odd Function**: a function with symmetry about the origin that satisfies the relationship $-f(x)=f(-x)$
- **Subtraction Identity for Cosine**: $\cos(x-y)=\cos x\cos y+\sin x\sin y$
- **Subtraction Identity for Sine**: $\sin(x-y)=\sin x\cos y-\cos x\sin y$
- **Subtraction Identity for Tangent**: $\tan(x-y)=\dfrac{\tan x-\tan y}{1+\tan x\tan y}$

## CLASSROOM ROUTINES

The importance of continuing the established classroom routines cannot be overstated. Daily routines must include such obvious activities as estimating, analyzing data, describing patterns, and answering daily questions. They should also include less obvious routines, such as how to select materials, how to use materials in a productive manner, how to put materials away, how to access classroom technology such as computers and calculators. An additional routine is to allow plenty of time for children to explore new materials before attempting any directed activity with these new materials. The regular use of routines is important to the development of students' number sense, flexibility, fluency, collaborative skills and communication. These routines contribute to a rich, hands-on standards based classroom and will support students' performances on the tasks in this unit and throughout the school year.

## STRATEGIES FOR TEACHING AND LEARNING

- Students should be actively engaged by developing their own understanding.

- Mathematics should be represented in as many ways as possible by using graphs, tables, pictures, symbols and words.

- Interdisciplinary and cross curricular strategies should be used to reinforce and extend the learning activities.

- Appropriate manipulatives and technology should be used to enhance student learning.

- Students should be given opportunities to revise their work based on teacher feedback, peer feedback, and metacognition which includes self-assessment and reflection.

- Students should write about the mathematical ideas and concepts they are learning.

- Consideration of all students should be made during the planning and instruction of this unit. Teachers need to consider the following:

  - What level of support do my struggling students need in order to be successful with this unit?

  - In what way can I deepen the understanding of those students who are competent in this unit?

  - What real life connections can I make that will help my students utilize the skills practiced in this unit?

## EVIDENCE OF LEARNING

By the conclusion of this unit, students should be able to demonstrate the following competencies:

- Demonstrate a method to prove addition or subtraction identities for sine, cosine, and tangent.
- Apply addition or subtraction identities for sine, cosine, and tangent.
- Use addition or subtraction identities to find missing values for sine, cosine and tangent functions.

## TASKS

The following tasks represent the level of depth, rigor, and complexity expected of all Pre-Calculus students. These tasks or tasks of similar depth and rigor should be used to demonstrate evidence of learning. It is important that all elements of a task be addressed throughout the learning process so that students understand what is expected of them. While some tasks are identified as a performance task, they may also be used for teaching and learning (learning task).

| | |
|---|---|
| **Scaffolding Task** | Tasks that build up to the learning task. |
| **Learning Task** | Constructing understanding through deep/rich contextualized problem solving tasks. |
| **Practice Task** | Tasks that provide students opportunities to practice skills and concepts. |
| **Performance Task** | Tasks which may be a formative or summative assessment that checks for student understanding/misunderstanding and or progress toward the standard/learning goals at different points during a unit of instruction. |
| **Culminating Task** | Designed to require students to use several concepts learned during the unit to answer a new or unique situation. Allows students to give evidence of their own understanding toward the mastery of the standard and requires them to extend their chain of mathematical reasoning. |
| **Formative Assessment Lesson (FAL)** | Lessons that support teachers in formative assessment which both reveal and develop students' understanding of key mathematical ideas and applications. These lessons enable teachers and students to monitor in more detail their progress towards the targets of the standards. |

| Task Name | Task Type *Grouping Strategy* | Content Addressed |
|---|---|---|
| Proving the Sine Addition and Subtraction Identities | Learning Task *Large Group/Partner Task* | Developing a proof for the Sine addition and subtraction identities. |
| Proving the Cosine Addition and Subtraction Identities | Learning Task *Large Group/Partner Task* | Developing a proof for the Cosine addition and subtraction identities. |
| A Distance Formula Proof for the Cosine Addition Identity | Learning Task *Individual/Partner Task* | Using the Distance Formula to develop a proof for the Cosine addition identity. |
| Proving the Tangent Addition and Subtraction Identities | Learning Task *Partner/Small Group Task* | Developing a proof for the Tangent addition and subtraction identities. |
| **Culminating Task:** How Many Angles Can You Find? | Performance Task *Individual/Partner Task* | Using trigonometric addition and subtraction identities to determine trigonometric values. |

## PROVING THE SINE ADDITION AND SUBTRACTION IDENTITIES

### Common Core Standards:

**MCC9-12.F.TF.9 (+)** Prove the addition and subtraction formulas for sine, cosine, and tangent and use them to solve problems.

### Introduction:

This task guides students through one possible proof of addition and subtraction identities for sine. It is designed to guide the student step by step through the process, with several stops along the way to make sure they are on the right track. The emphasis should not be on this particular method, but rather the act of proof itself. Ask students to look beyond the steps to see mathematics happening through proof.

## PROVING THE SINE ADDITION AND SUBTRACTION IDENTITIES

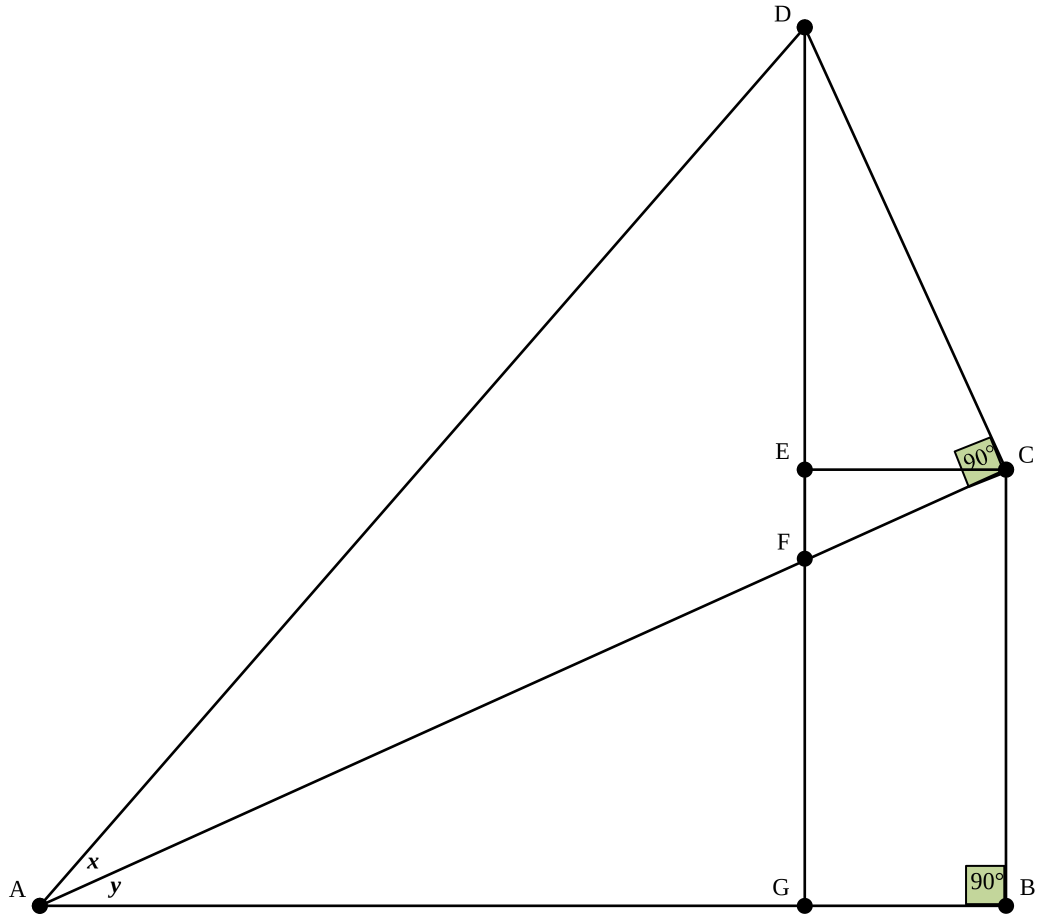
When it becomes necessary to add and subtract angles, finding the sine of the new angles is not as straight forward as adding or subtracting the sine of the individual angles. Convince yourself of this by performing the following operation:

$$\sin(30°) + \sin(60°) = \sin(90°)\,?$$

$$\frac{1}{2} + \frac{\sqrt{3}}{2} \neq 1$$

In this task, you will develop a formula for finding the sine of a sum or difference of two angles. Begin by looking at the diagram below. (note: $\overline{DG} \perp \overline{AB}$ ):

Use the diagram to answer the following questions:




1. What type of triangles are $\Delta ABC, \Delta ACD\ and\ \Delta DAG$?

   ***Right Triangles***

2. Write an expression for the measurement of $\angle DAB$.

   $m\angle DAB = x + y$

3. Write an expression for $\sin \angle DAB$.

   $$\sin(x+y) = \frac{\overline{DG}}{\overline{AD}}$$

4. $\overline{EG} \cong \overline{BC}$

Now we will use algebra to develop the formula for the sine of a sum of two angles.

a. Use your expression from #3.

$$\sin(x+y)=\frac{\overline{DG}}{\overline{AD}}$$

b. Rewrite DG as the sum of two segments, EG and DE, then split the fraction.

$$\sin(x+y)=\frac{\overline{DG}}{\overline{AD}}=\frac{\overline{EG}+\overline{DE}}{\overline{AD}}=\frac{\overline{EG}}{\overline{AD}}+\frac{\overline{DE}}{\overline{AD}}$$

c. Use the relationship from #4 to make a substitution.

$$\sin(x+y)=\frac{\overline{EG}}{\overline{AD}}+\frac{\overline{DE}}{\overline{AD}}=\frac{\overline{BC}}{\overline{AD}}+\frac{\overline{DE}}{\overline{AD}}$$

Did you arrive at this equation? $\sin(x+y)=\frac{\overline{BC}}{\overline{AD}}+\frac{\overline{DE}}{\overline{AD}}$ If not, go back and try again.

a. Now multiply the first term by $\frac{\overline{AC}}{\overline{AC}}$, and the second term by $\frac{\overline{CD}}{\overline{CD}}$. This will not change the value of our expressions, but will establish a link between some important pieces.

$$\sin(x+y)=\frac{\overline{BC}}{\overline{AD}}\cdot\frac{\overline{AC}}{\overline{AC}}+\frac{\overline{DE}}{\overline{AD}}\cdot\frac{\overline{CD}}{\overline{CD}}$$

***We are using AC and CD because they give us sine and cosine ratios when we multiply and divide them purposefully in each term.***

b. Now change the order on the denominators of both terms.

$$\sin(x+y)=\frac{\overline{BC}}{\overline{AC}}\cdot\frac{\overline{AC}}{\overline{AD}}+\frac{\overline{DE}}{\overline{CD}}\cdot\frac{\overline{CD}}{\overline{AD}}$$

Did you arrive at this equation? $\sin(x+y)=\frac{\overline{BC}}{\overline{AC}}\cdot\frac{\overline{AC}}{\overline{AD}}+\frac{\overline{DE}}{\overline{CD}}\cdot\frac{\overline{CD}}{\overline{AD}}$

Now is the most important step. Look at each factor. What relationship do each of those represent? For example, $\overline{BC}$ is the opposite side from $\angle X$, and $\overline{AC}$ is the hypotenuse in that triangle. This means that $\frac{\overline{BC}}{\overline{AC}} = \sin X$.

Substitute the other values and you have developed a formula for sin(*x*+*y*).

***Students may have trouble finding a substitution for $\frac{\overline{DE}}{\overline{CD}}$. Try to point them towards the fact that triangle CDE and triangle FCE are similar(CE is parallel to AB so angle ECA is congruent to angle BAC by alternate interior angles.)***

$$\sin(x+y) = \frac{\overline{BC}}{\overline{AC}} \cdot \frac{\overline{AC}}{\overline{AD}} + \frac{\overline{DE}}{\overline{CD}} \cdot \frac{\overline{CD}}{\overline{AD}} = \sin x \cos y + \cos x \sin y$$

Check your formula with the following exercises:

1. $\sin(45° + 45°) = \sin(90°)$?

$$\sin 45° \cos 45° + \cos 45° \sin 45° = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2} = 1 = \sin 90°$$

2. $\sin\left(\frac{\pi}{3} + \frac{\pi}{6}\right) = \sin\left(\frac{\pi}{2}\right)$?

$$\sin\frac{\pi}{3}\cos\frac{\pi}{6} + \cos\frac{\pi}{3}\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{1}{2} \cdot \frac{1}{2} = 1 = \sin\frac{\pi}{2}$$

Write your formula here:

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$

Now that we have a formula for the sine of a sum of angles, we can use it to develop the formula for the difference of angles.

We can think of the difference of two angles like this: $\sin(x-y) = \sin(x+(-y))$

Using the formula we developed above, we substitute in and get:

$\sin(x-y)=\sin x\cos(-y)+\cos x\sin(-y)$

But, cosine is an even function meaning that: $f(-x)=f(x)$ . Use this fact to simplify the equation.

$\sin(x-y)=\sin x\cos y+\cos x\sin(-y)$

Similarly, sine is an odd function meaning that: $f(-x)=-f(x)$. Use this fact to simplify the equation.

$\sin(x-y)=\sin x\cos y-\cos x\sin y$

Write your formula here:

$\sin(x-y)=\sin x\cos y-\cos x\sin y$

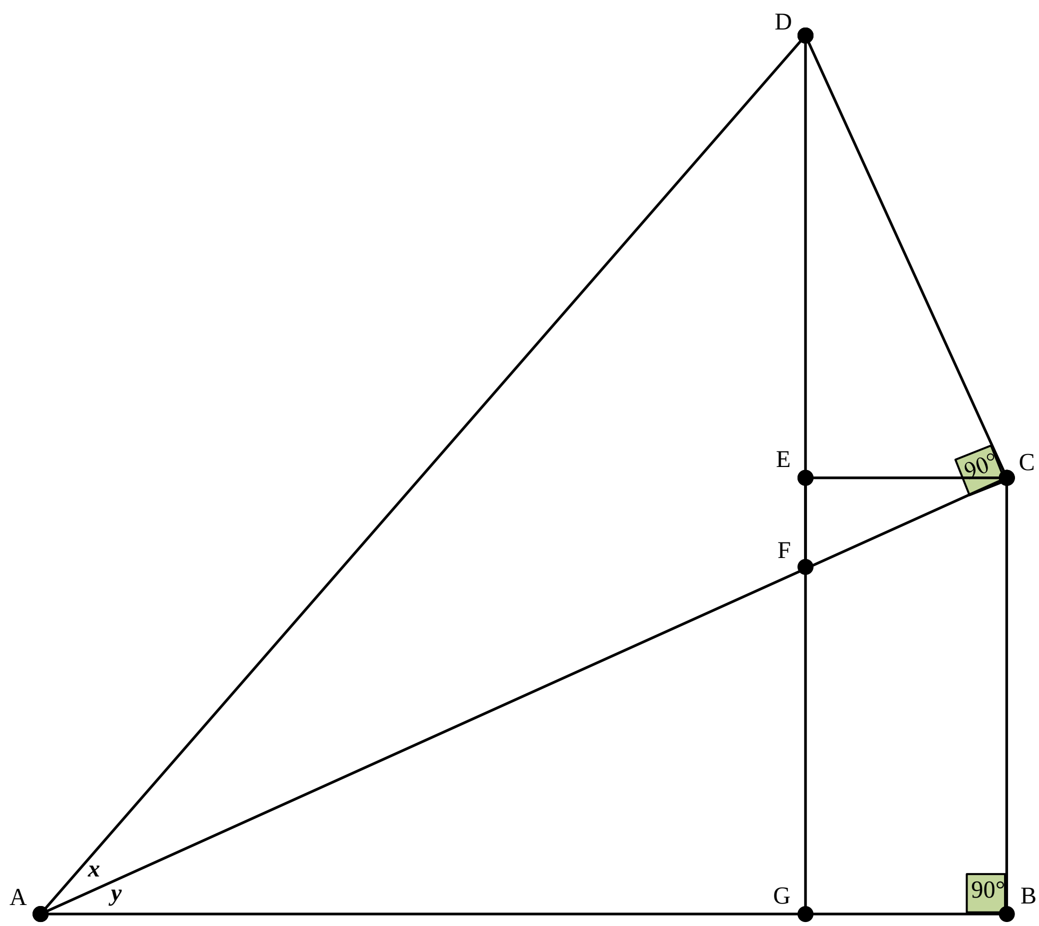
***A note to the teacher: This task only involves proving the addition and subtraction identities. The classroom teacher should build on this proof by providing problems that build competency and fluency in applying the identity. Such problems should include, but not be limited to: finding exact values, expressing as a trigonometric function of one angle, verifying identities and solving trigonometric equations.***

## PROVING THE SINE ADDITION AND SUBTRACTION IDENTITIES

When it becomes necessary to add and subtract angles, finding the sine of the new angles is not as straight forward as adding or subtracting the sine of the individual angles. Convince yourself of this by performing the following operation:

$\sin(30°)+\sin(60°)=\sin(90°)$ ?

In this task, you will develop a formula for finding the sine of a sum or difference of two angles. Begin by looking at the diagram below. (note: $\overline{DG} \perp \overline{AB}$ ):




Use the diagram to answer the following questions:

1. What type of triangles are $\Delta ABC, \Delta ACD$ *and* $\Delta DAG$?

2. Write an expression for the measurement of $\angle DAB$.

3. Write an expression for $\sin \angle DAB$.

4. $\overline{EG} \cong$ _____

Now we will use algebra to develop the formula for the sine of a sum of two angles.

a. Use your expression from #3.

b. Rewrite DG as the sum of two segments, EG and DE, then split the fraction.

c. Use the relationship from #4 to make a substitution.

Did you arrive at this equation? $\sin(x+y) = \frac{\overline{BC}}{\overline{AD}} + \frac{\overline{DE}}{\overline{AD}}$ If not, go back and try again.

a. Now multiply the first term by $\frac{\overline{AC}}{\overline{AC}}$, and the second term by $\frac{\overline{CD}}{\overline{CD}}$. This will not change the value of our expressions, but will establish a link between some important pieces.

b. Now change the order on the denominators of both terms.

Did you arrive at this equation? $\sin(x+y) = \frac{\overline{BC}}{\overline{AC}} \cdot \frac{\overline{AC}}{\overline{AD}} + \frac{\overline{DE}}{\overline{CD}} \cdot \frac{\overline{CD}}{\overline{AD}}$

Now is the most important step. Look at each factor. What relationship do each of those represent? For example, $\overline{BC}$ is the opposite side from $\angle X$, and $\overline{AC}$ is the hypotenuse in that triangle. This means that $\frac{\overline{BC}}{\overline{AC}} = \sin X$ .

Substitute the other values and you have developed a formula for sin(*x*+*y*).

Check your formula with the following exercises:

1. $\sin(45° + 45°) = \sin(90°)$?

2. $\sin\left(\frac{\pi}{3} + \frac{\pi}{6}\right) = \sin\left(\frac{\pi}{2}\right)$?

Write your formula here:

Now that we have a formula for the sine of a sum of angles, we can use it to develop the formula for the difference of angles.

We can think of the difference of two angles like this: $\sin(x - y) = \sin(x + (-y))$

Using the formula we developed above, we substitute in and get:

$$\sin(x - y) = \sin x \cos(-y) + \cos x \sin(-y)$$

But, cosine is an even function meaning that: $f(-x) = f(x)$ . Use this fact to simplify the equation.

Similarly, sine is an odd function meaning that: $f(-x) = -f(x)$ . Use this fact to simplify the equation.

Write your formula here:

## PROVING THE COSINE ADDITION AND SUBTRACTION IDENTITIES

### Common Core Standards:

**MCC9-12.F.TF.9 (+)** Prove the addition and subtraction formulas for sine, cosine, and tangent and use them to solve problems.

### Introduction:

This task follows the format of the first task. It is designed to guide the student step by step through the process, with several stops along the way to make sure they are on the right track. The emphasis should not be on this particular method, but rather the act of proof itself. Ask students to look beyond the steps to see mathematics happening through proof.

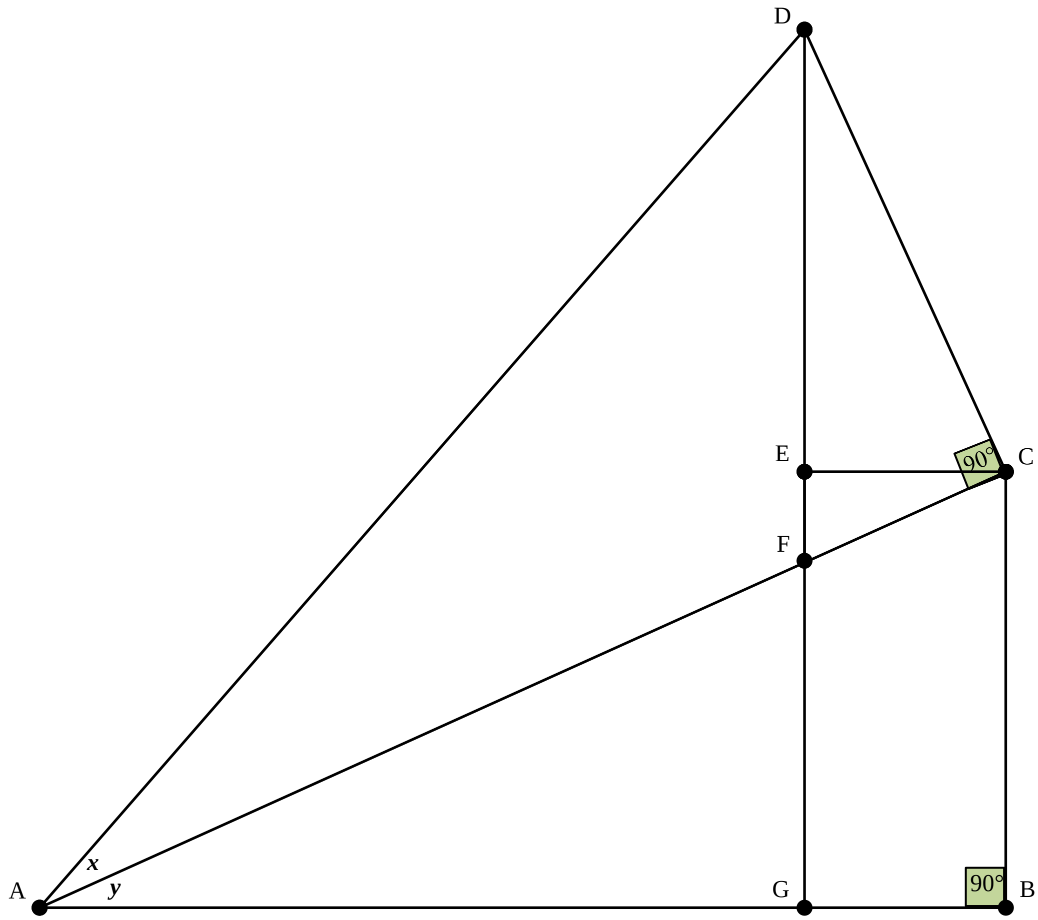
## PROVING THE COSINE ADDITION AND SUBTRACTION IDENTITIES

In the first task you saw that it is necessary to develop a formula for calculating the sine of sums or differences of angles. Can the same formula work for cosine? Experiment with your formula and see if you can use it to answer the exercise below:

$$\cos(30°)+\cos(60°)=\cos(90°)?$$

$$\frac{\sqrt{3}}{2}+\frac{1}{2}\neq 0$$

In this task, you will develop a formula for finding the sine of a sum or difference of two angles. Begin by looking at the diagram below. (note: $\overline{DG}\perp\overline{AB}$ ):




Use the diagram to answer the following questions:

1. What type of triangles are $\Delta ABC, \Delta ACD \text{ and } \Delta DAG$?

   ***Right Triangles***

2. Write an expression for the measurement of $\angle DAB$.

   $m\angle DAB = x + y$

3. Write an expression for $\cos \angle DAB$.

$$\cos(x+y) = \frac{\overline{AG}}{\overline{DA}}$$

4. $\overline{BG} \cong \overline{EC}$

5. $\angle ECA \cong \angle CAB$ by alternate interior angles.

6. $\angle CDE \cong \angle ECA$ because they are corresponding angles in similar triangles.

Now we will use algebra to develop the formula for the cosine of a sum of two angles.

a. Use your expression from #3.

$$\cos(x+y) = \frac{\overline{AG}}{\overline{DA}}$$

b. Rewrite AG as the difference of two segments, AB and BG, then split the fraction.

$$\cos(x+y) = \frac{\overline{AG}}{\overline{DA}} = \frac{\overline{AB}-\overline{BG}}{\overline{AD}} = \frac{\overline{AB}}{\overline{AD}} - \frac{\overline{BG}}{\overline{AD}}$$

c. Use the relationship from #4 to make a substitution.

$$\cos(x+y) = \frac{\overline{AB}}{\overline{AD}} - \frac{\overline{EC}}{\overline{AD}}$$

Did you arrive at this equation? $\cos(x+y) = \frac{\overline{AB}}{\overline{AD}} - \frac{\overline{EC}}{\overline{AD}}$ If not, go back and try again.

a. Now multiply the first term by $\frac{\overline{AC}}{\overline{AC}}$ , and the second term by $\frac{\overline{CD}}{\overline{CD}}$. This will not change the value of our expressions, but will establish a link between some important pieces.

$$\cos(x+y) = \frac{\overline{AB}}{\overline{AD}} \cdot \frac{\overline{AC}}{\overline{AC}} - \frac{\overline{EC}}{\overline{AD}} \cdot \frac{\overline{CD}}{\overline{CD}}$$

***We are using AC and CD because they give us sine and cosine ratios when we multiply and divide them purposefully in each term.***

b. Now change the order on the denominators of both terms.

$$\cos(x+y)=\frac{\overline{AB}}{\overline{AC}}\cdot\frac{\overline{AC}}{\overline{AD}}-\frac{\overline{EC}}{\overline{CD}}\cdot\frac{\overline{CD}}{\overline{AD}}$$

Did you arrive at this equation? $\cos(x+y)=\frac{\overline{AB}}{\overline{AC}}\cdot\frac{\overline{AC}}{\overline{AD}}-\frac{\overline{EC}}{\overline{CD}}\cdot\frac{\overline{CD}}{\overline{AD}}$

Now is the most important step. Look at each factor. What relationship do each of those represent? For example, $\overline{EC}$ is the opposite side from $\angle X$ (from similarity), and $\overline{CD}$ is a hypotenuse in that triangle. This means that $\frac{\overline{EC}}{\overline{CD}}=\sin X$ .

Substitute the other values and you have developed a formula for cos(*x*+*y*).

$$\cos(x+y)=\cos x\cdot\cos y-\sin x\cdot\sin y$$

Check your formula with the following exercises:

1. $\cos(30°+30°)=\cos(60°)$?

$$\cos(30°+30°)=\cos 30°\cdot\cos 30°-\sin 30°\cdot\sin 30°=\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}-\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{2}=\cos 60°$$

2. $\cos\left(\frac{\pi}{3}+\frac{\pi}{6}\right)=\cos\left(\frac{\pi}{2}\right)$?

$$\cos(\frac{\pi}{3}+\frac{\pi}{6})=\cos\frac{\pi}{3}\cdot\cos\frac{\pi}{6}-\sin\frac{\pi}{3}\cdot\sin\frac{\pi}{6}=\frac{1}{2}\cdot\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{2}\cdot\frac{1}{2}=0=\cos\frac{\pi}{2}$$

Write your formula here:

$\cos(x+y)=\cos x\cdot\cos y-\sin x\cdot\sin y$

Now that we have a formula for the cosine of a sum of angles, we can use it to develop the formula for the difference of angles.

We can think of the difference of two angles like this: $\cos(x-y)=\cos(x+(-y))$

Using the formula we developed above, we substitute in and get:

$\cos(x-y)=\cos x\cos(-y)-\sin x\sin(-y)$

But, cosine is an even function meaning that: $f(-x)=f(x)$ . Use this fact to simplify the equation.

$\cos(x-y)=\cos x\cdot\cos y-\sin x\cdot\sin\left(-y\right)$

Similarly, sine is an odd function meaning that: $f(-x)=-f(x)$. Use this fact to simplify the equation.

$\cos(x-y)=\cos x\cdot\cos y+\sin x\cdot\sin y$

Write your formula here:

$\cos(x-y)=\cos x\cdot\cos y+\sin x\cdot\sin y$

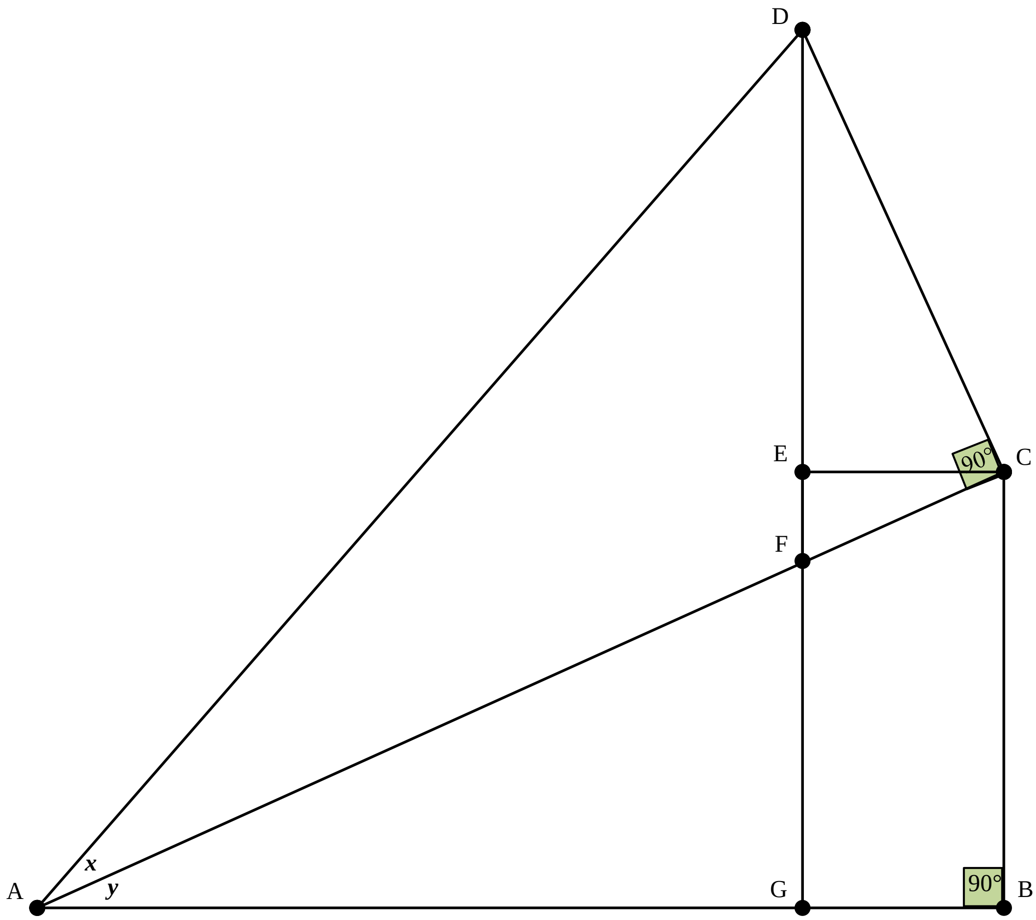
***A note to the teacher: This task only involves proving the addition and subtraction identities. The classroom teacher should build on this proof by providing problems that build competency and fluency in applying the identity. Such problems should include, but not be limited to: finding exact values, expressing as a trigonometric function of one angle, verifying identities and solving trigonometric equations.***

## PROVING THE COSINE ADDITION AND SUBTRACTION IDENTITIES

In the first task you saw that it is necessary to develop a formula for calculating the sine of sums or differences of angles. Can the same formula work for cosine? Experiment with your formula and see if you can use it to answer the exercise below:

$\cos(30°)+\cos(60°)=\cos(90°)$?

In this task, you will develop a formula for finding the sine of a sum or difference of two angles. Begin by looking at the diagram below. (note: $\overline{DG} \perp \overline{AB}$ ):




Use the diagram to answer the following questions:

1. What type of triangles are $\Delta ABC, \Delta ACD \text{ and } \Delta DAG$?

2. Write an expression for the measurement of $\angle DAB$.

3. Write an expression for $\cos \angle DAB$.

4. $\overline{BG} \cong$ _____

5. $\angle ECA \cong$ ______ by alternate interior angles.

6. $\angle CDE \cong$ ________ because they are corresponding angles in similar triangles.

Now we will use algebra to develop the formula for the cosine of a sum of two angles.

a. Use your expression from #3.

b. Rewrite AG as the difference of two segments, AB and BG, then split the fraction.

c. Use the relationship from #4 to make a substitution.

Did you arrive at this equation? $\cos(x+y) = \frac{\overline{AB}}{\overline{AD}} - \frac{\overline{EC}}{\overline{AD}}$ If not, go back and try again.

a. Now multiply the first term by $\frac{\overline{AC}}{\overline{AC}}$, and the second term by $\frac{\overline{CD}}{\overline{CD}}$. This will not change the value of our expressions, but will establish a link between some important pieces.

b. Now change the order on the denominators of both terms.

Did you arrive at this equation? $\cos(x+y)=\frac{\overline{AB}}{\overline{AC}}\cdot\frac{\overline{AC}}{\overline{AD}}-\frac{\overline{EC}}{\overline{CD}}\cdot\frac{\overline{CD}}{\overline{AD}}$

Now is the most important step. Look at each factor. What relationship do each of those represent? For example, $\overline{EC}$ is the opposite side from $\angle X$ (from similarity), and $\overline{CD}$ is a hypotenuse in that triangle. This means that $\frac{\overline{EC}}{\overline{CD}}=\sin X$ .

Substitute the other values and you have developed a formula for cos($x$+$y$).

Check your formula with the following exercises:

1. $\cos(30°+30°)=\cos(60°)$?

2. $\cos\left(\frac{\pi}{3}+\frac{\pi}{6}\right)=\cos\left(\frac{\pi}{2}\right)$?

Write your formula here:

Now that we have a formula for the cosine of a sum of angles, we can use it to develop the formula for the difference of angles.

We can think of the difference of two angles like this: $\cos(x-y)=\cos(x+(-y))$

Using the formula we developed above, we substitute in and get:

$$\cos(x-y)=\cos x\cos(-y)-\sin x\sin(-y)$$

But, cosine is an even function meaning that: $f(-x)=f(x)$ . Use this fact to simplify the equation.

Similarly, sine is an odd function meaning that: $f(-x) = -f(x)$. Use this fact to simplify the equation.

Write your formula here:

## A DISTANCE FORMULA PROOF FOR THE COSINE ADDITION IDENTITY

### Common Core Standards:

**MCC9-12.F.TF.9 (+)** Prove the addition and subtraction formulas for sine, cosine, and tangent and use them to solve problems.

### Introduction:

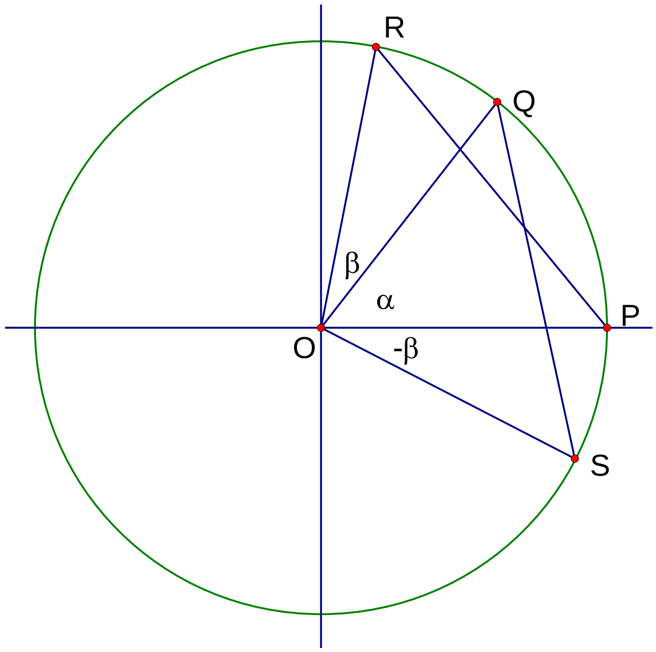
In this task, students derive the sum identity for the cosine function, in the process reviewing some of the geometric topics and ideas about proofs. This derivation also provides practice with algebraic manipulation of trigonometric functions that include examples of how applying the Pythagorean identities can often simplify a cumbersome trigonometric expression. Rewriting expressions in order to solve trigonometric equations is one of the more common applications of the sum and difference identities. This task is presented as an alternative proof to the ones the students performed earlier.

## A DISTANCE FORMULA PROOF FOR THE COSINE ADDITION IDENTITY

In this task, you will use the sum and difference identities to solve equations and find the exact values of angles that are not multiples of $\frac{\pi}{6}$ and $\frac{\pi}{4}$. Before you apply these identities to problems, you will first derive them. The first identity you will prove involves taking the cosine of the sum of two angles.

$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$

We can derive this identity by making deductions from the relationships between the quantities on the unit circle below.




**1.** Complete the following congruence statements:

a. $\angle ROP \cong \angle QOS$

b. $\overline{RO} \cong \overline{QO} \cong \overline{PO} \cong \overline{SO}$

c. By the ***SAS*** congruence theorem, $\Delta ROP \cong \Delta QOS$

d. $\overline{RP} \cong \overline{QS}$

**2.** Write the coordinates of each of the four points on the unit circle, remembering that the cosine and sine functions produce x- and y- values on the unit circle.

**a.** R = $(\cos(\alpha+\beta), \sin(\alpha+\beta))$

**b.** Q = $(\cos\alpha, \sin\alpha)$

**c.** P = (***1, 0***)

**d.** S = $(\cos(-\beta), \sin(-\beta))$

**3.** Use the coordinates found in problem 2 and the distance formula to find the length of chord $\overline{RP}$. ***Note: Students may not simplify here, but will need to in part 5.***

***Solution:***

$$\sqrt{\left(\cos(\alpha+\beta)-1\right)^2+\left(\sin(\alpha+\beta)-0\right)^2}$$

$$=\sqrt{\cos^2(\alpha+\beta)-2\cos(\alpha+\beta)+1+\sin^2(\alpha+\beta)}$$ ***squaring each binomial***

$$=\sqrt{\left(\cos^2(\alpha+\beta)+\sin^2(\alpha+\beta)\right)+1-2\cos(\alpha+\beta)}$$ ***rearranging terms***

$$=\sqrt{1+1-2\cos(\alpha+\beta)}$$ ***applying a Pythagorean identity***

$$=\sqrt{2-2\cos(\alpha+\beta)}$$

**4. a.** Use the coordinates found in problem 2 and the distance formula to find the length of chord $\overline{QS}$. ***Note: Students may not simplify here, but will need to in part 5.***

***Solution:***

$$\sqrt{\left(\cos\alpha-\cos(-\beta)\right)^2+\left(\sin\alpha-\sin(-\beta)\right)^2}$$

$$=\sqrt{\cos^2\alpha-2\cos\alpha\cos(-\beta)+\cos^2(-\beta)+\sin^2\alpha-2\sin\alpha\sin(-\beta)+\sin^2(-\beta)}$$ ***squaring each binomial***

$$=\sqrt{\left(\cos^2\alpha+\sin^2\alpha\right)+\left(\cos^2(-\beta)+\sin^2(-\beta)\right)-2\cos\alpha\cos(-\beta)-2\sin\alpha\sin(-\beta)}$$ ***rearranging terms***

$$=\sqrt{1+1-2\cos\alpha\cos(-\beta)-2\sin\alpha\sin(-\beta)}$$ ***applying a Pythagorean identity twice***

$$=\sqrt{2-2\cos\alpha\cos(-\beta)-2\sin\alpha\sin(-\beta)}$$

**b.** Two useful identities that you may choose to explore later are $\cos(-\theta)=\cos\theta$ and $\sin(-\theta)=-\sin\theta$. Use these two identities to simplify your solution to 4a so that your expression has no negative angles.

***Solution:***

$$\sqrt{2-2\cos\alpha\cos(-\beta)-2\sin\alpha\sin(-\beta)}$$

$$=\sqrt{2-2\cos\alpha\cos\beta-2\sin\alpha(-\sin\beta)}$$

$$=\sqrt{2-2\cos\alpha\cos\beta+2\sin\alpha\sin\beta}$$

**5.** From 1d, you know that $\overline{RP} \cong \overline{QS}$. You can therefore write an equation by setting the expressions found in problems 3 and 4b equal to one another. Simplify this equation and solve for $\cos(\alpha+\beta)$. Applying one of the Pythagorean Identities will be useful! When finished, you will have derived the angle sum identity for cosine.

***Solution:***

$$\sqrt{2-2\cos(\alpha+\beta)} = \sqrt{2-2\cos\alpha\cos\beta+2\sin\alpha\sin\beta}$$

$$2-2\cos(\alpha+\beta) = 2-2\cos\alpha\cos\beta+2\sin\alpha\sin\beta$$ ***squaring both sides***

$$2\cos(\alpha+\beta) = 2\cos\alpha\cos\beta+2\sin\alpha\sin\beta$$ ***adding 2 to both sides***

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta+\sin\alpha\sin\beta$$ ***dividing both sides by 2***

The other three sum and difference identities can be derived from the identity found in problem 5. These four identities can be summarized with the following two statements.

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$
$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \square \sin\alpha\sin\beta$$

Recall that so far, you can only calculate the exact values of the sines and cosines of multiples of $\frac{\pi}{6}$ and $\frac{\pi}{4}$. These identities will allow you to calculate the exact value of the sine and cosine of many more angles.

**6.** Evaluate $\sin 75°$ by applying the angle addition identity for sine and evaluating each trigonometric function:

$$\sin(30° + 45°) = \sin 30° \cos 45° + \cos 30° \sin 45°$$

***Solution:***

$$\frac{1}{2}\cdot\frac{\sqrt{2}}{2}+\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}=\frac{\sqrt{2}}{4}+\frac{\sqrt{6}}{4}=\frac{\sqrt{2}+\sqrt{6}}{4}$$

**7.** Similarly, find the exact value of the following trigonometric expressions:

**a.** $\cos(15°)$

***Solution:*** $\dfrac{\sqrt{2}+\sqrt{6}}{4}$

**b.** $\sin\left(\dfrac{\pi}{12}\right)$

***Solution:*** $\dfrac{\sqrt{6}-\sqrt{2}}{4}$

**c.** $\cos(345°)$

***Solution:*** $\dfrac{\sqrt{2}+\sqrt{6}}{4}$

**d.** $\sin\dfrac{19\pi}{12}$

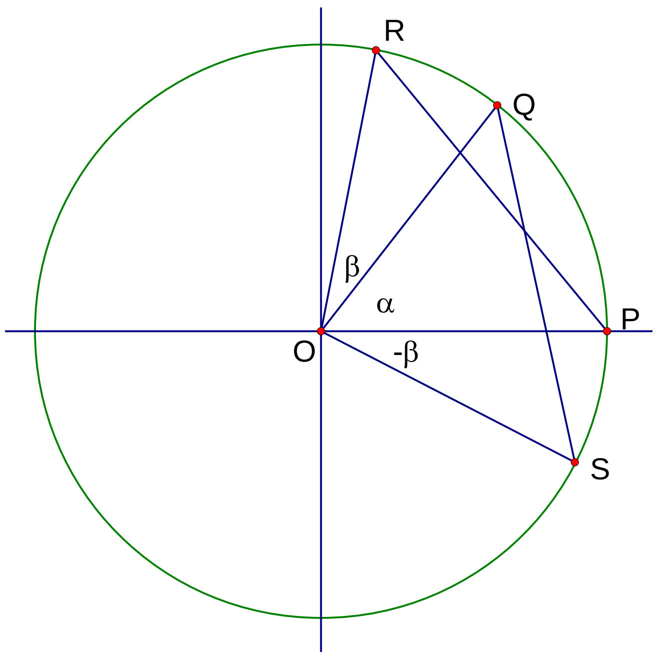
***Solution:*** $\dfrac{\sqrt{2}-\sqrt{6}}{4}$

## A DISTANCE FORMULA PROOF FOR THE COSINE ADDITION IDENTITY

In this task, you will use the sum and difference identities to solve equations and find the exact values of angles that are not multiples of $\frac{\pi}{6}$ and $\frac{\pi}{4}$. Before you apply these identities to problems, you will first derive them. The first identity you will prove involves taking the cosine of the sum of two angles.

$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$

We can derive this identity by making deductions from the relationships between the quantities on the unit circle below.




**1.** Complete the following congruence statements:

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**b.** $\overline{RO} \cong$ _______ $\cong$ ________ $\cong$ _________

**c.** By the __________ congruence theorem, $\Delta ROP \cong \Delta QOS$

**d.** $\overline{RP} \cong$ __________

**2.** Write the coordinates of each of the four points on the unit circle, remembering that the cosine and sine functions produce x- and y- values on the unit circle.

**a.** R =

**b.** Q =

**c.** P =

**d.** S =

**3.** Use the coordinates found in problem 2 and the distance formula to find the length of chord $\overline{RP}$.

**4.** Use the coordinates found in problem 2 and the distance formula to find the length of chord $\overline{QS}$.

**5.** From 1d, you know that $\overline{RP} \cong \overline{QS}$. You can therefore write an equation by setting the expressions found in problems 3 and 4b equal to one another. Simplify this equation and solve for $\cos(\alpha+\beta)$. Applying one of the Pythagorean Identities will be useful! When finished, you will have derived the angle sum identity for cosine.

The other three sum and difference identities can be derived from the identity found in problem 5. These four identities can be summarized with the following two statements.

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$
$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

Recall that so far, you can only calculate the exact values of the sines and cosines of multiples of $\frac{\pi}{6}$ and $\frac{\pi}{4}$. These identities will allow you to calculate the exact value of the sine and cosine of many more angles.

**6.** Evaluate $\sin 75°$ by applying the angle addition identity for sine and evaluating each trigonometric function:

$$\sin(30° + 45°) = \sin 30°\cos 45° + \cos 30°\sin 45°$$

**7.** Similarly, find the exact value of the following trigonometric expressions:

**a.** $\cos(15°)$

**b.** $\sin\left(\frac{\pi}{12}\right)$

**c.** $\cos(345°)$

**d.** $\sin\frac{19\pi}{12}$

## PROVING THE TANGENT ADDITION AND SUBTRACTION IDENTITIES

### Common Core Standards:

**MCC9-12.F.TF.9 (+)** Prove the addition and subtraction formulas for sine, cosine, and tangent and use them to solve problems.

### Introduction:

This task uses algebraic manipulation and the previously developed identities to derive tangent addition and subtraction identities. Again, the emphasis should be on the mathematical processes and not the method itself. Students should feel confident in manipulating algebraic expressions and simplifying trigonometric expressions using identities.

## PROVING THE TANGENT ADDITION AND SUBTRACTION IDENTITIES

By this point, you should have developed formulas for sine and cosine of sums and differences of angles. If so, you are already most of the way to finding a formula for the tangent of a sum of two angles.

Let's begin with a relationship that we already know to be true about tangent.

1. $\tan x = \dfrac{\sin x}{\cos x}$ so it stands to reason that $\tan\left(x+y\right) = \dfrac{\sin(x+y)}{\cos(x+y)}$

Use what you already know about sum and difference formulas to expand the relationship above.

2. $\tan\left(x+y\right) = \dfrac{\sin x\cos y + \cos x\sin y}{\cos x\cos y - \sin x\sin y}$

3. Now we want to simplify this. (Hint: Multiply numerator and denominator by $\frac{1}{\cos x}$ )

***The reason to multiply by*** $\frac{1}{\cos x}$ ***is to establish a tangent ratio and to divide out cos x. It is important that students see why to choose that value as a part of the proof.***

$$\tan(x+y) = \frac{\frac{\sin x \cos y}{\cos x} + \frac{\cos x \sin y}{\cos x}}{\frac{\cos x \cos y}{\cos x} - \frac{\sin x \sin y}{\cos x}} = \frac{\tan x \cos y + \sin y}{\cos y - \tan x \sin y}$$

4. You can simplify it some more. Think about step 3 for a hint.

***Students should multiply by*** $\frac{1}{\cos y}$ ***to establish a tangent ratio for the y variable and divide out cos y. Lead them back to #3 if they need help.***

$$\tan(x+y) = \frac{\frac{\tan x \cos y}{\cos y} + \frac{\sin y}{\cos y}}{\frac{\cos y}{\cos y} - \frac{\tan x \sin y}{\cos y}} = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

5. Write your formula here for the tangent of a sum:

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

6. Now that you have seen the process, develop a formula for the tangent of a difference.

***Encourage students to attempt this on their own without referring back to the proof of the addition identity. If they need help, they may reference it. Encourage them to persevere through this process and not give up easily. Mathematics and especially proof is meant to be a productive struggle in which students work hard to construct their own knowledge.***

***When they have finished, they should have:*** $\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$

Write your formula here:

$$\tan(x-y)=\frac{\tan x-\tan y}{1+\tan x\tan y}$$

## THE TANGENT ADDITION AND SUBTRACTION IDENTITIES

By this point, you should have developed formulas for sine and cosine of sums and differences of angles. If so, you are already most of the way to finding a formula for the tangent of a sum of two angles.

Let's begin with a relationship that we already know to be true about tangent.

1. $\tan x = \dfrac{\sin x}{\cos x}$ so it stands to reason that $\tan(x+y) =$ ________________

Use what you already know about sum and difference formulas to expand the relationship above.

2. $\tan(x+y) =$

3. Now we want to simplify this. (Hint: Multiply numerator and denominator by $\dfrac{1}{\cos x}$ )

4. You can simplify it some more. Think about step 3 for a hint.

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6. Now that you have seen the process, develop a formula for the tangent of a difference.

Write your formula here:

## CULMINATING TASK: HOW MANY ANGLES CAN YOU FIND?

### Common Core Standards:

**MCC9-12.F.TF.9 (+)** Prove the addition and subtraction formulas for sine, cosine, and tangent and use them to solve problems.

### Introduction:

The purpose of this culminating task is to give students an exercise in using the identities that they have developed. There are many possibilities for answers and students should demonstrate that they can apply each of the identities with other trigonometric relationships in order to find the answers.

You may consider suggesting different methods for students who need more guidance. For example, find 2 angles using the sine addition identity…etc.

## CULMINATING TASK: HOW MANY ANGLES CAN YOU FIND?

Using the following values, how many other trigonometric values of angles between 0 and 180 degrees can you find (without using a calculator)? Don't forget about complementary angles, co-terminal angles and other trig rules. Show work and justification for each value.

$$\sin 5° = 0.0872\ ;\ \cos 45° = 0.7071\ ;\ \sin 60° = 0.8660\ ;\ \cos 60° = 0.5$$

***Example: $\sin 50° = \sin(5° + 45°) = \sin 5° \cos 45° + \cos 5° \sin 45°$ and remembering that $\sin^2 5° + \cos^2 5° = 1$ allows us to find $\cos 5° = 0.9962$,***
***so $\sin 50° = 0.0872(0.7071) + 0.7071(0.9962) = 0.7661$ (which is quite close to the actual value of 0.7660).***

***EXTENSION: Have students develop an identity for sin(x+y+z), cos(x+y+z) and tan(x+y+z).***

## CULMINATING TASK: HOW MANY ANGLES CAN YOU FIND?

Using the following values, how many other trigonometric values of angles between 0 and 180 degrees can you find (without using a calculator)? Don't forget about complementary angles, co-terminal angles and other trig rules. Show work and justification for each value.

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