



# CCGPS Frameworks

## Mathematics

### CCGPS Pre-Calculus Unit 5: Matrices



**Dr. John D. Barge, State School Superintendent**  
*"Making Education Work for All Georgians"*

**Unit 5**  
**Matrices**

TABLE OF CONTENTS

|  |    |
|--|----|
| OVERVIEW .....                                 | 3  |
| STANDARDS ADDRESSED IN THIS UNIT .....         | 4  |
| ENDURING UNDERSTANDINGS .....                  | 8  |
| ESSENTIAL QUESTIONS .....                      | 8  |
| CONCEPTS AND SKILLS TO MAINTAIN .....          | 8  |
| SELECT TERMS AND SYMBOLS .....                 | 9  |
| CLASSROOM ROUTINES .....                       | 9  |
| STRATEGIES FOR TEACHING AND LEARNING .....     | 10 |
| EVIDENCE OF LEARNING .....                     | 10 |
| TASKS .....                                    | 11 |
| Central High Booster Club .....                | 12 |
| Walk Like a Mathematician .....                | 28 |
| Candy? What Candy?.....                        | 41 |
| An Okefenoke Food Web.....                     | 51 |
| Culminating Task: Vacationing in Georgia ..... | 63 |

## **OVERVIEW**

In this unit students will:

- represent and manipulate data using matrices
- define the order of a matrix as the number of rows by the number of columns
- add and subtract matrices and know these operations are possible only when the dimensions are equal
- recognize that matrix addition and subtraction are commutative
- multiply matrices by a scalar and understand the distributive and associative properties apply to matrices
- multiply matrices and know when the operation is defined
- recognize that matrix multiplication is not commutative
- understand and apply the properties of a zero matrix
- understand and apply the properties of an identity matrix
- find the determinant of a square matrix and understand that it is a nonzero value if and only if the matrix has an inverse
- use 2 X 2 matrices as transformations of a plane and determine the area of the plane using the determinant
- write a system of linear equations as a matrix equation and use the inverse of the coefficient matrix to solve the system
- write and use vertex-edge graphs to solve problems

In this unit, students learn to represent data rectangular arrangements of numbers. These arrangements of numbers into rows and columns are called matrices. Students should learn to compute with matrices and recognize the similarities and differences between the properties of real numbers and the properties of matrices. They will learn to use matrices in order to represent and solve more complex problems such as a system of equations and the area of a plane.

Students usually find matrix algebra operations to be very appealing since most operations can be done with a variety of calculators and/or computer programs. The tasks in this unit are designed to introduce matrix algebra and to provide practical applications for matrix transposes, determinants, inverses, and powers.

Although the units in this instructional framework emphasize key standards and big ideas at specific times of the year, routine topics such as estimation, mental computation, and basic computation facts should be addressed on an ongoing basis. Ideas related to the eight practice standards should be addressed constantly as well. To assure that this unit is taught with the appropriate emphasis, depth, and rigor, it is important that the tasks listed under “Evidence of Learning” be reviewed early in the planning process. A variety of resources should be utilized to supplement this unit. This unit provides much needed content information, but excellent learning

activities as well. The tasks in this unit illustrate the types of learning activities that should be utilized from a variety of sources.

### **STANDARDS ADDRESSED IN THIS UNIT**

Mathematical standards are interwoven and should be addressed throughout the year in as many different units and activities as possible in order to emphasize the natural connections that exist among mathematical topics.

#### **Perform operations on matrices and use matrices in applications.**

**MCC9-12.N.VM.6 (+)** Use matrices to represent and manipulate data, e.g., to represent payoffs or incidence relationships in a network.

**MCC9-12.N.VM.7 (+)** Multiply matrices by scalars to produce new matrices, e.g., as when all of the payoffs in a game are doubled.

**MCC9-12.N.VM.8 (+)** Add, subtract, and multiply matrices of appropriate dimensions.

**MCC9-12.N.VM.9 (+)** Understand that, unlike multiplication of numbers, matrix multiplication for square matrices is not a commutative operation, but still satisfies the associative and distributive properties.

**MCC9-12.N.VM.10 (+)** Understand that the zero and identity matrices play a role in matrix addition and multiplication similar to the role of 0 and 1 in the real numbers. The determinant of a square matrix is nonzero if and only if the matrix has a multiplicative inverse.

**MCC9-12.N.VM.12 (+)** Work with 2 X 2 matrices as transformations of the plane, and interpret the absolute value of the determinant in terms of area.

#### **Solve systems of equations**

**MCC9-12.A.REI.8 (+)** Represent a system of linear equations as a single matrix equation in a vector variable.

**MCC9-12.A.REI.9 (+)** Find the inverse of a matrix if it exists and use it to solve systems of linear equations (using technology for matrices of dimension  $3 \times 3$  or greater).

#### **Standards for Mathematical Practice**

The Standards for Mathematical Practice describe varieties of expertise that mathematics educators at all levels should seek to develop in their students. These practices rest on important “processes and proficiencies” with longstanding importance in mathematics education. The first

of these are the NCTM process standards of problem solving, reasoning and proof, communication, representation, and connections. The second are the strands of mathematical proficiency specified in the National Research Council's report *Adding It Up*: adaptive reasoning, strategic competence, conceptual understanding (comprehension of mathematical concepts, operations and relations), procedural fluency (skill in carrying out procedures flexibly, accurately, efficiently and appropriately), and productive disposition (habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy).

- 1. Make sense of problems and persevere in solving them.** High school students start to examine problems by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. By high school, students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. They check their answers to problems using different methods and continually ask themselves, "Does this make sense?" They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.
  
- 2. Reason abstractly and quantitatively.** High school students seek to make sense of quantities and their relationships in problem situations. They abstract a given situation and represent it symbolically, manipulate the representing symbols, and pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Students use quantitative reasoning to create coherent representations of the problem at hand; consider the units involved; attend to the meaning of quantities, not just how to compute them; and know and flexibly use different properties of operations and objects.
  
- 3. Construct viable arguments and critique the reasoning of others.** High school students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. High school students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that

which is flawed, and—if there is a flaw in an argument—explain what it is. High school students learn to determine domains to which an argument applies, listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

- 4. Model with mathematics.** High school students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. High school students making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.
- 5. Use appropriate tools strategically.** High school students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. High school students should be sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. They are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.
- 6. Attend to precision.** High school students try to communicate precisely to others by using clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context. By the time they reach high school they have learned to examine claims and make explicit use of definitions.
- 7. Look for and make use of structure.** By high school, students look closely to discern a pattern or structure. In the expression  $x^2 + 9x + 14$ , older students can see the 14 as  $2 \times 7$  and the 9 as  $2 + 7$ . They recognize the significance of an existing line in a geometric

figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see  $5 - 3(x - y)^2$  as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers  $x$  and  $y$ . High school students use these patterns to create equivalent expressions, factor and solve equations, and compose functions, and transform figures.

- 8. Look for and express regularity in repeated reasoning.** High school students notice if calculations are repeated, and look both for general methods and for shortcuts. Noticing the regularity in the way terms cancel when expanding  $(x - 1)(x + 1)$ ,  $(x - 1)(x^2 + x + 1)$ , and  $(x - 1)(x^3 + x^2 + x + 1)$  might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, derive formulas or make generalizations, high school students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

### **Connecting the Standards for Mathematical Practice to the Standards for Mathematical Content**

The Standards for Mathematical Practice describe ways in which developing student practitioners of the discipline of mathematics should engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle and high school years. Designers of curricula, assessments, and professional development should all attend to the need to connect the mathematical practices to mathematical content in mathematics instruction.

The Standards for Mathematical Content are a balanced combination of procedure and understanding. Expectations that begin with the word “understand” are often especially good opportunities to connect the practices to the content. Students who do not have an understanding of a topic may rely on procedures too heavily. Without a flexible base from which to work, they may be less likely to consider analogous problems, represent problems coherently, justify conclusions, apply the mathematics to practical situations, use technology mindfully to work with the mathematics, explain the mathematics accurately to other students, step back for an overview, or deviate from a known procedure to find a shortcut. In short, a missing mathematical knowledge effectively prevents a student from engaging in the mathematical practices.

In this respect, those content standards which set an expectation of understanding are potential “points of intersection” between the Standards for Mathematical Content and the Standards for Mathematical Practice. These points of intersection are intended to be weighted toward central and generative concepts in the school mathematics curriculum that most merit the time, resources, innovative energies, and focus necessary to qualitatively improve the curriculum, instruction, assessment, professional development, and student achievement in mathematics.

### **ENDURING UNDERSTANDINGS**

- Matrices provide an organizational structure in which to represent and solve complex problems.
- The commutative property applies to matrix addition but does not extend to matrix multiplication.
- A zero matrix behaves in addition, subtraction, and multiplication much like 0 in the real number system.
- An identity matrix behaves much like the number 1 in the real number system.
- The determinant of a matrix is nonzero if and only if the matrix has an inverse.
- 2 X 2 matrices can be written as transformations of the plane and can be interpreted as absolute value of the determinant in terms of area.
- Solving systems of linear equations can be extended to matrices and the methods we use can be justified.

### **ESSENTIAL QUESTIONS**

- How can we represent data in matrix form?
- How do we add and subtract matrices and when are these operations defined?
- How do we perform scalar multiplication on matrices?
- How do we multiply matrices and when is this operation defined?
- How do the commutative, associative, and distributive properties apply to matrices?
- What is a zero matrix and how does it behave?
- What is an identity matrix and how does it behave?
- How do we find the determinant of a matrix and when is it nonzero?
- How do we find the inverse of a matrix and when does a matrix not have an inverse defined?
- How do we solve systems of equations using inverse matrices?
- How do we find the area of a plane using matrices?
- How do we write and use vertex-edge graphs to solve problems?

### **CONCEPTS/SKILLS TO MAINTAIN**

It is expected that students will have prior knowledge/experience related to the concepts and skills identified below. It may be necessary to pre-assess in order to determine if time needs to be spent on conceptual activities that help students develop a deeper understanding of these ideas.

- Commutative Property
- Associative Property
- Distributive Property
- Identity Properties of Addition and Multiplication



- Inverse Properties of Addition and Multiplication
- Solving Systems of Equations Graphically and Algebraically

## **SELECT TERMS AND SYMBOLS**

The following terms and symbols are often misunderstood. These concepts are not an inclusive list and should not be taught in isolation. However, due to evidence of frequent difficulty and misunderstanding associated with these concepts, instructors should pay particular attention to them and how their students are able to explain and apply them.

**The definitions below are for teacher reference only and are not to be memorized by the students.** Students should explore these concepts using models and real life examples. Students should understand the concepts involved and be able to recognize and/or demonstrate them with words, models, pictures, or numbers.

The websites below are interactive and include a math glossary suitable for high school children. **Note – At the high school level, different sources use different definitions. Please preview any website for alignment to the definitions given in the frameworks**

<http://www.amathsdictionaryforkids.com/>

This web site has activities to help students more fully understand and retain new vocabulary.

<http://intermath.coe.uga.edu/dictionary/homepg.asp>

Definitions and activities for these and other terms can be found on the Intermath website.

- **Determinant:** the product of the elements on the main diagonal minus the product of the elements off the main diagonal
- **Dimensions or Order of a Matrix:** the number of rows by the number of columns
- **Identity Matrix:** the matrix that has 1's on the main diagonal and 0's elsewhere
- **Inverse Matrices:** matrices whose product ( in both orders) is the Identity matrix
- **Matrix:** a rectangular arrangement of numbers into rows and columns
- **Scalar:** in matrix algebra, a real number is called a scalar
- **Square Matrix:** a matrix with the same number of rows and columns
- **Zero Matrix:** a matrix whose entries are all zeros

## **CLASSROOM ROUTINES**

The importance of continuing the established classroom routines cannot be overstated. Daily routines must include such obvious activities as estimating, analyzing data, describing patterns, and answering daily questions. They should also include less obvious routines, such as how to

select materials, how to use materials in a productive manner, how to put materials away, how to access classroom technology such as computers and calculators. An additional routine is to allow plenty of time for children to explore new materials before attempting any directed activity with these new materials. The regular use of routines is important to the development of students' number sense, flexibility, fluency, collaborative skills and communication. These routines contribute to a rich, hands-on standards based classroom and will support students' performances on the tasks in this unit and throughout the school year.

### **STRATEGIES FOR TEACHING AND LEARNING**

- Students should be actively engaged by developing their own understanding.
- Mathematics should be represented in as many ways as possible by using graphs, tables, pictures, symbols and words.
- Interdisciplinary and cross curricular strategies should be used to reinforce and extend the learning activities.
- Appropriate manipulatives and technology should be used to enhance student learning.
- Students should be given opportunities to revise their work based on teacher feedback, peer feedback, and metacognition which includes self-assessment and reflection.
- Students should write about the mathematical ideas and concepts they are learning.
- Consideration of all students should be made during the planning and instruction of this unit. Teachers need to consider the following:
  - What level of support do my struggling students need in order to be successful with this unit?
  - In what way can I deepen the understanding of those students who are competent in this unit?
  - What real life connections can I make that will help my students utilize the skills practiced in this unit?

### **EVIDENCE OF LEARNING**

By the conclusion of this unit, students should be able to demonstrate the following competencies:

- use matrices to represent data
- multiply matrices by a scalar
- add, subtract, and multiply matrices of appropriate dimensions
- know when matrix operations are not defined
- apply the commutative, associative, and distributive properties to operations with matrices only when appropriate
- identify zero and identity matrices
- find the determinants of matrices
- use inverse matrices to solve systems of linear equations
- construct vertex-edge graphs and solve related problems
- find the area of a plane using matrices

**Georgia Department of Education**  
 Common Core Georgia Performance Standards Framework  
*CCGPS Pre-Calculus • Unit 5*

**TASKS**

The following tasks represent the level of depth, rigor, and complexity expected of all Pre-Calculus students. These tasks or tasks of similar depth and rigor should be used to demonstrate evidence of learning. It is important that all elements of a task be addressed throughout the learning process so that students understand what is expected of them. While some tasks are identified as a performance task, they may also be used for teaching and learning (learning task).

|  |   |
|--|---|
| <b>Scaffolding Task</b>                  | Tasks that build up to the learning task.   |
| <b>Learning Task</b>                     | Constructing understanding through deep/rich contextualized problem solving tasks.  |
| <b>Practice Task</b>                     | Tasks that provide students opportunities to practice skills and concepts.  |
| <b>Performance Task</b>                  | Tasks which may be a formative or summative assessment that checks for student understanding/misunderstanding and or progress toward the standard/learning goals at different points during a unit of instruction.  |
| <b>Culminating Task</b>                  | Designed to require students to use several concepts learned during the unit to answer a new or unique situation. Allows students to give evidence of their own understanding toward the mastery of the standard and requires them to extend their chain of mathematical reasoning. |
| <b>Formative Assessment Lesson (FAL)</b> | Lessons that support teachers in formative assessment which both reveal and develop students' understanding of key mathematical ideas and applications. These lessons enable teachers and students to monitor in more detail their progress towards the targets of the standards.   |

**Georgia Department of Education**  
 Common Core Georgia Performance Standards Framework  
*CCGPS Pre-Calculus • Unit 5*

| <b>Task Name</b>                                   | <b>Task Type</b><br><i>Grouping Strategy</i>       | <b>Content Addressed</b>   |
|--|--|--|
| Central High Booster Club                          | Learning Task<br><i>Individual/Partner Task</i>    | Write matrices and perform operations.   |
| Walk Like a Mathematician                          | Learning Task<br><i>Individual/Partner Task</i>    | Explore properties of matrices and solve area problems using determinants.   |
| Candy? What Candy?                                 | Learning Task<br><i>Partner/Small Group Task</i>   | Solve systems of equations problems using inverse matrices.  |
| An Okefenokee Food Web                             | Learning Task<br><i>Partner/Small Group Task</i>   | Use digraphs to represent and solve problems.  |
| The Gossip Chain<br><i>(Extension Task)</i>        | Performance Task<br><i>Individual/Partner Task</i> | Use digraphs to represent and solve problems.<br><i>(could be used as a formative assessment)</i>                      |
| <b>Culminating Task:</b><br>Vacationing in Georgia | Culminating Task<br><i>Individual</i>              | Write matrices, operate with matrices, and use matrices to solve problems involving systems of equations and digraphs. |

## **Central High Booster Club**

### **Mathematical Goals**

- Represent data in matrix form and determine the dimensions of matrices.
- Add and subtract matrices and know when these operations are possible.
- Perform scalar multiplication on matrices.
- Multiply matrices and know when matrix multiplication is defined.
- Solve problems using matrix operations

### **Common Core State Standards**

#### **Perform operations on matrices and use matrices in applications.**

**MCC9-12.N.VM.6 (+)** Use matrices to represent and manipulate data.

**MCC9-12.N.VM.7 (+)** Multiply matrices by scalars to produce new matrices, e.g., as when all of the payoffs in a game are doubled.

**MCC9-12.N.VM.8 (+)** Add, subtract, and multiply matrices of appropriate dimensions.

### **Standards for Mathematical Practice**

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Model with mathematics.
4. Look for and make use of structure.

### **Introduction**

*Central High Booster Club* introduces matrices as tools for organizing and storing information. Problems are based on a fund raising project where spirit items are made by booster club members and sold at the school store and at games. Cost of materials, time to produce each item, available inventory by month, needed inventory, and so forth provide material to create a variety of matrices, develop definitions, and basic matrix operations. Dimensions and dimension labels are used to provide rationale for addition and multiplication procedures. Emphasis is placed on interpreting entries as matrices are written, added, multiplied (scalar and regular) and transposed.

### **Materials**

- Pencil
- Handout

### **Central High Booster Club**

In order to raise money for the school, the Central High School Booster Club offered spirit items prepared by members for sale at the school store and at games. They sold stuffed teddy bears dressed in school colors, tote bags and tee shirts with specially sewn and decorated school insignias. The teddy bears, tote bags, and tee shirts were purchased from wholesale suppliers and decorations were cut, sewn and painted, and attached to the items by booster club parents. The wholesale cost for each teddy bear was \$4.00, each tote bag was \$3.50 and each tee shirt was \$3.25. Materials for the decorations cost \$1.25 for the bears, \$0.90 for the tote bags and \$1.05 for the tee shirts. Parents estimated the time necessary to complete a bear was 15 minutes to cut out the clothes, 20 minutes to sew the outfits, and 5 minutes to dress the bears. A tote bag required 10 minutes to cut the materials, 15 minutes to sew and 10 minutes to glue the designs on the bag. Tee shirts were made using computer generated transfer designs for each sport which took 5 minutes to print out, 6 minutes to iron on the shirts, and 20 minutes to paint on extra detailing.

The booster club parents made spirit items at three different work meetings and produced 30 bears, 30 tote bags, and 45 tee shirts at the first session. Fifteen bears, 25 tote bags, and 30 tee shirts were made during the second meeting; and, 30 bears, 35 tote bags and 75 tee shirts were made at the third session. They sold the bears for \$12.00 each, the tote bags for \$10.00 each and the tee shirts for \$10.00 each. In the first month of school, 10 bears, 15 tote bags, and 50 tee shirts were sold at the bookstore. During the same time period, Booster Club members sold 50 bears, 20 tote bags, and 100 tee shirts at the games.



**Georgia Department of Education**  
 Common Core Georgia Performance Standards Framework  
 CCGPS Pre-Calculus • Unit 5

The following is a **matrix**, a rectangular array of values, showing the wholesale cost of each item as well as the cost of decorations. "*wholesale*" and "*decorations*" are labels for the matrix **rows** and "*bears*", "*totes*", and "*shirts*" are labels for the matrix **columns**. The **dimensions of this matrix** called **A** are 2 rows and 3 columns and matrix **A** is referred to as a [2 x 3] **matrix**. Each number in the matrix is called an **entry**.

$$A = \begin{array}{c} \text{wholesale} \\ \text{decorations} \end{array} \begin{array}{c} \text{Cost per Item} \\ \text{bears} \quad \text{totes} \quad \text{shirts} \\ \left[ \begin{array}{ccc} 4.00 & 3.50 & 3.25 \\ 1.25 & .90 & 1.05 \end{array} \right] \end{array}$$

It is sometimes convenient to write matrices (plural of matrix) in a simplified format without labels for the rows and columns. Matrix **A** can be written as an array.

$$A = \begin{bmatrix} 4.00 & 3.50 & 3.25 \\ 1.25 & .90 & 1.05 \end{bmatrix} \text{ where the values can be identified as } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}.$$

In this system, the entry  $a_{22} = .90$ , which is the cost of decorations for tote bags.

1. Write and label matrices for the information given on the Central High School Booster Club's spirit project.
  - a. Let matrix **B** show the information given on the time necessary to complete each task for each item. Label the **rows** of the matrix *cut/print*, *sew/iron*, and *dress/decorate*. Label the **columns** *bears*, *totes*, *shirts*.
  - b. Find matrix **C** to show the numbers of bears, totes, and shirts produced at each of the three meetings. Label the **rows** of the matrix  $1^{st}$ ,  $2^{nd}$ , and  $3^{rd}$ . Label the **columns** *bears*, *totes*, *shirts*.
  - c. Matrix **D** should contain the information on items sold at the bookstore and at the game. Label the **rows** of the matrix *bears*, *totes*, *shirts*. Label the **columns** *bookstore*, *games*.
  - d. Let matrix **E** show the selling prices of the three items. Label the **row** of the matrix *selling price*. Label the **columns** *bears*, *totes*, *shirts*.

| <i>time required per item</i>   | <i>number produced by item</i>  |
|---|---|
| <i>bears totes shirts</i>   | <i>bears totes shirts</i>   |
| $B = \begin{array}{c} \text{cut / print} \\ \text{sew / iron} \\ \text{dress / decorate} \end{array} \left[ \begin{array}{ccc} 15 & 10 & 5 \\ 20 & 15 & 6 \\ 5 & 10 & 20 \end{array} \right]$ | $C = \begin{array}{c} 1st \\ 2nd \\ 3rd \end{array} \left[ \begin{array}{ccc} 30 & 30 & 45 \\ 15 & 25 & 30 \\ 30 & 35 & 75 \end{array} \right]$ |

|  |   |
|--|---|
| <p style="text-align: center;"><i>items sold</i></p> <p style="text-align: center;"><i>bkstore games</i></p> <p style="text-align: center;"><i>bears</i> <math>\left[ \begin{array}{cc} 10 &amp; 50 \\ 15 &amp; 20 \\ 50 &amp; 100 \end{array} \right]</math></p> <p style="text-align: center;"><i>totes</i></p> <p style="text-align: center;"><i>shirts</i></p> | <p style="text-align: center;"><i>bears totes shirts</i></p> <p style="text-align: center;"><math>E = \text{selling price} \left[ \begin{array}{ccc} 12.00 &amp; 10.00 &amp; 10.00 \end{array} \right]</math></p> |
|--|---|

2. Matrices are called **square matrices** when the number of rows equals the number of columns. A matrix with only one row or only one column is called a **row matrix** or a **column matrix**. Are any of the matrices from problem #1 square matrices or row matrices or column matrices? If so, identify them.

*B and C are square matrices. E is a row matrix*

Since matrices are arrays containing sets of discrete data with dimensions, they have a particular set of rules, or algebra, governing operations such as addition, subtraction, and multiplication. In order to **add two matrices**, the matrices must have the same dimensions. And, if the matrices have row and column labels, these labels must also match. Consider the following problem and matrices.

Several local companies wish to donate spirit items which can be sold along with the items made by the Booster Club at games help raise money for Central High School. J J's Sporting Goods store donates 100 caps and 100 pennants in September and 125 caps and 75 pennants in October. Friendly Fred's Food store donates 105 caps and 125 pennants in September and 110 caps and 100 pennants in October. How many items are available each month from both sources?

**To add two matrices, add corresponding entries.**

Let  $J = \begin{array}{c} \text{caps} \\ \text{pennants} \end{array} \begin{array}{cc} \text{Sept} & \text{Oct} \\ \left[ \begin{array}{cc} 100 & 125 \\ 100 & 75 \end{array} \right] \end{array}$  and  $F = \begin{array}{c} \text{caps} \\ \text{pennants} \end{array} \begin{array}{cc} \text{Sept} & \text{Oct} \\ \left[ \begin{array}{cc} 105 & 110 \\ 125 & 100 \end{array} \right] \end{array}$

then  $J + F = \begin{array}{c} \text{caps} \\ \text{pennants} \end{array} \begin{array}{cc} \text{Sept} & \text{Oct} \\ \left[ \begin{array}{cc} 100+105 & 125+110 \\ 100+125 & 75+100 \end{array} \right] \end{array}$  so

$$J + F = \begin{array}{c} \text{caps} \\ \text{pennants} \end{array} \begin{array}{cc} \text{Sept} & \text{Oct} \\ \left[ \begin{array}{cc} 205 & 235 \\ 225 & 175 \end{array} \right] \end{array}$$



Subtraction is handled like addition by subtracting corresponding entries.

3. Construct a matrix  $G$  with dimensions  $[1 \times 3]$  corresponding to the total production **cost** per item for *bears, totes, shirts*. Use this new matrix  $G$  and matrix  $E$  from #1 which corresponded to the **selling price** for each item to find matrix  $P$ , the **profit** the Booster Club can expect from the sale of each bear, tote bag, and tee shirt.

$$G = \text{Cost} \begin{bmatrix} 5.25 & 4.40 & 4.30 \end{bmatrix} \qquad P = E - G = \text{profit} \begin{bmatrix} 6.75 & 5.60 & 5.70 \end{bmatrix}$$

Another type of matrix operation is known as scalar multiplication. A **scalar** is a single number such as 3 and matrix **scalar multiplication** is done by multiplying each entry in a matrix by the same scalar.

$$\text{Let } M = \begin{bmatrix} -2 & 0 & 5 \\ 1 & -3 & 4 \end{bmatrix}, \text{ then } 3M = \begin{bmatrix} -6 & 0 & 15 \\ 3 & -9 & 12 \end{bmatrix}.$$

4. Use scalar multiplication to change matrix  $B$  (problem #1) from minutes required per item to hours required per item.

$$\left(\frac{1}{60}\right)B = \begin{matrix} & \begin{matrix} \text{bears} & \text{totes} & \text{shirts} \end{matrix} \\ \begin{matrix} \text{cut / print} \\ \text{sew / iron} \\ \text{dress / decorate} \end{matrix} & \left(\frac{1}{60}\right) \begin{bmatrix} 15 & 10 & 5 \\ 20 & 15 & 6 \\ 5 & 10 & 20 \end{bmatrix} \end{matrix} = \begin{matrix} & \begin{matrix} \text{bears} & \text{totes} & \text{shirts} \end{matrix} \\ \begin{matrix} \text{cut / print} \\ \text{sew / iron} \\ \text{dress / decorate} \end{matrix} & \begin{bmatrix} \frac{1}{4} & \frac{1}{6} & \frac{1}{12} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{10} \\ \frac{1}{12} & \frac{1}{6} & \frac{1}{3} \end{bmatrix} \end{matrix}$$

Matrices can also be multiplied together. Since each matrix represents an array of data, rules for multiplying them together depend on the position of each entry. Consider the following example.

At the beginning of November a stomach virus hits Central High School. Students in the Freshman and Sophomore classes are either well, a little sick, or really sick. The following tables show Freshmen and Sophomores according to their levels of sickness and their gender.

| Student Population |      |        | % of Sick Students |          |            |
|--------------------|------|--------|--------------------|----------|------------|
| Categories         | Male | Female | Categories         | Freshmen | Sophomores |
| Freshmen           | 250  | 300    | Well               | 20%      | 25%        |
| Sophomores         | 200  | 275    | Little Sick        | 50%      | 40%        |
|                    |      |        | Really Sick        | 30%      | 35%        |

Suppose school personnel needed to prepare a report and include the total numbers of well and sick male Freshmen and Sophomores in the school.

**Georgia Department of Education**  
 Common Core Georgia Performance Standards Framework  
 CCGPS Pre-Calculus • Unit 5

well Freshmen males + well Sophomore males = well males

$$(.2)(250) + (.25)(200) = 100$$

a little sick Freshmen males + a little sick Sophomore males = a little sick males

$$(.5)(250) + (.4)(200) = 205$$

really sick Freshmen males + really sick Sophomore males = really sick males

$$(.3)(250) + (.35)(200) = 145$$

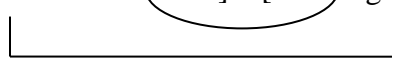
Notice the positions of the values in these products. We are multiplying rows by columns to get the information we want. Translating the tables to matrices and using the **rows by columns pattern of multiplication** we get the following result

$$\begin{array}{c} F \quad S \\ W \begin{bmatrix} .2 & .25 \\ .5 & .4 \\ .3 & .35 \end{bmatrix} \end{array} * \begin{array}{c} F \quad M \\ F \begin{bmatrix} 250 & 300 \\ 200 & 275 \end{bmatrix} \end{array} = \begin{array}{c} M \quad F \\ \begin{bmatrix} (.2)(250) + (.25)(200) & (.2)(300) + (.25)(275) \\ (.5)(250) + (.4)(200) & (.5)(300) + (.4)(275) \\ (.3)(250) + (.35)(200) & (.3)(300) + (.35)(275) \end{bmatrix} \end{array} = \begin{array}{c} M \quad F \\ W \begin{bmatrix} 100 & 128.75 \\ 205 & 260 \\ 145 & 186.25 \end{bmatrix} \end{array}$$

So

$$\begin{array}{c} F \quad S \\ W \begin{bmatrix} .2 & .25 \\ .5 & .4 \\ .3 & .35 \end{bmatrix} \end{array} * \begin{array}{c} F \quad M \\ F \begin{bmatrix} 250 & 300 \\ 200 & 275 \end{bmatrix} \end{array} = \begin{array}{c} M \quad F \\ W \begin{bmatrix} 100 & 128.75 \\ 205 & 260 \\ 145 & 186.25 \end{bmatrix} \end{array}$$

[level of sickness x class] \* [class] x gender = [level of sickness x gender]



$$[3 \times 2] * [2 \times 2] = [3 \times 2]$$



This procedure illustrates the multiplication of two matrices. In order to multiply two matrices, the number of columns of the matrix on the left must equal the number of rows of the matrix on the right. Also the labels of the columns of the left matrix must be the same as the labels of the rows of the right matrix. **If the dimensions of two matrices are not appropriately matched, it is not possible to multiply them.**

**Georgia Department of Education**  
Common Core Georgia Performance Standards Framework  
CCGPS Pre-Calculus • Unit 5

5. Given the following matrices, find their products if possible.

$$L = \begin{bmatrix} 1 & 3 \\ -5 & 4 \end{bmatrix} \quad M = \begin{bmatrix} -1 & 2 & 7 & -1 \\ 5 & 4 & 3 & 2 \end{bmatrix} \quad N = \begin{bmatrix} 3 & 0 \\ -2 & 1 \\ 5 & 5 \\ -1 & 2 \\ 6 & 3 \end{bmatrix} \quad T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$S = \begin{matrix} & \textit{male} & \textit{female} \\ \textit{well} & \begin{bmatrix} 60\% & 70\% \end{bmatrix} \\ \textit{sick} & \begin{bmatrix} 40\% & 30\% \end{bmatrix} \end{matrix}$$

$$C = \begin{matrix} & \textit{male} & \textit{female} \\ \textit{Jr} & \begin{bmatrix} 150 & 210 \end{bmatrix} \\ \textit{Sr} & \begin{bmatrix} 100 & 50 \end{bmatrix} \end{matrix}$$

- LM
- LN
- LT
- MN
- SC [Sometimes it is necessary to exchange the rows and columns of a matrix in order to make it possible to multiply. This process is called finding the **transpose** of a matrix and is most useful with labeled matrices.]
- Interpret  $SC^T_{1,2}$ .

$$LM = \begin{bmatrix} 14 & 14 & 16 & 5 \\ 25 & 6 & -23 & 13 \end{bmatrix}$$

$LN = \textit{dimension mismatch}$

$$LT = L = \begin{bmatrix} 1 & 3 \\ -5 & 4 \end{bmatrix} \quad (\textit{T is the identity matrix.})$$

$MN = \textit{dimension mismatch}$

$SC = \textit{dimension mismatch since the labels do not match. However you could multiply } SC^T$

$$\textit{and get } SC^T = \begin{matrix} & \textit{Jr} & \textit{Sr} \\ \textit{well} & \begin{bmatrix} 237 & 95 \end{bmatrix} \\ \textit{sick} & \begin{bmatrix} 123 & 55 \end{bmatrix} \end{matrix}$$

$SC^T_{1,2} = 123 \textit{ means 123 juniors are sick.}$

6. Using the matrices you wrote in problems #1 and #3 and matrix multiplication, find matrices to show

- the amount of profit made at the bookstore and at the games;

**Georgia Department of Education**  
 Common Core Georgia Performance Standards Framework  
 CCGPS Pre-Calculus • Unit 5

$$\begin{array}{r}
 \begin{array}{c} \text{bears} \quad \text{totes} \quad \text{shirts} \\ P = \text{profit} \end{array} \begin{bmatrix} 6.75 & 5.60 & 5.70 \end{bmatrix} \\
 \\
 \begin{array}{c} \text{bkstore} \quad \text{games} \\ PD = \text{profit} \end{array} \begin{bmatrix} 436.50 & 1019.50 \end{bmatrix}
 \end{array}
 \qquad
 \begin{array}{c}
 \text{items sold} \\
 \text{bkstore} \quad \text{games} \\
 \begin{array}{c} \text{bears} \\ \text{totes} \\ \text{shirts} \end{array} \begin{bmatrix} 10 & 50 \\ 15 & 20 \\ 50 & 100 \end{bmatrix}
 \end{array}$$

b. the amount of time (in minutes) it took to perform each task at the three work sessions.

$$\begin{array}{c}
 \begin{array}{c} \text{time required per item} \\ \text{bears} \quad \text{totes} \quad \text{shirts} \\ \text{cut / print} \\ \text{sew / iron} \\ \text{dress / decorate} \end{array} \begin{bmatrix} 15 & 10 & 5 \\ 20 & 15 & 6 \\ 5 & 10 & 20 \end{bmatrix} \\
 \\
 \begin{array}{c} \text{number produced by item} \\ \text{bears} \quad \text{totes} \quad \text{shirts} \\ \text{1st} \\ \text{2nd} \\ \text{3rd} \end{array} \begin{bmatrix} 30 & 30 & 45 \\ 15 & 25 & 30 \\ 30 & 35 & 75 \end{bmatrix} \\
 \\
 \begin{array}{c} \text{1st} \quad \text{2nd} \quad \text{3rd} \\ BC^T = \begin{array}{c} c / pr \\ s / ir \\ d / de \end{array} \begin{bmatrix} 975 & 625 & 1175 \\ 1320 & 855 & 1575 \\ 1350 & 925 & 2000 \end{bmatrix} \end{array} \quad \text{amount of time in minutes at each task at the work sessions}
 \end{array}$$

**REFERENCES:**

Crisler, N., Fisher, P., & Froelich, G. (1994). *Discrete mathematics Through Applications*. New York: W.H. Freeman and Company.  
 Department of Mathematics and Computer Science; North Carolina School of Science and Mathematics. (2000). *Contemporary Precalculus Through Applications, 2<sup>nd</sup> edition*. Chicago: Everyday Learning Corporation.

### **Central High Booster Club**

In order to raise money for the school, the Central High School Booster Club offered spirit items prepared by members for sale at the school store and at games. They sold stuffed teddy bears dressed in school colors, tote bags and tee shirts with specially sewn and decorated school insignias. The teddy bears, tote bags, and tee shirts were purchased from wholesale suppliers and decorations were cut, sewn and painted, and attached to the items by booster club parents. The wholesale cost for each teddy bear was \$4.00, each tote bag was \$3.50 and each tee shirt was \$3.25. Materials for the decorations cost \$1.25 for the bears, \$0.90 for the tote bags and \$1.05 for the tee shirts. Parents estimated the time necessary to complete a bear was 15 minutes to cut out the clothes, 20 minutes to sew the outfits, and 5 minutes to dress the bears. A tote bag required 10 minutes to cut the materials, 15 minutes to sew and 10 minutes to glue the designs on the bag. Tee shirts were made using computer generated transfer designs for each sport which took 5 minutes to print out, 6 minutes to iron on the shirts, and 20 minutes to paint on extra detailing.

The booster club parents made spirit items at three different work meetings and produced 30 bears, 30 tote bags, and 45 tee shirts at the first session. Fifteen bears, 25 tote bags, and 30 tee shirts were made during the second meeting; and, 30 bears, 35 tote bags and 75 tee shirts were made at the third session. They sold the bears for \$12.00 each, the tote bags for \$10.00 each and the tee shirts for \$10.00 each. In the first month of school, 10 bears, 15 tote bags, and 50 tee shirts were sold at the bookstore. During the same time period, Booster Club members sold 50 bears, 20 tote bags, and 100 tee shirts at the games.



**Georgia Department of Education**  
Common Core Georgia Performance Standards Framework  
*CCGPS Pre-Calculus • Unit 5*

The following is a **matrix**, a rectangular array of values, showing the wholesale cost of each item as well as the cost of decorations. "*wholesale*" and "*decorations*" are labels for the matrix **rows** and "*bears*", "*totes*", and "*shirts*" are labels for the matrix **columns**. The **dimensions of this matrix** called **A** are 2 rows and 3 columns and matrix **A** is referred to as a [2 x 3] **matrix**. Each number in the matrix is called an **entry**.

$$A = \begin{array}{c} \text{wholesale} \\ \text{decorations} \end{array} \begin{array}{c} \text{Cost per Item} \\ \text{bears} \quad \text{totes} \quad \text{shirts} \\ \left[ \begin{array}{ccc} 4.00 & 3.50 & 3.25 \\ 1.25 & .90 & 1.05 \end{array} \right] \end{array}$$

It is sometimes convenient to write matrices (plural of matrix) in a simplified format without labels for the rows and columns. Matrix **A** can be written as an array.

$$A = \begin{bmatrix} 4.00 & 3.50 & 3.25 \\ 1.25 & .90 & 1.05 \end{bmatrix} \text{ where the values can be identified as } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}.$$

In this system, the entry  $a_{22} = .90$ , which is the cost of decorations for tote bags.

1. Write and label matrices for the information given on the Central High School Booster Club's spirit project.

a. Let matrix **B** show the information given on the time necessary to complete each task for each item. Label the **rows** of the matrix *cut/print*, *sew/iron*, and *dress/decorate*. Label the **columns** *bears*, *totes*, *shirts*.

b. Find matrix **C** to show the numbers of bears, totes, and shirts produced at each of the three meetings. Label the **rows** of the matrix  $1^{st}$ ,  $2^{nd}$ , and  $3^{rd}$ . Label the **columns** *bears*, *totes*, *shirts*.

**Georgia Department of Education**  
Common Core Georgia Performance Standards Framework  
*CCGPS Pre-Calculus • Unit 5*

c. Matrix D should contain the information on items sold at the bookstore and at the game. Label the **rows** of the matrix *bears, totes, shirts*. Label the **columns** *bookstore, games*

d. Let matrix E show the selling prices of the three items. Label the **row** of the matrix *selling price*. Label the **columns** *bears, totes, shirts*.

2. Matrices are called **square matrices** when the number of rows equals the number of columns. A matrix with only one row or only one column is called a **row matrix** or a **column matrix**. Are any of the matrices from problem #1 square matrices or row matrices or column matrices? If so, identify them and state their dimensions.

Since matrices are arrays containing sets of discrete data with dimensions, they have a particular set of rules, or algebra, governing operations such as addition, subtraction, and multiplication. In order to **add two matrices**, the matrices must have the same dimensions. And, if the matrices have row and column labels, these labels must also match. Consider the following problem and matrices.

Several local companies wish to donate spirit items which can be sold along with the items made by the Booster Club at games help raise money for Central High School. J J's Sporting Goods store donates 100 caps and 100 pennants in September and 125 caps and 75 pennants in October. Friendly Fred's Food store donates 105 caps and 125 pennants in September and 110 caps and 100 pennants in October. How many items are available each month from both sources?

**To add two matrices, add corresponding entries.**

$$\text{Let } J = \begin{matrix} & \text{Sept} & \text{Oct} \\ \text{caps} & \begin{bmatrix} 100 & 125 \end{bmatrix} \\ \text{pennants} & \begin{bmatrix} 100 & 75 \end{bmatrix} \end{matrix} \quad \text{and} \quad F = \begin{matrix} & \text{Sept} & \text{Oct} \\ \text{caps} & \begin{bmatrix} 105 & 110 \end{bmatrix} \\ \text{pennants} & \begin{bmatrix} 125 & 100 \end{bmatrix} \end{matrix}$$

$$\text{then } J + F = \begin{matrix} & \text{Sept} & \text{Oct} \\ \text{caps} & \begin{bmatrix} 100+105 & 125+110 \end{bmatrix} \\ \text{pennants} & \begin{bmatrix} 100+125 & 75+100 \end{bmatrix} \end{matrix} \quad \text{so}$$

$$J + F = \begin{matrix} & \text{Sept} & \text{Oct} \\ \text{caps} & \begin{bmatrix} 205 & 235 \end{bmatrix} \\ \text{pennants} & \begin{bmatrix} 225 & 175 \end{bmatrix} \end{matrix}$$

Subtraction is handled like addition by subtracting corresponding entries.

3. Construct a matrix  $G$  with dimensions  $[1 \times 3]$  corresponding to the total production **cost** per item for *bears*, *totes*, *shirts*. Use this new matrix  $G$  and matrix  $E$  from #1 (which corresponded to the **selling price** for each item) to find matrix  $P$ , the **profit** the Booster Club can expect from the sale of each bear, tote bag, and tee shirt.

Another type of matrix operation is known as scalar multiplication. A **scalar** is a single number such as 3 and matrix **scalar multiplication** is done by multiplying each entry in a matrix by the same scalar.

$$\text{Let } M = \begin{bmatrix} -2 & 0 & 5 \\ 1 & -3 & 4 \end{bmatrix}, \text{ then } 3M = \begin{bmatrix} -6 & 0 & 15 \\ 3 & -9 & 12 \end{bmatrix}.$$

4. Use scalar multiplication to change matrix  $B$  (problem #1) from minutes required per item to hours required per item.



**Georgia Department of Education**  
 Common Core Georgia Performance Standards Framework  
 CCGPS Pre-Calculus • Unit 5

Matrices can also be multiplied together. Since each matrix represents an array of data, rules for multiplying them together depend on the position of each entry. Consider the following example.

At the beginning of November a stomach virus hits Central High School. Students in the Freshman and Sophomore classes are either well, a little sick, or really sick. The following tables show Freshmen and Sophomores according to their levels of sickness and their gender.

| Student Population |      |        | % of Sick Students |          |            |
|--------------------|------|--------|--------------------|----------|------------|
| Categories         | Male | Female | Categories         | Freshmen | Sophomores |
| Freshmen           | 250  | 300    | Well               | 20%      | 25%        |
| Sophomores         | 200  | 275    | Little Sick        | 50%      | 40%        |
|                    |      |        | Really Sick        | 30%      | 35%        |

Suppose school personnel needed to prepare a report and include the total numbers of well and sick male Freshmen and Sophomores in the school.

$$\text{well Freshmen males} + \text{well Sophomore males} = \text{well males}$$

$$(.2)(250) + (.25)(200) = 100$$

$$\text{a little sick Freshmen males} + \text{a little sick Sophomore males} = \text{a little sick males}$$

$$(.5)(250) + (.4)(200) = 205$$

$$\text{really sick Freshmen males} + \text{really sick Sophomore males} = \text{really sick males}$$

$$(.3)(250) + (.35)(200) = 145$$

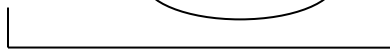
Notice the positions of the values in these products. We are multiplying rows by columns to get the information we want. Translating the tables to matrices and using the **rows by columns pattern of multiplication** we get the following result

$$\begin{matrix} & F & S & & M & F & & & M & F \\ W & \begin{bmatrix} .2 & .25 \\ .5 & .4 \\ .3 & .35 \end{bmatrix} & * & \begin{matrix} F \\ S \end{matrix} \begin{bmatrix} 250 & 300 \\ 200 & 275 \end{bmatrix} & = & \begin{bmatrix} (.2)(250) + (.25)(200) & (.2)(300) + (.25)(275) \\ (.5)(250) + (.4)(200) & (.5)(300) + (.4)(275) \\ (.3)(250) + (.35)(200) & (.3)(300) + (.35)(275) \end{bmatrix} & = & \begin{matrix} W \\ L \\ R \end{matrix} \begin{bmatrix} 100 & 128.75 \\ 205 & 260 \\ 145 & 186.25 \end{bmatrix} \end{matrix}$$

$$\begin{matrix} & F & S & & M & F \\ W & \begin{bmatrix} .2 & .25 \\ .5 & .4 \\ .3 & .35 \end{bmatrix} & * & \begin{matrix} F \\ S \end{matrix} \begin{bmatrix} 250 & 300 \\ 200 & 275 \end{bmatrix} & = & \begin{matrix} W \\ L \\ R \end{matrix} \begin{bmatrix} 100 & 128.75 \\ 205 & 260 \\ 145 & 186.25 \end{bmatrix} \end{matrix}$$

**Georgia Department of Education**  
 Common Core Georgia Performance Standards Framework  
*CCGPS Pre-Calculus • Unit 5*

$$[\text{level of sickness} \times \text{class}] * [\text{class} \times \text{gender}] = [\text{level of sickness} \times \text{gender}]$$



$$[3 \times 2] * [2 \times 2] = [3 \times 2]$$



This procedure illustrates the multiplication of two matrices. In order to multiply two matrices, the number of columns of the matrix on the left must equal the number of rows of the matrix on the right. Also the labels of the columns of the left matrix must be the same as the labels of the rows of the right matrix. **If the dimensions of two matrices are not appropriately matched, it is not possible to multiply them.**

5. Given the following matrices, find their products if possible.

$$L = \begin{bmatrix} 1 & 3 \\ -5 & 4 \end{bmatrix} \quad M = \begin{bmatrix} -1 & 2 & 7 & -1 \\ 5 & 4 & 3 & 2 \end{bmatrix} \quad N = \begin{bmatrix} 3 & 0 \\ -2 & 1 \\ 5 & 5 \\ -1 & 2 \\ 6 & 3 \end{bmatrix} \quad T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$S = \begin{matrix} & \textit{male} & \textit{female} \\ \textit{well} & \begin{bmatrix} 60\% & 70\% \end{bmatrix} \\ \textit{sick} & \begin{bmatrix} 40\% & 30\% \end{bmatrix} \end{matrix}$$

$$C = \begin{matrix} & \textit{male} & \textit{female} \\ \textit{Jr} & \begin{bmatrix} 150 & 210 \end{bmatrix} \\ \textit{Sr} & \begin{bmatrix} 100 & 50 \end{bmatrix} \end{matrix}$$

- a. LM
- b. LN
- c. LT
- d. MN

**Georgia Department of Education**  
Common Core Georgia Performance Standards Framework  
*CCGPS Pre-Calculus • Unit 5*

e. SC [Sometimes it is necessary to exchange the rows and columns of a matrix in order to make it possible to multiply. This process is called finding the **transpose** of a matrix and is most useful with labeled matrices.]

f. Interpret  $SC^T$  <sub>1,2</sub>

6. Using the matrices you wrote in problems #1 and #3 and matrix multiplication, find matrices to show

a. the amount of profit made at the bookstore and at the games

b. the amount of time (in minutes) it took to perform each task at the three work sessions.

REFERENCES:

Crisler, N., Fisher, P., & Froelich, G. (1994). *Discrete mathematics Through Applications*. New York: W.H. Freeman and Company.

Department of Mathematics and Computer Science; North Carolina School of Science and Mathematics. (2000). *Contemporary Precalculus Through Applications, 2<sup>nd</sup> edition*. Chicago: Everyday Learning Corporation.

## **Walk Like a Mathematician**

### **Mathematical Goals**

- Add, subtract, and multiply matrices and know when these operations are possible.
- Determine whether matrix addition is commutative.
- Determine whether matrix multiplication is commutative.
- Determine whether matrix addition is associative.
- Determine whether matrix multiplication is associative.
- Determine zero matrices.
- Determine identity matrices.
- Find inverse matrices.
- Confirm a matrix times its inverse equals the identity matrix.
- Find the determinant of matrices.
- Use the determinant of a matrix to find the area of a triangle.

### **Common Core State Standards**

#### **Perform operations on matrices and use matrices in applications.**

**MCC9-12.N.VM.8 (+)** Add, subtract, and multiply matrices of appropriate dimensions.

**MCC9-12.N.VM.9 (+)** Understand that, unlike multiplication of numbers, matrix multiplication for square matrices is not a commutative operation, but still satisfies the associative and distributive properties.

**MCC9-12.N.VM.10 (+)** Understand that the zero and identity matrices play a role in matrix addition and multiplication similar to the role of 0 and 1 in the real numbers. The determinant of a square matrix is nonzero if and only if the matrix has a multiplicative inverse.

**MCC9-12.N.VM.12 (+)** Work with  $2 \times 2$  matrices as transformations of the plane, and interpret the absolute value of the determinant in terms of area.

### **Standards for Mathematical Practice**

1. Reason abstractly and quantitatively.
2. Construct viable arguments and critique the reasoning of others.
3. Use appropriate tools strategically.
4. Look for and make use of structure.
5. Look for and express regularity in repeated reasoning.

**Introduction**

*Walk Like a Mathematician* is a more formal approach to matrix algebra. Properties of real numbers are examined to determine which properties are true in matrix operations. Students find inverses and determinants by hand and by using technology. Determinants are used to find areas of triangles and test for collinear points.

**Materials**

- Pencil
- Handout
- Graphing calculator

**Walk Like a Mathematician Learning Task:**

Matrices allow us to perform many useful mathematical tasks which ordinarily require large number of computations. Some types of problems which can be done efficiently with matrices include solving systems of equations, finding the area of triangles given the coordinates of the vertices, finding equations for graphs given sets of ordered pairs, and determining information contained in vertex edge graphs. In order to address these types of problems, it is necessary to understand more about matrix operations and properties; and, to use technology to perform some of the computations.

Matrix operations have many of the same properties as real numbers. There are more restrictions on matrices than on real numbers, however, because of the rules governing matrix addition, subtraction, and multiplication. Some of the real number properties which are more useful when considering matrix properties are listed below.

|                                 |  |  |
|---------------------------------|--|--|
| Let a, b, and c be real numbers |  |  |
|                                 | <b>ADDITION<br/>PROPERTIES</b>   | <b>MULTIPLICATION<br/>PROPERTIES</b>   |
| COMMUTATIVE                     | $a + b = b + a$  | $ab = ba$  |
| ASSOCIATIVE                     | $(a + b) + c = a + (b + c)$  | $(ab)c = a(bc)$  |
| IDENTITY                        | There exists a unique real number zero, 0, such that<br>$a + 0 = 0 + a = a$                                | There exists a unique real number one, 1, such that<br>$a * 1 = 1 * a = a$   |
| INVERSE                         | For each real number a,<br>there is a unique real number<br>- a such that<br><br>$a + (-a) = (-a) + a = 0$ | For each nonzero real number<br>a, there is a unique real<br>number $\frac{1}{a}$ such that<br><br>$a(\frac{1}{a}) = (\frac{1}{a})a = 1$ |

**Georgia Department of Education**  
Common Core Georgia Performance Standards Framework  
CCGPS Pre-Calculus • Unit 5

The following is a set of matrices without row and column labels. Use these matrices to complete the problems.

$$D = \begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix} \quad E = \begin{bmatrix} 2 & 4 & 1 \\ 0 & -1 & 3 \\ 1 & 0 & 2 \end{bmatrix} \quad F = \begin{bmatrix} 5 \\ -2 \end{bmatrix} \quad G = \begin{bmatrix} 0 & -1 \\ 1/3 & 2/3 \end{bmatrix} \quad H = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad J = \begin{bmatrix} 4 & 2 \\ -1 & 3 \end{bmatrix} \quad K = \begin{bmatrix} 3 & 4 & -1 \\ 0 & -2 & 3 \\ 5 & 1 & 2 \end{bmatrix} \quad L = \begin{bmatrix} 5 & -2 \\ -1 & 0 \end{bmatrix}$$

1. In this problem, you will determine whether matrix addition and matrix multiplication are commutative by performing operations and comparing the results.

- a. Does  $[D] + [J] = [J] + [D]$  ? *yes*
- b. Does  $[E] + [K] = [K] + [E]$  ? *yes*
- c. Is matrix addition commutative? Why or why not? *Any two matrices that can be added together will be commutative. The order in which you add the individual entries does not affect the result.*
- d. Does  $[D] * [J] = [J] * [D]$  ? *no*
- e. Does  $[E] * [H] = [H] * [E]$  ? *no and the product on the right is undefined*
- f. Is matrix multiplication commutative? Why or why not? *Matrix multiplication is not commutative.*

2. Are matrix addition and matrix multiplication associative?

- a. Does  $([D] + [J]) + [L] = [D] + ([J] + [L])$  ? *yes*
- b. Is matrix addition associative? Why or why not? *yes, changing the grouping does not affect the sum*
- c. Does  $([D] * [J]) * [L] = [D] * ([J] * [L])$  ? *yes*
- d. Is matrix multiplication associative? Why or why not? *yes, changing the grouping does not affect the product*

3. Is there a **zero** or **identity, 0**, for addition in matrices? If so, what does a **zero matrix** look like? Provide an example illustrating the additive identity property of matrices.

*A zero matrix is any matrix where all entries equal zero. There is no unique zero matrix.*

4. Do matrices have a **one** or an **identity, I**, for multiplication? If so what does an **identity matrix** look like; is it unique; and, does it satisfy the property  $a * I = I * a = a$ ?

*Identity matrices are square matrices in the form  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  or  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ . The main diagonal is filled with the number 1 and all other elements are 0's.*

a. Multiply  $[E]*[I]$  and  $[I]*[E]$ . Describe what you see. *The result of multiplying Matrix [E] times the Identity matrix, in either order, equals matrix [E]*

5. Find  $[D]*[G]$  and  $[G]*[D]$ . Describe what you see.

$$D * G = \begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1/3 & 2/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad G * D = \begin{bmatrix} 0 & -1 \\ 1/3 & 2/3 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

*The product of two inverse matrices should be the identity matrix, I.*

D and G are called **inverse matrices**.

In order for a matrix to have an inverse, it must satisfy two conditions.

1. The matrix must be a square matrix.

2. No row of the matrix can be a multiple of any other row.

Both D and G are 2x2 matrices; and, the rows in D are not multiples of each other.

The same is true of G.

The notation normally used for a matrix and its inverse is D and  $D^{-1}$  or G and  $G^{-1}$ .

6. Multiply G by 3 and look at the result. Can you see any relationship between D and the result?

*The numbers on the main diagonal have been switched and the other numbers have been multiplied by -1.*

The following formula can be used to find the inverse of a 2x2 matrix. Given matrix A where the rows of A are not multiples of each other:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{then} \quad A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

- a. Find the inverse of matrix J from the matrices listed above. Verify that J and J<sup>-1</sup> are inverses.

$$[J]^{-1} = \begin{bmatrix} \frac{3}{14} & \frac{-1}{7} \\ \frac{1}{14} & \frac{2}{7} \end{bmatrix} \quad \text{and} \quad [J]*[J]^{-1}=[J]^{-1}*[J]=[I]$$

*For higher order matrices, we will use technology to find inverses.*

Now your teacher will show you how to use technology to find the inverse of this matrix.

7. A unique number associated with every square matrix is called the determinant. Only square matrices have determinants.

- A. To find determinants of 2x2 matrices by hand use the following procedure.

$$\text{determinant}(A) = \det(A) = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

- a. Find  $|D| = 3$   
 b. Find  $|J| = 14$   
 c. Can you find  $|F|$ ? Why or why not? *no, it is not a square matrix*



**Georgia Department of Education**  
 Common Core Georgia Performance Standards Framework  
 CCGPS Pre-Calculus • Unit 5

B. One way to find determinants of 3x3 matrices use the following procedure: given matrix

$$B = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}, \text{ rewrite the matrix and repeat columns 1 and 2 to get } \begin{bmatrix} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{bmatrix}.$$

Now multiply and combine products according to the following patterns.

$$\begin{bmatrix} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{bmatrix} \text{ and } \begin{bmatrix} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{bmatrix} = \det(B) = aei + bfg + cdh - ceg - afh - bdi.$$

*Or you may use Expansion by Minors as the method of finding the determinant. Either method serves students, so you may want to show students both methods and let them choose which method they use. In addition, with accelerated students, you may give them some matrices larger than 3 X 3 and ask them which method extends to the larger matrices.*

a. Find  $|E| = 9$

b. Find  $|K| = 29$

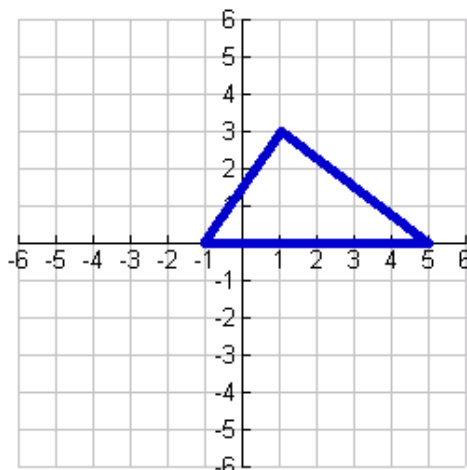
c. Now your teacher will show you how to use technology to find the determinants of these matrices.

8. The determinant of a matrix can be used to find the area of a triangle. If  $(x_1, y_1)$ ,  $(x_2, y_2)$ , and  $(x_3, y_3)$  are vertices of a triangle, the area of the triangle is

$$\text{Area} = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

a. Given a triangle with vertices  $(-1, 0)$ ,  $(1, 3)$ , and  $(5, 0)$ , find the area using the determinant formula. Verify that area you found is correct using geometric formulas.

*Using geometry, Area =  $\frac{1}{2}$  \* base \* height. Since this triangle's base is lined up on an axis, finding the area is very straightforward.  $A = \frac{1}{2} * 6 * 3 = 9$  square units.*



*Using matrices and a TI-84, the area is -9. Use +9 for the area.*

```
[A]
    [[-1  0  1]
     [ 1  3  1]
     [ 5  0  1]]
.5*det([A])
    -9
```

- b. Suppose you are finding the area of a triangle with vertices  $(-1, -1)$ ,  $(4, 7)$ , and  $(9, -6)$ . You find half the determinant to be  $-52.5$  and your partner works the same problem and gets  $+52.5$ . After checking both solutions, you each have done your work correctly. How can you explain this discrepancy?

*If the points of the triangle are chosen in clockwise order, the value of the determinant is negative. If the points are chosen in a counterclockwise order the value is positive.*

- c. Suppose another triangle with vertices  $(1, 1)$ ,  $(4, 2)$ , and  $(7, 3)$  gives an area of 0. What do you know about the triangle and the points?

*If the area of the triangle formed by the given points is 0, the points are collinear.*

- d. A gardener is trying to find a triangular area behind his house that encloses 1750 square feet. He has placed the first two fence posts at  $(0, 50)$  and at  $(40, 0)$ . The final fence post is on the property line at  $y = 100$ . Find the point where the gardener can place the final fence post.

$$1750 = \left| \frac{1}{2} \begin{vmatrix} 0 & 50 & 1 \\ 40 & 0 & 1 \\ x & 100 & 1 \end{vmatrix} \right| = \left| \frac{1}{2} \left( \begin{vmatrix} 0 & 1 \\ 100 & 1 \end{vmatrix} - 50 \begin{vmatrix} 40 & 1 \\ x & 1 \end{vmatrix} + 1 \begin{vmatrix} 40 & 0 \\ x & 100 \end{vmatrix} \right) \right| = \left| \frac{1}{2} (-50x - 2000 + 4000) \right|$$

$$1750 = 25x - 1000$$

$$25x = 750$$

*$x = 30$  so the third fence post should be placed at  $(30, 100)$  so that the garden area = 1750 ft<sup>2</sup>.*

**Walk Like a Mathematician Learning Task:**

Matrices allow us to perform many useful mathematical tasks which ordinarily require large number of computations. Some types of problems which can be done efficiently with matrices include solving systems of equations, finding the area of triangles given the coordinates of the vertices, finding equations for graphs given sets of ordered pairs, and determining information contained in vertex edge graphs. In order to address these types of problems, it is necessary to understand more about matrix operations and properties; and, to use technology to perform some of the computations.

Matrix operations have many of the same properties as real numbers. There are more restrictions on matrices than on real numbers, however, because of the rules governing matrix addition, subtraction, and multiplication. Some of the real number properties which are more useful when considering matrix properties are listed below.

| Let a, b, and c be real numbers | ADDITION<br>PROPERTIES   | MULTIPLICATION<br>PROPERTIES   |
|---------------------------------|--|--|
| COMMUTATIVE                     | $a + b = b + a$  | $ab = ba$  |
| ASSOCIATIVE                     | $(a + b) + c = a + (b + c)$  | $(ab)c = a(bc)$  |
| IDENTITY                        | There exists a unique real number zero, 0, such that<br>$a + 0 = 0 + a = a$                          | There exists a unique real number one, 1, such that<br>$a * 1 = 1 * a = a$   |
| INVERSE                         | For each real number a, there is a unique real number - a such that<br><br>$a + (-a) = (-a) + a = 0$ | For each nonzero real number a, there is a unique real number $\frac{1}{a}$ such that<br><br>$a(\frac{1}{a}) = (\frac{1}{a})a = 1$ |

**Georgia Department of Education**  
Common Core Georgia Performance Standards Framework  
*CCGPS Pre-Calculus • Unit 5*

The following is a set of matrices without row and column labels. Use these matrices to complete the problems.

$$D = \begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix} \quad E = \begin{bmatrix} 2 & 4 & 1 \\ 0 & -1 & 3 \\ 1 & 0 & 2 \end{bmatrix} \quad F = \begin{bmatrix} 5 \\ -2 \end{bmatrix} \quad G = \begin{bmatrix} 0 & -1 \\ 1/3 & 2/3 \end{bmatrix} \quad H = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad J = \begin{bmatrix} 4 & 2 \\ -1 & 3 \end{bmatrix} \quad K = \begin{bmatrix} 3 & 4 & -1 \\ 0 & -2 & 3 \\ 5 & 1 & 2 \end{bmatrix} \quad L = \begin{bmatrix} 5 & -2 \\ -1 & 0 \end{bmatrix}$$

1. In this problem, you will determine whether matrix addition and matrix multiplication are commutative by performing operations and comparing the results.

a. Does  $[D] + [J] = [J] + [D]$  ?

b. Does  $[E] + [K] = [K] + [E]$  ?

c. Is matrix addition commutative? Why or why not?

d. Does  $[D] * [J] = [J] * [D]$  ?

e. Does  $[E] * [H] = [H] * [E]$  ?

f. Is matrix multiplication commutative? Why or why not?

**Georgia Department of Education**  
Common Core Georgia Performance Standards Framework  
*CCGPS Pre-Calculus • Unit 5*

2. Are matrix addition and matrix multiplication associative?
- Does  $([D] + [J]) + [L] = [D] + ([J] + [L])$  ?
  - Is matrix addition associative? Why or why not?
  - Does  $([D] * [J]) * [L] = [D] * ([J] * [L])$  ?
  - Is matrix multiplication associative? Why or why not?
3. Is there a **zero** or **identity, 0**, for addition in matrices? If so, what does a **zero matrix** look like? Provide an example illustrating the additive identity property of matrices.
4. Do matrices have a **one** or an **identity, I**, for multiplication? If so what does an **identity matrix** look like; is it unique; and, does it satisfy the property  $a * I = I * a = a$ ?
- Multiply  $[E] * [I]$  and  $[I] * [E]$ . Describe what you see.
5. Find  $[D] * [G]$  and  $[G] * [D]$ . Describe what you see.
- D and G are called **inverse matrices**.

In order for a matrix to have an inverse, it must satisfy two conditions.

1. The matrix must be a square matrix.
  2. No row of the matrix can be a multiple of any other row.
- Both D and G are 2x2 matrices; and, the rows in D are not multiples of each other. The same is true of G.  
The notation normally used for a matrix and its inverse is D and  $D^{-1}$  or G and  $G^{-1}$ .

6. Multiply G by 3 and look at the result. Can you see any relationship between D and the result?

The following formula can be used to find the inverse of a 2x2 matrix. Given matrix A where the rows of A are not multiples of each other:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{then} \quad A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

- a. Find the inverse of matrix J from the matrices listed above. Verify that J and  $J^{-1}$  are inverses.
- b. Now your teacher will show you how to use technology to find the inverse of this matrix.

7. A unique number associated with every square matrix is called the determinant. Only square matrices have determinants.

A. To find determinants of 2x2 matrices by hand use the following procedure.

$$\text{determinant}(A) = \det(A) = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

a. Find  $|D|$

**Georgia Department of Education**  
 Common Core Georgia Performance Standards Framework  
 CCGPS Pre-Calculus • Unit 5

b. Find  $|J|$

c. Can you find  $|F|$ ? Why or why not?

B. One way to find determinants of 3x3 matrices use the following procedure: given matrix

$$B = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}, \text{ rewrite the matrix and repeat columns 1 and 2 to get } \begin{bmatrix} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{bmatrix}.$$

Now multiply and combine products according to the following patterns.

$$\begin{bmatrix} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{bmatrix} \text{ and } \begin{bmatrix} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{bmatrix} = \det(B) = aei + bfg + cdh - ceg - afh - bdi.$$

a. Find  $|E|$

b. Find  $|K|$

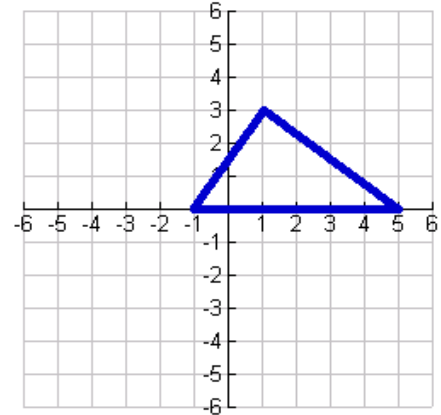
c. Now your teacher will show you how to use technology to find the determinants of these matrices.

8. The determinant of a matrix can be used to find the area of a triangle. If  $(x_1, y_1)$ ,  $(x_2, y_2)$ , and  $(x_3, y_3)$  are vertices of a triangle, the area of the triangle is

$$\text{Area} = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

**Georgia Department of Education**  
Common Core Georgia Performance Standards Framework  
*CCGPS Pre-Calculus • Unit 5*

- a. Given a triangle with vertices  $(-1, 0)$ ,  $(1, 3)$ , and  $(5, 0)$ , find the area using the determinant formula. Verify that area you found is correct using geometric formulas.



- b. Suppose you are finding the area of a triangle with vertices  $(-1, -1)$ ,  $(4, 7)$ , and  $(9, -6)$ . You find half the determinant to be  $-52.5$  and your partner works the same problem and gets  $+52.5$ . After checking both solutions, you each have done your work correctly. How can you explain this discrepancy?
- c. Suppose another triangle with vertices  $(1, 1)$ ,  $(4, 2)$ , and  $(7, 3)$  gives an area of 0. What do you know about the triangle and the points?
- d. A gardener is trying to find a triangular area behind his house that encloses 1750 square feet. He has placed the first two fence posts at  $(0, 50)$  and at  $(40, 0)$ . The final fence post is on the property line at  $y = 100$ . Find the point where the gardener can place the final fence post.



## **Candy? What Candy? Learning Task:**

### **Mathematical Goals**

- Multiply matrices.
- Write a system of linear equations as a single matrix equation.
- Find inverses of matrices.
- Use inverse matrices to solve systems of linear equations.
- Find equations of lines passing through two given points.
- Find equations of parabolas passing through three given points.

### **Common Core State Standards**

#### **Perform operations on matrices and use matrices in applications.**

**MCC9-12.N.VM.8 (+)** Add, subtract, and multiply matrices of appropriate dimensions.

**MCC9-12.N.VM.10 (+)** Understand that the zero and identity matrices play a role in matrix addition and multiplication similar to the role of 0 and 1 in the real numbers. The determinant of a square matrix is nonzero if and only if the matrix has a multiplicative inverse.

#### **Solve systems of equations**

**MCC9-12.A.REI.8 (+)** Represent a system of linear equations as a single matrix equation in a vector variable.

**MCC9-12.A.REI.9 (+)** Find the inverse of a matrix if it exists and use it to solve systems of linear equations (using technology for matrices of dimension  $3 \times 3$  or greater).

### **Standards for Mathematical Practice**

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Look for and make use of structure.
7. Look for and express regularity in repeated reasoning.

### **Introduction**

*Candy? What Candy?* gives students opportunities to find inverses of  $2 \times 2$  matrices by hand and inverses of larger matrices using calculators. Systems of equations are written as matrix equations and inverses of coefficient matrices are used to solve the systems. Students also learn how to find equations of lines passing through two given points by solving for coefficients in system of two equations in the form  $y = ax + b$  and equations of parabolas passing through three

given points by solving for coefficients in a system of three equations written in the form  $y = ax^2 + bx + c$ . This task includes a lab where students solve a system of three equations to determine what is hidden in a sealed lunch bag.

**Materials**

- Pencil
- Handout
- Graphing Calculator
- Candy
- Lunch Bags

1. A system of equations such as  $\begin{cases} ax + by = c \\ dx + ey = f \end{cases}$  can be written as a matrix equation

where  $\begin{bmatrix} a & b \\ d & e \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c \\ f \end{bmatrix}$  or  $[A][V] = [B]$  where  $[A]$  is the coefficient matrix,  $[V]$  is the variables matrix, and  $[B]$  is the matrix for the constant terms. The unknown variables matrix can be isolated with the following algebraic reasoning:

$[A][V] = [B]$       *Write the system of linear equations as a matrix equation*

$[A]^{-1}[A][V] = [A]^{-1}[B]$       *Multiply both sides of the equation by the **inverse** of matrix  $[A]$*

$[I][V] = [A]^{-1}[B]$       *The inverse of matrix  $[A]$  times matrix  $[A]$  equals the **identity** matrix*

$[V] = [A]^{-1}[B]$       *The unknown variables equal the inverse of  $[A]$  times  $[B]$*

a. To solve a system of equations such as  $\begin{cases} 5x - y = 7 \\ 2x + 3y = -1 \end{cases}$  write the matrix equation.

$$\begin{bmatrix} 5 & -1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \end{bmatrix}$$

$$\frac{1}{17} \begin{bmatrix} 3 & 1 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 5 & -1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{17} \begin{bmatrix} 3 & 1 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 7 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{17} \begin{bmatrix} 3 & 1 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 7 \\ -1 \end{bmatrix}$$

$$x = \frac{20}{17} \text{ and } y = \frac{-19}{17}$$

b. Solving systems of equations of higher order can be accomplished using a similar format and a TI-84 graphing calculator to find the inverse of the coefficient matrix and find the necessary products.

To solve the system  $\begin{cases} x - 2y + 3z = 3 \\ 2x + y + 5z = 8 \\ 3x - y - 3z = -22 \end{cases}$  write the coefficient matrix  $[A] = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & 5 \\ 3 & -1 & -3 \end{bmatrix}$  and

the answers matrix  $[B] = \begin{bmatrix} 3 \\ 8 \\ -22 \end{bmatrix}$ . Enter  $[A]$  and  $[B]$  into the TI-84, then multiply  $[A]^{-1}*[B]$ .

Your solution should be  $\begin{bmatrix} -4 \\ 1 \\ 3 \end{bmatrix}$  which represents the ordered triple  $(-4, 1, 3)$ . Check this

solution by substituting it into each of the three original equations. Does it work? *yes*

c. For the following systems of equations, write the matrix equation and solve for the variables.

$$\begin{cases} 2x + 3y = 2 \\ 4x - 9y = -1 \end{cases}$$

$$\begin{cases} 9x - 7y = 5 \\ 10x + 3y = -16 \end{cases}$$

$$\begin{cases} 5x - 4y + 3z = 15 \\ 6x + 2y + 9z = 13 \\ 7x + 6y - 6z = 6 \end{cases}$$

$$\left\{ \frac{1}{2}, \frac{1}{3} \right\}$$

$$\{-1, -2\}$$

$$\left\{ 2, -1, \frac{1}{3} \right\}$$

2. Use this information to write an equation of a line passing through the points (2, -5) and (-1, -4).

*Knowing the graphing form of the equation of a line is  $y = mx + b$ , you can*

*form a system of equations*  $\begin{cases} -5 = 2m + b \\ -4 = -1m + b \end{cases}$ . *Solving this system gives  $m = \frac{-1}{3}$  and  $b = -4\frac{1}{3}$*

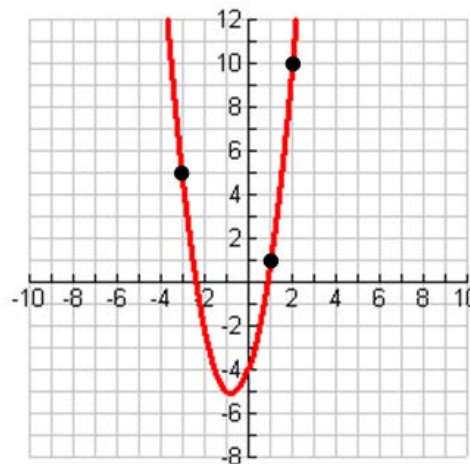
*and an equation of  $y = \frac{-1}{3}x - 4\frac{1}{3}$ .*

Consider the following graph. The parabola passes through the points  $(-3, 5)$ ,  $(1, 1)$ , and  $(2, 10)$ . Write a system of equations for this graph. Solve the system and write an equation for the parabola. Justify your answer.

Use the point values in  $y = ax^2 + bx + c$ .

Solve the system: 
$$\begin{cases} 1 = 1a + 1b + 1c \\ 5 = 9a - 3b + 1c \\ 10 = 4a + 2b + 1c \end{cases}$$

$$y = 2x^2 + 3x - 4$$



3. Suppose you walk into class one day and find a big stack of sealed lunch bags full of candy on a table just waiting for you to rip them open and devour their chocolaty contents. However, you cannot even touch them until you figure out how many pieces of each brand of candy are contained in each bag. Well, today is your lucky day! Each group of three gets one bag which must remain unopened until you can tell how many pieces of each type of candy W, X, Y, or Z. Each bag holds 3 different types of candy and a total of 9 pieces of candy. Your task is to determine exactly what is in your bag by writing a system of equations and solving that system using matrices.

3. Now, to find out about your bag of candy.

|   |
|---|
| <b>Bag Number</b> <u>10</u>                   |
| <b>Types of Snacks</b> <u>XYZ</u>             |
| <b>Total Number of Snacks Inside</b> <u>9</u> |
| <b>Total Number of Calories</b> <u>819.9</u>  |
| <b>Total Fat Grams</b> <u>43.1</u>            |
| <b>Total Weight</b> <u>152</u>                |
| <b>Total Carbohydrates</b> <u>97</u>          |
| <b>Total Protein</b> <u>13</u>                |

*The card attached to each bag given to a group of students should look like this card. It gives information about the contents of each bag. Using this information and the information from the Nutrition Table, students should write a system of equations for their bag of candy.*

**Georgia Department of Education**  
Common Core Georgia Performance Standards Framework  
*CCGPS Pre-Calculus • Unit 5*

a. Choose values from the Nutrition Chart and the totals given on the card attached to your bag and write a system of equations that describes the information regarding the snacks in your group's paper bag. List your equations below.

1. \_\_\_\_\_
2. \_\_\_\_\_
3. \_\_\_\_\_

*Using the values from the Nutrition Chart below, the possible equations for Bag #10 would be*

*number:  $x + y + z = 9$*

*fat:  $3.7x + 4y + 6z = 43.1$*

*carbohydrates:  $9x + 11y + 12z = 97$*

*protein:  $x + y + 2z = 13$*

*calories:  $73.3x + 80y + 110z = 819.9$*

*weight:  $14x + 13y + 21z = 152$*

b. Solve this system of equations using matrices. Explain how you determined how many pieces of each type of candy are in your bag.

*Students can use any three of the equations to form their system of equations. They should find that Bag #10 would contain  $x = 3$ ,  $y = 2$ , and  $z = 4$  pieces of candy. Each bag should contain a different assortment of candies.*

*Depending on particular school restrictions, it may not be possible to use candy or any food items for this lab. Any set of small items will do to put in the bags. (However, food items are particularly appealing to students and they will work diligently to solve this type of problem). Identify some characteristic measurements for the student to use to write their equations. A higher level problem would be to let students set up this activity for other classes as well as solving for their own bags of unknown items.*

c. Suppose you are working with a younger group of students and want to do this activity using two types of candy instead of three. Explain how you could use the nutrition information and matrix multiplication to label your bags of candy for other students.

### NUTRITION CHART

*Sample Nutrition Chart for Halloween sized Snickers, MM's, Twix, and Peanut Butter Cups. Students have 9 bars of 3 types of candy. Students use these values and the total values from their bags to write the equations. These values will vary according to the type of candy you select. A table of values like this should accompany each bag of candy.*

| <b>SNACK</b>              | <b>W</b> | <b>X</b> | <b>Y</b> | <b>Z</b> |
|---------------------------|----------|----------|----------|----------|
| <b>FAT GRAMS</b>          | 4g       | 3.7g     | 4g       | 6g       |
| <b>CARBOHYDRATE GRAMS</b> | 10.5g    | 9g       | 11g      | 12g      |
| <b>PROTEIN GRAMS</b>      | 1.5g     | 1g       | 1g       | 2g       |
| <b>CALORIES</b>           | 80       | 73.3     | 80       | 110      |
| <b>WEIGHT GRAMS</b>       | 17g      | 14g      | 13g      | 21g      |

## Candy? What Candy?

1. A system of equations such as  $\begin{cases} ax + by = c \\ dx + ey = f \end{cases}$  can be written as a matrix equation

where  $\begin{bmatrix} a & b \\ d & e \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c \\ f \end{bmatrix}$  or  $[A][V] = [B]$  where  $[A]$  is the coefficient matrix,  $[V]$  is the variables matrix, and  $[B]$  is the matrix for the constant terms. The unknown variables matrix can be

isolated with the following algebraic reasoning:

$$[A][V] = [B] \quad \text{Write the system of linear equations as a matrix equation}$$

$$[A]^{-1}[A][V] = [A]^{-1}[B] \quad \text{Multiply both sides of the equation by the **inverse** of matrix } [A]$$

$$[I][V] = [A]^{-1}[B] \quad \text{The inverse of matrix } [A] \text{ times matrix } [A] \text{ equals the **identity** matrix}$$

$$[V] = [A]^{-1}[B] \quad \text{The unknown variables equal the inverse of } [A] \text{ times } [B]$$

a. To solve a system of equations such as  $\begin{cases} 5x - y = 7 \\ 2x + 3y = -1 \end{cases}$  write the matrix equation.

$$\begin{bmatrix} 5 & -1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \end{bmatrix}$$

$$\frac{1}{17} \begin{bmatrix} 3 & 1 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 5 & -1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{17} \begin{bmatrix} 3 & 1 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 7 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{17} \begin{bmatrix} 3 & 1 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 7 \\ -1 \end{bmatrix}$$

$$x = \frac{20}{17} \text{ and } y = \frac{-19}{17}$$

b. Solving systems of equations of higher order can be accomplished using a similar format and a TI-84 graphing calculator to find the inverse of the coefficient matrix and find the necessary products.

**Georgia Department of Education**  
 Common Core Georgia Performance Standards Framework  
 CCGPS Pre-Calculus • Unit 5

To solve the system  $\begin{cases} x - 2y + 3z = 3 \\ 2x + y + 5z = 8 \\ 3x - y - 3z = -22 \end{cases}$  write the coefficient matrix  $[A] = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & 5 \\ 3 & -1 & -3 \end{bmatrix}$  and

the answers matrix  $[B] = \begin{bmatrix} 3 \\ 8 \\ -22 \end{bmatrix}$ . Enter  $[A]$  and  $[B]$  into the TI-84, then multiply  $[A]^{-1}*[B]$ .

Your solution should be  $\begin{bmatrix} -4 \\ 1 \\ 3 \end{bmatrix}$  which represents the ordered triple  $(-4, 1, 3)$ . Check this

solution by substituting it into each of the three original equations. Does it work?

c. For the following systems of equations, write the matrix equation and solve for the variables.

$$2x + 3y = 2$$

$$4x - 9y = -1$$

$$9x - 7y = 5$$

$$10x + 3y = -16$$

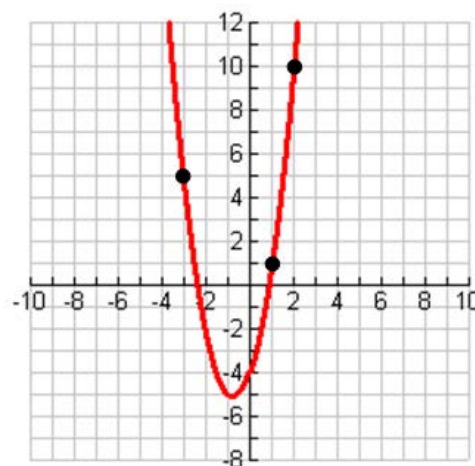
$$5x - 4y + 3z = 15$$

$$6x + 2y + 9z = 13$$

$$7x + 6y - 6z = 6$$

2. Use this information to write an equation of a line passing through the points  $(2, -5)$  and  $(-1, -4)$ .

Consider the following graph. The parabola passes through the points  $(-3, 5)$ ,  $(1, 1)$ , and  $(2, 10)$ . Write a system of equations for this graph. Solve the system and write an equation for the parabola. Justify your answer.





**Georgia Department of Education**  
Common Core Georgia Performance Standards Framework  
*CCGPS Pre-Calculus • Unit 5*

3. Suppose you walk into class one day and find a big stack of sealed lunch bags full of candy on a table just waiting for you to rip them open and devour their chocolaty contents. However, you cannot even touch them until you figure out how many pieces of each brand of candy are contained in each bag. Well, today is your lucky day! Each group of three gets one bag which must remain unopened until you can tell how many pieces of each type of candy W, X, Y, or Z. Each bag holds 3 different types of candy and a total of 9 pieces of candy. Your task is to determine exactly what is in your bag by writing a system of equations and solving that system using matrices.

3. Now, to find out about your bag of candy.

|  |
|--|
| <b>Bag Number</b> _____                    |
| <b>Types of Snacks</b> _____               |
| <b>Total Number of Snacks Inside</b> _____ |
| <b>Total Number of Calories</b> _____      |
| <b>Total Fat Grams</b> _____               |
| <b>Total Weight</b> _____                  |
| <b>Total Carbohydrates</b> _____           |
| <b>Total Protein</b> _____                 |

a. Choose values from the Nutrition Chart and the totals given on the card attached to your bag and write a system of equations that describes the information regarding the snacks in your group's paper bag. List your equations below.

1. \_\_\_\_\_
2. \_\_\_\_\_
3. \_\_\_\_\_

b. Solve this system of equations using matrices. Explain how you determined how many pieces of each type of candy are in your bag.

c. Suppose you are working with a younger group of students and want to do this activity using two types of candy instead of three. Explain how you could use the nutrition information and matrix multiplication to label your bags of candy for other students.

## NUTRITION CHART

| <b>SNACK</b>              | <b>W</b> | <b>X</b> | <b>Y</b> | <b>Z</b> |
|---------------------------|----------|----------|----------|----------|
| <b>FAT GRAMS</b>          |          |          |          |          |
| <b>CARBOHYDRATE GRAMS</b> |          |          |          |          |
| <b>PROTEIN GRAMS</b>      |          |          |          |          |
| <b>CALORIES</b>           |          |          |          |          |
| <b>WEIGHT GRAMS</b>       |          |          |          |          |

## **An Okefenokee Food Web**

### **Mathematical Goals**

- Represent data in matrix form.
- Understand and apply matrix representations of vertex-edge graphs.
- Use digraphs to represent realistic situations.
- Raise matrices to powers

### **Common Core State Standards**

#### **Perform operations on matrices and use matrices in applications.**

**MCC9-12.N.VM.6 (+)** Use matrices to represent and manipulate data

**MCC9-12.N.VM.7 (+)** Multiply matrices by scalars to produce new matrices

**MCC9-12.N.VM.8 (+)** Add, subtract, and multiply matrices of appropriate dimensions.

#### **Solve systems of equations**

**MCC9-12.A.REI.8 (+)** Represent a system of linear equations as a single matrix equation in a vector variable.

**MCC9-12.A.REI.9 (+)** Find the inverse of a matrix if it exists and use it to solve systems of linear equations (using technology for matrices of dimension  $3 \times 3$  or greater).

### **Standards for Mathematical Practice**

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Look for and make use of structure.
7. Look for and express regularity in repeated reasoning.

### **Introduction**

*An Okefenokee Food Web* uses vertex-edge digraphs to study various group interactions including food webs and gossiping. Predator prey graphs and powers of related adjacency matrices are interpreted to determine the effects of environmental changes on a food web.

### **Materials**

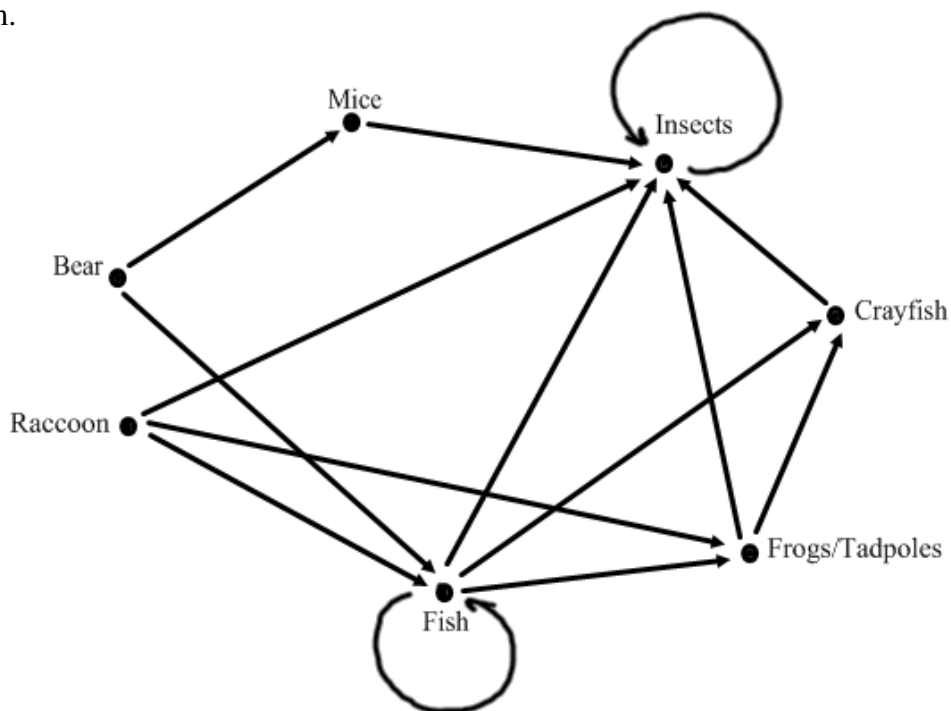
- Pencil
- Handout

- Graphing Calculator

### AN OKEFENOKEE FOOD WEB Learning Task

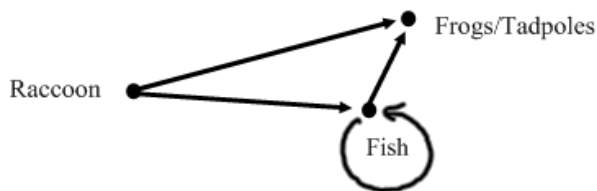
Recent weather conditions have caused a dramatic increase in the insect population of the Okefenokee Swamp area. The insects are annoying to people and animals and health officials are concerned there will be an increase in disease. Local authorities want to use an insecticide that would literally wipe out the entire insect population of the area. You, as an employee of the Environmental Protection Agency, must determine how detrimental this would be to the environment. Specifically, you are concerned on the effects on the food web of six animals known to populate the swamp.

Consider the following digraph of a food web for the six animals and the insects that are causing the problem.



A **digraph** is a directed vertex edge graph. Here each vertex represents an animal or insects. The direction of the edges indicates whether an animal preys on the linked animal. For example, raccoons eat fish. (Note: the food web shown is simplified. Initial producers of nutrients, plants, have not been included.)

Adjacency matrices can be used in conjunction with digraphs. If we consider just the relationships between raccoons, fish, and frogs in the food web shown, an adjacency matrix would be



$$\begin{array}{l}
 \text{R} \quad \text{FT} \quad \text{F} \\
 \text{R} \quad \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \\
 \text{FT} \quad \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \\
 \text{F} \quad \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}
 \end{array}$$

1. Label the rows and columns in alphabetical order and construct the associated matrix F to represent this web. What does a row containing a single one indicate? What does a column of zeros indicate?

|                 | <i>Bear</i> | <i>Cr</i> | <i>Fi</i> | <i>Fro</i> | <i>Ins</i> | <i>Mi</i> | <i>Rac</i> |
|-----------------|-------------|-----------|-----------|------------|------------|-----------|------------|
| <i>Bear</i>     | 0           | 0         | 1         | 0          | 0          | 1         | 0          |
| <i>Crayfish</i> | 0           | 0         | 0         | 0          | 1          | 0         | 0          |
| <i>Fish</i>     | 0           | 1         | 1         | 1          | 1          | 0         | 0          |
| <i>Frogs</i>    | 0           | 1         | 0         | 0          | 1          | 0         | 0          |
| <i>Insects</i>  | 0           | 0         | 0         | 0          | 1          | 0         | 0          |
| <i>Mice</i>     | 0           | 0         | 0         | 0          | 1          | 0         | 0          |
| <i>Raccoons</i> | 0           | 0         | 1         | 1          | 1          | 0         | 0          |

2. Which animals have the most direct sources of food? How can this be determined from the matrix? Find the number of direct food sources for each animal.

*Fish have the highest number of direct food sources. To find the number of direct food sources, add the entries in each row. Bears = 2; Mice = 1; Insects = 1; Crayfish = 1; Frogs/Tadpoles = 2; Fish = 4; Raccoons = 3.*

3. The insect column has the most ones. What does this suggest about the food web?

*The insect column has 6 ones indicating insects are prey (a direct food source) for 6 members of this food web. This suggests insects are very important in this ecosystem.*

4. The matrix  $F^2$  denotes indirect (through one intermediary) sources of food. For example, the fish relies on insects for food, and the bear relies on the fish for food, so the insect is an indirect source of food for the bear. Find  $F^2$ . Notice that insect column contains all nonzero numbers. What does this indicate?

|                 | <i>Bear</i> | <i>Cr</i> | <i>Fi</i> | <i>Fro</i> | <i>Ins</i> | <i>Mi</i> | <i>Rac</i> |
|-----------------|-------------|-----------|-----------|------------|------------|-----------|------------|
| <i>Bear</i>     | $F^2 =$     | 0         | 1         | 1          | 1          | 2         | 0          |
| <i>Crayfish</i> |             | 0         | 0         | 0          | 0          | 1         | 0          |
| <i>Fish</i>     |             | 0         | 2         | 1          | 1          | 4         | 0          |
| <i>Frogs</i>    |             | 0         | 0         | 0          | 0          | 2         | 0          |
| <i>Insects</i>  |             | 0         | 0         | 0          | 0          | 1         | 0          |
| <i>Mice</i>     |             | 0         | 0         | 0          | 0          | 1         | 0          |
| <i>Raccoons</i> |             | 0         | 2         | 1          | 1          | 3         | 0          |

5. Compute additional powers of the food web matrix to represent the number of direct and indirect sources of food for each animal. Which animal has the most food sources?

|         |   |         |   |
|---------|---|---------|---|
| $F^3 =$ | $\begin{bmatrix} 0 & 2 & 1 & 1 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 7 & 0 & 0 \end{bmatrix}$ | $F^4 =$ | $\begin{bmatrix} 0 & 2 & 1 & 1 & 9 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 12 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 11 & 0 & 0 \end{bmatrix}$ |
|---------|---|---------|---|

*Fish have the most food sources.*

**Georgia Department of Education**  
 Common Core Georgia Performance Standards Framework  
*CCGPS Pre-Calculus • Unit 5*

If an insecticide is introduced into the food web, killing the entire insect population, several animals will lose a source of food.

6. Construct a new matrix  $G$  to represent the food web with no insects. What effect does this have on the overall animal population? What has happened to the row sums? Compare these with those of the original matrix. What does a row sum of zero indicate?

|                 | <i>Bear</i> | <i>Cr</i> | <i>Fi</i> | <i>Fro</i> | <i>Mi</i> | <i>Rac</i> |
|-----------------|-------------|-----------|-----------|------------|-----------|------------|
| <i>Bear</i>     | 0           | 0         | 1         | 0          | 1         | 0          |
| <i>Crayfish</i> | 0           | 0         | 0         | 0          | 0         | 0          |
| <i>Fish</i>     | 0           | 1         | 1         | 1          | 0         | 0          |
| <i>Frogs</i>    | 0           | 1         | 0         | 0          | 0         | 0          |
| <i>Mice</i>     | 0           | 0         | 0         | 0          | 0         | 0          |
| <i>Raccoons</i> | 0           | 0         | 1         | 1          | 0         | 0          |

*Eliminating insects reduces the number of direct food sources for all animals. Some animals, mice and crayfish, no longer have a direct food source.*

7. Will all the animals be affected by the insecticide? Which animal(s) will be least affected?

*Mice, crayfish, and frogs and tadpoles have lost an important source of food. If these animals die off, and are taken out of the food web, then bears, fish, and raccoons are only left with fish as a primary food source.*

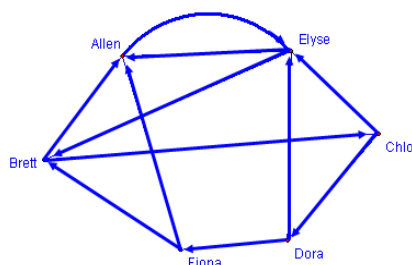
8. Organize and summarize your findings in a brief report to the health officials. Take and support a position on whether using an insecticide to destroy the insect population is harmful to the environment.

**Extension Problem:**

Within a small group of associates, different people are willing to share secrets selectively. Allen will share with Elyse. Brett will share with Chloe and Allen. Chloe will share with Elyse and Dora. Dora will share with Fiona and Elyse. Elyse will share with Allen and Brett. Fiona will share with Allen and Brett.

1. Show this information with a digraph and an adjacency matrix named  $S$ .

$$S = \begin{matrix} & \begin{matrix} A & B & C & D & E & F \end{matrix} \\ \begin{matrix} \text{Allen} \\ \text{Brett} \\ \text{Chloe} \\ \text{Dora} \\ \text{Elyse} \\ \text{Fiona} \end{matrix} & \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{matrix}$$



2. What do the zeros on the diagonal of the adjacency matrix indicate?

*A person cannot tell a secret to himself*

3. Find  $S^2$  and  $S^3$ .

$$S^2 = \begin{matrix} & \begin{matrix} A & B & C & D & E & F \end{matrix} \\ \begin{matrix} \text{Allen} \\ \text{Brett} \\ \text{Chloe} \\ \text{Dora} \\ \text{Elyse} \\ \text{Fiona} \end{matrix} & \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 2 & 2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \end{matrix}$$

$$S^3 = \begin{matrix} & \begin{matrix} A & B & C & D & E & F \end{matrix} \\ \begin{matrix} \text{Allen} \\ \text{Brett} \\ \text{Chloe} \\ \text{Dora} \\ \text{Elyse} \\ \text{Fiona} \end{matrix} & \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & 2 & 0 & 0 & 1 & 1 \\ 3 & 2 & 1 & 0 & 1 & 0 \\ 2 & 0 & 2 & 0 & 2 & 0 \\ 1 & 1 & 0 & 1 & 2 & 0 \\ 1 & 1 & 0 & 1 & 2 & 0 \end{bmatrix} \end{matrix}$$



**Georgia Department of Education**  
Common Core Georgia Performance Standards Framework  
*CCGPS Pre-Calculus • Unit 5*

4. Is it possible for Fiona to share a secret and that secret reach Chloe? What is the minimum number of times the secret is shared when Charles knows the secret?

*The first time a nonzero entry appears in the  $S_{\text{Fiona, Charles}}$  position is in  $S^2$ . So the minimum number of secret sharing is 2.*

5. How many ways is it possible for a secret to get from Brett to Elyse in 3 or fewer secret sharing episodes?

*In  $S$ , the  $S_{\text{Brett, Elyse}}$  value is 0 so there is no direct secret sharing between Brett and Elyse. In  $S^2$ , however, there are 2 two-person sharing events, and in  $S^3$  there is 1 three-person sharing event. It is possible for a secret to get from Brett to Elyse in  $0 + 2 + 1 = 3$  ways.*

REFERENCES:

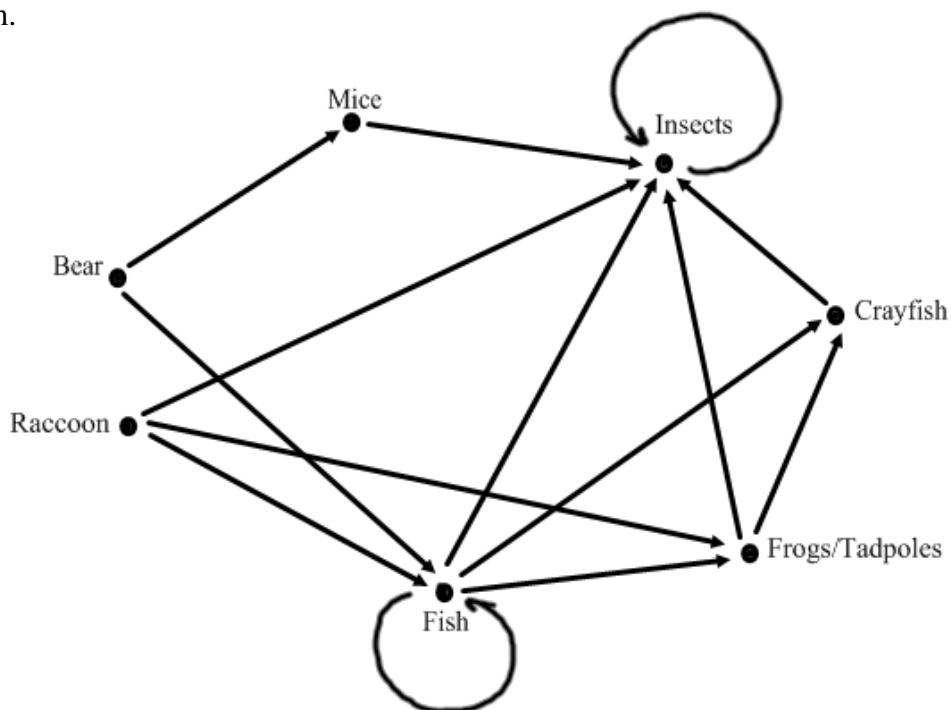
Crisler, N., Fisher, P., & Froelich, G. (1994). *Discrete mathematics Through Applications*. New York: W.H. Freeman and Company.

Peressini, Dominic; Project Director. *The Discrete Mathematics Project* (1997). Boulder, Colorado: University of Colorado through an Eisenhower Professional Development Grant.

### AN OKEFENOKEE FOOD WEB Learning Task

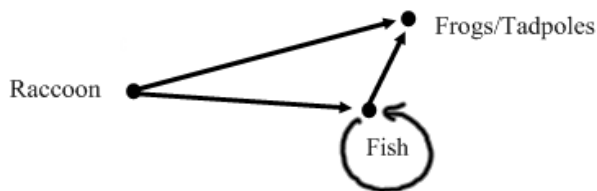
Recent weather conditions have caused a dramatic increase in the insect population of the Okefenokee Swamp area. The insects are annoying to people and animals and health officials are concerned there will be an increase in disease. Local authorities want to use an insecticide that would literally wipe out the entire insect population of the area. You, as an employee of the Environmental Protection Agency, must determine how detrimental this would be to the environment. Specifically, you are concerned on the effects on the food web of six animals known to populate the swamp.

Consider the following digraph of a food web for the six animals and the insects that are causing the problem.



A **digraph** is a directed vertex edge graph. Here each vertex represents an animal or insects. The direction of the edges indicates whether an animal preys on the linked animal. For example, raccoons eat fish. (Note: the food web shown is simplified. Initial producers of nutrients, plants, have not been included.)

Adjacency matrices can be used in conjunction with digraphs. If we consider just the relationships between raccoons, fish, and frogs in the food web shown, an adjacency matrix would be



|    | R   | FT | F |
|----|---|----|---|
| R  | $\left[ \begin{array}{ccc} 0 & 1 & 1 \end{array} \right]$ |    |   |
| FT | $\left[ \begin{array}{ccc} 0 & 0 & 0 \end{array} \right]$ |    |   |
| F  | $\left[ \begin{array}{ccc} 0 & 1 & 1 \end{array} \right]$ |    |   |

1. Label the rows and columns in alphabetical order and construct the associated matrix  $F$  to represent this web. What does a row containing a single one indicate? What does a column of zeros indicate?

2. Which animals have the most direct sources of food? How can this be determined from the matrix? Find the number of direct food sources for each animal.

3. The insect column has the most ones. What does this suggest about the food web?

**Georgia Department of Education**  
Common Core Georgia Performance Standards Framework  
*CCGPS Pre-Calculus • Unit 5*

4. The matrix  $F^2$  denotes indirect (through one intermediary) sources of food. For example, the fish relies on insects for food, and the bear relies on the fish for food, so the insect is an indirect source of food for the bear. Find  $F^2$ . Notice that insect column contains all nonzero numbers. What does this indicate?

*Bear Crayfish Fish Frogs Insects Mice Raccoons*

*Bear*

*Crayfish*

*Fish*

$F^2 =$  *Frogs*

*Insects*

*Mice*

*Raccoons*

5. Compute additional powers of the food web matrix to represent the number of direct and indirect sources of food for each animal. Which animal has the most food sources?

$F^3 =$

$F^4 =$

**Georgia Department of Education**  
Common Core Georgia Performance Standards Framework  
*CCGPS Pre-Calculus • Unit 5*

If an insecticide is introduced into the food web, killing the entire insect population, several animals will lose a source of food.

6. Construct a new matrix  $G$  to represent the food web with no insects. What effect does this have on the overall animal population? What has happened to the row sums? Compare these with those of the original matrix. What does a row sum of zero indicate?

7. Will all the animals be affected by the insecticide? Which animal(s) will be least affected?

8. Organize and summarize your findings in a brief report to the health officials. Take and support a position on whether using an insecticide to destroy the insect population is harmful to the environment.

**Extension Problem:**

Within a small group of associates, different people are willing to share secrets selectively. Allen will share with Elyse. Brett will share with Chloe and Allen. Chloe will share with Elyse and Dora. Dora will share with Fiona and Elyse. Elyse will share with Allen and Brett. Fiona will share with Allen and Brett.

1. Show this information with a digraph and an adjacency matrix named  $S$ .
  
  
  
  
  
  
  
  
  
  
2. What do the zeros on the diagonal of the adjacency matrix indicate?
  
  
  
  
  
  
  
  
  
  
3. Find  $S^2$  and  $S^3$ .
  
  
  
  
  
  
  
  
  
  
4. Is it possible for Fiona to share a secret and that secret reach Chloe? What is the minimum number of times the secret is shared when Charles knows the secret?
  
  
  
  
  
  
  
  
  
  
5. How many ways is it possible for a secret to get from Brett to Elyse in 3 or fewer secret sharing episodes?

**REFERENCES:**

Crisler, N., Fisher, P., & Froelich, G. (1994). *Discrete mathematics Through Applications*. New York: W.H. Freeman and Company.

Peressini, Dominic; Project Director. *The Discrete Mathematics Project* (1997). Boulder, Colorado: University of Colorado through an Eisenhower Professional Development Grant.

## **Culminating Task: Vacationing in Georgia**

### **Mathematical Goals**

- Represent data in matrix form.
- Perform operations with matrices.
- Solve problems using inverse matrices
- Use digraphs to represent realistic situations.
- Solve problems using digraphs

### **Common Core State Standards**

#### **Perform operations on matrices and use matrices in applications.**

**MCC9-12.N.VM.6 (+)** Use matrices to represent and manipulate

**MCC9-12.N.VM.7 (+)** Multiply matrices by scalars to produce new matrices

**MCC9-12.N.VM.8 (+)** Add, subtract, and multiply matrices of appropriate dimensions

#### **Solve systems of equations**

**MCC9-12.A.REI.8 (+)** Represent a system of linear equations as a single matrix equation in a vector variable.

**MCC9-12.A.REI.9 (+)** Find the inverse of a matrix if it exists and use it to solve systems of linear equations (using technology for matrices of dimension  $3 \times 3$  or greater).

### **Standards for Mathematical Practice**

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

### **Materials**

- Pencil
- Handout
- Graphing Calculator

## Culminating Task: Vacationing in Georgia

I. Your family is planning to spend a week’s vacation traveling in Georgia. First, your parents need to design a budget for the trip. The trip will begin in Atlanta and they estimate spending \$60 on gas, \$180 on food, \$200 on activities, and \$40 on souvenirs. Next, you will visit family in Albany. Your parents estimate the cost for this leg of their journey will be \$45 on gas, \$50 on food, and \$20 on activities. Finally, you all will visit Savannah and estimate spending \$90 on gas, \$200 on food, \$100 on activities, and \$75 on souvenirs.

1. Construct a  $3 \times 4$  matrix  $[V]$  to represent all the costs involved in this vacation.

$$[V] = \begin{matrix} & \begin{matrix} Gas & Food & Act & Souv \end{matrix} \\ \begin{matrix} Atl \\ Alb \\ Sav \end{matrix} & \begin{bmatrix} 60 & 180 & 200 & 40 \\ 45 & 50 & 20 & 0 \\ 90 & 200 & 100 & 75 \end{bmatrix} \end{matrix}$$

- b. Could a  $4 \times 3$  matrix be used instead? Explain.

*Yes, the rows would be the types of expenditures and the columns would be the cities.*

- c. The sales tax in Georgia is 7%. Write a matrix  $[T]$  representing the taxes for each expenditure.

$$[T] = \begin{matrix} & \begin{matrix} Gas & Food & Act & Souv \end{matrix} \\ \begin{matrix} Atl \\ Alb \\ Sav \end{matrix} & \begin{bmatrix} 4.20 & 12.60 & 14.00 & 2.80 \\ 3.15 & 3.50 & 1.40 & 0 \\ 6.30 & 14.00 & 7.00 & 5.25 \end{bmatrix} \end{matrix}$$

- d. Compute a cost including tax matrix  $[C]$  to represent their expenditures.

$$[C] = \begin{matrix} & \begin{matrix} Gas & Food & Act & Souv \end{matrix} \\ \begin{matrix} Atl \\ Alb \\ Sav \end{matrix} & \begin{bmatrix} 64.20 & 192.60 & 214.00 & 42.80 \\ 48.15 & 53.50 & 21.40 & 0 \\ 96.30 & 214.00 & 107.00 & 80.25 \end{bmatrix} \end{matrix}$$

- e. What is the entry in matrix  $[C]$  denoted as  $C_{3,1}$  and what is its meaning?  
*\$48.15 which indicates they are estimating the gas cost in Albany to cost \$48.15 when tax is included.*



- f. What is the sum of the third row in matrix [C] what does it mean?  
*Your parents estimate they will spend \$497.55 in Savannah*
- g. What is the sum of the first column in matrix [C] and what does it mean?  
*Your parents estimate they will spend \$208.65 on gas for this trip*

II. Everyone in your family knows you are an excellent math student. Therefore, whenever math problems arise on this vacation, they ask you to solve them. Solve the following and show or explain how you solved them.

8. Your father said he had earned the money for this vacation through interest earned in an investment portfolio. He had \$25,000 invested in two types of accounts: a municipal bond that earned 3% annual interest and a mutual fund that earned 9% annual interest. If he made \$1830.00 in interest for the year, how much was invested in each type of account.  
*\$7,000.00 in municipal bonds and \$18,000.00 in mutual funds*

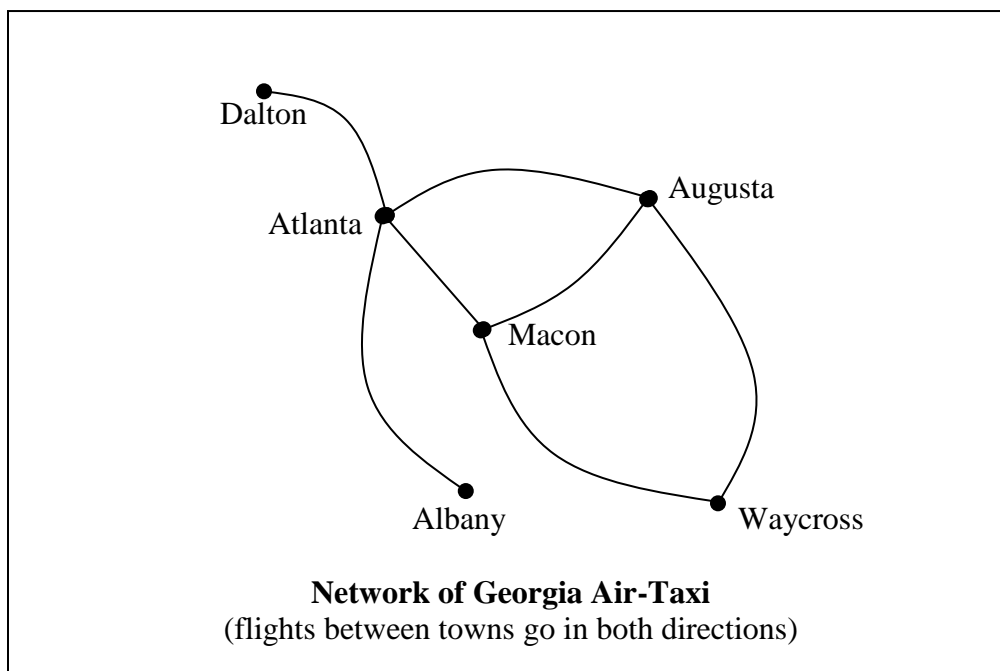
9. Your aunt and uncle in Albany are planning a birthday party for their youngest child at a skating rink. The cost of admission is \$3.50 per adult and \$2.25 per child, and there is a limit of 20 people. They have \$50 to spend. Determine how many adults and how many children can be invited.  
*4 adults and 16 children*

10. While in Savannah, you met a vendor selling frozen yogurt by the river. He said he had made \$565 and used 250 cones that day. If a single-scoop cone cost \$2 and a double-scoop cone cost \$2.50, how many of each type of cone did he sell?  
*120 single scoops and 130 double scoops*

11. At Lenox Mall in Atlanta, you and two other members of your family wanted to eat Chinese food. You found a Chinese restaurant and ordered three different luncheon combination platters. Mom ordered 2 portions of fried rice and 1 portion of chicken chow mein. Your sister ordered 1 portion of fried rice, 1 portion of chicken chow mein, and 1 eggroll. You ordered 2 portions of chicken chow mein and 1 eggroll. Your mother's platter cost \$5, your sister's cost \$5.25, and yours cost \$5.75. How much did 1 eggroll cost?  
*\$1.75*

12. You were elected the student council president for the next year at your school. After vacation is over, you have to plan a school carnival for the new school year. You are studying last year's event and know that 210 people attended last year's school carnival. The total amount of money collected for tickets was \$710. Prices were \$5 for regular admission, \$3 for students, and \$1 for children. The number of regular tickets sold was 10 more than twice the number of children's tickets sold. Determine how many of each kind of ticket were sold.  
*70 regular, 110 student, and 30 children tickets were sold.*

III. Although you enjoyed your vacation, riding in a car with your family was pretty annoying. You are hoping next time, you could fly. The next part of this task will have you analyze a Network of Georgia Air-Taxis.



The diagram above shows a map of the routes taken by Georgia Air-Taxi airline. On this diagram lengths and directions are irrelevant; all that matters is the connections between airports.

13. Construct a route matrix for this network in which 1 indicates that there is a direct route between two airports (nodes) and a 0 indicates that there is no direct flight between airports. (Please define the rows and columns in **alphabetical order**.)

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

14. Use Technology to find the square of the route matrix found in 13.

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 4 & 1 & 0 & 1 & 2 \\ 1 & 1 & 3 & 1 & 2 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 2 & 1 & 3 & 1 \\ 0 & 2 & 1 & 0 & 1 & 2 \end{bmatrix}$$

15. By multiplying a route matrix by itself the resulting matrix shows the number of two-stage routes connecting the airports. For example, the 1 route shown for Dalton to Dalton arises because it is possible to fly from Dalton to Atlanta back to Dalton. Your square matrix should have one element with a value of **4** and two with a value of **3**. Describe all the two-stage routes these numbers represent.

a. List all the two-stage routes that the element **4** represents in your square matrix:

*Atlanta to Dalton to Atlanta*

*Atlanta to Augusta to Atlanta*

*Atlanta to Macon to Atlanta*

*Atlanta to Albany to Atlanta*

b. List all the two-stage routes that the element **3** represents in your square matrix:

*Augusta to Atlanta to Augusta*

*Augusta to Macon to Augusta*

*Augusta to Waycross to Augusta*

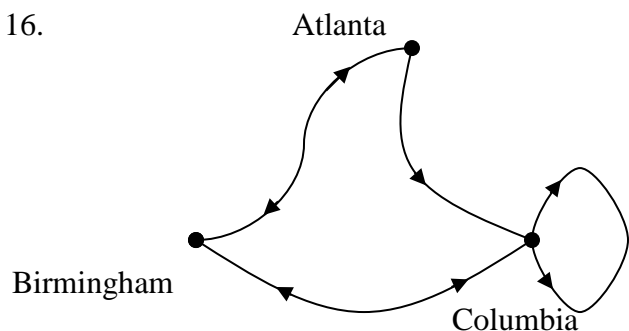
c. List all the two-stage routes that the **other** element **3** represents in your square matrix:

*Macon to Atlanta to Macon*

*Macon to Augusta to Macon*

*Macon to Waycross to Macon*

16.



- a. Construct a route matrix for the network shown in the diagram. Lines with only one arrow indicate that the flow of traffic is in one direction only. Note that it is possible to take a route from Columbia back to itself and that the journey can be taken in two directions.

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

- b. Find the square of the route matrix and describe a two stage route that the new matrix indicates.

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & 2 & 3 \\ 1 & 2 & 5 \end{bmatrix}$$

- c. Find the cube of the route matrix, what do you think this represents? Finally, take the element **4** in the cubed matrix and describe all the routes its represents.

*The element 4 represents all the 4-stage routes that begin in Atlanta and end in Birmingham. This means there would be have to be three different flights on these routes.*

*Atlanta to Birmingham to Atlanta to Birmingham*

*Atlanta to Columbia (North) to Columbia to Birmingham*

*Atlanta to Columbia (South) to Columbia to Birmingham*

*Atlanta to Birmingham to Columbia to Birmingham*



**Georgia Department of Education**  
Common Core Georgia Performance Standards Framework  
*CCGPS Pre-Calculus • Unit 5*

II. Everyone in your family knows you are an excellent math student. Therefore, whenever math problems arise on this vacation, they ask you to solve them. Solve the following and show or explain how you solved them.

8. Your father said he had earned the money for this vacation through interest earned in an investment portfolio. He had \$25,000 invested in two types of accounts: a municipal bond that earned 3% annual interest and a mutual fund that earned 9% annual interest. If he made \$1830.00 in interest for the year, how much was invested in each type of account.

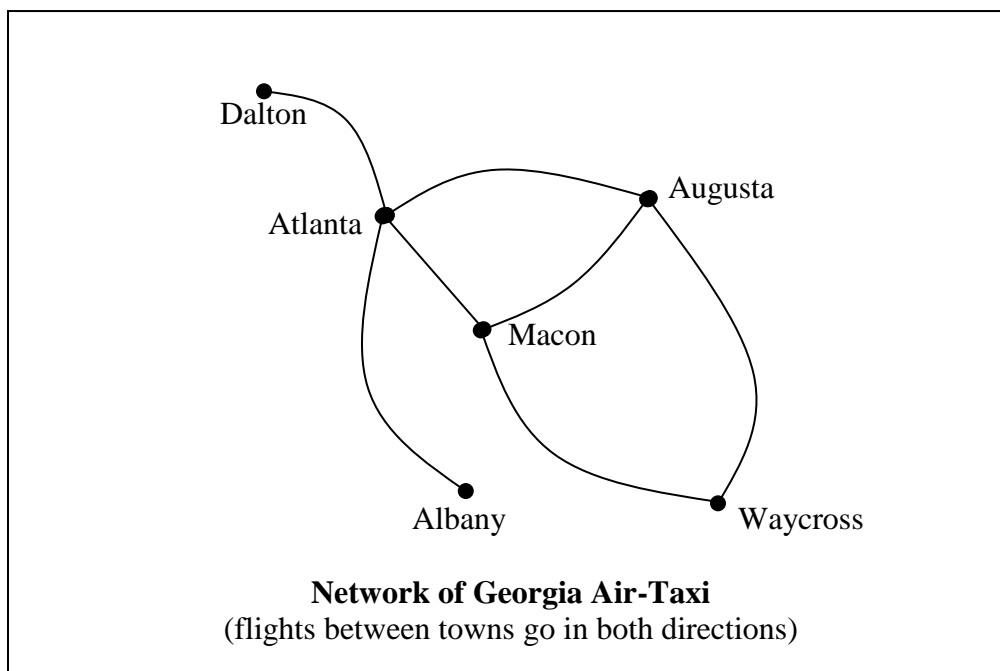
9. Your aunt and uncle in Albany are planning a birthday party for their youngest child at a skating rink. The cost of admission is \$3.50 per adult and \$2.25 per child, and there is a limit of 20 people. They have \$50 to spend. Determine how many adults and how many children can be invited.

10. While in Savannah, you met a vendor selling frozen yogurt by the river. He said he had made \$565 and used 250 cones that day. If a single-scoop cone cost \$2 and a double-scoop cone cost \$2.50, how many of each type of cone did he sell?

11. At Lenox Mall in Atlanta, you and two other members of your family wanted to eat Chinese food. You found a Chinese restaurant and ordered three different luncheon combination platters. Mom ordered 2 portions of fried rice and 1 portion of chicken chow mein. Your sister ordered 1 portion of fried rice, 1 portion of chicken chow mein, and 1 eggroll. You ordered 2 portions of chicken chow mein and 1 eggroll. Your mother's platter cost \$5, your sister's cost \$5.25, and yours cost \$5.75. How much did 1 eggroll cost?

12. You were elected the student council president for the next year at your school. After vacation is over, you have to plan a school carnival for the new school year. You are studying last year's event and know that 210 people attended last year's school carnival. The total amount of money collected for tickets was \$710. Prices were \$5 for regular admission, \$3 for students, and \$1 for children. The number of regular tickets sold was 10 more than twice the number of children's tickets sold. Determine how many of each kind of ticket were sold.

III. Although you enjoyed your vacation, riding in a car with your family was pretty annoying. You are hoping next time, you could fly. The next part of this task will have you analyze a Network of Georgia Air-Taxis.



The diagram above shows a map of the routes taken by Georgia Air-Taxi airline. On this diagram lengths and directions are irrelevant; all that matters is the connections between airports.

13. Construct a route matrix for this network in which 1 indicates that there is a direct route between two airports (nodes) and a 0 indicates that there is no direct flight between airports. (Please define the rows and columns in **alphabetical order**.)

14. Use Technology to find the square of the route matrix found in 13.

**Georgia Department of Education**  
Common Core Georgia Performance Standards Framework  
*CCGPS Pre-Calculus • Unit 5*

15. By multiplying a route matrix by itself the resulting matrix shows the number of two-stage routes connecting the airports. For example, the 1 route shown for Dalton to Dalton arises because it is possible to fly from Dalton to Atlanta back to Dalton. Your square matrix should have one element with a value of **4** and two with a value of **3**. Describe all the two-stage routes these numbers represent.

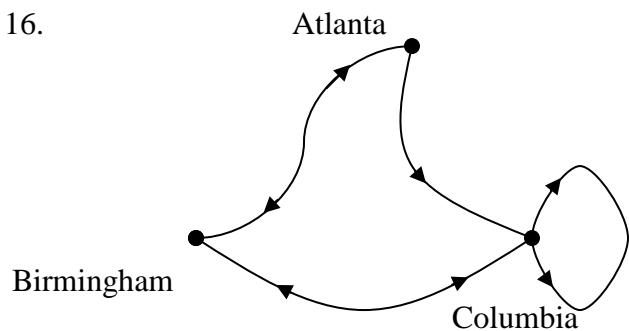
a. List all the two-stage routes that the element **4** represents in your square matrix:

b. List all the two-stage routes that the element **3** represents in your square matrix:

c. List all the two-stage routes that the **other** element **3** represents in your square matrix:



16.



- a. Construct a route matrix for the network shown in the diagram. Lines with only one arrow indicate that the flow of traffic is in one direction only. Note that it is possible to take a route from Columbia back to itself and that the journey can be taken in two directions.
  
  
  
  
  
  
  
  
  
  
- b. Find the square of the route matrix and describe a two stage route that the new matrix indicates.
  
  
  
  
  
  
  
  
  
  
- c. Find the cube of the route matrix, what do you think this represents? Finally, take the element **4** in the cubed matrix and describe all the routes its represents.