CCGPS Frameworks

Mathematics

CCGPS Pre-Calculus

Unit 7: Probability

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"Making Education Work for All Georgians"

Unit 7
Probability

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OVERVIEW

In this unit the student will:

- Calculate probabilities using the General Multiplication Rule and interpret the results in context
- Use permutations and combinations in conjunction with other probability methods to calculate probabilities of compound events and solve problems
- Define random variables, assign probabilities to its sample space, and graphically display the distribution of the random variable
- Calculate and interpret the expected value of random variables
- Develop the theoretical and empirical probability distribution and find expected values
- Set up a probability distribution for a random variable representing payoff values
- Make and explain in context decisions based on expected values

Although the units in this instructional framework emphasize key standards and big ideas at specific times of the year, routine topics such as estimation, mental computation, and basic computation facts should be addressed on an ongoing basis. Ideas related to the eight process standards should be addressed constantly as well. To assure that this unit is taught with the appropriate emphasis, depth, and rigor, it is important that the tasks listed under "Evidence of Learning" be reviewed early in the planning process. A variety of resources should be utilized to supplement this unit. This unit provides much needed content information, but excellent learning activities as well. The tasks in this unit illustrate the types of learning activities that should be utilized from a variety of sources.

STANDARDS ADDRESSED IN THIS UNIT

Mathematical standards are interwoven and should be addressed throughout the year in as many different units and activities as possible in order to emphasize the natural connections that exist among mathematical topics.

KEY & RELATED STANDARDS

Use the rules of probability to compute probabilities of compound events in a uniform probability model

MCC9-12.CP.8 (+) Apply the general Multiplication Rule in a uniform probability model, P(A and B)=[P(A)] x [P(B | A)] = [P(B)] x [P(A | B)], and interpret the answer in terms of the model.

MCC9-12.S.CP.9 (+) Use permutations and combinations to compute probabilities of compound events and solve problems.

Calculate expected values and use them to solve problems

MCC9-12.S.MD.1(+) Define a random variable for a quantity of interest by assigning a numerical value to each event in a sample space; graph the corresponding probability distribution using the same graphical displays as for data distributions.

MCC9-12.S.MD.2(+) Calculate the expected value of a random variable; interpret it as the mean of a probability distribution.

MCC9-12.S.MD.3(+) Develop a probability distribution for a random variable defined for a sample space in which theoretical probabilities can be calculated; find the expected value

MCC9-12.S.MD.4(+) Develop a probability distribution for a random variable defined for a sample space in which probabilities are assigned empirically; find the expected value

Use probabilities to evaluate outcomes of decisions

MCC9-12.S.MD.5(+) Weigh the possible outcomes of a decision by assigning probabilities to payoff values and finding expected values.

MCC9-12.S.MD.5a(+) Find the expected payoff for a game of chance

MCC9-12.S.MD.5b(+) Evaluate and compare strategies on the basis of expected values

MCC9-12.S.MD.6(+) Use probabilities to make fair decisions (e.g., drawing by lots, using a random number generator).

MCC9-12.S.MD.7(+) Analyze decisions and strategies using probability concepts (e.g., product testing, medical testing, pulling a hockey goalie at the end of a game).

RELATED STANDARD

MCC7.SP.8 Find probabilities of compound events using organized lists, tables, tree diagrams, and simulation.

MCC9-12.S.CP.2 Understand that two events A and B are independent if the probability of A and B occurring together is the product of their probabilities, and use this characterization to determine if they are independent. ★

MCC9-12.S.CP.3 Understand the conditional probability of A given B as P(A and B)/P(B), and interpret independence of A and B as saying that the conditional probability of A given B is the same as the probability of A, and the conditional probability of B given A is the same as the probability of B. ★

MCC9-12.S.CP.6 Find the conditional probability of A given B as the fraction of B's outcomes that also belong to A, and interpret the answer in terms of the model. ★

MCC9-12.S.CP.7 Apply the Addition Rule, P(A or B) = P(A) + P(B) – P(A and B), and interpret the answer in terms of the model. ★

Standards for Mathematical Practice

The Standards for Mathematical Practice describe varieties of expertise that mathematics educators at all levels should seek to develop in their students. These practices rest on important "processes and proficiencies" with longstanding importance in mathematics education. The first of these are the NCTM process standards of problem solving, reasoning and proof, communication, representation, and connections. The second are the strands of mathematical proficiency specified in the National Research Council's report *Adding It Up*: adaptive reasoning, strategic competence, conceptual understanding (comprehension of mathematical concepts, operations and relations), procedural fluency (skill in carrying out procedures flexibly, accurately, efficiently and appropriately), and productive disposition (habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy).

1. **Make sense of problems and persevere in solving them.** High school students start to examine problems by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. By high school, students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. They check their answers to problems using different methods and continually ask themselves, "Does this make sense?" They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

2. **Reason abstractly and quantitatively.** High school students seek to make sense of quantities and their relationships in problem situations. They abstract a given situation and represent it symbolically, manipulate the representing symbols, and pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Students use quantitative reasoning to create coherent representations of the problem at hand; consider the units involved; attend to the meaning of quantities, not just how to compute them; and know and flexibly use different properties of operations and objects.

3. **Construct viable arguments and critique the reasoning of others.** High school students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. High school students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. High school students learn to determine domains to which an argument applies, listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

4. **Model with mathematics.** High school students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. High school students making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

5. **Use appropriate tools strategically.** High school students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. High school students should be sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. They are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.

6. **Attend to precision.** High school students try to communicate precisely to others by using clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, specifying units of measure, and labeling axes to

clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context. By the time they reach high school they have learned to examine claims and make explicit use of definitions.

7. **Look for and make use of structure.** By high school, students look closely to discern a pattern or structure. In the expression $x^2 + 9x + 14$, older students can see the 14 as 2×7 and the 9 as $2 + 7$. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 - 3(x - y)^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers x and y. High school students use these patterns to create equivalent expressions, factor and solve equations, and compose functions, and transform figures.

8. **Look for and express regularity in repeated reasoning.** High school students notice if calculations are repeated, and look both for general methods and for shortcuts. Noticing the regularity in the way terms cancel when expanding $(x - 1)(x + 1)$, $(x - 1)(x^2 + x + 1)$, and $(x - 1)(x^3 + x^2 + x + 1)$ might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, derive formulas or make generalizations, high school students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

Connecting the Standards for Mathematical Practice to the Standards for Mathematical Content

The Standards for Mathematical Practice describe ways in which developing student practitioners of the discipline of mathematics should engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle and high school years. Designers of curricula, assessments, and professional development should all attend to the need to connect the mathematical practices to mathematical content in mathematics instruction.

The Standards for Mathematical Content are a balanced combination of procedure and understanding. Expectations that begin with the word “understand” are often especially good opportunities to connect the practices to the content. Students who do not have an understanding of a topic may rely on procedures too heavily. Without a flexible base from which to work, they may be less likely to consider analogous problems, represent problems coherently, justify conclusions, apply the mathematics to practical situations, use technology mindfully to work with the mathematics, explain the mathematics accurately to other students, step back for an overview, or deviate from a known procedure to find a shortcut. In short, a missing mathematical knowledge effectively prevents a student from engaging in the mathematical practices.

In this respect, those content standards which set an expectation of understanding are potential “points of intersection” between the Standards for Mathematical Content and the Standards for Mathematical Practice. These points of intersection are intended to be weighted toward central and generative concepts in the school mathematics curriculum that most merit the time, resources, innovative energies, and focus necessary to qualitatively improve the curriculum, instruction, assessment, professional development, and student achievement in mathematics.

ENDURING UNDERSTANDINGS

- Understand how to calculate probabilities using the General Multiplication Rule and interpret the results in context.
- Understand how to use permutations and combinations in conjunction with other probability methods to calculate probabilities of compound events and solve problems.
- Know how to define random variables, assign probabilities to its sample space, and graphically display the distribution of the random variable.
- Understand how to calculate and interpret the expected value of random variables.
- Understand hot to develop the theoretical and empirical probability distribution and find expected values.
- Know how to set up a probability distribution for a random variable representing payoff values.

ESSENTIAL QUESTIONS

- How do I use the General Multiplication Rule to calculate probabilities?
- How do I determine when to use a permutation or a combination to calculate a probability?
- How do I identify a random variable?
- How do I graphically display the probability distribution of a random variable?
- How do I calculate the expected value of a random variable?
- How do I calculate theoretical and empirical probabilities of probability distributions?
- How do I represent and calculate payoff values in a game of chance?
- How do I use expected values to make decisions?
- How do I explain the decisions I make using expected values?

CONCEPTS AND SKILLS TO MAINTAIN

In order for students to be successful, the following skills and concepts need to be maintained
- Understand the basic nature of probability
- Determine probabilities of simple and compound events
- Understand the Fundamental Counting Principle
- Organize and model simple situations involving probability

SELECT TERMS AND SYMBOLS

The following terms and symbols are often misunderstood. These concepts are not an inclusive list and should not be taught in isolation. However, due to evidence of frequent difficulty and misunderstanding associated with these concepts, instructors should pay particular attention to them and how their students are able to explain and apply them.

The definitions below are for teacher reference only and are not to be memorized by the students. Students should explore these concepts using models and real life examples. Students should understand the concepts involved and be able to recognize and/or demonstrate them with words, models, pictures, or numbers.

The websites below are interactive and include a math glossary suitable for high school children. **Note – At the high school level, different sources use different definitions. Please preview any website for alignment to the definitions given in the frameworks**
http://www.amathsdictionaryforkids.com/
This web site has activities to help students more fully understand and retain new vocabulary.
http://intermath.coe.uga.edu/dictnary/homepg.asp

Definitions and activities for these and other terms can be found on the Intermath website. Intermath is geared towards middle and high school students.

- **Conditional Probability.** $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$
- **Combinations.** A combination is an arrangement of objects in which order does NOT matter. ${}_nC_r = \dfrac{n!}{r!(n-r)!}$
- **Expected Value:** The mean of a random variable X is called the expected value of X. It can be found with the formula $\sum_{i=1}^{n} X_i P_i$ where P_i is the probability of the value of X_i. *For example: if you and three friends each contribute \$3 for a total of \$12 to be spent by the one whose name is randomly drawn, then one of the four gets the \$12 and three of the four gets \$0. Since everyone contributed \$3, one gains \$9 and the other three loses \$3. Then the expected value for each member of the group is found by (.25)(12) +(.75)(0) = 3. That is to say that each pays in the \$3 expecting to get \$3 in return. However, one person gets \$12 and the rest get \$0. A game or situation in which the expected value is equal to the cost (no net gain nor loss) is commonly called a "fair game." However, if you are allowed to put your name into the drawing twice, the expected value is (.20)(12)+(.80)(0) = \$2.40. That is to say that each pays in the \$3 expecting to get \$2.40 (indicating a loss of \$.60) in return. This game is not fair.*

- **Odds.** Typically expressed as a ratio of the likelihood that an event will happen to the likelihood that an event will not happen.

- **Permutations.** An ordered arrangement of *n* objects. The order of the objects matters – a different order creates a different outcome. ${}_nP_r = \frac{n!}{(n-r)!}$

- **Sample Space.** The set of all possible outcomes.

CLASSROOM ROUTINES

The importance of continuing the established classroom routines cannot be overstated. Daily routines must include such obvious activities as estimating, analyzing data, describing patterns, and answering daily questions. They should also include less obvious routines, such as how to select materials, how to use materials in a productive manner, how to put materials away, how to access classroom technology such as computers and calculators. An additional routine is to allow plenty of time for children to explore new materials before attempting any directed activity with these new materials. The regular use of routines is important to the development of students' number sense, flexibility, fluency, collaborative skills and communication. These routines contribute to a rich, hands-on standards based classroom and will support students' performances on the tasks in this unit and throughout the school year.

STRATEGIES FOR TEACHING AND LEARNING

The following strategies should be used to help students be successful in this unit.

- Students should be actively engaged by developing their own understanding.
- Mathematics should be represented in as many ways as possible by using graphs, tables, pictures, symbols and words.
- Interdisciplinary and cross curricular strategies should be used to reinforce and extend the learning activities.
- Appropriate manipulatives and technology should be used to enhance student learning.
- Students should be given opportunities to revise their work based on teacher feedback, peer feedback, and metacognition which includes self-assessment and reflection.
- Students should write about the mathematical ideas and concepts they are learning.
- Consideration of all students should be made during the planning and instruction of this unit. Teachers need to consider the following:
  - What level of support do my struggling students need in order to be successful with this unit?
  - In what way can I deepen the understanding of those students who are competent in this unit?
  - What real life connections can I make that will help my students utilize the skills practiced in this unit?

- Graphic organizers should be used to help students arrange ideas and tie mathematical thinking to situations involving real-world situations

EVIDENCE OF LEARNING

By the conclusion of this unit, students should be able to demonstrate the following competencies:

- Calculate probabilities using the general Multiplication Rule in a probability model.
- Use permutations and combinations to calculate probabilities of compound events and to solve problems.
- Given a probability situation, theoretical or empirical, understand how to define a random variable, assign probabilities to its sample space, and graph the probability distribution of the random variable.
- Calculate the expected value of a random variable
- Develop a theoretical and empirical probability distribution and find the expected value.
- Develop a probability distribution for a random variable representing payoff values in a game of chance.
- Make and explain decisions based on expected values.

TASKS

The following tasks represent the level of depth, rigor, and complexity expected of all *Pre-Calculus* students. These tasks or tasks of similar depth and rigor should be used to demonstrate evidence of learning. It is important that all elements of a task be addressed throughout the learning process so that students understand what is expected of them. While some tasks are identified as a performance task, they may also be used for teaching and learning (learning task).

Scaffolding Task	Tasks that build up to the learning task.
Learning Task	Constructing understanding through deep/rich contextualized problem solving tasks.
Practice Task	Tasks that provide students opportunities to practice skills and concepts.
Performance Task	Tasks which may be a formative or summative assessment that checks for student understanding/misunderstanding and or progress toward the standard/learning goals at different points during a unit of instruction.
Culminating Task	Designed to require students to use several concepts learned during the unit to answer a new or unique situation. Allows students to give evidence of their own understanding toward the mastery of the standard and requires them to extend their chain of mathematical reasoning.
Formative Assessment Lesson (FAL)	Lessons that support teachers in formative assessment which both reveal and develop students' understanding of key mathematical ideas and applications. These lessons enable teachers and students to monitor in more detail their progress towards the targets of the standards.

Task Name	**Task Type** ***Grouping Strategy***	**Content Addressed**
Permutations and Combinations	Learning Task *Partner/Individual*	Use the Fundamental Counting Principle to develop the permutations formula**;** use the permutations formula to develop the combinations formula**;** **i**dentify situations as appropriate for use of permutation or combination to calculate probabilities; use permutations and combinations in conjunction with other probability methods to calculate probabilities and solve problems.
Testing	Learning Task Partner/*Individual*	Calculate binomial probabilities and look at binomial distributions; graph probability distributions; calculate the mean probability distributions, expected values; calculate theoretical and empirical probabilities of probability distributions.
Please Be Discrete	Learning Task *Partner/Individual*	Define a random variable for a quantity of interest; graph the corresponding probability distributions; calculate the mean probability distributions, expected values; calculate theoretical and empirical probabilities of probability distributions.
Medical Testing	Formative Assessment Lesson (FAL) *Partner or Small Group*	Make sense of a real life situation and decide what math to apply to the problem; understand and calculate the conditional probability of an event A, given an event B, and interpret the answer in terms of a model.; represent events as a subset of a sample space using tables, tree diagrams, and Venn diagrams; interpret the results and

		communicate their reasoning clearly.
Georgia Lottery	Learning Task *Partner/Individual*	Determine the probability of winning in a game of chance; find the expected payoff for a game of chance; use expected values to compare benefits of playing a game of chance.
Mega Millions	Practice Task *Partner/Individual*	Identify situations as appropriate for use of a permutation or combination to calculate probabilities; calculate and interpret the expected value of a random variable; understand a probability distribution for a random variable representing payoff values in a game of chance; make decisions based on expected values.
Modeling Conditional Probabilities 2	Formative Assessment Lesson (FAL) *Partner or Small Group*	Representing events as a subset of a sample space using tables and tree diagrams; understanding when conditional probabilities are equal for particular and general situations.
Design a Lottery Game	Culminating Task	Identify situations as appropriate for use of a permutation or combination to calculate probabilities; calculate and interpret the expected value of a random variable; understand a probability distribution for a random variable representing payoff values in a game of chance; make decisions based on expected values

Permutations and Combinations Learning Task

Math Goals

- Use the Fundamental Counting Principle to develop the permutations formula
- Use the permutations formula to develop the combinations formula
- Identify situations as appropriate for use of permutation or combination to calculate probabilities
- Use permutations and combinations in conjunction with other probability methods to calculate probabilities and solve problems

Common Core State Standard

MCC9-12.S.CP. 9 (+) Use permutations and combinations to compute probabilities of compound events and solve problems.

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Model with mathematics
4. Attend to precision

Introduction

In this task students develop an understanding of combinatorial reasoning, using various types of diagrams and the fundamental counting principle to find numbers of outcomes. Students will also calculate the number of possible outcomes for a situation, recognizing and accounting for when items may occur more than once or when order is important.

Materials

- Calculators

Part 1: Fundamental Counting Principle

a) A deli has a lunch special which consists of a sandwich, soup, and a dessert for $4.99. They offer the following choices:

Sandwich – chicken salad, turkey, ham, or roast beef

Soup – tomato, chicken noodle, or broccoli cheddar

Dessert – cookie or pie

Use a diagram to determine the number of different lunch combinations. Then, use the Fundamental Counting Principle to determine the number of different lunch combinations.

Diagram:

Answers may vary: students may use table, list or tree diagram

Fundamental Counting Principle:

4*3*2 = 24

b) Karl has 5 shirts, 3 pairs of pants, and 2 sweaters in his closet. How many different outfits that consist of a shirt, pair of pants, and sweater can he make?

5 * 3 * 2 = 30

c) If you roll a dice, then toss a coin, how many different outcomes could you get?

6 * 2 = 12

d) A license plate in Canada consists of:

LETTER, LETTER, LETTER, NUMBER, NUMBER, NUMBER

How many different license plates can be created?

26 * 26 * 26 * 10 * 10 *10 = 17,576,000

e) A padlock has a 4-digit combination using the digits 0 – 9. How many different padlock combinations are there if repetition of the numbers is allowed?

10 * 10 * 10 * 10 = 10,000

Part 2: Permutations and Combinations

Permutations

Definition: A **permutation** is an ordered arrangement of *n* objects (people, numbers, letters, etc.) The **order** of the objects matters – **a different order creates a different outcome.**

a) There are 8 people running a race. How many different outcomes for the race are there?

Solution: There are 8 different people who can finish first. Once someone finishes first, there are only 7 people left competing for second place, then six left competing for third, and so on. So, to calculate all the different outcomes for the race use the Fundamental Counting Principle:

8*7*6*5*4*3*2*1= 40,320

There are 40,320 different outcomes for the race.

The example above requires you to multiply a series of descending natural numbers: 8*7*6*5*4*3*2*1. This can be written as 8! and read as "8 factorial".

8! means 8*7*6*5*4*3*2*1.

5! = 5*4*3*2*1 = 120

9! = 9*8*7*6*5*4*3*2*1 = 362,880

It is generally accepted that 0! = 1.

What is 6!? ***6*5*4*3*2*1=720***

Now, what if you had to calculate 20!? Do you want to enter all of those numbers into your calculator? The factorial key on your calculator can be found by:

Teachers will need to direct students to where to find the factorial key on the calculator based on what type of calculator they are using.

OK, now that we know what factorial means, let's revisit the race problem from above and change it a little bit.

b) There are 8 people competing in a race. In how many different ways can first, second, and third place medals be awarded?

Solution: There are 8 people eligible for first place. Once the first place winner finishes, there are only 7 people left to take second place, and then six left to take third place. Therefore, the number of different ways to award the medals would be:

8*7*6= 336

If we want to use the factorial notation described above, we would start with 8! or 8*7*6*5*4*3*2*1. However, we know that we want to stop multiplying after 6 so we divide by 5! or 5*4*3*2*1.

$$\frac{8*7*6*\cancel{5}*\cancel{4}*\cancel{3}*\cancel{2}*\cancel{1}}{\cancel{5}*\cancel{4}*\cancel{3}*\cancel{2}*\cancel{1}}$$

Let's look at the formula:

Permutation Formula (no repetition allowed)
${}_nP_r = P(n,r) = \frac{n!}{(n-r)!}$ where n is the number of things you choose from and r is the actual number of things you choose.

In our race example, there are 8 people to choose from which would represent *n* and we are choosing 3 of them to win first, second, and third place which would represent *r:*

$$_8P_3 = P(8,3) = \frac{8!}{(8-3)!} = \frac{8!}{5!} = \frac{8*7*6*\cancel{5}*\cancel{4}*\cancel{3}*\cancel{2}*\cancel{1}}{\cancel{5}*\cancel{4}*\cancel{3}*\cancel{2}*\cancel{1}} = 336$$

c) Twelve skiers are competing in the final round of the Olympic freestyle skiing aerial competition. In how many ways can 3 of the skiers finish first, second, and third to win the gold, silver, and bronze medals? $P(12,3) = \frac{12!}{(12-3)!} = \frac{12!}{9!} = 1320$

Using your calculator: To compute a permutation using your calculator, do the following:

Teachers will need to direct students to where to find the permutation key on the calculator based on what type of calculator they are using.

d) A relay race team has 4 runners who run different parts of the race. There are 16 students on your track team. How many different ways can your coach select students to compete in the race?

$$P(16,4) = \frac{16!}{(16-4)!} = \frac{16!}{12!} = 43,680$$

e) The school yearbook has an editor-in-chief and an assistant editor-in-chief. The staff of the yearbook has 15 students. In how many different ways can students be chosen for these 2 positions?

$$P(15,2) = \frac{15!}{(15-2)!} = \frac{15!}{13!} = 210$$

It is important to note that when you use the formula, repetition is not allowed. In other words, you can't have the same person win first and second place.

Another Case to Consider

f) In how many different ways can the letters HTAM be arranged to create four-letter "words"?

Solution: This is an example of a permutation because the order of the letters would produce a different "word" or outcome. So, we use the permutation formula:

$$_4P_4 = P(4,4) = \frac{4!}{(4-4)!} = \frac{4!}{0!} = \frac{4*3*2*1}{1} = 24$$

But, what if some of the letters repeated?

If you have different groups present the number of ways in which the letters of: "BALL", "LULL", "PENNY", "DADDY", "KAYAK" and "EEEEK" can be arranged, students can figure out that you must divide by the number of ways the repeated letters can be arranged and you won't have to tell them this property.

g) In how many ways can the letters in CLASSES be rearranged to create 7 letter "words"? Since the letter S repeats 3 times, some of the permutations will be the same so we will have to eliminate them.

There are 7 letters to choose from and we are choosing 7 of them, so we would have the following:

$${}_7P_7 = P(7,7) = \frac{7!}{(7-7)!} = \frac{7!}{0!} = \frac{7*6*5*4*3*2*1}{1} = 5040 \text{ if all the letters were different.}$$

Hold on – this is not our answer yet. We have to divide out our duplicate letters. As we mentioned earlier, the letter S repeats 3 times so we divide our answer by 3!: Therefore the answer is $\frac{5040}{3!} = 840$***different ways.***

h) How many ways can the letters in MISSISSIPPI be arranged to create 11-letter "words"?

If all the letters were all different, we'd have $P(11,11) = \frac{11!}{(11-11)!} = \frac{11!}{0!} = 39,916,800$ ***"words", but since there are 4 "S"s, 4 "I"s and 2 "P"s, we divide to get:*** $\frac{39{,}916{,}800}{4!4!2!} = 34,650$

Combinations

Definition: A **combination** is an arrangement of objects in which **order does NOT matter**.

Let's consider the following. You have three people – 1, 2, and 3. Here are the possibilities:

Order Does Matter (Permutation)	Order Does Not Matter (Combination)
1 2 3 1 3 2 2 1 3 2 3 1 3 1 2 3 2 1	1 2 3

The permutations have 6 times as many possibilities as the combinations.

Let's look at the formula we just learned and use it to calculate the number of permutations of the numbers 1, 2, and 3:

$${}_3P_3 = P(3,3) = \frac{3!}{(3-3)!} = \frac{3!}{0!} = \frac{3*2*1}{1} = 6 \text{ (as shown in the chart)}$$

So, to get the number of combinations we have to divide the number of permutations by 6 or 3!. Basically all we are doing is taking the permutation formula and reducing it by $r!$ to eliminate the duplicates. If you were to start with 4 numbers, there would be 24 or 4! times more permutations than combinations so you would start with the permutation formula and then divide by 4!.

This leads us to the combination formula:

Combination Formula

${}_nC_r = C(n,r) = \binom{n}{r} = \dfrac{n!}{r!(n-r)!}$ where n is the number of things you choose from and r is the actual number of things you choose. (no repetition allowed)

In our example above:

$$_3C_3 = C(43,3) = \frac{3!}{3!\,(3-3)!} = \frac{3*2*1}{3!*0!} = \frac{6}{3*2*1*1} = \frac{6}{6} = 1$$

a) A pizza shop offers twelve different toppings. How many different three-topping pizzas can be formed with the twelve toppings?

$$\boldsymbol{C(12,3) = \frac{12!}{3!\,(12-3)!} = \frac{12!}{3!\,9!} = 220}$$

Using your calculator: To compute a combination using your calculator, do the following:

Teachers will need to direct students to where to find the combination key on the calculator based on what type of calculator they are using.

b) Your English teacher has asked you to select 3 novels from a list of 10 to read as an independent project. In how many ways can you choose which books to read?

$$\boldsymbol{C(10,3) = \frac{10!}{3!\,(10-3)!} = \frac{10!}{3!\,7!} = 120}$$

A restaurant serves omelets that can be ordered with any of the ingredients shown:

Omelets $4
(plus $0.50 for each ingredient)

Vegetarian	**Meat**
green pepper	ham
red pepper	bacon
onion	sausage
mushroom	steak
tomato	
cheese	

c) Suppose you want *exactly* 2 vegetarian ingredients and 1 meat ingredient in your omelet. How many different types of omelets can you order?

C(6, 2)·C(4, 1) = 60

d) Suppose you can afford *at most* 3 ingredients in your omelet. How many different types of omelets can you order?

C(10, 0) +C(10, 1) +C(10, 2) + C(10, 3) = 176

Additional Practice

Directions: Simplify each expression to a single number or fraction.

1. $\dfrac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{5\cdot4\cdot3\cdot2\cdot1} =$
6

2. $\dfrac{8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{6\cdot5\cdot4\cdot3\cdot2\cdot1} =$
56

3. $3! =$
6

4. $6! =$
720

5. $\dfrac{4!}{3!} =$
4

6. $\dfrac{6!}{4!} =$
30

7. $\dfrac{101!}{99!} =$
10,100

8. ${}_5P_2 =$
20

9. ${}_5P_5 =$
120

10. $P(7,3) =$
210

11. ${}_6C_2 =$
15

12. $\dbinom{8}{3}$
56

Directions: Determine whether each is an example of a permutation or a combination.

13. The number of ways you can choose a group of 3 puppies to adopt from the animal shelter when there are 20 different puppies to choose from.

    Combination

14. The number of ways you could award 1^{st}, 2^{nd}, and 3^{rd} place medals for the science fair

    Permutation

15. The number of seven-digit phone numbers that can be made using the digits 0 – 9

    Permutation

16. The number of ways a committee of 3 could be chosen from a group of 20

    Combination

17. The number of ways a president, vice-president, and treasurer could be chosen from a group of 20

    Permutation

Directions: Solve each problem.

18. Little Caesars is offering a special where you can buy a large pizza with one cheese, one vegetable, and one meat for $7.99. There are 3 kinds of cheese, 9 vegetables, and 5 meats to choose from. How many different variations of the pizza special are possible? ***135***

19. If there are 11 people on a baseball team, determine how many different ways a pitcher and a catcher could be chosen. ***110***

20. There are eight seniors on the football team that are being considered as team captains. If there will be 3 team captains, how many different ways can 3 of these seniors be chosen as captains? ***56***

21. Nine people in your class want to be on a 5-person bowling team to represent the class. How many different teams can be chosen? ***126***

22. There are 5 people on a bowling team. How many different ways are there to arrange the order the people bowl in? ***120***

23. There are 5 people on a bowling team. How many ways can you choose your bowling team captain and team manager? ***20***

24. Determine how many ways a president, vice president, and treasurer can be chosen from a math club that has 7 members. ***210***

25. California license plates are: number, letter, letter, letter, number, number, number. For example: 3YNR975. How many possible license plate combinations are there in California? ***175,760,000***

26. There are 13 people on a softball team. How many ways are there to choose 10 players to take the field? ***286***

27. There are 13 people on a softball team. How many ways are there to assign them to play the 10 different positions on the field? ***1037836800***

28. A standard deck of cards has 52 playing cards. How many different 5-card hands are possible? ***2,598,960***

29. You are eating dinner at a restaurant. The restaurant offers 6 appetizers, 12 main dishes, 6 side orders, and 8 desserts. If you order one of each of these, how many different dinners can you order? ***3456***

30. A pizza parlor has a special on a three-topping pizza. How many different special pizzas can be ordered if the parlor has 8 toppings to choose from? ***56***

31. Find the number of possible committees of 4 people that could be chosen from a class of 30 students. ***27,405***

32. How many different 3-digit numbers can you make using the numbers 1, 2, 3, 4, and 5? Assume numbers can be repeated. ***125***

33. How many different seven-digit telephone numbers can be formed if the first digit cannot be zero or one? ***8,000,000***

34. How many different 5-digit zip codes are there if any of the digits 0 – 9 can be used?
    100,000

35. How many ways can you arrange the letters JORDAN to create 6-letter "words"?
    720

36. How many ways can you arrange the letters ILLINOIS to create 8-letter "words"?
    3360

37. A committee is to be formed with 5 girls and 5 boys. There are 8 girls to choose from and 12 boys. How many different committees can be formed? ***44,352***

38. You are buying a new car. There are 7 different colors to choose from and 10 different types of optional equipment you can buy. You can choose only 1 color for your car and can afford only 2 of the options. How many combinations are there for your car? ***315***

39. An amusement park has 20 different rides. You want to ride *at least* 15 of them. How many different combinations of rides can you go on? ***21700***

Reference:

Jordan School District. "The Fundamental Counting Principle, Permutations, and Combinations" (2012). departments.jordandistrict.org/.../**Lessons**%20to%20Upload/Algebra%202%20**Permutations**%20and%20**Combinations**.doc

Permutations and Combinations Learning Task

Name ____ Date______________________

Common Core State Standard

MCC9-12.S.CP.9 (+) Use permutations and combinations to compute probabilities of compound events and solve problems.

Standards for Mathematical Practice
1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Model with mathematics
4. Attend to precision

Part 1: Fundamental Counting Principle

a) A deli has a lunch special which consists of a sandwich, soup, and a dessert for $4.99. They offer the following choices:

Sandwich – chicken salad, turkey, ham, or roast beef

Soup – tomato, chicken noodle, or broccoli cheddar

Dessert – cookie or pie

Use a diagram to determine the number of different lunch combinations. Then, use the Fundamental Counting Principle to determine the number of different lunch combinations.

Diagram:

Fundamental Counting Principle:

b) Karl has 5 shirts, 3 pairs of pants, and 2 sweaters in his closet. How many different outfits that consist of a shirt, pair of pants, and sweater can he make?

c) If you roll a dice, then toss a coin, how many different outcomes could you get?

d) A license plate in Canada consists of:

LETTER, LETTER, LETTER, NUMBER, NUMBER, NUMBER

How many different license plates can be created?

e) A padlock has a 4-digit combination using the digits 0 – 9. How many different padlock combinations are there if repetition of the numbers is allowed?

Part 2: Permutations and Combinations

Permutations

<u>Definition</u>: A **permutation** is an <u>ordered</u> arrangement of *n* objects (people, numbers, letters, etc.) The **order** of the objects matters **– a different order creates a different outcome.**

a) There are 8 people running a race. How many different outcomes for the race are there?

The example above requires you to multiply a series of descending natural numbers: 8*7*6*5*4*3*2*1. This can be written as 8! and read as "8 factorial".

8! means 8*7*6*5*4*3*2*1.

5! = 5*4*3*2*1 = 120

9! = 9*8*7*6*5*4*3*2*1 = 362,880

It is generally accepted that 0! = 1.

What is 6!?

Now, what if you had to calculate 20!? Do you want to enter all of those numbers into your calculator? The factorial key on your calculator can be found by:

OK, now that we know what factorial means, let's revisit the race problem from above and change it a little bit.

b) There are 8 people competing in a race. In how many different ways can first, second, and third place medals be awarded?

If we want to use the factorial notation described above, we would start with 8! or 8*7*6*5*4*3*2*1. However, we know that we want to stop multiplying after 6 so we divide by 5! or 5*4*3*2*1.

$$\frac{8*7*6*\cancel{5}*\cancel{4}*\cancel{3}*\cancel{2}*\cancel{1}}{\cancel{5}*\cancel{4}*\cancel{3}*\cancel{2}*\cancel{1}}$$

Let's look at the formula:

Permutation Formula (no repetition allowed)

${}_nP_r = P(n,r) = \dfrac{n!}{(n-r)!}$ where *n* is the number of things you choose from and *r* is the actual number of things you choose.

In our race example, there are 8 people to choose from which would represent *n* and we are choosing 3 of them to win first, second, and third place which would represent *r:*

$${}_8P_3 = P(8,3) = \frac{8!}{(8-3)!} = \frac{8!}{5!} = \frac{8*7*6*\cancel{5}*\cancel{4}*\cancel{3}*\cancel{2}*\cancel{1}}{\cancel{5}*\cancel{4}*\cancel{3}*\cancel{2}*\cancel{1}} = 336$$

c) Twelve skiers are competing in the final round of the Olympic freestyle skiing aerial competition. In how many ways can 3 of the skiers finish first, second, and third to win the gold, silver, and bronze medals?

Using your calculator: To compute a permutation using your calculator, do the following:

d) A relay race team has 4 runners who run different parts of the race. There are 16 students on your track team. How many different ways can your coach select students to compete in the race?

e) The school yearbook has an editor-in-chief and an assistant editor-in-chief. The staff of the yearbook has 15 students. In how many different ways can students be chosen for these 2 positions?

It is important to note that when you use the formula, repetition is not allowed. In other words, you can't have the same person win first and second place.

Another Case to Consider

f) How many different ways can the letters HTAM be arranged to create four-letter "words"?

But, what if some of the letters repeated?

g) In how many ways can the letters in CLASSES be rearranged to create 7 letter "words"? Since the letter S repeats 3 times, some of the permutations will be the same so we will have to eliminate them.

h) How many ways can the letters in MISSISSIPPI be arranged to create 11-letter "words"?

Combinations

<u>Definition</u>: A **combination** is an arrangement of objects in which **order does NOT matter**.

Let's consider the following. You have three people – 1, 2, and 3. Here are the possibilities:

Order Does Matter (Permutation)	Order Does Not Matter (Combination)
1 2 3 1 3 2 2 1 3 2 3 1 3 1 2 3 2 1	1 2 3

The permutations have 6 times as many possibilities as the combinations.

Let's look at the formula we just learned and use it to calculate the number of permutations of the numbers 1, 2, and 3:

${}_3\mathrm{P}_3 = \mathrm{P}(3,3) = \frac{3!}{(3-3)!} = \frac{3!}{0!} = \frac{3*2*1}{1} = 6$ (as shown in the chart)

So, to get the number of combinations we have to divide the number of permutations by 6 or 3!. Basically all we are doing is taking the permutation formula and reducing it by r! to eliminate the duplicates. If you were to start with 4 numbers, there would be 24 or 4! times more permutations than combinations so you would start with the permutation formula and then divide by 4!.

This leads us to the combination formula:

Combination Formula

${}_nC_r = C(n,r) = \binom{n}{r} = \frac{n!}{r!(n-r)!}$ where n is the number of things you choose from and r is the actual number of things you choose. (no repetition allowed)

In our example above:

$${}_3\mathrm{C}_3 = \mathrm{C}(43,3) = \frac{3!}{3!\,(3-3)!} = \frac{3*2*1}{3!*0!} = \frac{6}{3*2*1*1} = \frac{6}{6} = 1$$

a) A pizza shop offers twelve different toppings. How many different three-topping pizzas can be formed with the twelve toppings?

Using your calculator: To compute a combination using your calculator, do the following:

b) Your English teacher has asked you to select 3 novels from a list of 10 to read as an independent project. In how many ways can you choose which books to read?

A restaurant serves omelets that can be ordered with any of the ingredients shown:

Omelets $4
(plus $0.50 for each ingredient)

Vegetarian	**Meat**
green pepper	ham
red pepper	bacon
onion	sausage
mushroom	steak
tomato	
cheese	

c) Suppose you want *exactly* 2 vegetarian ingredients and 1 meat ingredient in your omelet. How many different types of omelets can you order?

d) Suppose you can afford *at most* 3 ingredients in your omelet. How many different types of omelets can you order?

Additional Practice

Directions: Simplify each expression to a single number or fraction.

1. $\frac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{5\cdot4\cdot3\cdot2\cdot1}=$

2. $\frac{8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{6\cdot5\cdot4\cdot3\cdot2\cdot1}=$

3. $3! =$

4. $6! =$

5. $\frac{4!}{3!}=$

6. $\frac{6!}{4!}=$

7. $\frac{101!}{99!}=$

8. ${}_5P_2 =$

9. ${}_5P_5 =$

10. $P(7,3) =$

11. ${}_6C_2 =$

12. $\binom{8}{3}$

Directions: Determine whether each is an example of a permutation or a combination.

13. The number of ways you can choose a group of 3 puppies to adopt from the animal shelter when there are 20 different puppies to choose from.

14. The number of ways you could award 1^{st}, 2^{nd}, and 3^{rd} place medals for the science fair

15. The number of seven-digit phone numbers that can be made using the digits 0 – 9

16. The number of ways a committee of 3 could be chosen from a group of 20

17. The number of ways a president, vice-president, and treasurer could be chosen from a group of 20

Directions: Solve each problem.

18. Little Caesars is offering a special where you can buy a large pizza with one cheese, one vegetable, and one meat for $7.99. There are 3 kinds of cheese, 9 vegetables, and 5 meats to choose from. How many different variations of the pizza special are possible?

19. If there are 11 people on a baseball team, determine how many different ways a pitcher and a catcher could be chosen.

20. There are eight seniors on the football team that are being considered as team captains. If there will be 3 team captains, how many different ways can 3 of these seniors be chosen as captains?

21. Nine people in your class want to be on a 5-person bowling team to represent the class. How many different teams can be chosen?

22. There are 5 people on a bowling team. How many different ways are there to arrange the order the people bowl in?

23. There are 5 people on a bowling team. How many ways can you choose your bowling team captain and team manager?

24. Determine how many ways a president, vice president, and treasurer can be chosen from a math club that has 7 members.

25. California license plates are: number, letter, letter, letter, number, number, number. For example: 3YNR975. How many possible license plate combinations are there in California?

26. There are 13 people on a softball team. How many ways are there to choose 10 players to take the field?

27. There are 13 people on a softball team. How many ways are there to assign them to play the 10 different positions on the field?

28. A standard deck of cards has 52 playing cards. How many different 5-card hands are possible?

29. You are eating dinner at a restaurant. The restaurant offers 6 appetizers, 12 main dishes, 6 side orders, and 8 desserts. If you order one of each of these, how many different dinners can you order?

30. A pizza parlor has a special on a three-topping pizza. How many different special pizzas can be ordered if the parlor has 8 toppings to choose from?

31. Find the number of possible committees of 4 people that could be chosen from a class of 30 students.

32. How many different 3-digit numbers can you make using the numbers 1, 2, 3, 4, and 5? Assume numbers can be repeated.

33. How many different seven-digit telephone numbers can be formed if the first digit cannot be zero or one?

34. How many different 5-digit zip codes are there if any of the digits 0 – 9 can be used?

35. How many ways can you arrange the letters JORDAN to create 6-letter "words"?

36. How many ways can you arrange the letters ILLINOIS to create 8-letter "words"?

37. A committee is to be formed with 5 girls and 5 boys. There are 8 girls to choose from and 12 boys. How many different committees can be formed?

38. You are buying a new car. There are 7 different colors to choose from and 10 different types of optional equipment you can buy. You can choose only 1 color for your car and can afford only 2 of the options. How many combinations are there for your car?

39. An amusement park has 20 different rides. You want to ride *at least* 15 of them. How many different combinations of rides can you go on?

Reference:
Jordan School District. "The Fundamental Counting Principle, Permutations, and Combinations" (2012).
departments.jordandistrict.org/.../**Lessons**%20to%20Upload/Algebra%202%20**Permutations**%20and%20**Combinations**.doc

<u>Testing Learning Task</u>

Math Goals

- Calculate binomial probabilities and look at binomial distributions
- Graph probability distributions
- Calculate the mean of probability distributions, expected values
- Calculate theoretical and empirical probabilities of probability distributions

Common Core State Standard

MCC9-12.S.CP.9 (+) Use permutations and combinations to compute probabilities of compound events and solve problems.

MCC9-12.S.MD.1(+) Define a random variable for a quantity of interest by assigning a numerical value to each event in a sample space; graph the corresponding probability distribution using the same graphical displays as for data distributions.

MCC9-12.S.MD.2(+) Calculate the expected value of a random variable; interpret it as the mean of a probability distribution.

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Model with mathematics
4. Attend to precision

Introduction

In this task students develop an understanding of binomial probability distributions. Students will also compute conditional probabilities where they will distinguish between dependent and independent events. The will graph probability distributions and calculate expected values.

Materials

- Graphing calculators
- Graph paper

Today you are going to determine how well you would do on a true/false test if you guessed at every answer. Take out a sheet of paper. Type randint(1,2) on your calculator. If you get a 1, write “true.” If you get a 2, write “false.”

Do this 20 times.

Before the teacher calls out the answers, how many do you expect to get correct? Why?

If you have a small class, have students do this activity twice so you will have a decent sample size.

Some should say 10. Ask them why? Since the probability of "true" is ½, they should say that they should get ½ of 20 correct.

Teacher should randomly generate answers for the true-false test and call them out.

Grade your test. How many did you actually get correct? Did you do better or worse than you expected?

1) Make a dot plot of the **class distribution** of the total number correct on your graph paper.

Calculate the mean and median of your distribution. Which measure of center should be used based on the shape of your dot plot?

Discuss with the students which measure of center should be used based on the shape of the data. The mean should be a good measure because the dot plot is expected to be symmetrical (bell shaped) about 10.

3) Based on the class distribution, what percentage of students passed?

Solutions:
Answers will depend upon class data. Students should use the dot plot to answer the following questions.

4) Calculate the probabilities based on the dot plot:

a) What is the probability that a student got less than 5 correct?

b) What is the probability that a student got exactly 10 correct?

c) What is the probability that a student got between 9 and 11 correct (inclusive)?

d) What is the probability that a student got 10 or more correct?

e) What is the probability that a student got 15 or more correct?

f) What is the probability that a student passed the test?

g) Is it more likely to pass or fail a true/false test if you are randomly guessing?

h) Is it unusual to pass a test if you are randomly guessing?

It's important to note the assumptions for a binomial distribution. They are as follows:
****Each trial has two outcomes….success "p" or failure "1-p."***
****There is a fixed number of trials "n."***
****The probability of success does not change from trial to trial.***
****The trials are independent. The results for one trial do not depend on the results from another trial.***

For a situation to be considered as having a ***binomial distribution***, the following conditions must be satisfied:

- Each observation/trial has one of **two outcomes**. These two outcomes are referred to as "success" or "failure".
- There are a **fixed number of observations/trials**. The number of observations/trials is referred to as *n*.
- The observations/trials must be **independent.**
- The **probability of success**, referred to as *p*, **is the same** for each observation/trial.

5) Can this true-false test be considered a binomial setting? Why or why not?

Yes, it meets all of the conditions above

Binomial Probability

When X has the binomial distribution with n observations and probability p of success on each observation, the possible values of X are 0, 1, 2, . . ., n. The probability of X successes in this setting is computed with the formula:

$$P(X) = \binom{n}{r}(p)^X (1-p)^{n-X} \quad \text{or} \quad P(X) = \left(\frac{n!}{(n-X)!X!}\right)(p)^X (1-p)^{n-X}$$

Students will need you to model so calculate the probability that the student got exactly 5 correct on the test.

Explanation: If the student got 5 correct, then 15 were incorrect. P(correct) = ½ and the P(incorrect) = ½. So, if the first 5 were correct, and the last 15 were incorrect, then the student would have CCCCCIIIIIIIIIIIIIII graded on the quiz. The probability of getting that in that order is $\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^{15}$***. But, the problem did not specify that the first 5 were correct and the last 15 were incorrect. So that leads us to the question, "How many ways can we rearrange the 5 correct and 15 incorrect problems?" I usually remind students of how we rearranged the letters to the word "Mississippi." Then, they usually figure out that the answer is*** $\frac{20!}{5!15!}$ ***since there are five C's that repeat and fifteen I's that repeat. So our final answer to the question, "What is the probability that the student got exactly 5 correct is*** $\frac{20!}{5!15!}\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^{15}$***. At this point, you can relate*** $\frac{20!}{5!15!}$ ***to*** ${}_{20}C_5$ ***on the calculator.***

****Additional Problem: Calculate the probability that the student got fewer than 3 correct on the test.***

Explanation: First you need to make sure that they understand that fewer than 3 means 0, 1, or 2 correct. Since these outcomes are mutually exclusive (if you get 0 correct, then you will not get 1 correct, etc.), then you can add the probabilities. So, you would calculate each of the following probabilities:

P(0 correct) = ${}_{20}C_0\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^{20}$ ***or*** $\left(\frac{1}{2}\right)^{20}$

P(1 correct) = ${}_{20}C_1\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^{19}$

P(2 correct) = ${}_{20}C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^{18}$

Once your students have mastered this reasoning and formula, then you may want to show them the calculator buttons "binompdf(n, p, x) and binomcdf(n, p, x)." On the TI-84, you can find "binompdf" under "2nd" , "VARS", and then scroll down. If you want to calculate the probability of getting exactly 5 correct on the test, then you would type binompdf(20,1/2,5). If you want to calculate the probability of fewer than 3 are correct, then you would type binomcdf(20, ½, 2). Unlike the binompdf key, the binomcdf key adds up the probabilities starting at 0 and ending at x. The "c" stands for cumulative in binomcdf.

Now the students should have enough information to calculate the theoretical probabilities of the following:

6) Still considering that T-F test, calculate the following probabilities using the Binomial Distribution:

a) What is the probability that a student got less than 5 correct?

(P(0) + P(1) + P(2) + P(3) +P(4))*.5^{20} or binomcdf(20,.5,4)=.0059

b) What is the probability that a student got exactly 10 correct?

P(10)/(2^{20}) or binompdf(20, .5, 10)= .1762

c) What is the probability that a student got between 9 and 11 correct (inclusive)?

(P(9) + P(10) + P(11))*.5^{20} or binomcdf(20, .5, 11)- binomcdf(20, .5, 8) =.4966

d) What is the probability that a student got 10 or more correct?

(P(10) + P(11) + P(12)+ P(13) + P(15) + P(16) + P(17) + P(18) +P(19) + P(20))/(2^{10}) or

1-binomcdf(20, .5, 9) =.5881

e) What is the probability that a student got 15 or more correct?

(P(15) + P(16) + P(17) + P(18) +P(19) + P(20))*.5^{20} or 1-binomcdf(20, .5, 14)=.0207

f) What is the probability that a student passed the test?

(P(14) + P(15) + P(16) + P(17) + P(18) +P(19) + P(20))/2^{10} or 1-binomcdf(20, .5, 13)=.0577

Discuss: How do the theoretical probabilities compare to the experimental probabilities?

g) Is it more likely to pass or fail a true/false test if you are randomly guessing?

h) Is it unusual to pass a test if you are randomly guessing?

Testing Learning Task (Part 2)

Suppose there is a 5 question multiple choice test. Each question has 4 answers (A, B, C, or D).

1) Can this multiple choice test be considered a binomial setting? Why or why not?

Yes, it meets all of the conditions

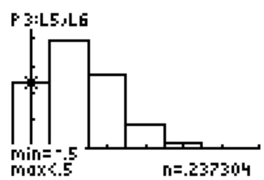
2) If you are strictly guessing, calculate the following probabilities:

a) P(0 correct) = ${}_5C_0\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^5$ ***= binompdf(5, .25,0)=.2373***

b) P(1 correct) = ***.3955***
c) P(2 correct) = ***.2637***
d) P(3 correct) = ***.0879***
e) P(4 correct) = ***.0146***
f) P(5 correct) = ***.0010***

3) Draw a histogram of the probability distribution for the number of correct answers on graph paper. Label the *x-axis* as the **number of correct answers**. The *y-axis* should be the **probability of x**.

The histogram should be skewed right as shown in this example.



4) Based on the distribution, how many problems do you expect to get correct?

Comment: the mean of a binomial distribution is np or 5(.25) = 1.25. From the graph, students should say that they expect one correct answer if guessing.

5) Based on the distribution, how likely is it that you would pass if you were strictly guessing? (*Calculate the probability of getting 4 or 5 correct.)*

P(4 or 5 correct) = .0146+.0010= .0156 So, it is not likely that you would pass by guessing alone.

6) What is the probability that you will get less than 3 correct? ***.8965***

7) What is the probability that you will get at least 3 correct? ***0.1035***

Now let's look at tests, such as the SAT, when you are penalized for guessing incorrectly. Suppose you have a multiple choice test with five answers (A, B, C, D, or E) per problem. The probability your guess is **correct = 1/5** and the probability that your guess is **incorrect = 4/5**.

Suppose the test that you are taking will award you one point for each question correct, but penalize you by ¼ of a point for each question you answer incorrectly. Test scores will be rounded to the nearest 10 percent.

8) If you strictly guess and get exactly 4 correct and 6 incorrect, what would be your score?

4 – (.25)6 = 2.5, so your rounded score would be 3/10 or 30%

9) If you take a 10 question test and know that 8 questions are correct, should you guess the answers for the other two questions?

8 – 0 .25(2) = 7.5 which rounds to 8. Yes, you have nothing to lose.

10) If you take a 10 question test and know that 6 questions are correct, should you guess the answers for the other 4 questions?

6 – 0.25(4) = 5. You could make a 50%; however, if you guessed one of the 4 correctly (which is expected), then you would still make a 60%.

11) Given that you answered all 10 questions and you knew that 6 were correct, answer the following questions:

a) If you can eliminate one of the answer choices for each of the 4 questions for which you are guessing, what would your expected score be?

The probability of guessing correctly for the 4 unknown problems is 1/4 since one answer is eliminated from 5 possible answers. Therefore, from the 4 questions which you did not know the answer, you expect 4(1/4) = 1 to be correct and 4(3/4)=3 to be incorrect. So, your expected score should be 6 + 1 – (.25)(3) = 6.25 which rounds to 6 (or 60%).

b) If you can eliminate two of the answer choices for each of the 4 questions for which you are guessing, what would your expected score be?

6+ 4(1/3) – 4(2/3)(.25) = 6.666 which rounds to 7 (or 70 %)

c) If you can eliminate three of the answer choices for each of the 4 questions for which you are guessing, what would your expected score be?

6 + 4(1/2)-4(1/2)(.25)=7.5 which rounds to 8 (or 80%)

Testing Learning Task (Part 3)

Earlier, we found the probability that the student passed a multiple choice test just by random guessing. However, we know that students usually have a little more knowledge than that, even when they do not study, and consequently do not guess for all problems.

Suppose that a student can retain about 30% of the information from class without doing any type of homework or studying. If the student is given a 15 question multiple choice test where each question has 4 answer choices (A, B, C, or D), then answer the following questions:

1. What is the probability that the student gives the correct answer on the test? What would be her percentage score on a 15 question test?

2. Given she provides the correct answer on the test, what is the probability that she strictly guessed?

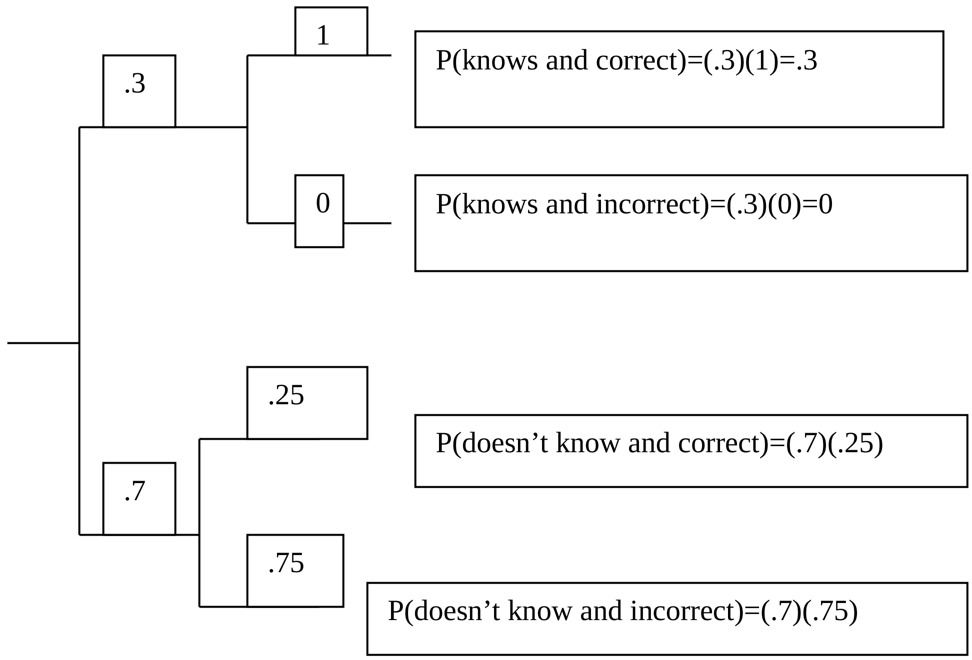
You may need to use the formula for conditional probability to do this problem. The formula is as follows:

P(A given B) = P(A and B)/P(B). It's symbolically written as such $P(A|B)=\frac{P(A\cap B)}{P(B)}$

where $\cap$ ***stands for the intersection of sets A and B.***

Note: By solving the above formula, $P(A\cap B)=P(A|B)\cdot P(B)$.

With conditional probability problems, I have found tree diagrams very helpful. To solve the problem, I would make a tree diagram. The first branches would be P(knows answer) and P(does not know answer). The next branches would be P(correct given she knows answer), P(incorrect given she knows the answer), P(correct given she does not know the answer), and P(incorrect given she does not know the answer). The product of the two branches would be P(A and B)…P(knows answer and correct), P(knows answer and incorrect), P(doesn't know answer and correct), P(doesn't know answer and incorrect). The tree diagram should look like the one below.



Using the tree diagram, and conditional probability formula, we can answer the following questions:

What is the probability that the student gives the correct answer on the test? The student can give the correct answer when she knows it or when she doesn't and guesses correctly. So the answer is (.3)(1) + (.7)(.25) = .475

What would be her percentage score on a 15 question test? 47.5%

Given she provides the correct answer on the test, what is the probability that she strictly guessed?
This is a conditional probability problem. You would use the following formula to solve the problem: $P(guess \mid correct) = \dfrac{P(guess\ and\ correct)}{P(correct)}$

From the tree diagram $P(guess\ and\ correct)$ ***= (.7)(.25)= .175. Note: she would not guess if she knew the answer. We already calculated*** $P(correct)$ ***=.475.***

Therefore, $P(guess \mid correct) = \dfrac{.175}{.475} \approx .368.$

Testing Learning Task

Name ____ Date________________

Common Core State Standard

MCC9-12.S.CP.9 (+) Use permutations and combinations to compute probabilities of compound events and solve problems.

MCC9-12.S.MD.1(+) Define a random variable for a quantity of interest by assigning a numerical value t each event in a sample space; graph the corresponding probability distribution using the same graphical displays as for data distributions.

MCC9-12.S.MD.2(+) Calculate the expected value of a random variable; interpret it as the mean of a probability distribution.

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Model with mathematics
4. Attend to precision

Today you are going to determine how well you would do on a true/false test if you guessed at every answer. Take out a sheet of paper. Type randint(1,2) on your calculator. If you get a 1, write "true." If you get a 2, write "false."

Do this 20 times.

Before the teacher calls out the answers, how many do you expect to get correct? Why?

Grade your test. How many did you actually get correct? Did you do better or worse than you expected?

1) Make a dot plot of the **class distribution** of the total number correct on your graph paper.

2) Calculate the mean and median of your distribution. Which measure of center should be used based on the shape of your dot plot?

3) Based on the class distribution, what percentage of students passed?

4) Calculate the probabilities based on the dot plot:

a) What is the probability that a student got less than 5 correct?

b) What is the probability that a student got exactly 10 correct?

c) What is the probability that a student got between 9 and 11 correct (inclusive)?

d) What is the probability that a student got 10 or more correct?

e) What is the probability that a student got 15 or more correct?

f) What is the probability that a student passed the test?

g) Is it more likely to pass or fail a true/false test if you are randomly guessing?

h) Is it unusual to pass a test if you are randomly guessing?

For a situation to be considered as having a binomial distribution, the following conditions must be satisfied:

- Each observation/trial has one of **two outcomes**. These two outcomes are referred to as "success" or "failure".
- There are a **fixed number of observations/trials**. The number of observations/trials is referred to as *n*.
- The observations/trials must be **independent.**
- The **probability of success**, referred to as *p*, **is the same** for each observation/trial.

5) Can this true-false test be considered a binomial setting? Why or why not?

Binomial Probability

When X has the binomial distribution with n observations and probability p of success on each observation, the possible values of X are 0, 1, 2, . . ., n. The probability of X successes in this setting is computed with the formula:

$$P(X) = \binom{n}{r}(p)^X (1-p)^{n-X} \quad \text{or} \quad P(X) = \left(\frac{n!}{(n-X)!X!}\right)(p)^X (1-p)^{n-X}$$

6) Still considering that T-F test, calculate the following probabilities using the Binomial Distribution:

a) What is the probability that a student got less than 5 correct?

b) What is the probability that a student got exactly 10 correct?

c) What is the probability that a student got between 9 and 11 correct (inclusive)?

d) What is the probability that a student got 10 or more correct?

e) What is the probability that a student got 15 or more correct?

f) What is the probability that a student passed the test?

g) Is it more likely to pass or fail a true/false test if you are randomly guessing?

h) Is it unusual to pass a test if you are randomly guessing?

Testing Learning Task (Part 2)

Suppose there is a 5 question multiple choice test. Each question has 4 answers (A, B, C, or D).

1) Can this multiple choice test be considered a binomial setting? Why or why not? 2) If you are strictly guessing, calculate the following probabilities:

a) P(0 correct) = ${}_5C_0\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^5$

b) P(1 correct) =

c) P(2 correct) =

d) P(3 correct) =

e) P(4 correct) =

f) P(5 correct) =

3) Draw a histogram of the probability distribution for the number of correct answers on graph paper. Label the *x-axis* as the **number of correct answers**. The *y-axis* should be the **probability of x**.

4) Based on the distribution, how many problems do you expect to get correct?

5) Based on the distribution, how likely is it that you would pass if you were strictly guessing? (*Calculate the probability of getting 4 or 5 correct.)*

6) What is the probability that you will get less than 3 correct?

7) What is the probability that you will get at least 3 correct?

Now let's look at tests, such as the SAT, when you are penalized for guessing incorrectly. Suppose you have a multiple choice test with five answers (A, B, C, D, or E) per problem. The probability your guess is **correct = 1/5** and the probability that your guess is **incorrect = 4/5**.

Suppose the test that you are taking will award you one point for each question correct, but penalize you by ¼ of a point for each question you answer incorrectly. Test scores will be rounded to the nearest 10 percent.

8) If you strictly guess and get exactly 4 correct and 6 incorrect, what would be your score?

9) If you take a 10 question test and know that 8 questions are correct, should you guess the answers for the other two questions?

10) If you take a 10 question test and know that 6 questions are correct, should you guess the answers for the other 4 questions?

11) Given that you answered all 10 questions and you knew that 6 were correct, answer the following questions:

a) If you can eliminate one of the answer choices for each of the 4 questions for which you are guessing, what would your expected score be?

b) If you can eliminate two of the answer choices for each of the 4 questions for which you are guessing, what would your expected score be?

c) If you can eliminate three of the answer choices for each of the 4 questions for which you are guessing, what would your expected score be?

Testing Learning Task (Part 3)

Earlier, we found the probability that the student passed a multiple choice test just by random guessing. However, we know that students usually have a little more knowledge than that, even when they do not study, and consequently do not guess for all problems.

Suppose that a student can retain about 30% of the information from class without doing any type of homework or studying. If the student is given a 15 question multiple choice test where each question has 4 answer choices (A, B, C, or D), then answer the following questions:

1. What is the probability that the student gives the correct answer on the test? What would be her percentage score on a 15 question test?

2. Given she provides the correct answer on the test, what is the probability that she strictly guessed?

Please Be Discrete Learning Task:

Math Goals

- Define a random variable for a quantity of interest
- Graph the corresponding probability distributions
- Calculate the mean probability distributions, expected values
- Calculate theoretical and empirical probabilities of probability distributions

Common Core State Standard

MCC9-12.S.MD.1(+) Define a random variable for a quantity of interest by assigning a numerical value to each event in a sample space; graph the corresponding probability distribution using the same graphical displays as for data distributions.

MCC9-12.S.MD.2(+) Calculate the expected value of a random variable; interpret it as the mean of a probability distribution.

MCC9-12.S.MD.3(+) Develop a probability distribution for a random variable defined for a sample space in which theoretical probabilities can be calculated; fin the expected value

MCC9-12.S.MD.4(+) Develop a probability distribution for a random variable defined for a sample space in which probabilities are assigned empirically; find the expected value

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Model with mathematics
4. Attend to precision

Introduction

In this task students will define a random variable and graph its corresponding probability distributions. Students will also compute theoretical and empirical probabilities and calculate expected values.

Materials

- Graphing calculators
- Graph paper
- Coin (one per group)

1) In October 1966, the United States Congress passed the Endangered Species Preservation Act. Subsequent legislation and international conventions are part of a worldwide effort to save endangered and threatened species. The U.S. Fish and Wildlife Service works to protect and recover these species and maintains data on endangered and threatened species. Between 1989 and 2008, 34 species were removed from the list of endangered or threatened species. Reasons for removal from the list include recovery, inaccurate original data and extinction. In this twenty year period, only three species have been removed due to extinction. (*source: U.S. Fish and Wildlife Service*)

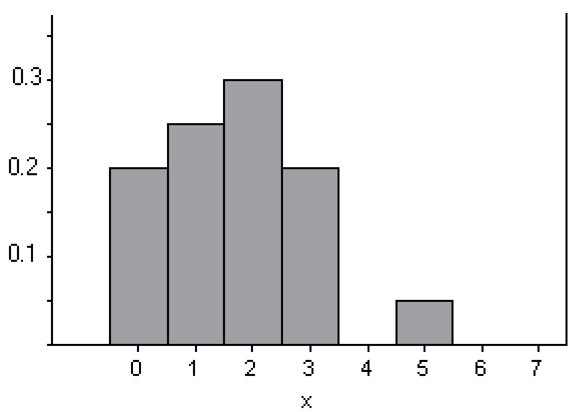
# of species removed from list	0	1	2	3	4	5
# of years in which that # of species was removed from the list.	4	5	6	4	0	1

That means that in 6 different years, 2 species were removed from the list. Note that the sum of the second row is 20 – the number of years for this study.

a) Construct the probability distribution of this data.

# of species removed from list	***0***	***1***	***2***	***3***	***4***	***5***
probability	$\frac{4}{20}=0.2$	$\frac{5}{20}=0.25$	$\frac{6}{20}=0.3$	$\frac{4}{20}=0.2$	$\frac{0}{20}=0$	$\frac{1}{20}=0.05$

b) Construct the probability histogram for this data.



c) Determine the mean of the probability distribution.

$$\mu = 0(0.2) + 1(0.25) + 2(0.3) + 3(0.2) + 4(0) + 5(0.05) = 1.7$$

d) Based on the data, how many species can the U.S. Fish and Wildlife Service expect to remove from the list per year? ***1.7***

2) A hurricane is a tropical cyclone with wind speeds that have reached at least 74 mph. Hurricanes are classified using the Saffir-Simpson scale, ranging from Category 1 to Category 5. Category 3 to 5 hurricanes are considered "major hurricanes." The table below lists the number of major hurricanes in the Atlantic Basin by year. *(source: National Climatic Data Center)*

Year	# of major hurricanes
1984	1
1985	3
1986	0
1987	1
1988	3
1989	2
1990	1
1991	2
1992	1

Year	# of major hurricanes
1993	1
1994	0
1995	5
1996	6
1997	1
1998	3
1999	5
2000	3

Year	# of major hurricanes
2001	4
2002	2
2003	3
2004	6
2005	7
2006	2
2007	2
2008	5

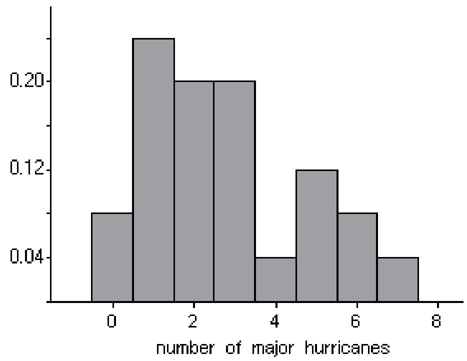
a) Construct a frequency table for this data.

# of major hurricanes, X	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*
frequency	*2*	*6*	*5*	*5*	*1*	*3*	*2*	*1*

b) Construct the probability distribution for this data.

# of major hurricanes, X	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*
P(X)	*0.08*	*0.24*	*0.2*	*0.2*	*0.04*	*0.12*	*0.08*	*0.04*

c) Construct a probability histogram for this data.



c) Determine the mean of the probability distribution.

$$\mu = 0(0.08) + 1(0.24) + 3(0.2) + 4(0.04) + 5(0.12) + 6(0.08) + 7(0.04) = 2.77$$

d) Based on the data, how many hurricanes can be expected in the Atlantic Basin in a year? ***2.77***

3) The first Olympic Winter Games were held in 1924. Between 1924 and 2006, twenty Winter Olympics have been held. The Winter Games were suspended in 1940 and 1944 due to World War II. The United States has participated in all twenty Olympic Winter Games. *(source: International Olympic Committee, http://www.olympics.org)*

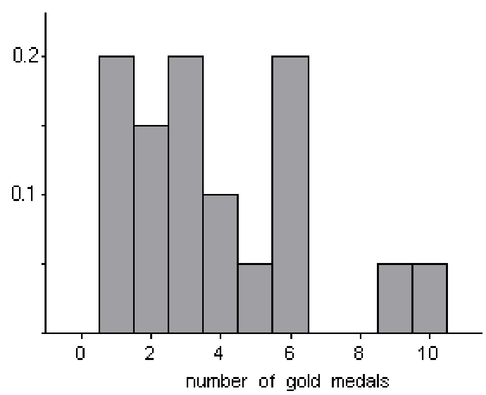
Year	Gold Medals	Silver Medals	Bronze Medals
2006	9	9	7
2002	10	13	11
1998	6	3	4
1994	6	5	2
1992	5	4	2
1988	2	1	3
1984	4	4	0
1980	6	4	2
1976	3	3	4
1972	3	2	3
1968	1	5	4

1964	1	2	3
1960	3	4	3
1956	2	3	2
1952	4	6	1
1948	3	4	2
1936	1	0	3
1932	6	4	2
1928	2	2	2
1924	1	2	1

a) Construct the discrete probability distribution for the number of gold medals.

# of Gold Metals, *x*	1	2	3	4	5	6	7	8	9	10
P(*x*)	*0.2*	*0.15*	*0.2*	*0.1*	*0.05*	*0.2*	*0*	*0*	*0.05*	*0.05*

b) Construct the probability histogram of this distribution.



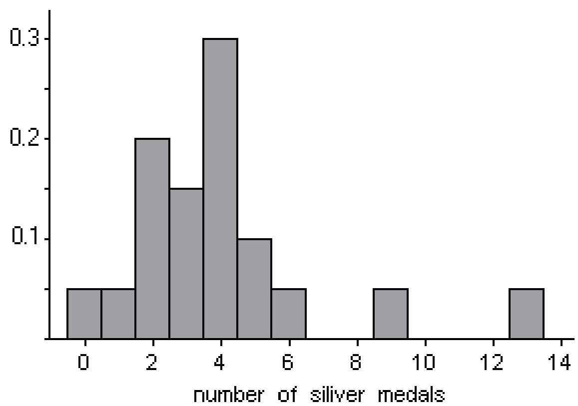
c) How many gold medals can the U.S. expect to win based on the given data?

$$\mu_{gold} = 1(0.2) + 2(0.15) + 3(0.2) + 4(0.1) + 5(0.05)$$
$$+ 6(0.2) + 9(0.05) + 10(0.05) = 3.9$$

d) Construct the probability distribution for the number of silver medals won.

# of Silver Metals, x	***0***	***1***	***2***	***3***	***4***	***5***	***6***	***9***	***13***
P(x)	***0.05***	***0.05***	***0.2***	***0.15***	***0.3***	***0.1***	***0.05***	***0.05***	***0.05***

e) Construct the probability histogram for this distribution.



f) How many silver medals can the U.S. expect to win per Olympics based on the given data?

$$\mu_{silver} = 0(0.05) + 1(0.05) + 2(0.2) + 3(0.15) + 4(0.3) + 5(0.1)$$
$$+ 6(0.05) + 9(0.05) + 13(0.05) = 4$$

4. Let's explore the probabilities associated with tossing a coin, focusing on the number of heads in four tosses of one coin. The theoretical probability distribution for this situation is below.

Number of heads, X	0	1	2	3	4
P(X)	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{16}$

a) Will experimental results match this theoretical distribution? Collect your own data by completing 32 trials. Compare the distribution of your data to the theoretical distribution.

Answers will vary. Make sure that students are completing 32 trials of 4 tosses and do not count each individual toss as a trial.

b) Combine the data from your entire class into one probability distribution.

Answers will vary. The easiest way to accomplish combining the data from all groups in the class is to create a large frequency table on your board and then have the class, or each group, create the probability distribution for the class.

c) How does this distribution compare to your group's distribution?

Answers will vary

d) How does this distribution compare to the theoretical distribution?

Answers will vary

Please Be Discrete Learning Task

Name ____ Date____________________

Common Core State Standard

MCC9-12.S.MD.1(+) Define a random variable for a quantity of interest by assigning a numerical value to each event in a sample space; graph the corresponding probability distribution using the same graphical displays as for data distributions.

MCC9-12.S.MD.2(+) Calculate the expected value of a random variable; interpret it as the mean of a probability distribution.

MCC9-12.S.MD.3(+) Develop a probability distribution for a random variable defined for a sample space in which theoretical probabilities can be calculated; fin the expected value

MCC9-12.S.MD.4(+) Develop a probability distribution for a random variable defined for a sample space in which probabilities are assigned empirically; find the expected value

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Model with mathematics
4. Attend to precision

1) In October 1966, the United States Congress passed the Endangered Species Preservation Act. Subsequent legislation and international conventions are part of a worldwide effort to save endangered and threatened species. The U.S. Fish and Wildlife Service works to protect and recover these species and maintains data on endangered and threatened species. Between 1989 and 2008, 34 species were removed from the list of endangered or threatened species. Reasons for removal from the list include recovery, inaccurate original data and extinction. In this twenty year period, only three species have been removed due to extinction. (*source: U.S. Fish and Wildlife Service*)

# of species removed from list	0	1	2	3	4	5
# of years in which that # of species was removed from the list.	4	5	6	4	0	1

a) Construct the probability distribution of this data.

b) Construct the probability histogram for this data on your graph paper.

c) Determine the mean of the probability distribution.

d) Based on the data, how many species can the U.S. Fish and Wildlife Service expect to remove from the list per year?

2) A hurricane is a tropical cyclone with wind speeds that have reached at least 74 mph. Hurricanes are classified using the Saffir-Simpson scale, ranging from Category 1 to Category 5. Category 3 to 5 hurricanes are considered "major hurricanes." The table below lists the number of major hurricanes in the Atlantic Basin by year. *(source: National Climatic Data Center)*

Year	# of major hurricanes
1984	1
1985	3
1986	0
1987	1
1988	3
1989	2
1990	1
1991	2
1992	1

Year	# of major hurricanes
1993	1
1994	0
1995	5
1996	6
1997	1
1998	3
1999	5
2000	3

Year	# of major hurricanes
2001	4
2002	2
2003	3
2004	6
2005	7
2006	2
2007	2
2008	5

a) Construct a frequency table for this data.

b) Construct the probability distribution for this data.

c) Construct a probability histogram for this data on your graph paper.

d) Determine the mean of the probability distribution.

e) Based on the data, how many hurricanes can be expected in the Atlantic Basin in a year?

3) The first Olympic Winter Games were held in 1924. Between 1924 and 2006, twenty Winter Olympics have been held. The Winter Games were suspended in 1940 and 1944 due to World War II. The United States has participated in all twenty Olympic Winter Games. *(source: International Olympic Committee, http://www.olympics.org)*

Year	Gold Medals	Silver Medals	Bronze Medals
2006	9	9	7
2002	10	13	11
1998	6	3	4
1994	6	5	2
1992	5	4	2
1988	2	1	3
1984	4	4	0
1980	6	4	2
1976	3	3	4
1972	3	2	3
1968	1	5	4
1964	1	2	3
1960	3	4	3
1956	2	3	2
1952	4	6	1
1948	3	4	2
1936	1	0	3
1932	6	4	2
1928	2	2	2
1924	1	2	1

a) Construct the discrete probability distribution for the number of gold medals.

# of Gold Metals, *x*										
P(*x*)										

b) Construct the probability histogram of this distribution on your graph paper.

c) How many gold medals can the U.S. expect to win per Olympics based on the given data?

d) Construct the probability distribution for the number of silver medals won.

# of Silver Metals, *x*									
P(*x*)									

e) Construct the probability histogram for this distribution on your graph paper.

f) How many silver medals can the U.S. expect to win per Olympics based on the given data?

4. Let's explore the probabilities associated with tossing a coin, focusing on the number of heads in four tosses of one coin. The theoretical probability distribution for this situation is below.

Number of heads, X	0	1	2	3	4
P(X)	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{16}$

a) Will experimental results match this theoretical distribution? Collect your own data by completing 32 trials. Compare the distribution of your data to the theoretical distribution.

b) Combine the data from your entire class into one probability distribution.

c) How does this distribution compare to your group's distribution?

d) How does this distribution compare to the theoretical distribution?

Formative Assessment Lesson: Medical Testing

Source: Formative Assessment Lesson Materials from Mathematics Assessment Project
http://www.map.mathshell.org/materials/download.php?fileid=1275

ESSENTIAL QUESTIONS:

- How do I calculate the probability of Event A, given the probability of Event B?
- How do I use the Multiplication Rule to calculate the probability of an event?
- How do I interpret a probability into the context of a model?

TASK COMMENTS:
Tasks and lessons from the Mathematics Assessment Project are specifically designed to help teachers effectively formatively assess their students. The way the tasks and lessons are designed gives the teacher a clear understanding of what the students are able to do and not do. Within the lesson, teachers will find suggestions and question prompts that will help guide students towards understanding. For more information, access the MAP website:
http://www.map.mathshell.org/materials/index.php

The task, *Medical Testing*, is a Formative Assessment Lesson (FAL) that can be found at the website: http://www.map.mathshell.org/materials/lessons.php?taskid=438&subpage=problem

The FAL document provides a clear lesson design, from the opening of the lesson to the closing of the lesson.

The PDF version of the task can be found at the link below:
http://www.map.mathshell.org/materials/download.php?fileid=1275

STANDARDS ADDRESSED IN THIS TASK:

MCC9-12.CP.8 (+) Apply the general Multiplication Rule in a uniform probability model, P(A and B)=[P(A)] x [P(B | A)] = [P(B)] x [P(A | B)], and interpret the answer in terms of the model.

MCC9-12.S.MD.7(+) Analyze decisions and strategies using probability concepts (e.g., product testing, medical testing, pulling a hockey goalie at the end of a game).

Standards for Mathematical Practice
This lesson uses all of the practices with emphasis on:
1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics

Georgia Lottery Learning Task:

Please feel free to modify the context of this task to meet your needs.

Math Goals

- Determine the probability of winning in a game of chance
- Find the expected payoff for a game of chance
- Use expected values to compare benefits of playing a game of chance

Common Core State Standard

MCC9-12.S.MD.5(+) Weigh the possible outcomes of a decision by assigning probabilities to payoff values and finding expected values.

MCC9-12.S.MD.5a(+) Find the expected payoff for a game of chance

MCC9-12.S.MD.5b(+) Evaluate and compare strategies on the basis of expected values

MCC9-12.S.MD.6(+) Use probabilities to make fair decisions (e.g., drawing by lots, using a random number generator).

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Model with mathematics
4. Attend to precision

Introduction

In this task students will determine the chance of winning a 3 and 4-number lottery game. Students will then be asked to determine the chances of winning different payouts. Students will have to make decisions based on the expected payouts.

Materials

- Graphing calculators

Probabilities in Cash 4 Lottery Game

The Georgia state lottery offers several variations of lottery games. In one of the simplest games, Cash 4, a player picks any four-digit number and places a bet ranging from $0.50 to $5.00. The player wins if his or her number is selected in the daily drawing. The payouts for lottery games are determined by one's odds of winning. The odds in favor of an event are typically expressed as a ratio of the likelihood that an event will happen to the likelihood that an event will not happen.

There are several different betting strategies for Cash 4 listed below.

Calculate the probability of winning the Cash 4 lottery for each strategy.

In order to calculate the odds of winning, one must know the total number of different 4-digit numbers. There are 10 ways to pick the first number, 10 ways to pick the second number, 10 ways to pick the third number, and 10 ways to pick the fourth number. The number of different 4-digit numbers is 10 x 10 x 10 x 10 = 10,000.

1. ***Play It Straight:*** Player plays 4 different digits. Player wins only with exact match.

1/10,000: Since the order of the numbers matters, there is exactly 1 chance to win.
The number may only be in the order: abcd.

2. ***Play It Boxed:*** Player plays 3 of the same digit and 1 other digit. Player wins if the number is drawn in any order.

4/10,000: Since the number may be drawn in any order, there are 4 chances to win. The numbers may be in any of the following orders: abbb, babb, bbab, or bbba.
If students discovered how to arrange words with repeated letters earlier, this is analogous to "LULL".

3. ***Box 2 Pairs:*** Player picks 2 pairs of numbers. Player wins if the number is drawn in any order.

6/10,000: Since the number may be drawn in any order, there are 6 chances to win. The numbers may be in any of the following orders: aabb, bbaa, abab, baba, abba, or baab.
If students discovered how to arrange words with repeated letters earlier, this is analogous to "TOOT".

4. ***Box 1 Pair + 2 Digits****:* Player picks 1 pair of number and 2 other digits. Player wins if the number is drawn in any order.

12/10,000: Since the number may be drawn in any order, there are 12 chances to win. The numbers may be in any of the following orders: aabc, aacb, bcaa, cbaa, abca, acba, baac, caab, abac, acab, baca, caba.
If students discovered how to arrange words with repeated letters earlier, this is analogous to "BALL".

5. ***Box 4 Different Digits****:* Player plays 4 different digits. Player wins if the number is drawn in any order.

24/10,000: Since the number may be drawn in any order, there are 24 chances to win. The order of the 4 numbers does not matter, so the total number of permutations is 4 x 3 x 2 x 1 = 24.

Expected Value

Expected value is the most likely value of a random variable. In the case of an investment decision, it is the average value of all possible payoffs. In order to find the expected value of a variable, you must first multiply each possible payoff by its probability of occurring, followed by adding all of the products together.

For example, let's say that you roll a 6-sided die. If you roll a 3, then you win $5.00. If you don't roll a 3, then you have to pay $1.00. What is the expected value of the game?

You should recognize that since the probability of rolling a 3 is 1/6, the probability of not rolling a 3 is 5/6. Therefore, expected value = P(3) • ($5) + P(not 3) • (–$ 1) = (1/6) • ($5) + (5/6) • (– $1) = 5/6 – 5/6 = 0. If the expected value is 0, we say the game is fair.

Expected Value of the Lottery

In a certain state lottery, a player chooses three digits, which must be in a specific order. Note that the numbers may lead with the digit 0, so numbers such as 056 or 009 are acceptable numbers. Digits may also be repeated. In each lottery drawing, a three-digit sequence is selected. Any player with a lottery pick matching all three digits, in the correct order, receives a payout of $500.

1. Determine the probability of winning this lottery game.

Let's say that we chose the number 2, 9, and 2, in that order. On the first draw, the probability of drawing a 2 out of 10 possible numbers is 1/10. Because digits may be repeated, the 2 is replaced. On the second draw, the probability of now drawing a 9 is still 1/10. The 9 is then replaced. On the third draw, the probability of drawing a 2 is still 1/10. We can now multiply the probabilities together to determine the overall probability of winning: 1/10 x 1/10 x 1/10 = 1/1000

2. Calculate the expected value of winning if its costs $3 to play one game and explain what it means.

The expected value of winning is:
P(winning) • $497 + P(not winning) • $-3 =(1/1000) • (497) + (999/1000) • (-3) = 0.497 – 2.997 = -2.5
In other words, you are expected to lose $2.50 on average when playing the game. Note that it is impossible to actually lose $2.50 when playing this game, but expected value is the theoretical average of what should happen over time.

3. Determine a fair cost for an individual to play this lottery game.

The expected value for a fair game should be 0. The probability of winning the game is still 1/1000.
We can set the equation for expected value equal to 0 to solve for a fair cost of the game:

$(1/1000) \cdot (500 - x) + (999/1000) \cdot (-x) = 0$

This simplifies to:
$0.5 - 0.001x - 0.999x = 0$***, which may be further simplified to*** $0.5 - x = 0$

The above equation may be solved to produce a value of 0.50. This means that the game must cost 50 cents to play in order for it to be a fair game! Teachers can take this opportunity to talk about the "fairness" of the lottery.

Reference:
Adapted from: Langholtz, Jessica. "Probability of Winning the Lottery".
http://www.tip.sas.upenn.edu/curriculum/units/2011/02/11.02.07.pdf

Georgia Lottery Learning Task

Name ____ Date________________

Common Core State Standard

MCC9-12.S.MD.5(+) Weigh the possible outcomes of a decision by assigning probabilities to payoff values and finding expected values.

MCC9-12.S.MD.5a(+) Find the expected payoff for a game of chance

MCC9-12.S.MD.5b(+) Evaluate and compare strategies on the basis of expected values

MCC9-12.S.MD.6(+) Use probabilities to make fair decisions (e.g., drawing by lots, using a random number generator).

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Model with mathematics
4. Attend to precision

Probabilities in Cash 4 Lottery Game

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There are several different betting strategies for Cash 4 listed below.

Calculate the probability of winning the Cash 4 lottery for each strategy.

1. ***Play It Straight:*** Player plays 4 different digits. Player wins only with exact match.

2. ***Play It Boxed:*** Player plays 3 of the same digit and 1 other digit. Player wins if the number is drawn in any order.

3. ***Box 2 Pairs:*** Player picks 2 pairs of numbers. Player wins if the number is drawn in any order.

4. ***Box 1 Pair + 2 Digits****:* Player picks 1 pair of number and 2 other digits. Player wins if the number is drawn in any order.

5. ***Box 4 Different Digits****:* Player plays 4 different digits. Player wins if the number is drawn in any order.

Expected Value

Expected value is the most likely value of a random variable. In the case of an investment decision, it is the average value of all possible payoffs. In order to find the expected value of a variable, you must first multiply each possible payoff by its probability of occurring, followed by adding all of the products together.

For example, let's say that you roll a 6-sided die. If you roll a 3, then you win $5.00. If you don't roll a 3, then you have to pay $1.00. What is the expected value of the game?

You should recognize that since the probability of rolling a 3 is 1/6, the probability of not rolling a 3 is 5/6. Therefore, expected value = P(3) • (5) + P(not 3) • (– 1) = (1/6) • (5) + (5/6) • (– 1) = 5/6 – 5/6 = 0. If the expected value is 0, we say the game is fair.

Expected Value of the Lottery

In a certain state lottery, a player chooses three digits, which must be in a specific order. Note that the numbers may lead with the digit 0, so numbers such as 056 or 009 are acceptable numbers. Digits may also be repeated. In each lottery drawing, a three-digit sequence is selected. Any player with a lottery pick matching all three digits, in the correct order, receives a payout of $500.

1. Determine the probability of winning this lottery game.

2. Calculate the expected value of winning if its costs $3 to play one game and explain what it means.

3. Determine a fair cost for an individual to play this lottery game.

Formative Assessment Lesson: Modeling Conditional Probabilities 2

Source: Formative Assessment Lesson Materials from Mathematics Assessment Project
http://www.map.mathshell.org/materials/download.php?fileid=1200

ESSENTIAL QUESTIONS:

- How do I calculate conditional probabilities?
- How do I use probabilities to determine if a game of chance is fair?
- How do I interpret a probability into the context of a model?

TASK COMMENTS:
Tasks and lessons from the Mathematics Assessment Project are specifically designed to help teachers effectively formatively assess their students. The way the tasks and lessons are designed gives the teacher a clear understanding of what the students are able to do and not do. Within the lesson, teachers will find suggestions and question prompts that will help guide students towards understanding. For more information, access the MAP website:
http://www.map.mathshell.org/materials/index.php

The task, *Modeling Conditional Probabilities 2*, is a Formative Assessment Lesson (FAL) that can be found at the website: http://www.map.mathshell.org/materials/lessons.php?taskid=405#task405

The FAL document provides a clear lesson design, from the opening of the lesson to the closing of the lesson.

The PDF version of the task can be found at the link below:
http://www.map.mathshell.org/materials/download.php?fileid=1200

STANDARDS ADDRESSED IN THIS TASK:

MCC9-12.CP.8 (+) Apply the general Multiplication Rule in a uniform probability model, P(A and B)=[P(A)] x [P(B | A)] = [P(B)] x [P(A | B)], and interpret the answer in terms of the model.

MCC9-12.S.MD.5b(+) Evaluate and compare strategies on the basis of expected values

MCC9-12.S.MD.6(+) Use probabilities to make fair decisions (e.g., drawing by lots, using a random number generator).

Standards for Mathematical Practice
This lesson uses all of the practices with emphasis on:
1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics

Mega Millions Practice Task

Please feel free to modify the context of this task to meet your needs.

Math Goals:

- Identify situations as appropriate for use of a permutation or combination to calculate probabilities
- Calculate and interpret the expected value of a random variable
- Understand a probability distribution for a random variable representing payoff values in a game of chance
- Make decisions based on expected values

Common Core State Standards

MCC9-12.CP.8 (+) Apply the general Multiplication Rule in a uniform probability model, P(A and B)=[P(A)] x [P(B | A)] = [P(B)] x [P(A | B)], and interpret the answer in terms of the model.

MCC9-12.S.CP.9 (+) Use permutations and combinations to compute probabilities of compound events and solve problems.

MCC9-12.S.MD.2(+) Calculate the expected value of a random variable; interpret it as the mean of a probability distribution.

MCC9-12.S.MD.5(+) Weigh the possible outcomes of a decision by assigning probabilities to payoff values and finding expected values.

MCC9-12.S.MD.5a(+) Find the expected payoff for a game of chance

MCC9-12.S.MD.5b(+) Evaluate and compare strategies on the basis of expected values

MCC9-12.S.MD.6(+) Use probabilities to make fair decisions (e.g., drawing by lots, using a random number generator).

Standards for Mathematical Practice

1. **Make sense of problems and persevere in solving them**
2. **Reason abstractly and quantitatively**
3. **Construct viable arguments and critique the reasoning of others**
4. **Model with mathematics**
5. **Use appropriate tools strategically**
6. **Attend to precision**
7. **Look for and make use of structure**

8. Look for and express regularity in repeated reasoning

Materials:

- Calculator

Introduction

In this task, students will be asked to combine what they have learned about probability throughout this unit. Students, individually or with a group, will investigate a real-world application of probability by determining the probability of winning the lottery. Students will calculate pay-off amounts and decide whether playing the game is to their advantage.

Mega Millions is a national lottery game that is known for its large rollover jackpot payouts. Each player picks 6 total numbers: 5 different numbers from 1 to 56 (the "white numbers") and 1 number from 1 to 46 (the "Mega number"). A player wins the jackpot if he or she matches all six numbers selected in a drawing. A player can also win by having one or more numbers match.

1) Calculate the probability of winning the Mega Millions jackpot.

The numbers of ways to select 5 numbers from a pool of 56 numbers may be calculated using combinations: ${}_{56}C_5$ = 3,819,816.
The number of ways to select 1 number from a pool of 46 numbers is 46.
Thus, the total number of Mega Millions combinations is 3,819,816 x 46 = 175,711,536.

There is only one way that the first five numbers on your lottery ticket can match the five selected white numbers. There is also only one way for the sixth number on your lottery ticket to match the Mega Number. Therefore, there is one way to win the jackpot.

The probability of winning the jackpot is 1/175,711,536 = $5.6911 x 10^{-9}$

2) Calculate the probabilities of other winning combinations.

a) Match all 5 white numbers but not the Mega number (payout =$250,000)

There is only one way that the first five numbers on your lottery ticket can match the five selected white numbers (${}_5C_5$). There are 45 ways for your sixth number to match any of the 45 losing Mega numbers (${}_{45}C_1$). The number of ways to achieve this combination is 1 x 45 = 45, which leads to a probability of 45/175,711,536. This simplifies to about $2.5610 X 10^{-7}$ or "one chance in 3,904,701."

b) Match 4 out of 5 white numbers and the Mega number (payout =$10,000)

There are five ways that four of the first five numbers on your lottery ticket can match the five selected white numbers ($_5C_4$).There are 51 ways for your fifth white number to match any of the 51 losing white numbers ($_{51}C_1$). There is one way for your sixth number to match the winning Mega number ($_1C_1$). The number of ways to achieve this combination is 5 x 51 x 1 = 255, which leads to a probability of 255/175,711,536. This simplifies to about 1.4512 X 10^{-6} or "one chance in 689,065."

c) Match 4 out of 5 white numbers but not the Mega number (payout =$150)

There are five ways that four of the first five numbers on your lottery ticket can match the five selected white numbers ($_5C_4$). There are 51 ways for your fifth white number to match any of the 51 losing white numbers ($_{51}C_1$). There are 45 ways for your sixth number to match any of the 45 losing Mega numbers ($_{45}C_1$). The number of ways to achieve this combination is 5 x 51 x 45 = 11,475, which leads to a probability of 11,475/175,711,536. This simplifies to about 6.5306 X 10^{-5} or "one chance in 15,313."

d) Match 3 out of 5 white numbers and the Mega number (payout = $150):

There are ten ways that three of the first five numbers on your lottery ticket can match the five selected white numbers ($_5C_3$). There are 1,275 ways for two of your white numbers to match any of the 51 losing white numbers ($_{51}C_2$). There is one way for your sixth number to match the winning Mega number ($_1C_1$). The number of ways to achieve this combination is 10 x 1,275 x 1 = 12,750, which leads to a probability of 12,750/175,711,536. This simplifies to about 7.2562 X 10^{-5} or "one chance in 13,781."

e) Match 3 out of 5 white numbers but not the Mega number (payout =$7)

There are ten ways that three of the first five numbers on your lottery ticket can match the five selected white numbers ($_5C_3$). There are 1,275 ways for two of your white numbers to match any of the 51 losing white numbers ($_{51}C_2$). There are 45 ways for your sixth number to match any of the 45 losing Mega numbers ($_{45}C_1$). The number of ways to achieve this combination is 10 x 1,275 x 45 = 573,750, which leads to a probability of 573,750/175,711,536. This simplifies to about 0.00327 or "one chance in 306."

f) Match 2 out of 5 white numbers and the Mega number (payout = $10)

There are ten ways that two of the first five numbers on your lottery ticket can match the five selected white numbers ($_5C_2$). There are 20,825 ways for three of your white numbers to match any of the 51 losing white numbers ($_{51}C_3$). There is one way for your sixth number to match the winning Mega number ($_1C_1$). The number of ways to achieve this combination is 10 x 20,825 x 1 = 208,250, which leads to a probability of 208,250/175,711,536. This simplifies to about 0.001185 or "one chance in 844."

g) Match 1 out of 5 white numbers and the Mega number (payout = $3)

There are five ways that one of the first five numbers on your lottery ticket can match the five selected white numbers ($_5C_1$). There are 249,900 ways 12 for three of your white numbers to match any of the 51 losing white numbers ($_{51}C_4$). There is one way for your sixth number to match the winning Mega number ($_1C_1$). The number of ways to achieve this combination is 5 x 249,900 x 1 = 1,249,500, which leads to a probability of 1,249,500/175,711,536. This simplifies to about 0.00711 or "one chance in 141."

h) Match 0 out of 5 white numbers and the Mega number (payout = $2)

There is one way that none of the first five numbers on your lottery ticket can match the five selected white numbers ($_5C_0$). There are 2,349,060 ways for five of your white numbers to match any of the 51 losing white numbers ($_{51}C_5$). There is one way for your sixth number to match the winning Mega number ($_1C_1$). The number of ways to achieve this combination is 1 x 2,349,060 x 1 = 2,349,060, which leads to a probability of 2,349,060/175,711,536. This simplifies to about 0.01337 or "one chance in 75."

3) Calculate the expected value of the game to determine if it is worth purchasing a $1 lottery ticket for a chance to win the large jackpot prize. Use a jackpot value of $42 million for your calculation.

The expected value of winning is:

P(jackpot) • $42,000,000 + P(match 5 white, not Mega) • $250,000 + P(match 4 white and Mega) • $10,000 + P(match 4 white, not Mega) • $150 + P(match 3 white and Mega) • $150 + P(match 3 white, not Mega) • $7 + P(match 2 white and Mega) • $10 + P(match 1 white and Mega) • $3 + P(match 0 white and Mega) • $2 P(not winning) • $ –1 = –0.55

In other words, you are expected to lose on average 55¢ in this game.

Reference:
Adapted from: Langholtz, Jessica. "Probability of Winning the Lottery".
http://www.tip.sas.upenn.edu/curriculum/units/2011/02/11.02.07.pd

Mega Millions Practice Task

Name ____ Date_________________________

Common Core State Standards

MCC9-12.CP.8 (+) Apply the general Multiplication Rule in a uniform probability model, P(A and B)=[P(A)] x [P(B | A)] = [P(B)] x [P(A | B)], and interpret the answer in terms of the model.

MCC9-12.S.CP.9 (+) Use permutations and combinations to compute probabilities of compound events and solve problems.

MCC9-12.S.MD.2(+) Calculate the expected value of a random variable; interpret it as the mean of a probability distribution.

MCC9-12.S.MD.5(+) Weigh the possible outcomes of a decision by assigning probabilities to payoff values and finding expected values.

MCC9-12.S.MD.5a(+) Find the expected payoff for a game of chance

MCC9-12.S.MD.5b(+) Evaluate and compare strategies on the basis of expected values

MCC9-12.S.MD.6(+) Use probabilities to make fair decisions (e.g., drawing by lots, using a random number generator).

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them
2. Reason abstractly and quantitatively
3. Construct viable arguments and critique the reasoning of others
4. Model with mathematics
5. Use appropriate tools strategically
6. Attend to precision
7. Look for and make use of structure
8. Look for and express regularity in repeated reasoning

Mega Millions is a national lottery game that is known for its large rollover jackpot payouts. Each player picks 6 total numbers: 5 different numbers from 1 to 56 (the "white numbers") and 1 number from 1 to 46 (the "Mega number"). A player wins the jackpot if he or she matches all six numbers selected in a drawing. A player can also win by having one or more numbers match.

1) Calculate the probability of winning the Mega Millions jackpot.

2) Calculate the probabilities of other winning combinations:

a) Match all 5 white numbers but not the Mega number (payout =$250,000)

b) Match 4 out of 5 white numbers and the Mega number (payout =$10,000)

c) Match 4 out of 5 white numbers but not the Mega number (payout =$150)

d) Match 3 out of 5 white numbers and the Mega number (payout = $150):

e) Match 3 out of 5 white numbers but not the Mega number (payout =$7)

f) Match 2 out of 5 white numbers and the Mega number (payout = $10)

g) Match 1 out of 5 white numbers and the Mega number (payout = $3)

h) Match 0 out of 5 white numbers and the Mega number (payout = $2)

3) Calculate the expected value of the game to determine if it is worth purchasing a $1 lottery ticket for a chance to win the large jackpot prize. Use a jackpot value of $42 million for your calculation.

Culminating Task: Design a Lottery Game

Please feel free to modify the context of this task to meet your needs.

Math Goals:

- Identify situations as appropriate for use of a permutation or combination to calculate probabilities
- How the normal distribution uses area to make estimates of frequencies which can be expressed as probabilities and recognizing that only some data are well described by a normal distribution.
- Calculate and interpret the expected value of a random variable
- Understand a probability distribution for a random variable representing payoff values in a game of chance
- Make decisions based on expected values

Common Core State Standards

MCC9-12.CP.8 (+) Apply the general Multiplication Rule in a uniform probability model, P(A and B)=[P(A)] x [P(B | A)] = [P(B)] x [P(A | B)], and interpret the answer in terms of the model.

MCC9-12.S.CP.9 (+) Use permutations and combinations to compute probabilities of compound events and solve problems.

MCC9-12.S.MD.2(+) Calculate the expected value of a random variable; interpret it as the mean of a probability distribution.

MCC9-12.S.MD.5(+) Weigh the possible outcomes of a decision by assigning probabilities to payoff values and finding expected values.

MCC9-12.S.MD.5a(+) Find the expected payoff for a game of chance

MCC9-12.S.MD.5b(+) Evaluate and compare strategies on the basis of expected values

MCC9-12.S.MD.6(+) Use probabilities to make fair decisions (e.g., drawing by lots, using a random number generator).

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6. Attend to precision
7. Look for and make use of structure
8. Look for and express regularity in repeated reasoning

Materials:

- Calculator

Introduction

In this task, students will be asked to combine what they have learned about probability throughout this unit. Students, individually or with a group, will investigate a real-world application of probability by designing a lottery games and determining the probability of winning the lottery. Students will determine pay-off amounts and determine whether the game is "fair".

Individuals who design lottery games must consider a variety of factors in order to ensure that players have a decent chance of winning and that the house will likely make a profit. As a result, it is important for lottery games to be carefully designed so that they are not too easy (where the house would not profit) or too hard (where there is little incentive to play).

1. Students should read the first page of a research paper published in the *Journal of the Operational Research Society*. The paper discusses various factors that must be considered in the design of lottery games, including the incentive of a rollover jackpot.

*The first page is available at: http://www.jstor.org/pss/822753 (a copy is included in the back of this task)

2. You will serve as an advisor to a state-sponsored lottery game. You will need to:

a) Design the rules of the "fair" game

b) Name the game and design the ticket

c) Decide cost to play and payoff amounts for winning

d) Calculate the probability of winning

e) Prove that the game is a "fair game"
Answers will vary. Students should be encouraged to be creative but be reasonable.

Reference:

Adapted from: Langholtz, Jessica. "Probability of Winning the Lottery".
http://www.tip.sas.upenn.edu/curriculum/units/2011/02/11.02.07.pdf

Culminating Task: Design a Lottery Game

Name ____ Date____________________

Common Core State Standards

MCC9-12.CP.8 (+) Apply the general Multiplication Rule in a uniform probability model, P(A and B)=[P(A)] x [P(B | A)] = [P(B)] x [P(A | B)], and interpret the answer in terms of the model.

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Individuals who design lottery games must consider a variety of factors in order to ensure that players have a decent chance of winning and that the house will likely make a profit. As a result, it is important for lottery games to be carefully designed so that they are not too easy (where the house would not profit) or too hard (where there is little incentive to play).

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a) Design the rules of the "fair" game

b) Name the game and design the ticket

c) Decide cost to play and payoff amounts for winning

d) Calculate the probability of winning

e) Prove that the game is a "fair game"

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www.palgrave-journals.com/jors

On the design of lottery games

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We describe a model of participation in lottery games designed to address the optimisation of tax revenue in state-sponsored lotteries. The model treats participants dynamically and examines a long-run equilibrium. A novel high frequency approximation is used to turn the problem into a static, state-contingent deterministic programming problem. We demonstrate that the solution of this problem has qualitatively plausible properties and then calibrate the model against the United Kingdom National Lottery (UKNL). The results suggest that the current design of the UKNL may not be maximising tax revenue.
Journal of the Operational Research Society (2003) **54**, 89–100. doi:10.1057/palgrave.jors.2601484

Keywords: lottery; dynamic optimisation; simulations

Introduction

State-sponsored lotteries are extensively used to raise revenue for good causes such as education and the arts. Worldwide the annual value of tickets bought in 1998 exceeded $51bn (see La Fleur and La Fleur[1] for references). States often contract out the running of such games to an operating company with a proportion of the cost of each ticket going to good causes or to general tax revenue and the contractor taking a further proportion as operating costs; most of the remainder is returned to participants as prizes. For example, in 1998 out of every pound spent on the UK National Lottery (UKNL), about 50 pence went to prizes, 28 pence to good causes, 13 pence to Customs and Excise as excise duties, 5 pence to the retailer as commission, 3 pence to cover the operating company (Camelot) operating costs, and Camelot kept 1 pence as profits. The tax rate and the level of participation will affect the total good-causes revenue. Participation depends on the tax rate and the design of the lottery, in particular, the probability of winning a jackpot prize. In order to study the effects of changing such parameters on the good-causes revenue it is necessary to model their effect on participation. In this paper we present such a model and use it to address the issue of whether the current tax rate and jackpot probability are the appropriate ones to maximize good-causes revenue in the UKNL.

A common feature of lotteries is that, if there are no winners in a given draw, the jackpot prize pool from that draw is added to the pool for the next draw: a rollover. The increased pool typically induces a higher level of participation. This suggests good-causes revenue may be increased by making rollovers more likely and this can be achieved by reducing the probability of winning a jackpot. However, doing so may also have a disincentive effect on participation in draws that do not feature a rollover and thus offset any increase in revenue. To capture such a trade-off effectively requires a dynamic model of individual participation. One possible approach would be to fit a 'black box' model (eg, regression-based) using standard statistical methods. However, the similarity and stability of lottery designs means that the available data offers little natural variation and poor prospects of a good fit, particularly if one controls for differences in country-specific propensities to gamble. Our approach is to work from first principles and build a model of individual participation based on standard inter-temporal models of individual behaviour under uncertainty. Lack of variation in the data forces us to adopt the simplest model parameterization. Nonetheless, this is sufficient to capture the principal features of observed behaviour. Since quantitative verification of the model is not possible, we check that the qualitative predictions of the model are plausible and consistent with casual observation. Our final step is to aggregate the behaviour of individual agents into a full model of demand for lottery tickets, which enables us to examine the effects of changing lottery design and tax rates.

There is little literature on the issues discussed here and (to our knowledge) none which bases the optimal design problem on a dynamic model of individual behaviour. Farrell *et al*[2] equate the effective price of a ticket in the UKNL to the ticket price less the expected value of the prizes and use the variation in the latter in rollover weeks to estimate the elasticity of demand. Farrell *et al*[3] analyse the same data using use time-series methods. Although both these papers find values for the short- and long-run elasti-

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