GPS Transition Frameworks

Mathematics I → GPS Geometry

Students who are transitioning from GPS Mathematics I to GPS Geometry will be missing critical content from GPS Algebra. This document provides the critical content in both web-based and print formats. The standards are the same in either format, though there are differences in the delivery.

The printed content is provided in pages that follow

Web-based content available at:
http://www.gavirtuallearning.org/Resources/shared_math2.aspx
Content specific to the GPS Transition Frameworks can be accessed through the following links:

Quadratic Functions Introductions

Finding Real Solutions of Quadratic Equations: Learning Tasks

Finding Real Solutions of Quadratic Equations: Assessments

Graphing and Solving Quadratic Equations and Inequalities: Learning Tasks

Graphing and Solving Quadratic Equations and Inequalities: Assessments

Finding Complex Solutions of Quadratic Equations and Models: Learning Tasks

Finding Complex Solutions of Quadratic Equations and Models: Assessments

Quadratic Functions Project

Piecewise, Exponential, and Inverses: Introduction

Representing Relations and Functions: Learning Tasks

Representing Relations and Functions: Assessments

Finding the Best Model: Introduction

Absolute Value Functions: Learning Tasks

Absolute Value Functions: Assessments

Data Analysis Revisited: Learning Tasks

Data Analysis Revisited: Assessments

Finding the Best Model: Project

Georgia Department of Education

GPS Transition Frameworks

Mathematics I → GPS Geometry
Quadratics and Complex Numbers

1st Edition
May, 2011
Georgia Department of Education

Table of Contents:

Mathematics I → GPS Geometry
Transition Unit:
Quadratics and Complex Numbers

INTRODUCTION:

Students who are transitioning from GPS Mathematics I to GPS Geometry will be missing content from GPS Algebra mainly those dealing with some of the more advanced algebra topics. This transition unit will provide students with tasks to further develop their understanding of quadratics and introduce them to complex numbers.

Students are first introduced to quadratic functions in GPS Mathematics I, where they study the characteristics of the basic function $f(x) = x^2$ and learn to solve simple quadratic equations that can be put in the form $x^2 + bx + c = 0$. In this transitional unit, through exploration of many real world situations which are represented by quadratic functions, students will extend their previous study to include an in-depth analysis of general quadratic functions in both standard form, $f(x) = ax^2 + bx + c$, and vertex form. In this transitional unit, students extend their knowledge of solving quadratic equations through factoring and learn the quadratic formula, which can be used to solve any quadratic equation. Study of the quadratic formula introduces complex numbers and students learn about the arithmetic of complex numbers. Students make connections between algebraic results and characteristics of the graphs of quadratic function and apply this understanding in solving quadratic inequalities. In addition, they explore sums of terms of arithmetic sequences as examples of quadratic functions. This work provides a foundation for modeling data with quadratic functions, a topic that will be explored later in the next transition unit.

ENDURING UNDERSTANDINGS:

- The graph of any quadratic function is a vertical and/or horizontal shift of a vertical stretch or shrink of the basic quadratic function $f(x) = x^2$.
- The vertex of a quadratic function provides the maximum or minimum output value of the function and the input at which it occurs.
- Every quadratic equation can be solved using the Quadratic Formula.
- The discriminant of a quadratic equation determines whether the equation has two real roots, one real root, or two complex conjugate roots.
- The complex numbers are an extension of the real number system and have many useful applications.

KEY STANDARDS ADDRESSED:

MM2N1. Students will represent and operate with complex numbers.

a. Write square roots of negative numbers in imaginary form.

b. Write complex numbers in the form $a + bi$.
c. Add, subtract, multiply and divide complex numbers.
d. Simplify expressions involving complex numbers.

MM2A3. Students will analyze quadratic functions in the forms $f(x) = ax^2 + bx + c$ and $f(x) = a(x - h)^2 + k$.

a. Convert between standard and vertex form.
b. Graph quadratic functions as transformations of the function $f(x) = x^2$.
c. Investigate and explain characteristics of quadratic functions, including domain, range, vertex, axis of symmetry, zeros, intercepts, extrema, intervals of increase and decrease, and rates of change.
d. Explore arithmetic sequences and various ways of computing their sums.
e. Explore sequences of partial sums of arithmetic series as examples of quadratic functions.

MM2A4. Students will solve quadratic equations and inequalities in one variable.

a. Solve equations graphically using appropriate technology.
b. Find real and complex solutions of equations by factoring, taking square roots, and applying the quadratic formula.
c. Analyze the nature of roots using technology and using the discriminant.
d. Solve quadratic inequalities both graphically and algebraically, and describe the solutions using linear inequalities.

<u>RELATED STANDARDS ADDRESSED:</u>

MM1P1. Students will solve problems (using appropriate technology).

a. Build new mathematical knowledge through problem solving.
b. Solve problems that arise in mathematics and in other contexts.
c. Apply and adapt a variety of appropriate strategies to solve problems.
d. Monitor and reflect on the process of mathematical problem solving.

MM1P2. Students will reason and evaluate mathematical arguments.

a. Recognize reasoning and proof as fundamental aspects of mathematics.
b. Make and investigate mathematical conjectures.
c. Develop and evaluate mathematical arguments and proofs.
d. Select and use various types of reasoning and methods of proof.

MM1P3. Students will communicate mathematically.

a. Organize and consolidate their mathematical thinking through communication.
b. Communicate their mathematical thinking coherently and clearly to peers, teachers, and others.
c. Analyze and evaluate the mathematical thinking and strategies of others.
d. Use the language of mathematics to express mathematical ideas precisely.

MM1P4. Students will make connections among mathematical ideas and to other disciplines.

a. Recognize and use connections among mathematical ideas.
b. Understand how mathematical ideas interconnect and build on one another to produce a coherent whole.
c. Recognize and apply mathematics in contexts outside of mathematics.

MM1P5. Students will represent mathematics in multiple ways.

a. Create and use representations to organize, record, and communicate mathematical ideas.
b. Select, apply, and translate among mathematical representations to solve problems.
c. Use representations to model and interpret physical, social, and mathematical phenomena.

<u>Unit Overview:</u>

This unit focuses on quadratic functions, equations, and inequalities and requires understanding of many topics from Mathematics I. In completing the unit, students will use all of the skills and concepts acquired in achieving the algebra standards in Mathematics I and will apply their understanding of the coordinate plane and use of mathematical argument from the geometry standard as well, students will need to have had experience expressing the solutions to linear inequalities in one variable and working with sequences, both in recursive and closed form. Tasks in the unit assume an understanding of the characteristics of functions studied in Mathematics I, especially of the basic function $f(x) = x^2$ and transformations involving vertical shifts, stretches, and shrinks, as well as reflections across the x- and y-axes. The tasks build on the ability to solve simple non-linear equations especially simple rational equations, quadratic equations that can be solved by finding square roots, and quadratic equations of the form $x^2 + bx + c = 0$.

The unit begins with an applied problem that explores a new type of function transformation, the horizontal shift, and combinations of this transformation with those studied in Unit 4. Students learn to factor general quadratic expressions completely over the integers and to solve general quadratic equations by factoring by working with quadratic functions that model the behavior of objects that are thrown in the air and allowed to fall subject to the force of gravity. Students then continue their study of objects in free fall to learn to find the vertex of the graph of any polynomial function and to convert the formula for a quadratic function from standard to vertex form. Next, students explore quadratic inequalities graphically, apply the vertex form of a quadratic function to find real solutions of quadratic equations that cannot be solved by factoring, and then use exact solutions of quadratic equations to give exact values for the endpoints of the intervals in the solutions of quadratic inequalities.

After students have learned to find the real solutions of any quadratic equation, they develop the concept of discriminant of a quadratic equation, learn the quadratic formula, and then explore complex numbers as non-real solutions of quadratic equations. Basic arithmetic operations on complex numbers are investigated so that students have the ability to verify complex solutions to quadratic equations and understand that they come in conjugate pairs. The unit ends with an exploration of arithmetic series, students develop formulas for calculating the sum of arithmetic series, and apply the concepts to counting possible pairs from a set of objects, to counting the number of diagonals of a polygon, and to understanding the definition of polygonal numbers.

Throughout the unit there is an emphasis on problem solving and mathematical reasoning. The applications in the unit include most of the standard types of application problems involving quadratic functions and equations. The order of progress allows students to develop mathematical arguments for the theorems of the unit, specifically to explain why the graph of every quadratic function is a translation of the graph of the basic function $f(x) = x^2$ and to justify the quadratic formula.

Throughout the unit there is a focus on the integration of algebraic and graphical viewpoints and multiple representations of the same mathematical concepts. Therefore, it is important to constantly:

- Promote student use of multiple representations of concepts and require students to explain how to translate information from one representation to another. Unit activities require that students explain how their equations represent the physical situation they are intended to model.
- Emphasize sketching quadratic graphs by hand to reveal important features and to use the understanding of characteristics of quadratic graphs to select appropriate viewing windows when using graphing technology.
- Distinguish between solving the equation in the mathematical model and solving the global mathematical problem.

As a final note, we observe that completing the square is a topic for GPS Advanced Algebra and is not used in this unit.

Formulas and Definitions:

Horizontal shift: A rigid transformation of a graph in a horizontal direction, either left or right.

Complete factorization over the integers: Writing a polynomial as a product of polynomials so that none of the factors is the number 1, there is at most one factor of degree zero, each polynomial factor has degree less than or equal to the degree of the product polynomial, each polynomial factor has all integer coefficients, and none of the factor polynomial can written as such a product.

<u>Vertex form of a quadratic function</u>: A formula for a quadratic equation of the form $f(x) = a(x-h)^2 + k$, where a is a nonzero constant and the vertex of the graph is the point (h, k).

<u>Discriminant of a quadratic equation</u>: The discriminant of a quadratic equation of the form $ax^2 + bx + c = 0,\ a \neq 0,$ is the number $b^2 - 4ac$.

<u>Theorems:</u>

For $h = \frac{-b}{2a}$ and $k = f\left(\frac{-b}{2a}\right)$, $f(x) = a(x-h)^2 + k$ is the same function as $f(x) = ax^2 + bx + c$.

The graph of any quadratic function can be obtained from transformations of the graph of the basic function $f(x) = x^2$.

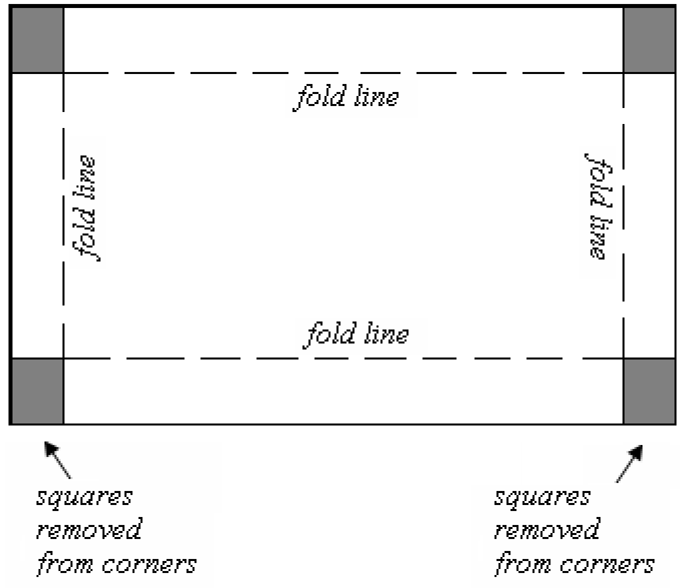
Quadratic formula: The solution(s) of the quadratic equation of the form $ax^2 + bx + c = 0$, where a, b, and c are real numbers with $a \neq 0$, is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The discriminant of a quadratic equation is positive, zero, or negative if and only if the equation has two real solutions, one real solution, or two complex conjugate number solutions respectively.

<u>TASKS</u>:
The remaining content of this framework consists of student tasks or activities. The first is intended to launch the unit. Each activity is designed to allow students to build their own algebraic understanding through exploration. The last task is a culminating task, designed to assess student mastery of the unit. There is a student version, as well as a Teacher Edition version that includes notes for teachers and solutions.

Henley's Chocolates Learning Task

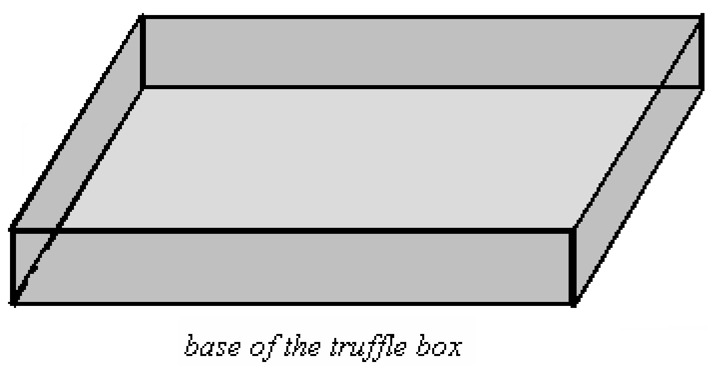


Henley Chocolates is famous for its mini chocolate truffles, which are packaged in foil covered boxes. The base of each box is created by cutting squares that are 4 centimeters on an edge from each corner of a rectangular piece of cardboard and folding the cardboard edges up to create a rectangular prism 4 centimeters deep. A matching lid is constructed in a similar manner, but, for this task, we focus on the base, which is illustrated in the diagrams below.

For the base of the truffle box, paper tape is used to join the cut edges at each corner. Then the inside and outside of the truffle box base are covered in foil.

Henley Chocolates sells to a variety retailers and creates specific box sizes in response to requests from particular clients. However, Henley Chocolates requires that their truffle boxes always be 4 cm deep and that, in order to preserve the distinctive shape associated with Henley Chocolates, the bottom of each truffle box be a rectangle that is two and one-half times as long as it is wide.

1. Henley Chocolates restricts box sizes to those which will hold plastic trays for a whole number of mini truffles. A box needs to be at least 2 centimeters wide to hold one row of mini truffles. Let L denote the length of a piece of cardboard from which a truffle box is made. What value of L corresponds to a finished box base for which the bottom is a rectangle that is 2 centimeters wide?



2. Henley Chocolates has a maximum size box of mini truffles that it will produce for retail sale. For this box, the bottom of the truffle box base is a rectangle that is 50 centimeters long. What are the dimensions of the piece of cardboard from which this size truffle box base is made?

3. Since all of the mini truffle boxes are 4 centimeters deep, each box holds two layers of mini truffles. Thus, the number of truffles that can be packaged in a box depends on the number of truffles that can be in one layer, and, hence, on the area of the bottom of the box. Let $A(x)$ denote the area, in square centimeters, of the rectangular bottom of a truffle box base. Write a formula for $A(x)$ in terms of the length L, in centimeters, of the piece of cardboard from which the truffle box base is constructed.

4. Although Henley Chocolates restricts truffle box sizes to those that fit the plastic trays for a whole number of mini truffles, the engineers responsible for box design find it simpler to study the function A on the domain of all real number values of L in the interval from the minimum value of L found in item 1 to the maximum value of L found in item 2. State this interval of L values as studied by the engineers at Henley Chocolates.

The next few items depart from Henley Chocolates to explore graph transformations that will give us insight about the function A for the area of the bottom of a mini truffle box. We will return to the function A in item 9.

5. **Use technology to graph** each of the following functions on the same axes with the graph of the function f defined by $f(x) = x^2$. Use a new set of axes for each function listed below, but repeat the graph of f each time. For each function listed, describe a rigid transformation of the graph of f that results in the graph of the given function. Make a conjecture about the graph of $y = (x-h)^2$, where h is any real number.
   a. $y = (x-3)^2$
   b. $y = (x-6)^2$
   c. $y = (x-8)^2$

6. **Use technology to graph** each of the following functions on the same axes with the graph of the function *f* defined by $f(x) = x^2$. Use a new set of axes for each function listed below, but repeat the graph of *f* each time. For each function listed, describe a rigid transformation of the graph of *f* that results in the graph of the given function.
   a. $y = (x+2)^2$
   b. $y = (x+5)^2$
   c. $y = (x+9)^2$

7. We can view the exercises in item 5 as taking a function, in this case the function, $f(x) = x^2$, and replacing the "*x*" in the formula with "$x - h$". We can view the exercises in item 6 as replacing the "*x*" in the formula with "$x + h$", but we can also view these exercises as replacing the "*x*" in the formula with "$x - h$".
   a. How can this be done?

   b. Does your conjecture from item 5 agree with the transformations you described for item 6? If so, explain how it works. If not, adjust the statement of your conjecture to include these examples also.

   c. What do you think this will happen if we replace the "*x*" in the formula with "$x - h$" for other functions in our basic family of functions? Have you seen any examples of such replacements before?

8. For each pair of functions below, predict how you think the graphs will be related and then **use technology to graph** the two functions on the same axes and check your prediction.
   a. $y = x^2$ and $y = 3x^2$
   b. $y = 3x^2$ and $y = 3(x-4)^2$
   c. $y = \frac{1}{2}x^2$ and $y = -\frac{1}{2}x^2 + 5$
   d. $y = -0.75x^2$ and $y = 0.75(x+6)^2$
   e. $y = 2x^2$ and $y = -2(x-5)^2 + 7$

Now we return to the function studied by the engineers at Henley Chocolates.

9. Let g be the function with the same formula as the formula for function A but with domain all real numbers. Describe the transformations of the function f, the square function, that will produce the graph of the function g. Use **technology to graph** f and g on the same axes to check that the graphs match your description of the described transformations.

10. Describe the graph of the function A in words and make a hand drawn sketch. Remember that you found the domain of the function in item 4. What is the range of the function A?

11. The engineers at Henley Chocolates responsible for box design have decided on two new box sizes that they will introduce for the next winter holiday season.
    a. The area of the bottom of the larger of the new boxes will be 640 square centimeters. Use the function A to write and solve an equation to find the length L of the cardboard need to make this new box.

    b. The area of the bottom of the smaller of the new boxes will be 40 square centimeters. Use the function A to write and solve an equation to find the length L of the cardboard need to make this new box.

12. How many mini-truffles do you think the engineers plan to put in each of the new boxes?

The Protein Bar Toss Learning Task

Blake and Zoe were hiking in a wilderness area. They came up to a scenic view at the edge of a cliff. As they stood enjoying the view, Zoe asked Blake if he still had some protein bars left, and, if so, could she have one? Blake said, "Here's one; catch!" As he said this, he pulled a protein bar out of his backpack and threw it up to toss it to Zoe. But the bar slipped out of his hand sooner than he intended, and the bar went straight up in the air with his arm out over the edge of the cliff. The protein bar left Blake's hand moving straight up at a speed of 24 feet per second. If we let t represent the number of seconds since the protein bar left the Blake's hand and let $h(t)$ denote the height of the bar, in feet above the ground at the base of the cliff, then, assuming that we can ignore the air resistance, we have the following formula expressing $h(t)$ as a function of t,

$$h(t) = -16t^2 + 24t + 160.$$

In this formula, the coefficient on the t^2-term is due to the effect of gravity and the coefficient on the t-term is due to the initial speed of the protein bar caused by Blake's throw. In this task, you will explore, among many things, the source of the constant term.

1. In Unit 3, you considered a formula for the distance fallen by an object dropped from a high place. List some ways in which this situation with Blake and the protein bar differs from the situation previously studied.

2. **Use technology to graph the equation** $y = -16t^2 + 24t + 160$. Find a viewing window that includes the part of this graph that corresponds to the situation with Blake and his toss of the protein bar. What viewing window did you select?

3. What was the height of the protein bar, measured from the ground at the base of the cliff, at the instant that it left Blake's hand? What special point on the graph is associated with this information?

4. If Blake wants to reach out and catch the protein bar on its way down, how many seconds does he have to process what happened and position himself to catch it? Justify your answer graphically and algebraically.

5. If Blake does not catch the falling protein bar, how long it take for the protein bar to hit the ground below the cliff? Justify your answer graphically. Then write a quadratic equation that you would need to solve to justify the answer algebraically.

The equation from item 5 can be solved by factoring, but it requires factoring a quadratic polynomial where the coefficient of the x^2-term is not 1. Our next goal is to learn about factoring this type of polynomial. We start by examining products that lead to such quadratic polynomials.

6. For each of the following, perform the indicated multiplication and use a rectangular model to show a geometric interpretation of the product as area for positive values of *x*.
   a. $(2x+3)(3x+4)$
   b. $(x+2)(4x+11)$
   c. $(2x+1)(5x+4)$

7. For each of the following, perform the indicated multiplication.
   a. $(2x-3)(9x+2)$
   b. $(3x-1)(x-4)$
   c. $(4x-7)(2x+9)$

The method for factoring general quadratic polynomial of the form $ax^2 + bx + c$, with *a, b,* and *c* all non-zero integers, is similar to the method learned in Mathematics I for factoring quadratics of this form but with the value of *a* restricted to *a* = 1. The next item guides you through an example of this method.

8. Factor the quadratic polynomial $6x^2 + 7x - 20$ using the following steps.
   a. Think of the polynomial as fitting the form $ax^2 + bx + c$.

      What is *a*? ____ What is *c*? ____ What is the product *ac*? ____

   b. List all possible pairs of integers such that their product is equal to the number *ac*. It may be helpful to organize your list in a table. Make sure that your integers are chosen so that their product has the same sign, positive or negative, as the number *ac* from above, and make sure that you list all of the possibilities.

   c. What is *b* in the quadratic polynomial given? _____ Add the integers from each pair listed in part b. Which pair adds to the value of *b* from your quadratic polynomial? We'll refer to the integers from this pair as *m* and *n*.

   d. Rewrite the polynomial replacing *bx* with $mx + nx$. [*Note either m or n could be negative; the expression indicates to add the terms mx and nx including the correct sign.*]

   e. Factor the polynomial from part d by grouping.

   f. Check your answer by performing the indicated multiplication in your factored polynomial. Did you get the original polynomial back?

9. Use the method outlined in the steps of item 8 to factor each of the following quadratic polynomials. Is it necessary to always list all of the integer pairs whose product is *ac*? Explain your answer.
   a. $2x^2 + 3x - 54$
   b. $4w^2 - 11w + 6$
   c. $3t^2 - 13t - 10$
   d. $8x^2 + 5x - 3$
   e. $18z^2 + 17z + 4$
   f. $6p^2 - 49p + 8$

10. If you are reading this, then you should have factored all of the quadratic polynomials listed in item 9 in the form $(Ax+B)(Cx+D)$, where the *A, B, C,* and *D* are all integers.
   a. Compare your answers with other students, or other groups of students. Did everyone in the class write their answers in the same way? Explain how answers can look different but be equivalent.

   b. Factor $24q^2 - 4q - 8$ completely.

   c. Show that $24q^2 - 4q - 8$ can be factored in the form $(Ax+B)(Cx+D)$, where the *A, B, C,* and *D* are all integers, using three different pairs of factors.

   d. How should answers to quadratic factoring questions be expressed so that everyone who works the problem correctly lists the same factors, just maybe not in the same order?

11. If a quadratic polynomial can be factored in the form $(Ax+B)(Cx+D)$, where the *A, B, C,* and *D* are all integers, the method you have been using will lead to the answer, specifically called the correct ***factorization***. As you continue your study of mathematics, you will learn ways to factor quadratic polynomials using numbers other than integers. For right now, however, we are interested in factors that use integer coefficients. Show that each of the

quadratic polynomials below <u>cannot</u> be factored in the form $(Ax+B)(Cx+D)$, where the *A, B, C,* and *D* are all integers.

a. $4z^2+z-6$

b. t^2+2t+8

c. $3x^2+15x-12$

12. Now we return to our goal of solving the equation from item 5. Recall that you solved quadratic equations of the form $ax^2+bx+c=0$, with a = 1, in Unit 4. The method required factoring the quadratic polynomial and using the Zero Factor Property. The same method still applies when $a \neq 1$, its just that the factoring is more involved, as we have seen above. Use your factorizations from items 9 and 10 as you solve the quadratic equations below.

    a. $2x^2+3x-54=0$

    b. $4w^2+6=11w$

    c. $3t^2-13t=10$

    d. $2x(4x+3)=3+x$

    e. $18z^2+21z=4(z-1)$

    f. $8-13p=6p(6-p)$

    g. $24q^2=4q+8$

13. Solve the quadratic equation from item 5. Explain how the solution gives an algebraic justification for your answer to the question.

14. Suppose the cliff had been 56 feet higher. Answer the following questions for this higher cliff.

    a. What was the height of the protein bar, measured from the ground at the base of the cliff, at the instant that it left Blake's hand? What special point on the graph is associated with this information?

    b. What is the formula for the height function in this situation?

c. If Blake wants to reach out and catch the protein bar on its way down, how many seconds does he have to process what has happened and position himself to catch it? Justify your answer algebraically.

d. If Blake does not catch the falling protein bar, how long it take for the protein bar to hit the ground below the cliff? Justify your answer algebraically.

15. Suppose the cliff had been 88 feet lower. Answer the following questions for this lower cliff.

a. What was the height of the protein bar, measured from the ground at the base of the cliff, at the instant that it left Blake's hand? What special point on the graph is associated with this information?

b. What is the formula for the height function in this situation?

c. If Blake wants to reach out and catch the protein bar on its way down, how many seconds does he have to process what has happened and position himself to catch it? Justify your answer algebraically.

d. If Blake does not catch the falling protein bar, how long it take for the protein bar to hit the ground below the cliff? Justify your answer algebraically.

Protein Bar Toss, Part 2, Learning Task

In the first part of the learning task about Blake attempting to toss a protein bar to Zoe, you found how long it took for the bar to go up and come back down to the starting height. However, there is a question we did not consider: How high above its starting point did the protein bar go before it started falling back down? We're going to explore that question now.

1. So far in Units 3 and 4, you have examined the graphs of many different quadratic functions. Consider the functions you graphed in the Henley Chocolates task and in the first part of the Protein Bar Toss. Each of these functions has a formula that is, or can be put in, the form $y = ax^2 + bx + c$ with $a \neq 0$. When we consider such formulas with domain all real numbers, there are some similarities in the shapes of the graphs. The shape of each graph is called a *parabola*. List at least three characteristics common to the parabolas seen in these graphs.

2. The question of how high the protein bar goes before it starts to come back down is related to a special point on the graph of the function. This point is called the *vertex* of the parabola. What is special about this point?

3. In the first part of the protein bar task you considered three different functions, each one corresponding to a different cliff height. Let's rename the first of these functions as h_1, so that

$$h_1(t) = -16t^2 + 24t + 160.$$

   a. Let $h_2(t)$ denote the height of the protein bar if it is thrown from a cliff that is 56 feet higher. Write the formula for the function h_2.

   b. Let $h_3(t)$ denote the height of the protein bar if it is thrown from a cliff that is 88 feet lower. Write the formula for the function h_3.

   c. **Use technology to graph** all three functions, h_1, h_2, and h_3, on the same axes.

d. Estimate the coordinates of the vertex for each graph.

e. What number do the coordinates have in common? What is the meaning of this number in relation to the toss of the protein bar?

f. The other coordinate is different for each vertex. Explain the meaning of this number for each of the vertices.

4. Consider the formulas for h_1, h_2, and h_3.
   a. How are the formulas different?

   b. Based on your answer to part a, how are the three graphs related? Do you see this relationship in your graphs of the three functions on the same axes? If not, restrict the domain in the viewing window so that the part of each graph you see corresponds to the same set of t-values.

5. In the introduction above we asked the question: How high above its starting point did the protein bar go before it started falling back down?
   a. Estimate the answer to the question for the original situation represented by the function h_1.

   b. Based on the relationship of the graphs of h_2 and h_3 to h_1, answer the question for the functions h_2 and h_3.

Estimating the vertex from the graph gives us an approximate answer to our original question, but an algebraic method for finding the vertex would give us an exact answer. The answers to the questions in item 5 suggest a way to use our understanding of the graph of a quadratic function to develop an algebraic method for finding the vertex. We'll pursue this path next.

6. For each of the quadratic functions below, find the *y*-intercept of the graph. Then find all the points with this value for the *y*-coordinate.
   a. $f\left(x\right)=x^2-4x+9$
   b. $f\left(x\right)=4x^2+8x-5$
   c. $f\left(x\right)=-x^2-6x+7$
   d. $f\left(x\right)=ax^2+bx+c,\ \ a\neq 0$

7. One of the characteristics of a parabola graph is that the graph has a line of symmetry.
   a. For each of the parabolas considered in item 6, use what you know about the graphs of quadratic functions in general with the specific information you have about these particular functions to find an equation for the line of symmetry.

   b. The line of symmetry for a parabola is called the *axis of symmetry*. Explain the relationship between the axis of symmetry and the vertex of a parabola. Then, find the *x*-coordinate of the vertex for each quadratic function listed in item 6.

   c. Find the *y*-coordinate of the vertex for the quadratic functions in item 6, parts a, b, and c, and then state the vertex as a point.

   d. Describe a method for finding the vertex of the graph of any quadratic function given in the form $f\left(x\right)=ax^2+bx+c,\ \ a\neq 0$.

8. Return to height functions h_1, h_2, and h_3.
   a. Use the method you described in item 7, part d, to find the exact coordinates of the vertex of each graph.

   b. Find the exact answer to the question: How high above its starting point did the protein bar go before it started falling back down?

9. Each part below gives a list of functions. Describe the geometric transformation of the graph of the first function that results in the graph of the second, and then describe the transformation of the graph of the second that gives the graph of the third, and, where applicable, describe the transformation of the graph of the third that yields the graph of the last function in the list. For the last function in the list, expand its formula to the form $\boldsymbol{f(x) = ax^2 + bx + c}$ and compare to the function in the corresponding part of item 6 with special attention to the vertex of each.
   a. $f(x) = x^2,\ f(x) = x^2 + 5,\ f(x) = (x-2)^2 + 5$
   b. $f(x) = x^2,\ f(x) = 4x^2,\ f(x) = 4x^2 - 9,\ f(x) = 4(x+1)^2 - 9,$
   c. $f(x) = x^2,\ f(x) = -x^2,\ f(x) = -x^2 + 16,\ f(x) = -(x+3)^2 + 16$

10. For any quadratic function of the form $\boldsymbol{f(x) = ax^2 + bx + c}$:
    a. Explain how to get a formula for the same function in the form $f(x) = a(x-h)^2 + k$.

    b. What do the *h* and *k* in the formula of part a represent relative to the function?

11. Give the **vertex form** of the equations for the functions h_1, h_2, and h_3 and verify algebraically the equivalence with the original formulas for the functions. Remember that you found the vertex for each function in item 8, part a.

12. For the functions given below, put the formula in the vertex form $f(x) = a(x-h)^2 + k$, give the equation of the axis of symmetry, and describe how to transform the graph of $y = x^2$ to create the graph of the given function.
    a. $f(x) = 3x^2 + 12x + 13$
    b. $f(x) = x^2 - 7x + 10$
    c. $f(x) = -2x^2 + 12x - 24$

13. Make a hand-drawn sketch of the graphs of the functions in item 12. Make a dashed line for the axis of symmetry, and plot the vertex, y-intercept and the point symmetric with the y-intercept.

14. Which of the graphs that you drew in item 13 have x-intercepts?

    a. Find the x-intercepts that do exist by solving an appropriate equation and then add the corresponding points to your sketch(es) from item 13.

    b. Explain geometrically why some of the graphs have x-intercepts and some do not.

    c. Explain how to use the vertex form of a quadratic function to decide whether the graph of the function will or will not have x-intercepts. Explain your reasoning.

Paula's Peaches Revisited Learning Task

In this task, we revisit Paula, the peach grower who wanted to expand her peach orchard last year. In the established part of her orchard, there are 30 trees per acre with an average yield of 600 peaches per tree. Data from the local agricultural experiment station indicated that if Paula chose to plant more than 30 trees per acre in the expanded section of orchard, when the trees reach full production several years from now, the average yield of 600 peaches per tree would decrease by 12 peaches per tree for each tree over 30 per acre. In answering the questions below, remember that the data is expressed in averages and does mean that each tree produces the average number of peaches.

1. Let x be the number of trees Paula might plant per acre in her new section of orchard and let $Y(x)$ represent the predicted average yield in peaches per acre. Write the formula for the function Y. Explain your reasoning, and sketch a graph of the function on an appropriate domain.

2. Paula wanted to average at least as many peaches per acre in the new section of orchard as in the established part.
   a. Write an inequality to express the requirement that, for the new section, the average yield of peaches per acre should be at least as many peaches as in the established section.

   b. Solve the inequality graphically. Your solution should be an inequality for the number of trees planted per acre.

   c. Change your inequality to an equation, and solve the equation algebraically. How are the solutions to the equation related to the solution of your inequality?

3. Suppose that Paula wanted to a yield of at least 18900 peaches.

   a. Write an inequality to express the requirement of an average yield of 18900 peaches per acre.

b. Solve the equation $x^2 - 80x + 1575 = 0$ by factoring.

c. Solve the inequality from part a. Explain how and why the solutions from part b are related to the solution of the inequality?

Solving $x^2 - 80x + 1575 = 0$ by factoring required you to find a pair of factors of 1575 with a particular sum. With a number as large as 1575, this may have taken you several minutes. Next we explore an alternative method of solving quadratic equations that applies the vertex form of quadratic functions. The advantages of this method are that it can save time over solving equations by factoring when the right factors are hard to find and that it works with equations involving quadratic polynomials that cannot be factored over the integers.

4. Consider the quadratic function $f(x) = x^2 - 80x + 1575$.

   a. What are the *x*-intercepts of the graph? Explain how you know.

   b. Rewrite the formula for the function so that the *x*-intercepts are obvious from the formula.

   c. There is a third way to express the formula for the function, the vertex form. Rewrite the formula for the function in vertex form.

   d. Use the vertex form and take a square root to solve for the *x*-intercepts of the graph. Explain why you should get the same answers as part a.

   e. Explain the relationship between the vertex and the *x*-intercepts.

5. Consider the quadratic equation $x^2 + 4x - 3 = 0$.

   a. Show that the quadratic polynomial $x^2 + 4x - 3$ cannot be factored over the integers.

   b. Solve the equation by using the vertex form of the related quadratic function and taking a square root.

   c. Approximate the solutions to four decimal places and check them in the original equation.

   d. How are the *x*-intercepts and axis of symmetry related? Be specific.

6. Suppose that Paula wanted to grow at least 19000 peaches.

   a. Write an inequality for this level of peach production.

   b. Solve the inequality graphically.

   c. Solve the corresponding equation algebraically. Approximate any non-integer solutions to four decimal places. Explain how to use the solutions to the equation to solve the inequality.

7. Suppose that Paula wanted to grow at least 20000 peaches.

   a. Write an inequality for this level of peach production.

   b. What happens when you solve the corresponding equation algebraically?

c. Solve the inequality graphically.

d. Explain the connection between parts b and c.

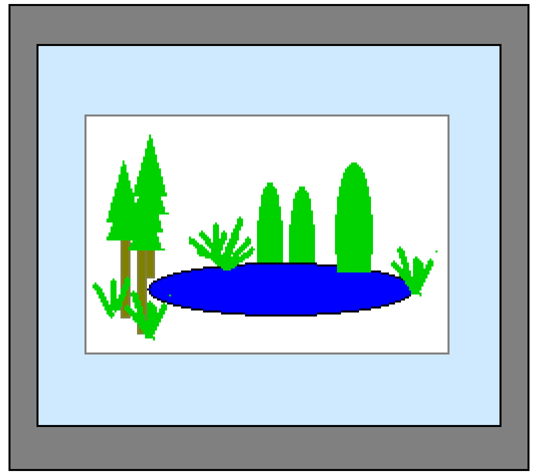
The method you used to solve the equations in items 5 and 6 above can be applied to solve: $a \cdot x^2 + b \cdot x + c = 0$ for any choice of real number coefficients for *a, b,* and *c* as long as $a \neq 0$. When this is done in general, we find that the solutions have the form $x = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $x = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$. This result is called the quadratic formula and usually stated in summary form as follows.

<u>The Quadratic Formula</u>: If $a \cdot x^2 + b \cdot x + c = 0$ with $a \neq 0$, then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad .$$

Using the Quadratic Formula is more straightforward and, hence, more efficient than the method of putting the quadratic expression in vertex form and solving by taking square roots. The next task will explore using it to solve quadratic equations.

Just the Right Border Learning Task



1. Hannah is an aspiring artist who enjoys taking nature photographs with her digital camera. Her mother, Cheryl, frequently attends estate sales in search of unique decorative items. Last month Cheryl purchased an antique picture frame that she thinks would be perfect for framing one of Hannah's recent photographs. The frame is rather large, so the photo needs to be enlarged. Cheryl wants to mat the picture. One of Hannah's art books suggest that mats should be designed so that the picture takes up 50% of the area inside the frame and the mat covers the other 50%. The inside of the picture frame is 20 inches by 32 inches. Cheryl wants Hannah to enlarge, and crop if necessary, her photograph so that it can be matted with a mat of uniform width and, as recommended, take up 50% of the area inside the mat. See the image at the right.

   a. Let *x* denote the width of the mat for the picture. Write an equation in *x* that models this situation.

   b. Put the equation from part a in the standard form $ax^2 + bx + c = 0$.

   c. Show that the equation from part b cannot be solved by factoring over the integers.

   d. The quadratic formula can be used to solve quadratic equations that cannot be solved by factoring over the integers. Using the simplest equivalent equation in standard form, identify *a*, *b,* and *c* from the equation in part b and find $b^2 - 4ac$; then substitute these values in the quadratic formula to find the solutions for *x*. Give exact answers for *x* and approximate the solutions to two decimal places.

   e. To the nearest tenth of an inch, what should be the width of the mat and the dimensions for the photo enlargement?

2. The quadratic formula can be very useful in solving problems. Thus, it should be practiced enough to develop accuracy in using it and to allow you to commit the formula to memory. Use the quadratic formula to solve each of the following quadratic equations, even if you could solve the equation by other means. Begin by identifying *a*, *b,* and *c* and finding $b^2 - 4ac$; then substitute these values into the formula.

   a. $4z^2 + z - 6 = 0$
   b. $t^2 + 2t + 8 = 0$
   c. $3x^2 + 15x = 12$
   d. $25w^2 + 9 = 30w$
   e. $7x^2 = 10x$
   f. $\frac{t}{2} + \frac{7}{t} = 2$
   g. $3\left(2p^2 + 5\right) = 23p$
   h. $12z^2 = 90$

3. The expression $b^2 - 4ac$ in the quadratic formula is called the **discriminant** of the quadratic equation in standard form. All of the equations in item 2 had values of *a*, *b*, and *c* that are rational numbers. Answer the following questions for quadratic equations in standard form when ***a*, *b*, and *c* are rational numbers**. Make sure that your answers are consistent with the solutions from item 2.

   a. What is true of the discriminant when there are two real number solutions to a quadratic equation?

   b. What is true of the discriminant when the two real number solutions to a quadratic equation are rational numbers?

   c. What is true of the discriminant when the two real number solutions to a quadratic equation are irrational numbers?

   d. Could a quadratic equation with rational coefficients have one rational solution and one irrational solution? Explain your reasoning.

   e. What is true of the discriminant when there is only one real number solution? What kind of number do you get for the solution?

   f. What is true of the discriminant when there is no real number solution to the equation?

Solve $\frac{2}{3}q^2 + \frac{1}{4}q = \frac{1}{6}$ using the quadratic formula.

4. There are many ways to prove the quadratic formula. One that relates to the ideas you have studied so far in this unit comes from considering a general quadratic function of the form $f(x) = ax^2 + bx + c$, putting the formula for the function in vertex form, and then using the vertex form to find the roots of the function. Such a proof does not require that *a*, *b*, and *c* be restricted to rational numbers; *a*, *b*, and *c* can be any real numbers with $a \neq 0$. Why is the restriction $a \neq 0$ needed?

<u>Alternate item 4</u>

There are many ways to show why the quadratic formula always gives the solution(s) to any quadratic equation with real number coefficients. You can work through one of these by responding to the parts below. Start by assuming that each of *a*, *b*, and *c* is a real number, that $a \neq 0$, and then consider the quadratic equation $ax^2 + bx + c = 0$.

a. Why do we assume that $a \neq 0$?

b. Form the corresponding quadratic function, $f(x) = ax^2 + bx + c$, and put the formula for *f*(x) in vertex form, expressing *k* in the vertex form as a single rational expression.

c. Use the vertex form to solve for *x*-intercepts of the graph and simplify the solution. Hint: Consider two cases, *a* > 0 and *a* < 0, in simplifying $\sqrt{a^2}$.

5. Use the quadratic formula to solve the following equations with real number coefficients. Approximate each real, but irrational, solution correct to hundredths.
   a. $x^2 + \sqrt{5}x + 1 = 0$
   b. $3q^2 - 5q + 2\pi = 0$
   c. $3t^2 + 11 = 2\sqrt{33}t$
   d. $9w^2 = \sqrt{13}w$

6. Verify each answer for item 5 by using a graphing utility to find the *x*-intercept(s) of an appropriate quadratic function.

   a. Put the function for item 5, part c, in vertex form. Use the vertex form to find the *t*-intercept.

   b. Solve the equation from item 5, part d, by factoring.

7. Answer the following questions for quadratic equations in standard from where ***a*, *b*, and *c* are real numbers**.

   a. What is true of the discriminant when there are two real number solutions to a quadratic equation?

   b. Could a quadratic equation with real coefficients have one rational solution and one irrational solution? Explain your reasoning.

   c. What is true of the discriminant when there is only one real number solution?

   d. What is true of the discriminant when there is no real number solution to the equation?

   e. Summarize what you know about the relationship between the determinant and the solutions of a quadratic of the form $ax^2 + bx + c = 0$ where *a*, *b*, and *c* are real numbers with $a \neq 0$ into a formal statement using biconditionals.

8. A landscape designer included a cloister (a rectangular garden surrounded by a covered walkway on all four sides) in his plans for a new public park. The garden is to be 35 feet by 23 feet and the total area enclosed by the garden and walkway together is 1200 square feet. To the nearest inch, how wide does the walkway need to be?

9. In another area of the park, there will be an open rectangular area of grass surrounded by a flagstone path. If a person cuts across the grass to get from the southeast corner of the area to the northwest corner, the person will walk a distance of 15 yards. If the length of the area is 5 yards more than the width, to the nearest foot, how much distance is saved by cutting across rather than walking along the flagstone path?

Imagining a New Number Learning Task

In other learning tasks of this unit, you encountered some quadratic equations for which the discriminants are negative numbers. For each of these equations, there is no real number solution because a solution requires that we find the square root of the discriminant and no real number can be the square root of a negative number. If there were a real number answer, the square of that number would have to be negative, but the square of every real number is greater than or equal to zero.

The problem of taking the square root of a negative number was ignored or dismissed as impossible by early mathematicians who encountered it. In his 1998 book, *The Story of* $\sqrt{-1}$, Paul J. Nahin describes the situation that most historians of mathematics acknowledge as the first recorded encounter with the square root of a negative number. Nahin quotes W. W. Beman, a professor of mathematics and mathematics historian from the University of Michigan, in a talk he gave at an 1897 meeting of the American Association for the Advancement of Science:

> We find the square root of a negative quantity appearing for the first time in the *Stetreometria* of Heron of Alexandria [*c.* 75 A.D.]. . . . After having given a correct formula of the determination of the volume of a frustum of a pyramid with square base and applied it successfully to the case where the side of the lower base is 10, of the upper 2, and the edge 9, the author endeavors to solve the problem where the side of the lower base is 28, the upper 4, and the edge 15. Instead of the square root of 81 – 144 required by the formula, he takes the square root of 144 – 81 . . . , i.e., he replaces $\sqrt{-1}$ by 1, and fails to observe that the problem as stated is impossible. Whether this mistake was due to Heron or to the ignorance of some copyist cannot be determined.

Nahin then observes that "so Heron missed being the earliest known scholar to have derived the square root of a negative number in a mathematical analysis of a physical problem. If Heron really did fudge his arithmetic then he paid dearly for it in lost fame."

Nahin then reports on Diophantus of Alexandria, who most likely wrote his famous book, *Arithmetica,* about the year 250 A.D. Problem 22 of book 6 of the *Arithmetica* posed the question of finding the length of the legs of a right triangle with area 7 and perimeter 12, measured in appropriate units. Diophantus reduced this problem to that of solving the quadratic equation $336x^2 + 24 = 172x$. Diophantus knew a method of solving quadratic equations equivalent to the quadratic formula, but, quoting Nahin, "What he wrote was simply that the quadratic equation was not possible."

1. Follow the steps below to see how Diophantus arrived at the equation $336x^2 + 24 = 172x$.

a. Let *a* and *b* denote the lengths of the legs of a right triangle with area 7 and perimeter 12. Explain why $ab = 14$ and $a + b + \sqrt{a^2 + b^2} = 12$.

b. Let *x* be a number so that $a = \frac{1}{x}$ and $b = 14x$. Based on the meaning of *a* and *b,* explain why there must be such a number *x*.

c. Replace *a* and *b* in the equation $a + b + \sqrt{a^2 + b^2} = 12$ with the expressions in terms of *x*, and write an equivalent equation with the square root expression on the left side of the equation.

d. Square both sides of the final equation from part c and simplify to obtain Diophantus' equation: $336x^2 + 24 = 172x$.

2. What happens when you use the quadratic formula to solve $336x^2 + 24 = 172x$?

According to Eugene W. Hellmich writing in Capsule 76 of *Historical Topics for the Mathematics Classroom, Thirty-first Yearbook of the National Council of Teachers of Mathematics*, 1969:

> The first clear statement of difficulty with the square root of a negative number was given in India by Mahavira (*c*. 850), who wrote: "As in the nature of things, a negative is not a square, it has no square root." Nicolas Chuquet (1484) and Luca Pacioli (1494) in Europe were among those who continued to reject imaginaries.

However, there was a break in the rejection of square roots of negative numbers in 1545 when Gerolamo (or Girolamo) Cardano, known in English as Jerome Cardan, published his important book about algebra, *Ars Magna* (Latin for "The Great Art"). Cardano posed the problem of dividing ten into two parts whose product is 40.

3. Note that, when Cardano stated his problem about dividing ten into two parts, he was using the concept of "divide" in the sense of dividing a line segment of length 10 into two parts of shorter length.

   a. Show that Cardano's problem leads to the quadratic equation $x^2 - 10x + 40 = 0$.

b. Find the solutions to this equation given by the quadratic formula even though they are not real numbers.

4. Rather than reject the solutions to $x^2 - 10x + 40 = 0$ as impossible, Cardano simplified them to obtain $5+\sqrt{-15}$ and $5-\sqrt{-15}$, stated that these solutions were "manifestly impossible", but plunged ahead by saying "nevertheless, we will operate." He "operated" by treating these expressions as numbers that follow standard rules of algebra and checked that they satisfied his original problem.

   a. Assuming that these numbers follow the usual rules of algebra, verify that their sum is 10.

   b. Assuming in addition that $\left(\sqrt{-15}\right)\left(\sqrt{-15}\right) = -15$, verify that the product of the numbers is 40.

So, Cardano was the first to imagine that there might be some numbers in addition to the real numbers that we represent as directed lengths. However, Cardano did not pursue this idea. According to Hellmich in his mathematics history capsule, "Cardano concludes by saying that these quantities are 'truly sophisticated' and that to continue working with them would be 'as subtle as it would be useless.'" Cardano did not see any reason to continue working with the numbers because he was unable to see any physical interpretation for numbers. However, other mathematicians saw that they gave useful algebraic results and continued the development of what today we call ***complex numbers***.

Cardano's *Ars Magna* drew much attention among mathematicians of his day, not because of his computations with the numbers $5+\sqrt{-15}$ and $5-\sqrt{-15}$, but because it contained a formula, which came to be known as the Cardan formula, for solving any cubic equation.

5. State the standard form for a general cubic equation.

   a. Create some examples of cubic equations in standard form.

   b. Write formulas for cubic functions whose *x*-intercepts are the solutions of the cubic equations from part a.

c. Use a graphing utility to graph the functions from part b. How many *x*-intercepts does each of the functions from part b have?

d. How many real number solutions does each of the equations from part a have?

The Italian engineer Rafael Bombelli continued Cardano's work. In some cases, Cardan's formula gives roots of cubic equations expressed using the square root of a negative number. In his book, *Algebra,* published in 1572, Bombelli showed that the roots of the cubic equation $x^3 = 15x + 4$ are 4, $-2+\sqrt{3}$, and $-2-\sqrt{3}$ but observed that Cardan's formula expresses the root of $x = 4$ as $x = \sqrt[3]{2+\sqrt{-121}} + \sqrt[3]{2-\sqrt{-121}}$. Quoting Nahin in *The Story of* $\sqrt{-1}$, "It was Bombelli's great insight to see that the weird expression that Cardan's formula gives for *x* is real, but expressed in a very unfamiliar manner." This realization led Bombelli to develop the theory of numbers complex numbers.

Today we recognize Bombelli's "great insight," but many mathematicians of his day (and some into the twentieth century) remained suspicious of these new numbers. Rene Descartes, the French mathematician who gave us the Cartesian coordinate system for plotting points, did not see a geometric interpretation for the square root of a negative number so in his book *La Geometrie* (1637) he called such a number "imaginary." This term stuck so that we still refer to the square root of a negative number as "imaginary." By the way, Descartes is also the one who coined the term "real" for the real numbers.

We now turn to the mathematics of these "imaginary" numbers.

In 1748, Leonard Euler, one of the greatest mathematicians of all times, started the use of the notation "*i*" to represent the square root of –1, that is, $i = \sqrt{-1}$. Thus,

$$i^2 = \sqrt{-1}\sqrt{-1} = -1,$$

since *i* represents the number whose square is – 1.

This definition preserves the idea that the square of a square root returns us to the original number, but also shows that one of the basic rules for working with square roots of real numbers,

for any real numbers *a* and *b,* $\sqrt{a}\sqrt{b} = \sqrt{ab}$

does not hold for square roots of negative numbers because, if that rule were applied we would not get -1 for i^2.

6. As we have seen, the number *i* is not a real number; it is a new number. We want to use it to expand from the real numbers to a larger system of numbers.

   a. What meaning could we give to $2i, 3i, 4i, 5i, ...$?

   b. Find the square of each of the following: $2i,\ 7i,\ 10i,\ 25i$.

   c. How could we use *i* to write an expression for each of the following: $\sqrt{-4}, \sqrt{-25}, \sqrt{-49}$?

   d. What meaning could we give $-i, -2i, -3i, -4i, -5i, ...$?

   e. Write an expression involving *i* for each of the following: $-\sqrt{-9}, -\sqrt{-16}, -\sqrt{-81}$.

7. We are now ready to be explicit about imaginary numbers. An ***imaginary number*** is any number that can be written in the form bi, where *b* is a real number and $i = \sqrt{-1}$. Imaginary numbers are also sometimes called ***pure*** imaginary numbers. Write each of the following imaginary numbers in the standard form *bi*:
   $\sqrt{-\frac{1}{36}},\ \sqrt{-11},\ \sqrt{\frac{-5}{64}},\ -\sqrt{-7},\ -\sqrt{-18}$.

8. Any number that can be written in the form $a + bi$, where *a* and *b* are real numbers, is called a ***complex number***. We refer to the form $a + bi$ as the ***standard form*** of a complex number and call *a* the ***real part*** and *b* the ***imaginary part***. Write each of the following as a complex number in standard form and state its real part and its imaginary part. $6 - \sqrt{-1},\ -12 + \sqrt{-100},\ 31 - \sqrt{-20}$

As defined above, the set of complex numbers includes all of the real numbers (when the imaginary part is 0), all of the imaginary numbers (when the real part is 0), and lots of other numbers that have nonzero real and imaginary parts.

We say that two complex numbers are ***equal*** if their real parts are equal and their imaginary parts are equal, that is,

if $a+bi$ and $c+di$ are complex numbers,

then $a+bi=c+di$ if and only if $a=c$ and $b=d$.

In order to define operations on the set of complex numbers in a way that is consistent with the established operations for real numbers, we define addition and subtraction by combining the real and imaginary parts separately:

if $a+bi$ and $c+di$ are complex numbers,

then $\boldsymbol{(a+bi)+(c+di)=(a+c)+(b+d)i}$.

$\boldsymbol{(a+bi)-(c+di)=(a-c)+(b-d)i}$

9. Apply the above definitions to perform the indicated operations and write the answers in standard form.

a. $(3+5i)+(2-6i)$

b. $(5-4i)-(-3+5i)$

c. $13i-(3-i)$

d. $(-7-9i)-(-2)$

e. $(5.4+8.3i)+(-3.7+4.6i)$

f. $\left(\frac{3}{7}-\frac{5}{7}i\right)-\left(\frac{4}{7}+\frac{4}{7}i\right)$

To multiply complex numbers, we use the standard form of complex numbers, multiply the expressions as if the symbol i were an unknown constant, and then use the fact that $i^2 = -1$ to continue simplifying and write the answer as a complex number in standard form.

10. Perform each of the following indicated multiplications, and write your answer as a complex number in standard form.

    a. $6i(2-i)$

    b. $(2-i)(3+4i)$

    c. $\frac{2}{7}(9-6i)$

    d. $\left(4-i\sqrt{13}\right)\left(2+i\sqrt{13}\right)$ -- Note writing i in front of an imaginary part expressed as a root is standard practice to make the expression easier to read.

    e. $\left(i\sqrt{5}\right)\left(i\sqrt{5}\right)$

    f. $\left(\sqrt{-6}\right)\left(-\sqrt{-12}\right)$

11. For each of the following, use substitution to determine whether the complex number is a solution to the given quadratic equation.

    a. Is $-2+3i$ a solution of $x^2+4x+13=0$? Is $-2-3i$ a solution of this same equation?

    b. Is $2+i$ a solution of $x^2-3x+3=0$? Is $2-i$ a solution of this same equation?

12. Find each of the following products.

    a. $(1+i)(1-i)$

    b. $(5-2i)(5+2i)$

    c. $\left(\sqrt{7}+8i\right)\left(\sqrt{7}-8i\right)$

    d. $\left(-\frac{1}{2}-\frac{2}{3}i\right)\left(-\frac{1}{2}+\frac{2}{3}i\right)$

e. $\left(4+i\sqrt{6}\right)\left(4-i\sqrt{6}\right)$

13. The complex numbers $a+bi$ and $a-bi$ are called ***complex conjugates***. Each number is considered to be the complex conjugate of the other.

a. Review the results of your multiplications in item 12, and make a general statement that applies to the product of any complex conjugates.

b. Make a specific statement about the solutions to a quadratic equation in standard form when the discriminant is a negative number.

So far in our work with operations on complex numbers, we have discussed addition, subtraction, and multiplication and have seen that, when we start with two complex numbers in standard form and apply one of these operations, the result is a complex number that can be written in standard form.

14. Now we come to division of complex numbers. Consider the following quotient: $\dfrac{-2+3i}{3-4i}$.

This expression is not written as a complex number in standard form; in fact, it is not even clear that it can be written in standard form. The concept of complex conjugates is the key to carrying out the computation to obtain a complex number in standard form.

a. Find the complex conjugate of $3-4i$, which is the denominator of $\dfrac{-2+3i}{3-4i}$.

b. Multiply the numerator and denominator of $\dfrac{-2+3i}{3-4i}$ by the complex conjugate from part a.

c. Explain why $\dfrac{-2+3i}{3-4i}=\dfrac{-2+3i}{3-4i}\cdot\dfrac{3+4i}{3+4i}$ and then use your answers from part b to obtain $-\dfrac{18}{25}+\dfrac{1}{25}i$ as the complex number in standard form equal to the original quotient.

d. For real numbers, if we multiply the quotient by the divisor we obtain the dividend. Does this relationship hold for the calculation above?

15. Use the same steps as in item 14 to simplify each of the following quotients and give the answer as a complex number in standard form.

a. $(5-6i)\div(2+i)=\dfrac{5-6i}{2+i}$

b. $\dfrac{3\sqrt{2}}{7-i\sqrt{2}}$

c. $\dfrac{14+2i}{2i}$

d. $\dfrac{2+\sqrt{-8}}{-5+\sqrt{-18}}$

Item 15 completes our discussion of basic numerical operations with complex numbers. In our discussion of the early history of the development of complex numbers, we noted that Cardano did not continue work on complex numbers because he could not envision any geometric interpretation for them. It was over two hundred years until, in 1797, the Norwegian surveyor Caspar Wessel presented his ideas for a very simple geometric model of the complex numbers. For the remainder of the task you will investigate the modern geometric representation of complex numbers, based on Wessel's work and that of Wallis, Argand, Gauss, and other mathematicians, some of whom developed the same interpretation as Wessel independently.

In representing the complex numbers geometrically, we begin with a number line to represent the pure imaginary numbers and then place this number line perpendicular to a number line for the real numbers. The number $0i = 0$ is both imaginary and real, so it should be on both number lines. Therefore, the two number lines are drawn perpendicular and intersecting at 0, just as we do in our standard coordinate system. Geometrically, we represent the complex number $a + bi$ by the point (a, b) in the coordinate system. When we use this representation, we refer to a complex number as a point in the ***complex number plane.*** Note that this is a very different interpretation for points in a plane that the one we use for graphing functions whose domain and range are subsets of the real numbers as we did in considering cubic functions in item 5.

16. Use a complex number plane to graph and label each of the following complex numbers:
$2+3i$, $-8-6i$, $-5i$, -2, $7-3i$, $6i$, $-3+4i$, 3.5

17. Geometrically, what is the meaning of the absolute value of a real number? We extend this idea and define the ***absolute value of the complex number*** $\boldsymbol{a + b\,i}$ to be the distance in the complex number plane from the number to zero. We use the same absolute value symbol as we did with real numbers so that $|a + b\,i|$ represents the absolute value of the complex number $a + b\,i$.

    a. Find the absolute value of each of the complex numbers plotted in item 16.

    b. Verify your calculations from part a geometrically.

    c. Write a formula in terms of a and b for calculating $|a + b\,i|$.

    d. What is the relationship between the absolute value of a complex number and the absolute value of its conjugate? Explain.

    e. What is the relationship between the absolute value of a complex number and the product of that number with its conjugate? Explain.

18. The relationship you described in part e of item 17 is a specific case of a more general relationship involving absolute value and products of complex numbers. Find each of the following products of complex numbers from item 16 and compare the absolute value of each product to the absolute values of the factors in the product. (Note that you found the length of each of the factors below in item 17, part a. Finally, make and prove a conjecture about the absolute value of the product of two complex numbers.

    a. $(2 + 3i)(7 - 3i)$

    b. $(-3 + 4i)(5i)$

c. $(-2)(-8-6i)$

d. $(6i)(3.5)$

19. By definition, $i^1 = i$ and $i^2 = -1$.

a. Find $i^3,\ i^4,\ i^5,\ \ldots,\ i^{12}$. What pattern do you observe?

b. How do the results of your calculations of powers of i relate to item 18?

c. Devise a way to find any positive integer power of i, and use it to find the following powers of i: $i^{26},\ i^{55},\ i^{136},\ i^{373}$.

20. We conclude this task with an exploration of the geometry of multiplying a complex number by i.

a. Graph $i,\ i^2,\ i^3,\ i^4$ in the complex number plane. How does the point move each time it is multiplied by i?

b. Let $z = 5 + 12i$. What is iz?

c. Plot each of the complex numbers in part b and draw the line segment connecting each point to the origin.

d. Explain why multiplying by i does not change the absolute value of a complex number.

e. What geometric effect does multiplying by i seem to have on a complex number? Verify your answer for the number z from part b.

Geometric Connections Learning Task

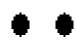
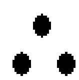
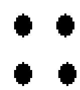
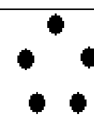
Scenario 1: A group of college freshmen attend a freshmen orientation session. Each is given a numbered "Hello" nametag. Students are told to shake hands with every other student there, and they do. How many handshakes are exchanged?

Scenario 2: An airline has several hub cities and flies daily non-stop flights between each pair of these cities. How many different non-stop routes are there?

Scenario 3: A research lab has several computers that share processing of important data. To insure against interruptions of communication, each computer is connected directly to each of the other computers. How many computer connections are there?

The questions asked at the end of the scenarios are really three specific versions of the same purely mathematical question: Given a set of objects, how many different pairs of objects can we form? One of the easiest ways to approach this problem involves thinking geometrically. The objects, whether they are college freshmen, cities, computers, etc., can be represented as points, and each pairing can be represented by drawing a line segment between the points. Such a representation is called a ***vertex-edge graph***. You will explore vertex-edge graphs further in GPS Advanced Algebra, but for now our focus is counting pairs of objects.

1. Use a vertex-edge graph and an actual count of all possible edges between pairs of points to fill in the table below. When there are more than two vertices, it helps to arrange the points as if they are vertices of regular polygons as shown in the table.

Number of points/vertices	1	2	3	4	5
Example diagram of vertices					
Number of line segments/edges					

2. Our goal is to find a formula that gives the number of pairs of objects as a function of *n*, the number of objects to be paired. Before we try to find the general formula, let's find the answers for a few more specific values.

   a. Draw a set of 6 points and all possible edges among 5 of the points. How many edges do you have so far? How many additional edges do you draw to complete the diagram to include all possible edges between two points? What is the total number of edges?

b. Draw a set of 7 points and all possible edges among 6 of the points. How many edges do you have so far? How many additional edges do you draw to complete the diagram to include all possible edges between two points? What is the total number of edges?

c. We are trying to find a formula for the number of pairs of objects as a function of the number of objects. What is the domain of this function? Why does this domain allow us to think of the function as a sequence?

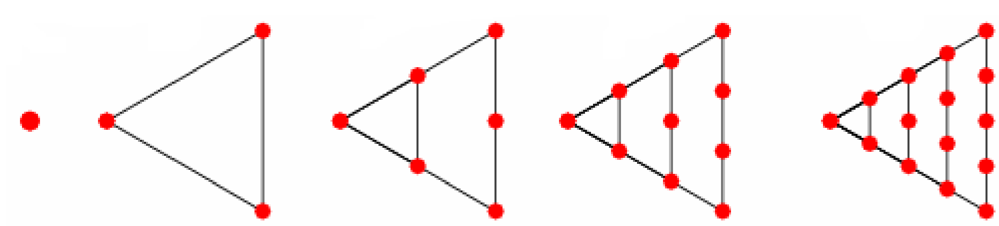
d. Denote the number of pairs of *n* objects by p_n, so that the sequence is $p_1, p_2, p_3, \ldots$. Remember that a recursive sequence is one that is defined by giving the value of at least one beginning term and then giving a recursive relation that states how to calculate the value of later terms based on the value(s) of one or more earlier terms. Generalize the line of reasoning from parts a and b to write a recursive definition of the sequence $p_1, p_2, p_3, \ldots$. Which beginning term(s) do you need to specify explicitly? What is the recursive relation to use for later terms?

e. Use your recursive definition to complete the table below and verify that it agrees with the previously found values.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
p_n															

f. Explain how the recursive definition of $p_1, p_2, p_3, \ldots$ also leads to expressing p_n as a sum of integers.

3. The numbers in the sequence $p_2, p_3, p_4 \ldots$ are known as the *triangular numbers*, so called because that number of dots can be arranged in a triangular pattern as shown below. (Image from Weisstein, Eric W. "Triangular Number." From *MathWorld*--A Wolfram Web Resource. http://mathworld.wolfram.com/TriangularNumber.html)



a. The standard notation for the triangular numbers is $T_1,\ T_2,\ T_3,\ \ldots.$ Through exploration of patterns and/or geometric representations of these numbers, find a closed form formula for the n^{th} triangular number, T_n.

b. Explain the relationship between the sequences $p_1, p_2, p_3, \ldots$ and $T_1,\ T_2,\ T_3,\ \ldots.$

c. Use the relationship explained in part b to write a closed form formula for p_n

4. Now we return to the scenarios.

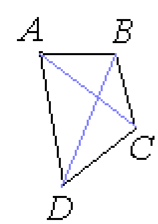
a. In Scenario 1, if there are 40 students at the orientation session, how many handshakes are exchanged?

b. In Scenario 2, if there are 190 airline routes between pairs of cities, how many cities are there in the group of hub cities?

c. In Scenario 3, if there are 45 computer connections, how many computers does the research lab use?

The formula you found in item 3, part d, and then applied in answering the questions of item 4, is well-known as the formula for counting ***combinations of n objects taken two at a time.*** We'll next explore another counting problem with geometric connections.

5. Remember that a ***diagonal*** of a polygon is a segment that connects non-adjacent vertices of a polygon. For example, in the quadrilateral *ABCD*, *AC* and *BD* are the diagonals.



a. Based on the definition above, complete the table below.

Number of sides/vertices in the polygon	3	4	5	6	7	8	9	*n*
Number of diagonals from each vertex								
Total number of diagonals in the polygon								

b. The last entry in the table provides a formula for the total number of diagonals in a polygon with *n* sides. Explain your reasoning for this formula.

6. Let $d_0, d_1, d_2, \ldots$ denote the sequence such that d_k is number of diagonals in a polygon with $k + 3$ sides.

   a. Use the table form item 5, part a, to give the values for the first seven terms of the sequence: $d_0, d_1, d_2, d_3, d_4, d_5, d_6$.

   b. We next seek a recursive definition of the sequence $d_0, d_1, d_2, \ldots$. This recursive definition requires understanding another sequence known as the sequence of ***first differences***. To aid our discussion of this other sequence, let f_1 denote the value added to d_0 to obtain d_1, let f_2 denote the value added to d_1 to obtain d_2, let f_3 denote the value added to d_2 to obtain d_3, and so forth. Give the values for the first six terms of the sequence $f_1, f_2, f_3, \ldots$.

   c. The sequence $f_1, f_2, f_3, \ldots$ is an arithmetic sequence. Explain why.

   d. The standard formula for the k^{th} term of an arithmetic sequence is $a + (k-1)d$, where *a* is the first term and *d* is the common difference. What are *a* and *d* for the sequence $f_1, f_2, f_3, \ldots$? Write a formula for the k^{th} term of the sequence.

   e. Write a recursive definition for the k^{th} term of the sequence $d_0, d_1, d_2, \ldots$.

   f. Explain why, for $k \geq 1$, we have the relationship that $d_k = f_1 + f_2 + \ldots + f_k$.

   g. Use your formula for the number of diagonals of a polygon with *n* sides to write a formula for d_k.

   h. Combine parts f and g to write a formula for sum of the first *k* terms of the sequence $f_1, f_2, f_3, \ldots$.

7. So far in this task, you've worked with two formulas: one that gives the number of combinations of *n* objects taken 2 at a time and the other that counts the number of diagonals of a polygon with *n* sides. Since the first formula also gives the numbers of edges in the vertex-edge graph containing *n* vertices and all possible edges, there is a relationship between the two formulas. Explain the relationship geometrically and algebraically.

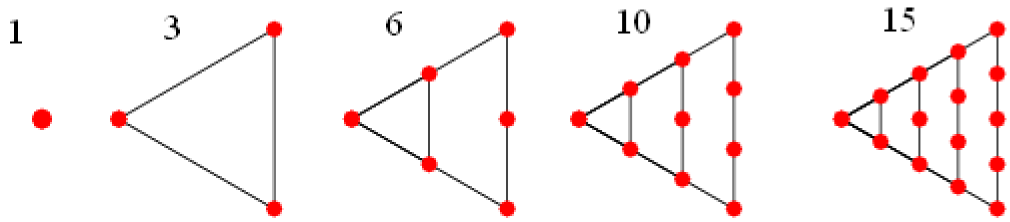
8. Each of the formulas studied so far provides a way to calculate certain sums of integers. One of the formulas sums the terms of the arithmetic sequence $f_1, f_2, f_3, \ldots$. Look back at the sequence $p_1, p_2, p_3, \ldots$ from items 2 and 3.

   a. What is the sequence of first differences for the sequence $p_1, p_2, p_3, \ldots$?

   b. Is the sequence of first differences arithmetic? Explain.

   c. What does this tell you about each term of the sequence $p_1, p_2, p_3, \ldots$ relative to arithmetic series?

In item 3, we recalled that summing up the terms of the sequence 1, 2, 3, … gives the *triangular numbers*, so called because that number of dots can be arranged in a triangular pattern as shown below. (Image from Weisstein, Eric W. "Triangular Number." From *MathWorld*--A Wolfram Web Resource. http://mathworld.wolfram.com/TriangularNumber.html)



The sequence 1, 2, 3, … starts with 1 and adds 1 each time. Next we consider what happens if we start with 1 and add 2 each time.

9. The sequence that starts with 1 and adds 2 each time is very familiar sequence.

   a. List the first five terms of the sequence and give its common mathematical name.

   b. Why is the sequence an arithmetic sequence?

   c. Complete the table below for summing terms of this sequence.

Number of terms to be summed	Indicated sum of terms	Value of the sum
1	1	1
2	1 + 3	
3		
4		
5		
6		

   d. Give a geometric interpretation for the sums in the table.

   e. Fill in the last row of the table above when the number of terms to be summed is *n*, including a conjecture about the value for the last column. Use ". . . " in the indicated sum but include an expression, in terms of *n*, for the last term to be summed.

Our next goal is justification of the conjecture you just made in item 9, part e. However, rather than work on this specific problem, it is just as easy to consider summing up any arithmetic sequence. The sum of the terms of a sequence is called a series so, as we proceed, we'll be exploring closed form formulas for ***arithmetic series.***

10. Let *a* and *d* be real numbers and let $a_1, a_2, a_3, \ldots$ be the arithmetic sequence with first term *a* and common difference *d*. We'll use some standard notation for distinguishing between a sequence and the corresponding series which sums the terms of the sequence.

    a. Fill in the table below with expressions in terms of *a* and *d*.

Term of the sequence	Expression using *a* and *d*
a_1	
a_2	
a_3	
a_4	
a_5	
a_6	

    b. Let $s_1, s_2, s_3, \ldots$ be the sequence defined by the following pattern:

$$s_1 = a_1$$
$$s_2 = a_1 + a_2$$
$$s_3 = a_1 + a_2 + a_3$$
$$s_4 = a_1 + a_2 + a_3 + a_4$$

•
•
•

$$s_n = a_1 + a_2 + \cdots + a_n$$

    Conjecture and prove a closed form formula for the arithmetic series s_n.

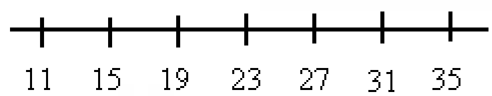
    c. Use the sum of an arithmetic series formula to verify the conjecture for the sum of the arithmetic series in item 9.

11. Next we'll explore a different approach to a closed form formula for arithmetic series, one that is quite useful when we are examining the type of series that occurs in a problem like the following. (This item is adapted from the "Common Differences" Sample Secondary

Task related to K-12 Mathematics Benchmarks of The American Diploma Project, copyright 2007, Charles A. Dana Center at the University of Texas at Austin.)

Sam plays in the marching band and is participating in the fund raiser to finance the spring trip. He's taking orders for boxes of grapefruit to be delivered from a Florida grove just in time for the Thanksgiving and Christmas holiday season. With the help of his parents and grandparents, he is trying to win the prize for getting the most orders. The number of orders he got for each day of the last week are, respectively, 11, 15, 19, 23, 27, 31, 35. What is the total number of orders?

a. The total number of orders is the arithmetic series $11+15+19+23+27+31+35$. The numbers in the corresponding arithmetic sequence are plotted on a number line below.

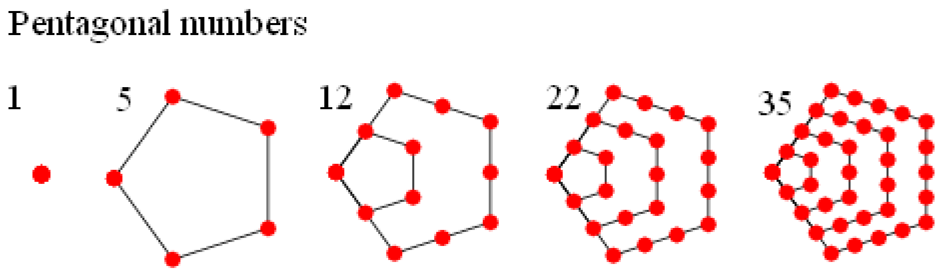


Find a shortcut for finding the sum of these seven terms. State your shortcut as a mathematical conjecture, and give a justification that your conjecture works.

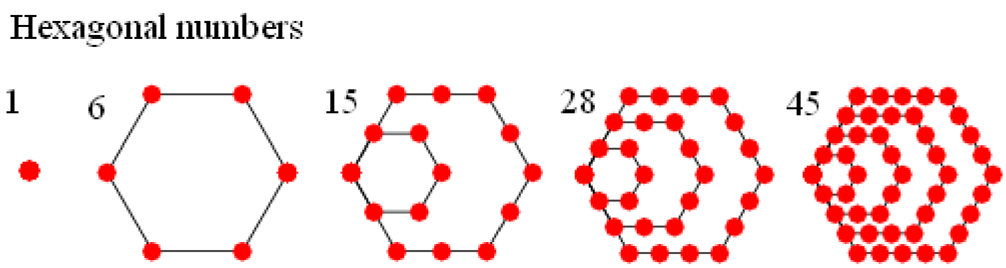
b. Test your conjecture from part a on an arithmetic series with exactly 5 terms. Does it work for a series with exactly 9 terms? Does your conjecture apply to a series with exactly 6 terms?

c. Modify your conjecture, if necessary, so that it describes a general method for finding any finite sum of consecutive terms from an arithmetic sequence. Give a justification that your method will always work.

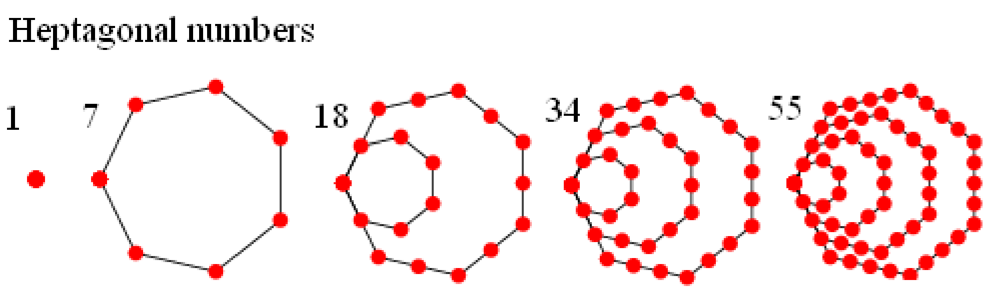
In addition to the well known triangular and square numbers, there are ***polygonal numbers*** corresponding to each type of polygon. The names relate to the figures that can be made from that number of dots. For example, the first five pentagonal, hexagonal, and heptagonal numbers are shown below.



(Image from Weisstein, Eric W. "Pentagonal Number." From *MathWorld*--A Wolfram Web Resource. http://mathworld.wolfram.com/PentagonalNumber.html)



(Image from Weisstein, Eric W. "Hexagonal Number." From *MathWorld*--A Wolfram Web Resource. http://mathworld.wolfram.com/HexagonalNumber.html)



(Image from Weisstein, Eric W. "Heptagonal Number." From *MathWorld*--A Wolfram Web Resource. http://mathworld.wolfram.com/HeptagonalNumber.html)

12. As shown above, the number 1 is considered to be a polygonal number for each size polygon. The actual numbers in the sequence of each type of polygonal numbers are sums of an arithmetic series; for example, the n^{th} triangular number is the sum of the arithmetic series $1+2+\cdots+n$. Use your knowledge of arithmetic sequences and series and of polygonal numbers complete the table below.

Polygonal number name	First five terms of the sequence of such numbers	Arithmetic sequence whose terms sum to form the numbers	Values of a, d	Formula for n^{th} polygonal number
triangular				
square				
pentagonal				
hexagonal				
heptagonal				
octagonal				

13. In 2001, the owner of two successful restaurants in south metropolitan Atlanta developed a long term expansion plan that led to the opening of five more restaurants in 2002. In 2003, eight new restaurants opened; in 2004, eleven new restaurants opened. Assume that the owner continued with this plan of increasing the number of new restaurants by three each year.

a. How many restaurants did the owner have in operation by the end of 2007?

b. If the owner continues with this plan, in what year will the 500^{th} restaurant open?

Student Edition: Non-stop Sports Culminating Task

Peachtree Plains High School field's teams in all Georgia High School Association sports so on any day of the school year many students are practicing, playing, or competing.

1. Monica is on the golf team. When she hits the ball with enough force to give it an initial speed of 112 feet per second, the height of the ball in feet is given by

$$h(t) = -16t^2 + 112t,$$

where t is the number of seconds after Monica's club hits the ball.

   a. Convert the formula for the function h into vertex form.

   b. Graph the function, state its domain and range, intervals of increase and decrease, intercepts, vertex, and axis of symmetry. Interpret each of these features of the graph in terms of the path of the golf ball.

2. One day at football practice, Darrell, the kicker, punted the ball so that its height in feet above the ground was given by

$$h(t) = -16t^2 + 40t + 4,$$

where t is the number of seconds since the ball was punted.

   a. At what time was the football 20 feet or higher above the ground?

   b. At what time was the football less than its height when punted?

3. Based on enrollment projections school officials have decided that, for the next school year, the classroom trailers will be moved to the current practice football field and a new practice field will be located behind the school parking lot. Including end zones, the practice field will be 120 yards by 53 yards in order to closely approximate a standard field. However, the owner of a local nursery has donated enough grass seed to plant 81,000 square feet. Since they have more than enough grass seed for the practice field, school officials would like to plant a uniform border around the field. What are the dimensions of the 81,000 square foot rectangular area that should planted for the practice field and uniform border?

4. Last season the volleyball team played in an invitational tournament held at the Emory University. During the tournament, each invited team played one match against each other participating team. In volleyball, the first team to win three games, or sets, wins a match. At this tournament every match required five sets to determine the winner.

a. Write a formula for the function that expresses the total number of sets as a function as the total number of teams in the tournament.

b. If there were a total of 140 sets played during the tournament, how many teams participated?

5. The Peachtree Plains High School softball field is a regulation field so that the home plate and first, second, and third bases are the vertices of a square with side length of 60 feet.

   a. What is the distance from home plate to second base?

   b. First base is 36 feet east and 48 feet north of home plate. Set up a coordinate system with each unit representing 1 foot, east represented by the direction of the positive *x*-axis, and north represented by the positive *y*-axis. Assume that the home plate of the softball field is located at the origin and regard the system as a complex plane. What complex number is identified with first base?

   c. Find the complex number corresponding to third base. Justify your reasoning.

   d. Find the sum of the complex numbers representing first and third bases and show that this number represents the location of second base.

6. Gary is on the baseball team. During a crucial game, the bases were loaded , and Gary was at bat with a full count of 3 balls and 2 strikes when he connected with the next pitch. The ball was 3 feet above the ground when it left Gary's bat, and it reached its greatest height of 28 feet when it was above the head of the center fielder, who was 200 feet from home plate at the time.

   a. The outfield fence is 8 feet high and at center field is 375 feet from home plate. If the ball cleared the fence and went out of the baseball field, Gary had a grand slam home run. Was Gary's hit a home run? Justify your answer algebraically and use a graphing utility to verify it graphically.

   b. The height of the ball is a function of its distance, x, from home plate. Denote this function by H, sketch the graph based on the given information, and then describe how to the graph H using transformations of the graph of the function $f(x) = x^2$.

   c. Write the formula for $H(x)$ in standard form.

7. Alexa is on the junior varsity basketball team and hopes to be on the varsity team next year. She has decided on a training plan to get her ready for tryouts next year. Part of her plan involves work on free throws, one of her weaknesses. Starting four weeks before tryouts she will complete 20 free throws on the first day, 25 on the second day, 30 on the third, and

so forth. What is the total number of free-throws that she will toss during this four weeks of focused free-throw practice?

8. When a player shoots a free throw, its height above the floor is determined by the force of gravity, the initial velocity of the ball, and the height at the moment it leaves the shooter's hand. It is standard to represent the initial velocity, that is the velocity of the ball at the moment it leaves the shooters hand, by v_0. Typically, when Alexa shoots a free throw, the ball is 7.5 feet above the floor when it leaves her hand. When this is so, the height of Alexa's free throw, in feet above the floor, is given by the function

$$h(t) = -16t^2 + v_0 t + 7.5,$$

where t is the number of seconds from the instant the ball leaves her hand.

The rim of the basketball net is 10 feet above the floor. There are many factors that determine whether a free throw goes through the net, but a basic requirement is that it get at least 10 feet high.

a. For what initial velocity will one of Alexa's free throws go exactly 10 feet high? Explain your reasoning.

b. What happens to the ball for an initial velocity greater than the one in part a? Explain your reasoning.

c. What happens to the ball for an initial velocity less than the one in part a? Explain your reasoning.

GPS Transition Frameworks

Mathematics I → GPS Geometry
Piecewise Functions and Curve Fitting

1st Edition
May, 2011
Georgia Department of Education

Table of Contents

Mathematics I → GPS Geometry
Transition Unit:
Piecewise Functions and Curve Fitting

INTRODUCTION:

Students who are transitioning from GPS Mathematics I to GPS Geometry will be missing content from GPS Algebra mainly those dealing with some of the more advanced algebra topics. This transition unit will provide students with tasks to introduce them to piecewise functions and curve fitting.

In GPS Mathematics I, students built on their knowledge of functions by exploring quadratic, cubic, absolute value, and rational functions. They also developed ways of discussing and notation for functions. The Transition Unit on Quadratics and Complex Numbers deepened students' knowledge of quadratic functions. This unit wraps up the study of basic functions with piecewise and step functions. Discontinuous and continuous piecewise functions, and their utility, are discussed. GPS Mathematics I introduced students to data through the study of summary statistics. This unit builds on that knowledge and extends it to linear and quadratic regression. Regression and different basic functions are combined as further illustration of the utility of these mathematical ideas.

ENDURING UNDERSTANDINGS:

- Functions can be defined using multiple rules or statements of correspondence for the inputs and outputs; these are piecewise functions. The domain and rule for the individual pieces must be specified for most piecewise functions.
- Piecewise functions can be defined by combining different families of functions.
- Absolute value functions can be defined as a piecewise function, using linear functions with opposite slopes that intersect at the vertex.
- Absolute value equations and inequalities can be solved both algebraically and graphically.
- Step functions are specific piecewise functions that can be defined using a single rule or correspondence.
- Choosing an appropriate model for a set of data requires examining a plot of the data and analyzing the fit of the data to the model.
- Correlation provides information about the strength and direction of a linear relationship, but it does not indicate causation between the two quantitative variables.
- Most data does not perfectly fit any one function rule; some amount of error usually exists. Piecewise functions can be employed to provide the best model of a set of data.

KEY STANDARDS ADDRESSED:

MM2A1. Students will investigate step and piecewise functions, including greatest integer and absolute value functions.

a. Write absolute value functions as piecewise functions.

b. Investigate and explain characteristics of a variety of piecewise functions including domain, range, vertex, axis of symmetry, zeros, intercepts, extrema, points of discontinuity, intervals over which the function is constant, intervals of increase and decrease, and rates of change.
c. Solve absolute value equations and inequalities analytically, graphically, and by using appropriate technology.

MM2D2. Students will determine an algebraic model to quantify the association between two quantitative variables.

a. Gather and plot data that can be modeled with linear and quadratic functions.
b. Examine the issues of curve fitting by finding good linear fits to data using simple methods such as the median-median line and "eyeballing".
c. Understand and apply the process of linear and quadratic regression for curve fitting using appropriate technology.
d. Investigate issues that arise when using data to explore the relationship between two variables, including confusion between correlation and causation.

RELATED STANDARDS ADDRESSED:

MA1A1. Students will explore and interpret the characteristics of functions, using graphs, tables, and simple algebraic techniques.

a. Represent functions using function notation.
b. Graph the basic functions $f(x) = x^n$ where $n = 1$ to 3, $f(x) = \sqrt{x}$, $f(x) = |x|$, and $f(x) = 1/x$.
c. Graph transformations of basic functions including vertical shifts, stretches, and shrinks, as well as reflections across the x- and y-axes. [*Previewed in this unit.*]
d. Investigate and explain the characteristics of a function: domain, range, zeros, intercepts, intervals of increase and decrease, maximum and minimum values, and end behavior.
e. Relate to a given context the characteristics of a function, and use graphs and tables to investigate its behavior.
f. Explore rates of change, comparing constant rates of change (i.e., slope) versus variable rates of change. Compare rates of change of linear, quadratic, square root, and other function families.
g. Determine graphically and algebraically whether a function has symmetry and whether it is even, odd, or neither.

MM1P1. Students will solve problems (using appropriate technology).

a. Build new mathematical knowledge through problem solving.
b. Solve problems that arise in mathematics and in other contexts.
c. Apply and adapt a variety of appropriate strategies to solve problems.
d. Monitor and reflect on the process of mathematical problem solving.

MM1P2. Students will reason and evaluate mathematical arguments.

a. Recognize reasoning and proof as fundamental aspects of mathematics.
b. Make and investigate mathematical conjectures.

c. Develop and evaluate mathematical arguments and proofs.
d. Select and use various types of reasoning and methods of proof.

MM1P3. Students will communicate mathematically.
a. Organize and consolidate their mathematical thinking through communication.
b. Communicate their mathematical thinking coherently and clearly to peers, teachers, and others.
c. Analyze and evaluate the mathematical thinking and strategies of others.
d. Use the language of mathematics to express mathematical ideas precisely.

MM1P4. Students will make connections among mathematical ideas and to other disciplines.
a. Recognize and use connections among mathematical ideas.
b. Understand how mathematical ideas interconnect and build on one another to produce a coherent whole.
c. Recognize and apply mathematics in contexts outside of mathematics.

MM1P5. Students will represent mathematics in multiple ways.
a. Create and use representations to organize, record, and communicate mathematical ideas.
b. Select, apply, and translate among mathematical representations to solve problems.
c. Use representations to model and interpret physical, social, and mathematical phenomena.

Unit Overview:

The unit begins with developing the idea of piecewise functions through a semi real-world context. Students use graphs to write equations of piecewise functions and the graph piecewise functions using a variety of function families. Students graph continuous and discontinuous functions.

The second task continues the idea of determining equations for a piecewise function using real-world data, from a turnpike toll schedule, to determine the equations of the lines. This context provides a forum for the discussion of lines of best fit, including "eyeballing" (or visual approximation), linear regression, and median-median lines. This task continues by defining absolute value functions as piecewise functions and reviewing previous work with graphical shifts, stretches, and reflections. Finally, solving the absolute values equations and inequalities associated with the context wrap up the task.

The third task introduces another real world context, taxes. In addition to examining a real-world scenario, students look at the intersection of a line and a piecewise function graphically, determining the point of intersection algebraically and graphically. They also write persuasive arguments using mathematical ideas. This task includes discontinuous piecewise functions, including step functions.

The fourth task explicitly addresses step functions, specifically exploring greatest and least integer functions. Connections are drawn between these functions and absolute value functions. Students will continue working extensively with transformations of functions.

The last learning task explores linear and quadratic regression through the context of studying the orbital debris in space. The ideas of constant differences and correlation are also addressed.

Finally, the culminating task pulls all the standards together in the context of a fictional major league baseball team. In addition to fictional models, actual data from present major league teams is employed. The last task is designed to demonstrate the type of assessment activities students should be comfortable with by the end of the unit.

Throughout this unit, including the culminating task, it is assumed that students have access to a graphing utility that calculates regression lines, correlation, and quadratic regression. If students are to graph data by hand, it is assumed that they will superimpose curves over the data, on the same grid. Also, the unit emphasizes the reasonableness of results and models. Students should be encouraged to communicate their reasoning both in writing and in class discussions.

Vocabulary and Formulas:

Correlation coefficient – The correlation, *r*, measures the direction and strength of a *linear* relationship between two variables. Formula: $r = \frac{1}{n-1}\sum\left(\frac{x_i - \bar{x}}{s_x}\right)\left(\frac{y_i - \bar{y}}{s_y}\right)$.

Extrapolation – the use of the regression curve to make predictions outside the domain of values of the independent variable.

Greatest integer function (floor function) – The greatest integer function is determined by locating the *greatest integer less than or equal to* the x-value in question. Common notations: $f(x) = \lfloor x \rfloor$, $f(x) = [x]$ or $f(x) = [|x|]$.

Interpolation – Interpolation is used to make predictions within the domain of values of the independent variable.

Least integer function (ceiling function) – The least integer function is determined by locating the *least integer greater than or equal to* the x-value in question. Notation: $f(x) = \lceil x \rceil$.

Least squares regression line (LSRL) – This method is commonly called linear regression. The least-squares method minimizes the sum of the squared distance each data point is from the line. We can also think of the regression line as the line that minimizes the sum of the vertical distances between the data points and the line.

Linear regression – A straight line that describes how a dependent variable *y* changes as the independent variable *x* changes.

Median-median line –This method of linear regression separates the data into 3 equal groups, from smallest x-values to largest. The median x-values and y-values of each group are calculated. These medians, from smallest x-values to largest, are named (x_1, y_1), (x_2, y_2), (x_3, y_3). Then a line through the first and third medians is found. Finally, a line parallel to this line, 1/3 of the distance between the line and the remaining median is formed. The resulting line is of the following form $y = ax + b$, $a = \dfrac{y_3 - y_1}{x_3 - x_1}$, $b = \dfrac{y_1 + y_2 + y_3 - a(x_1 + x_2 + x_3)}{3}$. This method of regression is more resistant to outliers than the least squares regression line.

Method of finite differences – Method for determining if data points with equally spaced *x*-values form a linear, quadratic, etc., relationship.

Piecewise functions – a function whose output is determined by input rules specified over different intervals; that is, it is a function with two or more "pieces" over different domains. The pieces may consist of similar or different families of functions.

Quadratic Regression – A quadratic equation that describes how a dependent variable *y* changes as the independent variable *x* changes.

Regression – A curve that describes how a dependent variable *y* changes as the independent variable *x* changes. (Linear and quadratic regression are explored in this unit.)

Step function – Piecewise functions that forms as a set of discontinuous steps.

Training for a Race Learning Task

Saundra is a personal trainer at a local gym. Earlier this year, three of her clients asked her to help them train for an upcoming 5K race. Though Saundra had never trained someone for a race, she developed plans for each of her clients that she believed would help them perform their best.

She wanted to see if her plans were effective, so when she attended the race to cheer them on, she collected data at regular intervals along the race. Her plan was to create graphs for each of the runners and compare their performances.

Since each had an individualized strategy, each runner ran a different plan during the race. One of her clients - Sue, the oldest client - was supposed to begin slowly, increasing over the first kilometer until she hit a speed which she believed she could maintain over the rest of the race.

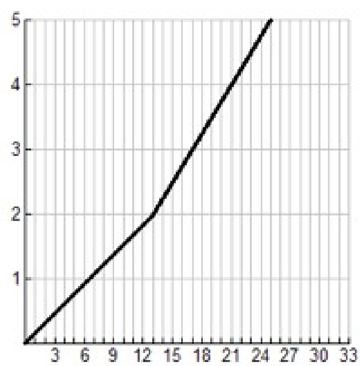
Her second client, Jim, was supposed to begin with a strong burst for the first kilometer, then slow to a steady pace until the final kilometer when he would finish with a strong burst.

Her third client, Jason, is a very experienced runner. His plan was to run at a steady pace for the first two kilometers, then run at his maximum speed for the final 3 kilometers.

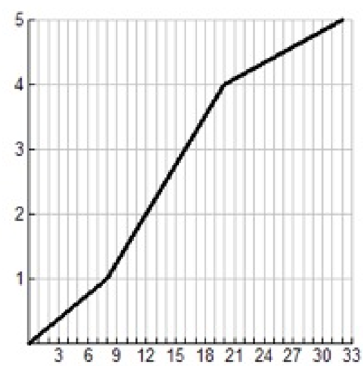
Each of the clients came close to performing as they planned.

1. Saundra created graphs for two of the clients, but she set them aside without labeling the graphs. Now she cannot remember whose graphs she has. Can you identify the client based on these graphs? Explain how you know.

Graph 1



Graph 2



2. Describe how the runner in Graph #1 performed. For what distance did the runner increase speed, decrease speed, or maintain speed? Be sure to specify the units for discussing speed.

3. Compare the performance of the runner in Graph #2 to the runner in Graph #1.

4. Saundra found the data for her third client on her desk.

a. Graph the data for this runner.

Time	***Km***
4:00	***1***
8:30	***2***
13:00	***3***
22:00	***4***
26:00	***5***

b. Is the data linear? Quadratic? How can you tell if data is linear or quadratic? Does it fit any of the models you have studied?

c. What do you notice about the graphs of the three runners? Can any of their graphs be modeled by a single function?

Sometimes, a single function is not adequate for modeling data. In this case, a single line does not really show how each runner performed at each interval. A **piecewise function** is a function whose output is determined by input rules specified over different intervals; that is, it is a function with two or more "pieces" over different domains. The pieces may consist of similar or different families of functions.

5a. Connect the points in the third graph to show the "pieces" of different performance levels by the runner. Why does it make sense to connect the points in this situation?

b. We now want to determine the piecewise functions that describe the graphs of the runners' performances. What types of functions are present in each of the three graphs?

6. For each runner, write the equations of the "pieces," or segments, of the graph. Be sure to indicate the appropriate interval for each piece (for which x-values that equation yields the correct graph).

a. Graph #1: Make a chart of the functions involved and the domains for the specific functions.

Function	Domain

Because the performance of the runner in Graph #1 can be modeled by a piecewise **function**, we should use notation that indicates we have a single function.

We write $f(x)=\begin{cases} \text{rule for function 1, domain for function 1} \\ \text{rule for function 2, domain for function 2} \\ \ldots \\ \text{rule for last function, domain for last function} \end{cases}$.

b. Write the piecewise function, f, that models the performance of the runner in Graph #1. How is the slope of each piece related to the runner's speed?

c. Write the piecewise function, g, for the performance of the runner in Graph #2 and h, for the performance of the runner in Graph #3..

Piecewise functions do not always have to be line segments. The "pieces" could be pieces of any kind of graph.

7. Try to graph some of these piecewise functions. You may find it helpful to use what you already know about transformations of the parent functions. (Recall from your previous work with inequalities the implications of the different inequality signs.)

Traveling on the Turnpike Learning Task

Consider the toll schedule given by the Ohio Turnpike Commission. In this task, we will explore the relationship between the distance you travel (based on the number of the exit you enter and leave the turnpike) and the amount of the toll.

1. Suppose we got on the turnpike at Sandusky-Norwalk.
a. Graph the ordered pairs (exit #, fare) for 5 exits whose numbers are larger than 118 (Sandusky-Norwalk).

b. We need to write a linear equation for the points in this scatter plot. For what values of x will this graph and its equation be appropriate?

c. Looking at the five data points you chose and plotted, approximate the line of best fit for the points and explain the reason for your line. Draw the line on your plot.

In addition to approximating the line of best fit, we can also use a strategy called the median-median line.[1] This method of regression by the median-median method separates the data into 3 equal groups, from smallest x-values to largest. The median x-values and y-values of each group are found. These medians, from smallest x-values to largest, can be named (x_1, y_1), (x_2, y_2), (x_3, y_3). Then a line through the first and third medians is found. Finally, a line parallel to this line, 1/3 of the distance between the line and the remaining median is formed. The resulting line is of the following form y = ax + b, $a = \dfrac{y_3 - y_1}{x_3 - x_1}$, $b = \dfrac{y_1 + y_2 + y_3 - a(x_1 + x_2 + x_3)}{3}$.

Although this method can be accomplished by hand, graphing technology can also be used to determine the median-median line.

d. Determine the median-median line for the five points you've been using and draw the line on the same axes as your approximated line.

[1] Information on the median-median line obtained from Walters, E. J., Morrell, C. H., & Auer, R. E. (2006). An investigation of the median-median method of linear regression. *Journal of Statistics Education*, *14*(2). Available online at www.amstat.org/publications/jse/v14n2/morrell.html.
Teachers and students interested in learning more about median-median methods, including the history, are encouraged to access the article.

A third way to determine the line of best fit is using the **least-squares method of linear regression**, commonly called linear regression. The least-squares method minimizes the sum of the squared distance each point is from the line. We can also think of the regression line as the line that minimizes the sum of the vertical distances between the data points and the line.

e. Use the linear regression capabilities of your calculator (TI-Interactive, Fathom, Excel, etc.) to determine the line of best fit for your five data points. Graph this new line on the plot with your data and previous best-fit lines.

f. To see that this technology-generated line does, indeed, minimize the sum of the vertical distances between the data points and line, make 3 tables, similar to the one below. Then find the sum of the distances.

Exit Number	**Toll**	**Predicted Toll from Line**	**Vertical Distance between Approximation and Toll**

g. The median-median method of linear regression is based on finding the median values of data points. Calculating the least squares method by hand requires using means and standard deviations. From what you already know about medians and means, what might be one reason to use the median-median line instead of the more common line of regression, the least squares line?

2. Now, let's look at the equation for the tolls if we began the turnpike at Exit 118 and drove in the other direction.

a. Graph the ordered pairs (exit #, fare) for 5 exits whose numbers are smaller than 118 (Sandusky-Norwalk).

b. Approximate the line of best fit (without using median-median or least-squares techniques) and explain your method. Draw the line on your scatter plot.

c. Using appropriate technology (or by hand), construct the median-median line and draw it on your scatter plot.

d. Using appropriate technology, determine the least-squares regression equation and draw it on your scatter plot.

e. Consider your least-squares regression line. What does the slope mean in the context of this problem? Does the y-intercept have a contextual meaning? If so, what does it represent?

3. Our next goal is to write the equation of the function that models the toll schedule whenever we begin at the Sandusky-Norwalk exit and travel in either direction.

a. Graph your two least-squares regression equations (from #1e and #2d) on the same axes. Find the intersection of the lines algebraically and verify it graphically. What does that intersection represent? Is this what you expected? Explain.

b. We want to write a single function to represent this graph. Using the equations and intersection you found, write a piecewise function for the tolls to be paid when entering the Ohio Turnpike from the Sandusky-Norwalk exit.

c. The above graph may remind you of the graph of the absolute value function. In Unit One, you encountered the absolute value function. This function is a special case of the piecewise function and can be defined accordingly. We write an absolute value function as $f(x) = |x|$.

- Draw the graph of $f(x) = |x|$, over the domain $-5 \le x \le 5$.

- Identify the pieces of the function that could be defined using different linear functions. As you did in Part 1 #6, list the functions and their respective domains. Then write the absolute value function as a piecewise function.

Function	Domain

- In Unit One, you also explored the impact of transformations on the graphs of functions. For any general function, $g(x) = a * f(b(x - c)) + d$, explain how each variable impacts the graph of $y = f(x)$.

d. Similarly, we can approximate the function for the tolls required when traveling the Ohio Turnpike from the Sandusky-Norwalk exit as a single absolute value function.

- Considering the context, where should the vertex of the absolute value function be located? Why? How does the value of the vertex influence the equation of the function?

- One of the characteristics of absolute value functions is its symmetry. Where is the line of symmetry in $f(x) = |x|$? Where should the line of symmetry for our graph be? Why? How do we know that there is symmetry in this graph?

- Consider both parts of the Sandusky-Norwalk toll graph. Determine a vertical stretch for the absolute value function that you think will best fit this data. Explain your choice.

- Using absolute value notation, write the equation of the function describing the tolls owed when traveling the turnpike from Exit 118.

- Using piecewise notation, write the equation of the function describing the tolls owed when traveling the turnpike from Exit 118.

- Graph your new function on the same axes with your data points and your regression lines. How well do you think your approximated function models the actual data?

e. How would you use your function from part d to determine the amount of the toll if a new exit were added between two existing exits? Describe your strategy. Add an exit between any two existing exits and test your strategy. Where did you place the exit, and how much was the toll?[2] Explain if your prediction is reasonable.

4. Use your absolute value function to answer the following. Assume that all whole numbers are possible exit numbers.
a. For what values is the toll exactly \$2.00? Write the equation or inequality whose solution would answer this question. Then solve algebraically.

We can solve the same problem graphically.
b. Graph the equations $y = |0.06315\ (x - 118)|$ and y = 2 on the same axes. What do you notice about the points of intersection?

c. Now, go back to the equation or inequality from part a. Move all terms to one side of the equation, leaving only a 0 on the other side. How can you use this equation and a graph to determine the answer to the question of where the toll equals \$2.00? Explain. Then draw the graph.

d. Zeke and Zelda want to take the Ohio Turnpike to avoid traffic, but they realize that they only have \$3.00 in cash. If they enter the turnpike at Sandusky-Norwalk, how far can they travel in either direction? For what values is the toll at most \$3.00? Again, write the equation or inequality. Then solve algebraically.

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[2] Question adapted from the NCTM Illuminations lesson "Taking Its Toll." The entire activity can be found at http://illuminations.nctm.org/LessonDetail.aspx?ID=L571.

e. Let's solve this same problem graphically. Let's first graph the equations $y = |0.06315(x - 118)|$ and $y = 3$ on the same axes. What do you notice about the points of intersection?

f. For what values is the toll charged more than $1.50? Again, write the equation or inequality. Then solve both algebraically and graphically.

g. Zoe, Zeke and Zelda's mother, told her children that if they wanted to take the Ohio Turnpike to the mall, they would need at least $1.50 but no more than $3.00. Based on this information, at which exits could the mall be located? Write an equation or inequality to model the question. Then solve either algebraically or graphically. Explain why you chose your method.

SCHEDULE of TOLLS CLASS 2

Effective January 1, 2007

Class 2 Interchange	MP	2	13	25	34	39	52	59	64	71	81	91	110	118	135	140	142	145	151	152	161	173	180	187	193	209	215
Westgate	2		0.75	1.50	2.25	2.50	3.25	3.75	4.00	4.50	5.00	5.75	6.75	7.25	8.50	8.75	9.00	9.00	9.50	9.50	10.00	10.75	11.25	11.75	12.00	13.00	13.25
Bryan-Montpelier	13	0.75		0.75	1.25	1.75	2.50	2.75	3.25	3.50	4.25	4.75	6.00	6.50	7.50	8.00	8.00	8.25	8.50	8.75	9.25	10.00	10.25	10.75	11.25	12.25	12.50
Archbold-Fayette	25	1.50	0.75		0.50	1.00	1.75	2.00	2.50	2.75	3.50	4.00	5.25	5.75	6.75	7.25	7.25	7.50	7.75	7.75	8.50	9.25	9.50	10.00	10.50	11.50	11.75
Wauseon	34	2.25	1.25	0.50		0.50	1.00	1.50	1.75	2.25	3.00	3.50	4.75	5.25	6.25	6.50	6.75	7.00	7.25	7.25	8.00	8.50	9.00	9.50	10.00	10.75	11.25
Delta-Lyons	39	2.50	1.75	1.00	0.50		0.75	1.25	1.50	2.00	2.50	3.25	4.50	5.00	6.00	6.25	6.50	6.50	7.00	7.00	7.50	8.25	8.75	9.25	9.50	10.50	11.00
Toledo Airport-Swanton	52	3.25	2.50	1.75	1.00	0.75		0.50	0.75	1.25	1.75	2.50	3.50	4.00	5.25	5.50	5.50	5.75	6.25	6.25	6.75	7.50	8.00	8.25	8.75	9.75	10.00
Maumee-Toledo	59	3.75	2.75	2.00	1.50	1.25	0.50		0.50	0.75	1.50	2.00	3.25	3.75	4.75	5.00	5.25	5.25	5.75	5.75	6.25	7.00	7.50	8.00	8.25	9.25	9.75
Perrysburg-Toledo	64	4.00	3.25	2.50	1.75	1.50	0.75	0.50		0.50	1.00	1.75	2.75	3.25	4.50	4.75	4.75	5.00	5.50	5.50	6.00	6.75	7.25	7.50	8.00	9.00	9.25
Stony Ridge-Toledo	71	4.50	3.50	2.75	2.25	2.00	1.25	0.75	0.50		0.75	1.25	2.50	3.00	4.00	4.25	4.50	4.50	5.00	5.00	5.50	6.25	6.75	7.25	7.50	8.50	9.00
Elmore-Woodville-Gibsonburg	81	5.00	4.25	3.50	3.00	2.50	1.75	1.50	1.00	0.75		0.50	1.75	2.25	3.25	3.75	3.75	4.00	4.25	4.50	5.00	5.75	6.25	6.50	7.00	8.00	8.25
Fremont-Port Clinton	91	5.75	4.75	4.00	3.50	3.25	2.50	2.00	1.75	1.25	0.50		1.25	1.75	2.75	3.00	3.25	3.25	3.75	3.75	4.25	5.00	5.50	6.00	6.25	7.25	7.75
Sandusky-Bellevue	110	6.75	6.00	5.25	4.75	4.50	3.50	3.25	2.75	2.50	1.75	1.25		0.50	1.50	2.00	2.00	2.25	2.50	2.50	3.25	4.00	4.25	4.75	5.25	6.25	6.50
Sandusky-Norwalk	118	7.25	6.50	5.75	5.25	5.00	4.00	3.75	3.25	3.00	2.25	1.75	0.50		1.00	1.25	1.50	1.75	2.00	2.00	2.75	3.50	3.75	4.25	4.75	5.75	6.00
Vermilion	135	8.50	7.50	6.75	6.25	6.00	5.25	4.75	4.50	4.00	3.25	2.75	1.50	1.00		0.50	0.50	0.50	1.00	1.00	1.50	2.25	2.75	3.25	3.50	4.50	5.00
Amherst-Oberlin	140	8.75	8.00	7.25	6.50	6.25	5.50	5.00	4.75	4.25	3.75	3.00	2.00	1.25	0.50		0.50	0.50	0.75	0.75	1.25	2.00	2.50	3.00	3.25	4.25	4.75
Lorain Co. West	142	9.00	8.00	7.25	6.75	6.50	5.50	5.25	4.75	4.50	3.75	3.25	2.00	1.50	0.50	0.50		N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A
Lorain-Eylria	145	9.00	8.25	7.50	7.00	6.50	5.75	5.25	5.00	4.50	4.00	3.25	2.25	1.75	0.50	0.50	N/A		0.50	0.50	1.00	1.75	2.25	2.50	3.00	4.00	4.25
N.Ridgeville-Cleveland	151	9.50	8.50	7.75	7.25	7.00	6.25	5.75	5.50	5.00	4.25	3.75	2.50	2.00	1.00	0.75	N/A	0.50		N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A
N.Olmsted-Cleveland	152	9.50	8.75	7.75	7.25	7.00	6.25	5.75	5.50	5.00	4.50	3.75	2.50	2.00	1.00	0.75	N/A	0.50	N/A		0.50	1.25	1.75	2.25	2.50	3.50	4.00
Strongsville-Cleveland	161	10.00	9.25	8.50	8.00	7.50	6.75	6.25	6.00	5.50	5.00	4.25	3.25	2.75	1.50	1.25	N/A	1.00	N/A	0.50		0.75	1.25	1.50	2.00	3.00	3.25
Cleveland	173	10.75	10.00	9.25	8.50	8.25	7.50	7.00	6.75	6.25	5.75	5.00	4.00	3.50	2.25	2.00	N/A	1.75	N/A	1.25	0.75		0.50	0.75	1.25	2.25	2.50
Akron	180	11.25	10.25	9.50	9.00	8.75	8.00	7.50	7.25	6.75	6.25	5.50	4.25	3.75	2.75	2.50	N/A	2.25	N/A	1.75	1.25	0.50		0.50	0.75	1.75	2.25
Streetsboro	187	11.75	10.75	10.00	9.50	9.25	8.25	8.00	7.50	7.25	6.50	6.00	4.75	4.25	3.25	3.00	N/A	2.50	N/A	2.25	1.50	0.75	0.50		0.50	1.25	1.75
Ravenna	193	12.00	11.25	10.50	10.00	9.50	8.75	8.25	8.00	7.50	7.00	6.25	5.25	4.75	3.50	3.25	N/A	3.00	N/A	2.50	2.00	1.25	0.75	0.50		1.00	1.25
Warren	209	13.00	12.25	11.50	10.75	10.50	9.75	9.25	9.00	8.50	8.00	7.25	6.25	5.75	4.50	4.25	N/A	4.00	N/A	3.50	3.00	2.25	1.75	1.25	1.00		0.50
Lordstown-West	215	13.25	12.50	11.75	11.25	11.00	10.00	9.75	9.25	9.00	8.25	7.75	6.50	6.00	5.00	4.75	N/A	4.25	N/A	4.00	3.25	2.50	2.25	1.75	1.25	0.50	

A Taxing Situation Learning Task

1. Piecewise functions are used to describe a wide variety of data sets. One good example of a piecewise function is income tax. The 2007 Federal Tax Rate Schedule for a single person filing taxes is as follows.

Taxable Income	Tax
$0 - $7,825	10%
$7,825 - $31,850	782.50 plus 15% of amount over $7,825
$31,850 - $77,100	$4,386.25 plus 25% of the amount over $31,850
$77,100 - $160,850	$15,698.75 plus 28% of the amount over $77,100
$160,850 - $349,700	$39,148.75 plus 33% of the amount over $160,850
$349,700 +	$101,469.25 plus 35% of the amount over $349,700

a. Write the equation for a piecewise function, c, that would accurately represent the income tax for a single person in the United States according to this current tax plan.

b. Compare the salaries and the taxes owed by each of these single US taxpayers in 2007. Include in your discussion the percent of their income they retain after taxes.

1. A teacher who made $36,000
2. An attorney who made $80,000
3. A dental assistant who made $28,000
4. A radiologist who made $200,000
5. A professional athlete who made $1.5 million

c. Graph the function. Be sure to label your axes. Is this a continuous or discontinuous function? Explain how you know.

2. Jacob Jones has made a proposal for a flat tax for US taxpayers. He has proposed that every taxpayer should pay 17% of their taxable income in taxes.

a. Write an equation for the function, f, to represent Mr. Jones's proposal.

b. Graph this equation on the same coordinate plane as #1c.

c. At what income level would a flat tax be the same as our current tax rate? Explain.

3. Jill Jackson proposed a different type of tax, a hybrid between the current system and a flat tax. This tax would impose a different flat tax depending on one's income. We'll call this a progressive flat tax. Here is her plan, similar in structure to the tax plan in Australia.

Taxable Income	Tax
$0 - $7,825	10% of income
$7,826 - $31,850	15% of income
$31,851 - $77,100	17% of income
$77,101 - $160,850	20% of income
$160,851 - $349,700	22% of income
$349,701 +	25% of income

(Note: When completing tax returns, taxpayers round to the nearest dollar. How should we alter the taxable incomes above to include all likely incomes? This will be important when we graph our function.)

a. Write the equation for a piecewise function, p, that would accurately represent the income tax for a single person in the United States under Jill's plan.

b. Graph this equation on the same coordinate plane as #1c and #2b.

c. Determine the domain and range for the progressive flat tax graph. Is this a true function? Explain. Is this a continuous or discontinuous graph? Explain how you know.

4. Maria Middleton, an aspiring politician, decided to investigate a fourth tax plan. Maria wanted to look at a variation of Jill's plan. For each tax bracket, she found the midpoint. She then used the tax amount from that point as the owed taxes for each person in that tax bracket.

a. Determine the function, m , that models Maria's tax plan.

b. Graph Maria's plan with the graphs of the other three plans. How is the graph of Maria's plan similar to or different from the other three tax plans? Be sure to discuss rates of change and intervals in which the functions are increasing, decreasing, or constant.

c. Determine the domain and range of Maria's plan (up through the fifth tax bracket). Is this a continuous or discontinuous function?

d. As Maria's campaign manager and economic advisor, what feedback would you give Maria about her plan? Justify your feedback with mathematical evidence.

5. Consider the taxpayers from question 1b.

a. Make a chart comparing the salaries and taxes paid under the first three plans.

b. Pick two taxpayers who would prefer different tax plans. Write a persuasive argument from each perspective. For both arguments, provide mathematical justifications. (Each argument should be approximately a paragraph.)

c. According to the US Census Bureau, the median income in the US for the year 2006 was $48,201 and about 81% of the working population of the US had an income of less than $100,000. Look at the graphs and the above table. Which type of tax do you believe a majority of US taxpayers would prefer? Explain.

Parking Deck Pandemonium Learning Task

The type of piecewise function proposed by Maria Middleton in “A Taxing Situation” is called a **step function**. Although there are many different kinds of step functions, two common ones are the **least integer function**, or the “ceiling function,” and the **greatest integer function**, sometimes called the “floor function.”

The fee schedule at parking decks is often modeling using a step function. Let’s look at a few different parking deck rates to see the step functions in action. (Most parking decks have a maximum daily fee. However, for our exploration, we will assume that this maximum does not exist.)

1. As you drive through town, Pete’s Parking Deck advertises free parking up to the first hour (i.e., the first 59 minutes). Then, the cost is $1 each additional hour or part of an hour. (If you park for 1 ½ hours, you owe $1; 2 hours costs $2.)

a. Make a table listing the fees for parking at Pete’s for up to 5 hours. Be sure to include some non-integer values. Then draw the graph that illustrates the fee schedule at Pete’s.

b. Use your graph to determine the fee if you park for 3 ½ hours. What about 3 hours, 55 minutes? 4 hours, 5 minutes?

c. What are the x- and y-intercepts of this graph? What do they represent?

d. What do you notice about the time and the corresponding fees? Make a conjecture about the fee if you were to park at Pete’s Parking Deck for 10 ½ hours (assuming no maximum fee).

Similar to the other situations explored in this unit, this graph could be represented by a piecewise function.

e. Write a piecewise function to model the fee schedule at Pete’s Parking Deck.

The fee schedule at Pete's Parking Deck is more efficiently modeled by the **greatest integer**, or floor, function. A number of different notations are used for the greatest integer function. Here are the three most common: $f(x) = \lfloor x \rfloor$, $f(x) = [x]$, or $f(x) = [|x|]$. The greatest integer function is determined by locating the *greatest integer less than or equal to* the x-value in question.

f. Evaluate each of the following by determining the greatest integer less than or equal to the x-value. That is, $f(x) = [x]$.
   1. $f(4.5)$ 2. $f(7)$ 3. $f(-1.2)$ 4. $f(-.1)$ 5. $f(.6)$

g. Draw the graph of the greatest integer function over the domain $-10 \le x \le 10$. What are the domain and range of this graph? How is this graph different from graphs of other functions you have studied?

2. Paula's Parking Deck is down the street from Pete's. Paula recently renovated her deck to make the parking spaces larger, so she charges more per hour than Pete. Paula's Parking Deck offers free parking up to the first hour (i.e., the first 59 minutes). Then, the cost is $2 each additional hour or part of an hour. (If you park for 1 ½ hours, you owe $2.)

a. Graph the fee schedule for Paula's Parking Deck for up to the first 5 hours.

b. How does this graph compare with the graph of Pete's Parking Deck? To what graphical transformation does this change correspond?

c. If you were to connect the left endpoints of the steps in 1a, what would be the equation of the resulting function? If you connected the left endpoints of the steps in 2a, what would be the equation of the resulting function? How do these answers relate to your answer in 2b?

d. Write the function, *g*, in terms of the greatest integer function, that represents this graph.

e. Draw the graph of $y = g(x)$ over the domain $-10 \le x \le 10$. What are the domain and range for this graph?

3. Pablo's Parking Deck is across the street from Paula's deck. Pablo decided to make his fee schedule even more straightforward than Pete's and Paula's. Rather than provide any free parking, Pablo charges $1 for each 0 – 59 minutes. (If you park for 59 minutes, you owe $1; if you park for 1 hour, you owe $2; etc.)

a. Graph the fee schedule for Pablo's Parking Deck for up to the first 5 hours.

b. How does this graph compare with the graph of Pete's Parking Deck? To what two different graphical transformations does this change correspond?

c. Write the function, *h*, in terms of the greatest integer function, that represents this graph. (What are the two different forms that this function could take?)

d. Draw the graph of $y = h(x)$ over the domain $-10 \le x \le 10$. What are the domain and range for this graph?

4. Padma's Parking Deck is the last deck on the street. To be a bit more competitive, Padma decided to offer parking for each full hour at $1/hour. (If you park for 59 minutes or exactly 1 hour, you owe $1; if you park for up to and including 2 hours, you owe $2.)

a. Graph the fee schedule for Padma's Parking Deck for up to the first 5 hours.

b. To which of the graphs of the other parking deck rates is this most similar? How are the graphs similar? How are they different?

The fee schedule at Padma's Parking Deck is modeled by the **least integer**, or ceiling, function. Least integer functions are written with the following notation: $f(x) = \lceil x \rceil$. The least integer function is determined by locating the *least integer greater than or equal to* the x-value in question.

c. Evaluate each of the following by determining the least integer greater than or equal to the x-value. That is, $f(x) = \lceil x \rceil$.

1. $f(4.5)$ 2. $f(7)$ 3. $f(-1.2)$ 4. $f(-.1)$ 5. $f(.6)$

d. Draw the graph of $y = f(x)$ over the domain $-10 \le x \le 10$. What are the domain and range for this graph?

e. Suppose Padma chose to offer the first full hour. After that, patrons would be charged $1 for up to each full hour. In terms of the least integer function, what function would model this plan?

5. How would the graphs and function rules for the parking deck fee schedules be different in each of the following situations? Make a conjecture and then verify by testing a variety of domain values.

a. The fee increased each half hour instead of each hour.
b. The fee increased every 2 hours instead of every hour.
c. The fee remained the same for up to the first 5 hours. The hourly rate was then reduced by half for each additional hour.

6. As additional practice with step functions, graph each of the following. State the transformations from the parent function (either $f(x) = [x]$ or $g(x) = \lceil x \rceil$), domain, range, and y-intercept. As a challenge, determine an additional function rule from the same family that would yield the same graph.

a. $h(x) = 2[x] - 1$ b. $j(x) = -[x] + 2$ c. $k(x) = \lceil x - 2 \rceil$

Orbital Debris and Space Collisions[3] Learning Task

Each time a space shuttle is launched mission-related debris is released, separated, or dispensed of as part of the mission. Satellites that have exhausted their missions remain in orbit. Paint chips off spacecraft continue to float in orbit. All of these man-made items contribute to space pollution in the LEO (low Earth orbit). The LEO comprises the space within 2000 km of Earth's surface; the most concentrated orbital debris is in the LEO. Telecommunications satellites are found in the geostationary orbit, above the LEO; orbital debris specialists believe that there is little debris in the geostationary orbit.

In addition to polluting the LEO, orbital debris run the risk of colliding with current and future spacecraft. Although all spacecraft collide with small particles during a mission, a collision with a particle 10 cm in diameter or larger can cause catastrophic damage to a craft. There are approximately 17,000 pieces of debris (man-made and natural) of this size in the LEO.

In response to the growing space pollution issue, the United States began close monitoring of orbital space debris, and, along with the United Nations, put into effect a plan to reduce the amount of orbital debris produced and left in space each year. In the US, a plan for minimizing the creation of new orbital debris was proposed in 1997 and approved in 2001. One provision of this plan included the re-entry of mission-related and satellite fragment debris.

The work of monitoring and reporting on orbital debris is carried out at the NASA Orbital Debris Program Office in Houston, Texas. Complex software and mathematical models are used to track the debris and the collisions; however, we can use simpler models to get an idea of what is happening. In this task, we will use linear and quadratic functions to model space collision data.

1. In March 1998, there were 8644 objects (mission-related debris, rocket bodies, spacecraft, fragmentation debris) in the LEO. By April 2008, this number had climbed to 12,637 objects. Approximately 1500 new objects were added in 2007 as a result of the breakup of the Fengyun-1C and Upper Atmospheric Research Satellites, making 2007 the worst ever year for producing new debris.

[3] This task was inspired by NCTM Illuminations: Modeling Orbital Debris Problems, available at http://illuminations.nctm.org/LessonDetail.aspx?id=L376. Additional information was obtained from the website of the NASA Orbital Debris Program Office (http://orbitaldebris.jsc.nasa.gov).

a. Create a scatterplot for the amount of orbital debris in the LEO from 1998 to 2008. Explain any patterns or trends in the data. What are the form (linear, quadratic, etc.), direction (positive or negative), and strength (strong, moderate, or weak) of the relationship between year and debris total? (Debris totals provided here are from the end of the first quarter each year.)

Year	Debris Total
1998	8644
1999	8694
2000	8587
2001	8793
2002	8973
2004	9051
2005	9316
2006	9458
2007	11,141
2008	12,637

b. Eliminate the last two data points and create a new scatterplot. Again, explain any patterns or trends in the data. What are the form, direction, and strength of the relationship between year and debris total?

c. Using appropriate technology (graphing calculator, Excel, TI-Interactive, etc.), determine the least-squares regression line for the data. Interpret the slope and y-intercept of your equation.

d. Draw the regression line on your graph from 1b. How well does the line appear to fit the data?

Rather than just "eyeballing" how well the line fits the data, statisticians often look at the **correlation coefficient**, *r*, to determine the goodness of fit. The correlation measures the direction and strength of a *linear* relationship between two variables. The formula for correlation is a bit complex: $r = \frac{1}{n-1}\sum\left(\frac{x_i - \bar{x}}{s_x}\right)\left(\frac{y_i - \bar{y}}{s_y}\right)$. In practice, we use calculators or software to calculate the correlation. Here are some facts about correlation:

- Positive correlation indicates a positive association between the variables; negative correlation indicates a negative association between variables.
- The correlation is always between -1 and 1. The closer the correlation is to 1, the stronger the positive linear relationship. The closer the correlation is to -1, the stronger the negative linear relationship. The closer the correlation is to 0, the weaker the linear relationship.
- However, a strong linear relationship, i.e. a correlation of 1 or -1, does not indicate a causal relationship between the independent and dependent variables.

e. Find the correlation coefficient using a calculator or other technology. Based on the value of the correlation coefficient, what is the strength and direction of the data?

Linear equations are not the only types of functions we can use to model data. Moreover, most real-life phenomena are not modeled exactly by any specific function. However, we can use mathematical models to provide information about the trends of the data. During this course, you have studied a number of different functions. In this course, we will only look at linear and quadratic regression. Modeling data with other polynomial functions, as well as other types of functions, will be addressed in future courses.

f. Using appropriate technology (graphing calculator, Excel, TI-Interactive, etc.), determine the quadratic regression equation for the data. Draw this curve on your graph from 1b. How well does the parabola appear to fit the data?

g. Based on the data from 1998 to 2006, which curve best models the data? Explain your answer. (Consider the characteristics of the least-squares line that make it a best-fitting line. Can you apply the same idea to the quadratic regression equation?)

h. Data for the amount of debris in 2003 was unavailable. Use your model from 1f to predict the amount of debris in the year 2003.

Question 1h is an example of **interpolation**. Interpolation is used to make predictions within the domain of values of the independent variable. **Extrapolation** is the use of the regression curve to make predictions outside the domain of values of the independent variable. Care must be taken to determine the feasibility of extrapolation. That is, sufficient evidence should exist that the trend described by the regression curve would continue outside the present domain.

i. Would you be confident using your model above to predict the debris in the LEO in 2010? Explain.

2. In addition to compiling data on the amount of debris in the LEO, the NASA Orbiting Debris Program also tracks collisions in the LEO. The data below are the approximate counts of cumulative collisions in the LEO. These collisions include both man-made and natural debris.

Cumulative Number of LEO Collisions by Year	
Year	**Approx. Cum. Coll.**
1965	0
1970	1
1975	2
1980	3
1985	5
1990	10
1995	19
2000	28
2005	40

a. Create a scatterplot for the amount of orbital debris in the LEO from 1998 to 2008. Explain any patterns or trends in the data. What are the form, direction, and strength of the relationship between year and cumulative collisions?

Using appropriate technology (graphing calculator, Excel, TI-Interactive, etc.), determine the least-squares regression line and the correlation coefficient for the data. What does the correlation tell you about the strength of the linear relationship?

b. Draw the regression line on your graph from 2a. How well does the line appear to fit the data? How does this observation compare with the correlation coefficient?

c. Using appropriate technology (graphing calculator, Excel, TI-Interactive, etc.), determine the quadratic regression equation for the data. Draw this curve on your graph from 2a. How well does the parabola appear to fit the data?

3. The choice for which type of curve should be used to model a set of data can often be determined by the **method of finite differences**.

a. Consider a basic linear equation: $y = 2x - 3$. We know that the slope of this line is 2 and the y-intercept is -3. Let's look at a table of values that satisfy this equation. Then determine the difference between successive y-values (the change in the y-coordinates is denoted Δy). What do you notice about the successive differences? How does this relate to the equation of the line?

X	**y**	Δy
-2		
-1		
0		
1		
2		

b. Let's use the same equation to generate a different set of values. If we were to use these y-values, how would we get the same constant difference from 3a?

x	**Y**	Δy	**Constant difference**
-5			
-2			
0			
3			
7			

c. Let's see why this works! Consider the most generic linear equation, $y = ax + b$. First, create a table like that in 3a. If $y = ax + b$, complete column 2 in the table. Then find the differences between successive y-values.

X	$y = ax + b$	Δy
-2		
-1		
0		
1		
2		

d. Now, let's look at a situation like what we encountered in 3b. Once again, complete the following chart. What do expect the slope to equal?

x	$y = ax + b$	**Slope between points**
-5		
-2		
0		
3		
7		

In linear equations, the *first differences* or *constant differences* are equivalent to the slope of the line. So, if a set of data has constant slope (or nearly constant slope), then it can be adequately modeled with a linear function. This brings up the question of other polynomials. Is there a similar rule for quadratic, cubic, etc. functions?

e. Consider the basic quadratic function $y = ax^2 + bx + c$. Complete a table of values as you did in part c. This time, however, add a fourth column for the second differences, the differences of the first differences. What do you notice?

X	$y = ax^2 + bx + c$	**First differences**	**Second differences**
0			
1			
2			
3			
4			

f. Again, consider $y = ax^2 + bx + c$. This time, create a table of values but choose x-values with a *constant difference* other than 1. Is there still a common second difference?

X	$y = ax^2 + bx + c$	**First differences**	**Second differences**

Just as linear functions have constant *first* differences (i.e. constant slope), quadratic functions have constant *second* differences. Likewise, cubic functions have constant *third* differences; etc.

Let's look at a simple quadratic function before we return to our orbital collision data. Consider the equation $y = x^2 + 2x + 1$ and a table of values that satisfy the equation.

X	**y**	**First differences**	**Second differences**
-2	1		
-1	0	-1	
0	1	1	2
1	4	3	2
2	9	5	2

g. Create a new table of values, again using x-values with at constant difference other than 1. Then find the first differences and the second differences.

X	**Y**	**First Differences**	**Second differences**

4. Let's return to the collision data.
   a. Calculate the first differences and second differences for the collision data. Also calculate the slope between each successive pair of data points.

Cumulative Number of LEO Collisions by Year				
Year	Approx. Cum. Coll.	Slope	First Differences	Second differences
1965	0			
1970	1			
1975	2			
1980	3			
1985	5			
1990	10			
1995	19			
2000	28			
2005	40			

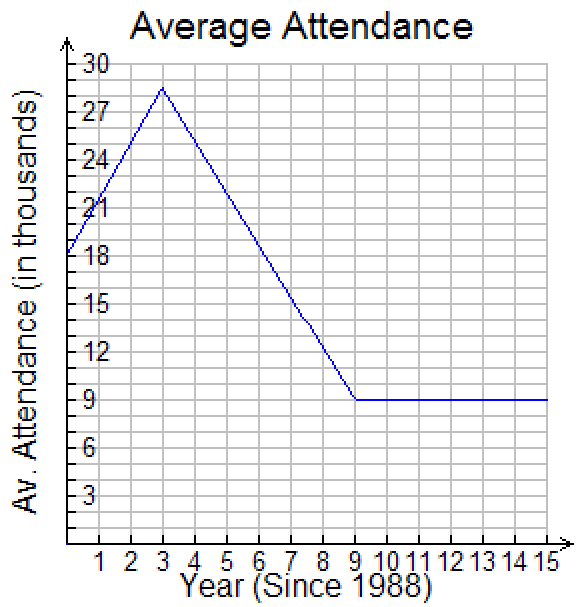
   b. Based on 4a and your regression curves and plots in problem 2, which curve is a better fit for the data? Explain your answer.

   c. Use your choice from 4b to predict the cumulative number of LEO collisions in 2010.

Student Version : Culminating Task: The Raleigh Rottweilers[4]

Raphael Rodriguez, upon his retirement from major league baseball, has decided to invest his millions into an existing MLB franchise. After much deliberation, he bought the declining Raleigh Rottweilers. At one point in time, the Rotts had been a thriving team. Raphael decided to look into the history of the franchise to see what he could change that might improve team performance.

1. The first thing Raphael researched was the attendance at Rottweiler games. The graph below shows the average attendance at home games by year.

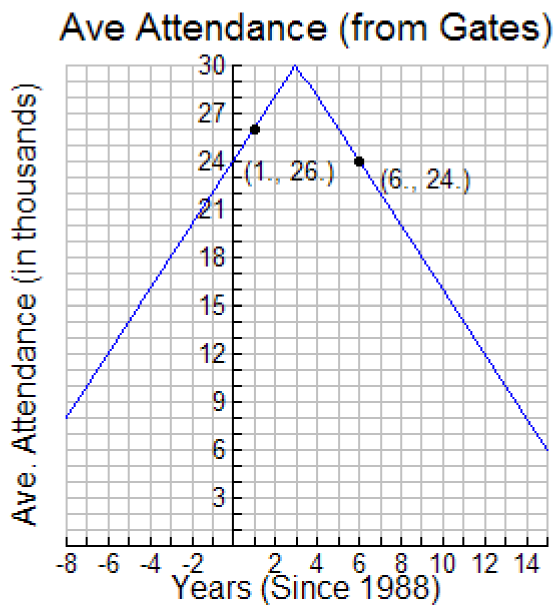


a. Write the function that describes the average attendance at the games.

b. Interpret the y-intercept of the above graph in terms of the context. Also explain what the slopes indicate about the average attendance.

[4] All statistics about real players and teams were obtained from the ESPN website (www.espn.com).

2. What Raphael soon discovered, by talking to the players and administrative staff who had been around for a while, was that the attendance model above did not represent *actual* attendance. Instead, the graph above represented the average *ticket sales* per year. Data from entrance gate scanners revealed the following graph for the *actual average attendance*.



a. Contextually, what does the vertex of the graph represent? Describe any symmetry in the graph. Provide mathematical justification that the symmetry exists.

b. Write the function that describes this graph. Express the function in two different ways (as a piecewise and in terms of transformations of the absolute value parent function).

c. According to this model, in what years was the average attendance 26,000 fans per game? Explain your answer both graphically and algebraically.

d. According to this model, in what years was the average attendance less than 20,000 fans per game? Write your answers using interval notation. Again, explain your answer graphically and algebraically.

3. Katy Karey, the promotions director for the Raleigh Rottweilers, was concerned with the graph of actual attendance in 1b. She realized that the model did not account for the baseball fans who watched the games from the Executive Dining Room or the private suites, options that were added for fans during the 1988 season. After adding in these patrons, she derived the following model.

$$f(x) = \begin{cases} \dfrac{-10}{x-1} + 15, & x \le 0 \\ -\dfrac{1}{5}(x-4)^2 + 33, & 0 < x \le 9 \\ 55 - 3x, & x > 9 \end{cases}$$

a. Identify the parent graphs and transformations for each of the first two pieces of the above function.

b. Graph her function.

c. Identify the maximum value of the graph. What does this represent?

d. Is this a continuous or discontinuous function? If it is discontinuous, state the location(s) of discontinuity and provide a potential cause for the discontinuity.

4. The previous owner, Joe "Clueless" Jacobson, had established the following, non-negotiable pay scale for his players. He used the ratio, *r*, of the previous year's runs to hits to determine the appropriate salary. For each tenth increase in the decimal value of *r*, the player would earn $100,000 more. (For a player with 75 runs and 94 hits, $r = .798$. He would make $800,000.) So the function is given as follows, where the salary is in thousands of dollars. (Rookies were paid based on their final year of college or minor league play.)

$$s(r) = \begin{cases} 100, & 0 < r \le .1 \\ 200, & .1 < r \le .2 \\ 300, & .2 < r \le .3 \\ \ldots & \\ 1{,}000, & .9 < r \le 1 \end{cases}$$

a. Draw the graph of his pay scale. Write the formula for a step function that represents this pay scale. (Use transformations.) Is this a least integer or greatest integer function? If so, which one? Justify your answer.

b. Determine the present pay for each of the following players.

Player Number	2007 Runs	2007 Hits	Salary
23	45	82	
47	10	11	
08	112	173	

c. Under this pay scale, what is the most that any player could get paid?

5. The players' elected captain, David Christenson, volunteered to speak to Raphael for the team. During his discussion with Raphael, David claimed that teams that pay their players more money have greater success.

	2008 Average Salary (in millions)	2007 Regular Season Games Won (out of 162)
New York Yankees	6.74	94
New York Mets	4.61	88
Boston Red Sox	4.77	96
Chicago White Sox	4.49	72
Chicago Cubs	4.39	85
Atlanta Braves	3.41	84
Toronto Blue Jays	3.52	83
Colorado Rockies	2.64	90
Texas Rangers	2.35	75
Arizona Diamondbacks	2.36	90
Minnesota Twins	2.49	79
Kansas City Royals	2.24	69

a. Based on the data presented in the table below, is David's claim substantiated? That is, is there a linear relationship between a team's average salary and the number of games they win? Provide a plot, equation, and correlation coefficient to support your answer.

b. According to this model, if a team had a 2008 average salary of $6 million, how many games would you expect the team to have won in 2007?

c. According to this model, if a team had a 2008 average salary of $500,000, how many games would you expect the team to have won in 2007?

6. Yusuf "Base Running" Betemi asked for a private meeting with Raphael. Yusuf boasted that he had the highest on-base percentage of any other player on the team and should, therefore, receive the highest pay. Raphael promised Yusuf he would consider the proposal and look into the pay for players in the National League with high on-base percentages.

Raphael found the following 2008 salaries for the players in the National League with the highest on-base percentages in 2007. (Note: The player with the highest on-base percentage retired after 2007.)

2007 OBP	2008 Salary (in millions)
0.434	16.6
0.429	13.9
0.425	15
0.416	5.25
0.410	7.8
0.405	9.5
0.401	11.3
0.400	13.25
0.400	14.25

a. Plot the data. Describe the form, direction, and strength of the relationship. Of linear or quadratic, which curve best fits this data? Provide mathematical justification based on vertical distances.

b. Draft a memo from Raphael Rodriguez to Yusuf Betemi summarizing your findings and responding to his request to be the highest paid based on having the highest on-base percentage.

7. Next, Steve Kitch, the pitcher who had been in the league the longest, approached Raphael. He said, “Raph, pitchers get better with age and experience. I deserve a higher salary. All the good pitchers earn higher salaries from one year to the next.” Again, Raphael agreed to do some research. He found the following data for a long-time pitcher in the National League.

Years in NL	Salary (in millions)
1	6
2	7
3	8.5
4	10.2
5	12
6	12.6
7	12.2
8	11.6
9	11
10	10.4
11	9

a. Plot the data. Comment on the form, direction, and shape. Use the method of successive differences to justify which model (linear or quadratic) is a better fit for the data.

b. Use the regression capabilities of your graphing calculator to determine the appropriate regression equation. Also, plot your new curve on the scatterplot. How well does the curve appear to fit the data?

c. According to this model, if Steve has been pitching in the league for 8 years, what should his salary be? What would it be his 10th year?

8. Raphael hopes that one day the Raleigh Rottweilers will reach the top and go to the World Series. As a result, he wants to treat his players like they are championship quality and he wants to pay them appropriately, despite their past salaries. However, he did not have time to individually assess each player before they had to sign contracts. So, in an effort to begin paying his players more reasonable salaries, he decided to somewhat base the next year's salaries on the present year's salary. For guidance on how the salaries should change from one year to the next, Raphael looked to the Atlanta Braves. He used the 2007 and 2008 salaries to determine a model for year-to-year salary changes.

Below are the salaries for Braves players who were on the roster in both 2007 and 2008.

2007 Salary (in thousands)	2008 Salary (in thousands)
300	400
336	390
2350	2363
7500	8000
15,475	15,000
8500	15,500
8000	14,000
667	967
1300	1400
380	430
12,333	15,000
9000	12,500
428	460
8000	7000

a. Which model (linear or quadratic) best fits this data? Explain using any method you choose.

b. Use both models to predict next year's salaries of the following players.

Player Number	2007 Salary
25	$100,000
46	$300,000
10	$800,000

c. In part b, you were required to use extrapolation and interpolation. Is the extrapolation safe in this case? If so, explain why. If not, explain what it means to extrapolate and why it was not safe.

d. As Raphael's business partner, which prediction equation would you choose and why?

9. Raphael realized that player morale was not the only thing he needed to improve if he wanted to make the team successful again. In an effort to involve the fans, Raphael decided to hold a Logo designing contest. However, he has established specific requirements that must be met for the logo. They are as follows:

> The logo must use combinations of vertical and horizontal shifts, vertical and horizontal stretches or shrinks, and reflections through the *x*-axis or *y*-axis of the basic functions listed below. Horizontal and vertical lines may also be used.

$f(x) = x$, $f(x) = x^2$, $f(x) = x^3$, $f(x) = \sqrt{x}$, $f(x) = |x|$, $f(x) = \frac{1}{x}$, $f(x) = \lfloor x \rfloor$, and $f(x) = \lceil x \rceil$

The logo relate to one or all of the following: rottweilers, baseball, or Raleigh, NC. Additionally, it must be aesthetically appealing and must include the following:

- at least one step function (greatest integer or least integer)
- at least one absolute value function
- at least four different families of equations (four of the above)
- at least two examples of vertical shifts, vertical stretches and/or shrinks
- at least two examples of horizontal shifts, stretches and/or shrinks
- at least one reflection

When you have completed your logo, write a cover letter to Raphael Rodriguez explaining how your logo meets each of the requirements listed above.