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INTRODUCTION:
In seventh and eighth grade, students learned about functions generally and about linear functions specifically. This unit explores properties of basic quadratic, cubic, absolute value, square root, and rational functions and new language and notation for talking about functions. The discussion of function characteristics includes further development of the language of mathematical reasoning to include formal discussion of the logical relationships between an implication and its converse, inverse, and contrapositive.

ENDURING UNDERSTANDINGS:
- Functions have three parts: (i) a domain, which is the set of inputs to the function, (ii) a range, which is the set of outputs, and (iii) some rule or statement of correspondence indicating how each input determines a unique output.
- The domain and rule of correspondence determine the range.
- Graphs are geometric representations of functions.
- Functions are equal if and only if they have the same domain and rule of correspondence.
- Function notation provides an efficient way to talk about functions, but notation is just that, an efficient way to talk about functions. The variables used to represent domain values, range values, and the function as a whole are arbitrary. Changing variable names does not change the function.
- Logical equivalence is a concept that applies to the form of a conditional statement. Every conditional statement and its contrapositive are logically equivalent. Given a true conditional statement, whether the converse or inverse of the conditional statement is also true depends on the content of the statement. The converse and inverse forms are not logically equivalent to the original conditional form.

KEY STANDARDS ADDRESSED:
MM1A1. Students will explore and interpret the characteristics of functions, using graphs, tables, and simple algebraic techniques.
   a. Represent functions using function notation.
   b. Graph the basic functions $f(x) = x^n$ where $n = 1$ to 3, $f(x) = \sqrt{x}$, $f(x) = |x|$, and $f(x) = 1/x$.
   c. Graph transformations of basic functions including vertical shifts, stretches, and shrinks, as well as reflections across the x- and y-axes. [Previewed in this unit.]
   d. Investigate and explain the characteristics of a function: domain, range, zeros, intercepts, intervals of increase and decrease, maximum and minimum values, and end behavior.
e. Relate to a given context the characteristics of a function, and use graphs and tables to investigate its behavior.

f. Recognize sequences as functions with domains that are whole numbers.

g. Explore rates of change, comparing constant rates of change (i.e., slope) versus variable rates of change. Compare rates of change of linear, quadratic, square root, and other function families.

**MM1G2. Students will understand and use the language of mathematical argument and justification.**

a. Use conjecture, inductive reasoning, deductive reasoning, counterexamples, and indirect proof as appropriate.

b. Understand and use the relationships among a statement and its converse, inverse, and contrapositive.

**Unit Overview:**

Prior to this unit students need to have worked extensively with operations on integers, rational numbers, and square roots of nonnegative integers as indicated in the Grade 6 – 8 standards for Number and Operations.

In the unit students will apply and extend the Grade 7 – 8 standards related to writing algebraic expressions, evaluating quantities using algebraic expressions, understanding inequalities in one variable, and understanding relations and linear functions as they develop much deeper and more sophisticated understanding of relationships between two variables. Students are assumed to have a deep understanding of linear relationships between variable quantities. Students should understand how to find the areas of triangles, rectangles, squares, and circles and the volumes of rectangular solids.

The unit begins with intensive work with function notation. Students learn to use function notation to ask and answer questions about functional relationships presented tabular, graphical, and algebraic form. The distinction between discrete and continuous domains is explored through comparing and contrasting functions which have the same rule of correspondence, but different domains. Through extensive work with reading and drawing graphs, students learn to view graphs of functional relationships as whole objects rather than collections of individual points and to apply standard techniques used to draw representative graphs of functions with unbounded domains.

The unit includes an introduction to propositional logic of conditional statements, their converses, inverses, and contrapositives. Conditional statements about the absolute value function and vertical translations of the absolute value function give students concrete examples that can be classified as true or false by examining graphs. Combining analysis of conditional statements with further exploration of basic functions demonstrates that conditional statements are important throughout the study of mathematics, not just in geometry where this material has traditionally been introduced.
Average rate of change of a function is introduced through the interpretation of average speed, but is extended to the context of rate of change of revenue for varying quantities of a product sold. Students contrast constant rates of change to variable ones through the concept of average rate of change. The basic function families are introduced primarily through real-world contexts that are modeled by these functions so that students understand that these functions are studied because they are important for interpreting the world in which we live. The work with vertical shifts, stretches, and shrinks and reflections of graphs of basic functions is integrated throughout the unit rather than studied as a topic in isolation. Functions whose graphs are related by one of these transformations arise from the context and the context promotes understanding of the relationship between a change in a formula and the corresponding change in the graph.

Students need facility in working with functions given via tables, graphs, or algebraic formulas, using function notation correctly, and learning to view a function as an entity to be analyzed and compared to other functions. Throughout this unit, it is important to:

- Begin exploration of a new function by generating a table of values using a variety of numbers from the domain. Decide, based on the context, what kinds of numbers can be in the domain, and make sure you choose negative numbers or numbers expressed as fractions or decimals if such numbers are included in the domain.
- Do extensive graphing by hand. Once students have a deep understanding of the relationships between formulas and graphs, regular use of graphing technology will be important. For this introductory unit, graphing by hand is necessary to develop understanding.
- Be extremely careful in the use of language. Always use the name of the function, for example $f$, to refer to the function as a whole and use $f(x)$ to refer to the input when the input is $x$. For example, when language is used correctly, a graph of the function $f$ in the $x$, $y$-plane is the graph of the equation $y = f(x)$ since we graph those points, and only those points, of the form $(x, y)$ where the $y$-coordinates is equal to $f(x)$.

**TASKS:** The remaining content of this framework consists of student learning tasks designed to allow students to learn by investigating situations with a real-world context. The first learning task is intended to launch the unit. Its primary focus is introducing function notation and the more formal approach to functions characteristic of high school mathematics. The second through seventh learning tasks extend students knowledge of functions through in depth consideration of domain, range, average rate of change, and other characteristics of functions basic to the study of high school mathematics. The last task is designed to demonstrate the type of assessment activities students should be comfortable with by the end of the unit.
**Exploring Functions with Fiona Learning Task**

1. While visiting her grandmother, Fiona Evans found markings on the inside of a closet door showing the heights of her mother, Julia, and her mother’s brothers and sisters on their birthdays growing up. From the markings in the closet, Fiona wrote down her mother’s height each year from ages 2 to 16. Her grandmother found the measurements at birth and one year by looking in her mother’s baby book. The data is provided in the table below, with heights rounded to the nearest inch.

| Age (yrs.) | x  | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|------------|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| Height (in.) | y  | 21 | 30 | 35 | 39 | 43 | 46 | 48 | 51 | 53 | 55 | 59 | 62 | 64 | 65 | 66 | 66 | 66 |

a. Which variable is the independent variable, and which is the dependent variable? Explain your choice.

b. Make a graph of the data.

c. Should you connect the dots on your graph? Explain.

d. Describe how Julia’s height changed as she grew up.

e. How tall was Julia on her 11th birthday? Explain how you can see this in both the graph and the table.

f. What do you think happened to Julia’s height after age 16? Explain. How could you show this on your graph?
In Mathematics I and all advanced mathematics, function notation is used as an efficient way to describe relationships between quantities that vary in a functional relationship. In the remaining parts of this investigation, we’ll explore function notation as we look at other growth patterns and situations.

2. Fiona has a younger brother, Tyler, who attends a pre-kindergarten class for 4-year olds. One of the math activities during the first month of school was measuring the heights of the children. The class made a large poster to record the information in a bar graph that had a bar for the height of each child. The children used heights rounded to the nearest whole number of inches; however, the teacher also used an Excel spreadsheet to record their heights to the nearest half inch, as shown below.

Table A: Class Height Information
August 2007

<table>
<thead>
<tr>
<th>Student Number</th>
<th>Last name</th>
<th>First name</th>
<th>Height in inches</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Barnes</td>
<td>Joshua</td>
<td>42.0</td>
</tr>
<tr>
<td>2</td>
<td>Coleman</td>
<td>David</td>
<td>40.5</td>
</tr>
<tr>
<td>3</td>
<td>Coleman</td>
<td>Diane</td>
<td>40.0</td>
</tr>
<tr>
<td>4</td>
<td>Drew</td>
<td>Keisha</td>
<td>37.5</td>
</tr>
<tr>
<td>5</td>
<td>Evans</td>
<td>Tyler</td>
<td>39.5</td>
</tr>
<tr>
<td>6</td>
<td>Hagan</td>
<td>Emily</td>
<td>38.0</td>
</tr>
<tr>
<td>7</td>
<td>Nguyen</td>
<td>Violet</td>
<td>37.0</td>
</tr>
<tr>
<td>8</td>
<td>Rader</td>
<td>Joshua</td>
<td>38.5</td>
</tr>
<tr>
<td>9</td>
<td>Ruiz</td>
<td>Alina</td>
<td>38.5</td>
</tr>
<tr>
<td>10</td>
<td>Vogel</td>
<td>Zach</td>
<td>39.5</td>
</tr>
</tbody>
</table>

After making the Excel table, the teacher decided to also make an Excel version of the bar graph. While she was working on the bar graph, she had the idea of also graphing the information in the rectangular coordinate system, using the Student Number as the x-value and the height to the nearest half inch as the y-value. Here is her graph.
The relationship that uses Student Number as input and the height of the student with that student number as output describes a function because, for each student number, there is exactly one output, the height of the student with that student number.

a. The graph was drawn with an Excel option named *scatter plot*. This option allows graphs of relationships whether or not the graphs represent functions. Sketch a scatter plot using student last names as inputs and heights of students as outputs. Explain why this relationship is not a function.

b. Sketch a scatter plot using student first names as inputs and heights of students as outputs. Is this relationship a function? Explain your answer.

c. In Graph A, the pre-kindergarten teacher chose an Excel format that did not connect the dots. Explain why the dots should not be connected.

d. Using the information in the Excel spreadsheet about the relationship between student number and height of the corresponding student, fill in each of the following blanks.

The height of student 2 is equal to ____ inches.
The height of student 6 is equal to ____ inches.
The height of student _____ is equal to 37 inches.
For what student numbers is the height equal to 38.5 inches? ____
e. We are now ready to discuss function notation. First we need to give our function a mathematical name. Since the outputs of the functions are heights, we name the function $h$. In Mathematics I and following courses, we’ll use one letter names for functions as a way to refer to the whole relationship between inputs and outputs. So, for this example, we mean that $h$ consists of all the input-output pairs of student number and corresponding height in inches; thus, we can say that Graph A above is the graph of the function $h$ because the graph shows all of these input-output pairs.

Using the function name $h$, we write $h(1) = 42$ to indicate that the height of student 1 is equal to 42 inches. We read $h(1)$ as “$h$ of 1”. This wording is similar to the phrase “height of student 1”. Without referring to the specific meaning of function $h$, we say that $h(1)$ means the output value of function $h$ when the input value is 1. Since the output value is the number 42, we write $h(1) = 42$.

More examples: $h(3) = 39$ (read “$h$ of 3 equals 39”) means that student 3 is 39 in. tall, and $h(4) = 37.5$ (read “$h$ of 4 equals 37.5”) means that student 4 is 37.5 in. tall.

The following fill-in-the-blank questions repeat the questions from part d) in function notation. Fill in these blanks too.

\[ h(2) = \underline{\hphantom{10}} \quad h(6) = \underline{\hphantom{10}} \quad h(\underline{\hphantom{10}}) = 37 \]

For what values of $x$ does $h(x) = 38.5$? \underline{\hphantom{10}}

3. We now return to the function in #1 above and name this function $J$ (for Julia’s height). Consider the notation $J(2)$. We note that function notation gives us another way to write about ideas that you began learning in middle school, as shown in the table below.

<table>
<thead>
<tr>
<th>Statement</th>
<th>Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>At age 2, Julia was 35 inches tall.</td>
<td>Natural language</td>
</tr>
<tr>
<td>When $x$ is 2, $y$ is 35.</td>
<td>Statement about variables</td>
</tr>
<tr>
<td>When the input is 2, the output is 35.</td>
<td>Input-output statement</td>
</tr>
<tr>
<td>$J(2) = 35$.</td>
<td>Function notation</td>
</tr>
</tbody>
</table>

The notation $J(x)$ is typically read “$J$ of $x$,” but thinking “$J$ at $x$” is also useful since $J(2)$ can be interpreted as “height at age 2,” for example.

Note: Function notation looks like a multiplication calculation, but the meaning is very different. To avoid misinterpretation, be sure you know which letters represent functions. For example, if $g$ represents a function, then $g(4)$ is not multiplication of $g$ and 4 but is rather the value of “$g$ at 4,” that is, the output value of the function $g$ when the input is value is 4.
a. What is \( J(11) \)? What does this mean?

b. When \( x \) is 3, what is \( y \)? Express this fact using function notation.

c. Find an \( x \) so that \( J(x) = 53 \). Explain your method. What does your answer mean?

d. From your graph or your table, estimate \( J(6.5) \). Explain your method. What does your answer mean?

e. Estimate a value for \( x \) so that \( J(x) \) is approximately 60. Explain your method. What does your answer mean?

f. Describe what happens to \( J(x) \) as \( x \) increases from 0 to 16.

4. Fiona attends Peachtree Plains High School. When the school opened five years ago, a few teachers and students put on Fall Fest, featuring contests, games, prizes, and performances by student bands. To raise money for the event, they sold Fall Fest T-shirts. The event was very well received, and so Fall Fest has become a tradition. This year Fiona is one of the students helping with Fall Fest and is in charge of T-shirt sales. She gathered information about the growth of T-shirt sales for the Fall Fests so far and created the graph below that shows the function \( S \).
a. What are the independent and dependent variables shown in the graph?

b. For which years does the graph provide data?

c. Does it make sense to connect the dots in the graph? Explain.

d. What were the T-shirt sales in the first year? Use function notation to express your result.

e. Find \( S(3) \), if possible, and explain what it means or would mean.

f. Find \( S(6) \), if possible, and explain what it means or would mean.

g. Find \( S(2.4) \), if possible, and explain what it means or would mean.

h. If possible, find a \( t \) such that \( S(t) = 65 \). Explain.

i. If possible, find a \( t \) such that \( S(t) = 62 \). Explain.

j. Describe what happens to \( S(t) \) as \( t \) increases, beginning at \( t = 1 \).

k. What can you say about \( S(t) \) for values of \( t \) greater than 6?
Note: As you have seen above, functions can be described by tables and by graphs. In high school mathematics, functions are often given by formulas. In the remaining items for this task, we develop, or are given, a formula for each function under consideration, but it is important to remember that not all functions can be described by formulas.

5. Fiona’s T-shirt committee decided on a long sleeve T-shirt in royal blue, one of the school colors, with a FallFest logo designed by the art teacher. The committee needs to decide how many T-shirts to order. Fiona was given the job of collecting price information so she checked with several suppliers, both local companies and some on the Web. She found the best price at Peachtree Plains Promotions, a local company owned by parents of a Peachtree Plains High School senior.

The salesperson for Peachtree Plains Promotions told Fiona that there would be a $50 fee for setting up the imprint design and different charges per shirt depending on the total number of shirts ordered. For an order of 50 to 250 T-shirts, the cost is $9 per shirt. Based on sales from the previous five years, Fiona is sure that they will order at least 50 T-shirts and will not order more than 250. If \( x \) is the number of T-shirts to be ordered for this year’s FallFest, and \( y \) is the total dollar cost of these shirts, then \( y \) is a function of \( x \). Let’s name this function \( C \), for cost function. Fiona started the table below.

<table>
<thead>
<tr>
<th>( x )</th>
<th>50</th>
<th>100</th>
<th>150</th>
<th>200</th>
<th>250</th>
</tr>
</thead>
<tbody>
<tr>
<td>9(x)</td>
<td>450</td>
<td>900</td>
<td>1350</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( y = C(x) )</td>
<td>500</td>
<td>950</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

a. Fill in the missing values in the table above.

b. Make a graph to show how the cost depends upon the number of T-shirts ordered. You can start by plotting the points corresponding to values in the table. What points are these? Should you connect these points? Explain. Should you extend the graph beyond the first or last point? Explain.

c. Write a formula showing how the cost depends upon the number of T-shirts ordered. For what numbers of T-shirts does your formula apply? Explain.

d. What does \( C(70) \) mean? What is the value of \( C(70) \)? Did you use the table, the graph, or the formula?

e. If the T-shirt committee decides to order only the 67 T-shirts that are pre-paid, how much will it cost? Show how you know. Express the result using function notation.

f. If the T-shirt committee decides to order the 67 T-shirts that are pre-paid plus 15 more, how much will it cost? Show how you know. Express the result using function notation.
6. Fiona is taking physics. Her sister, Hannah, is taking physical science. Fiona decided to use functions to help Hannah understand one basic idea related to gravity and falling objects. Fiona explained that, if a ball is dropped from a high place, such as the Tower of Pisa in Italy, then there is a formula for calculating the distance the ball has fallen. If $y$, measured in meters, is the distance the ball has fallen and $x$, measured in seconds, is the time since the ball was dropped, then $y$ is a function of $x$, and the relationship can be approximated by the formula $y = d(x) = 5x^2$. Here we name the function $d$ because the outputs are distances.

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>…</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^2$</td>
<td>0</td>
<td>1</td>
<td>4</td>
<td>9</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$y = d(x) = 5x^2$</td>
<td>0</td>
<td>5</td>
<td>20</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

a. Fill in the missing values in the table above.

b. Suppose the ball is dropped from a building at least 100 meters high. Measuring from the top of the building, draw a picture indicating the position of the ball at times indicated in your table of values.

c. Draw a graph of $x$ versus $y$ for this situation. Should you connect the dots? Explain.

d. What is the relationship between the picture (part b) and the graph (part c)?

e. You know from experience that the speed of the ball increases as it falls. How can you “see” the increasing speed in your table? How can you “see” the increasing speed in your picture?

f. What is $d(4)$? What does this mean?

g. Estimate $x$ such that $d(x) = 50$. Explain your method. What does it mean?

h. In this context, $y$ is proportional to $x^2$. Explain what that means. How can you see this in the table?
7. Fiona is paid $7 per hour in her part-time job at the local Dairy Stop. Let \( t \) be the amount of time that she works, in hours, during the week, and let \( P(t) \) be her gross pay (before taxes), in dollars, for the week.
   a. Make a table showing how her gross pay depends upon the amount of time she works during the week.

   b. Make a graph illustrating how her gross pay depends upon the amount of time that she works. Should you connect the dots? Explain.

   c. Write a formula showing how her gross pay depends upon the amount of time that she works.

   d. What is \( P(9) \)? What does it mean? Explain how you can use the graph, the table, and the formula to compute \( P(9) \).

   e. If Fiona works 11 hours and 15 minutes, what will her gross pay be? Show how you know. Express the result using function notation.

   f. If Fiona works 4 hours and 50 minutes, what will her gross pay be? Show how you know. Express the result using function notation.

   g. One week Fiona’s gross pay was $42. How many hours did she work? Show how you know.

   h. Another week Fiona’s gross pay was $57.19. How many hours did she work? Show how you know.
Fences and Functions Learning Task

1. Claire decided to plant a rectangular garden in her back yard using 30 pieces of fencing that were given to her by a friend. Each piece of fencing was a vinyl panel 1 yard wide and 6 feet high. Claire wanted to determine the possible dimensions of her garden, assuming that she would use all of the fencing and did not cut any of the panels. She began by placing ten panels (10 yards) parallel to the back side of her house and then calculated that the other dimension of her garden then would be 5 yards, as shown in the diagram below.

Claire looked at the 10 fencing panels laying on the ground and decided that she wanted to consider other possibilities for the dimensions of the garden. In order to organize her thoughts, she let \( x \) be the garden dimension parallel to the back of her house, measured in yards, and let \( y \) be the other dimension, perpendicular to the back of the house, measured in yards. She recorded the first possibility for the dimensions of the garden as follows: When \( x = 10 \), \( y = 5 \).

\[
\begin{array}{|c|}
\hline
10 \text{ yds} \\
\hline
\text{Garden} \quad 5 \text{ yds} \\
\hline
\end{array}
\]

\[
\begin{array}{|c|}
\hline
\text{House} \\
\hline
\end{array}
\]

a. Explain why \( y \) must be 5 when \( x \) is 10.

b. Make a table showing the possibilities for \( x \) and \( y \).

c. Find the perimeter of each of the possible gardens you listed in part b. What do you notice? Explain why this happens.

d. Did you consider \( x = 15 \) in part b? If \( x = 15 \), what must \( y \) be? What would Claire do with the fencing if she chose \( x = 15 \)?

e. Can \( x \) be 16? What is the maximum possible value for \( x \)? Explain.
f. Write a formula relating the $y$-dimension of the garden to the $x$-dimension.

g. Make a graph of the possible dimensions of Claire’s garden.

What would it mean to connect the dots on your graph? Does connecting the dots make sense for this context? Explain.

h. As the $x$-dimension of the garden increases by 1 yard, what happens to the $y$-dimension? Does it matter what $x$-value you start with? How do you see this in the graph? In the table? In your formula? What makes the dimensions change together in this way?

2. After listing the possible rectangular dimensions of the garden, Claire realized that she needed to pay attention to the area of the garden, because area determines how many plants can be grown.

a. Does the area of the garden change as the $x$-dimension changes? Make a prediction, and explain your thinking.

b. Use grid paper to make accurate sketches for at least three possible gardens. How is the area of each garden represented on the grid paper?

c. Make a table listing all the possible $x$-dimensions for the garden and the corresponding areas. (To facilitate your calculations, you might want to include the $y$-dimensions in your table or add an area column to your previous table.)

d. Make a graph showing the relationship between the $x$-dimension and the area of the garden. Should you connect the dots? Explain.
3. Because the area of Claire’s garden depends upon the \( x \)-dimension, we can say that the area is a function of the \( x \)-dimension. Let’s use \( G \) for the name of the function that uses each \( x \)-dimension an input value and gives the resulting garden area as the corresponding output value.

a. Use function notation to write the formula for the garden area function. What does \( G(11) \) mean? What is the value of \( G(11) \)? What line of your table, from \#2, part c, and what point on your graph, from \#2, part d, illustrate this same information?

b. The set of all possible input values for a function is called the \textit{domain} of the function. What is the domain of the garden area function \( G \)? How is the question about domain related to the question about connecting the dots on the graph you drew for \#2, part d?

c. The set of all possible output values is called the \textit{range} of the function. What is the range of the garden area function \( G \)? How can you see the range in your table? In your graph?

d. As the \( x \)-dimension of the garden increases by 1 yard, what happens to the garden area? Does it matter what \( x \)-dimension you start with? How do you see this in the graph? In the table? Explain what you notice.

e. What is the maximum value for the garden area, and what are the dimensions when the garden has this area? How do you see this in your table? In your graph?

f. What is the minimum value for the garden area, and what are the dimensions when the garden has this area? How do you see this in your table? In your graph?

g. In deciding how to lay out her garden, Claire made a table and graph similar to those you have made in this investigation. Her neighbor Javier noticed that her graph had symmetry. Your graph should also have symmetry. Describe this symmetry by indicating the line of symmetry. What about the context of the garden situation causes this symmetry?
h. After making her table and graph, Claire made a decision, put up the fence, and planted her garden. If it had been your garden, what dimensions would you have used and why?

4. Later that summer, Claire’s sister-in-law Kenya mentioned that she wanted to use 30 yards of chain-link fence to build a new pen for her two pet pot bellied pigs. Claire experienced déjà vu and shared how she had analyzed how to fence her garden. As Claire explained her analysis, Kenya realized that her fencing problem was very similar to Claire’s.

a. Does the formula that relates the \( y \)-dimension of Claire’s garden to the \( x \)-dimension of her garden apply to the pen for Kenya’s pet pigs? Why or why not?

b. Does the formula that relates the area of Claire’s garden to its \( x \)-dimension apply to the pen for Kenya’s pet pigs? Why or why not?

c. Write a formula showing how to compute the area of the pen for Kenya’s pet pigs given the \( x \)-dimension for the pen.

d. Does the \( x \)-dimension of the pen for Kenya’s pet pigs have to be a whole number? Explain.

e. Let \( P \) be the function which uses the \( x \)-dimension of the pen for Kenya’s pet pigs as input and gives the area of the pen as output. Write a formula for \( P(x) \).

f. Make a table of input and output values for the function \( P \). Include some values of the \( x \)-dimension for the pen that that could not be used as the \( x \)-dimension for Claire’s garden.

g. Make a graph of the function \( P \).
h. Does your graph show any points with x-value less than 1? Could Kenya have made a pen with \( \frac{1}{2} \) yard as the x-dimension? If so, what would the other dimension be? How about a pen with an x-dimension 0.1 of a yard? How big is a potbellied pig? Would a potbellied pig fit into either of these pens?

i. Is it mathematically possible to have a rectangle with x-dimension equal to \( \frac{1}{2} \)? How about a rectangle with x-dimension 0.1?

j. Of course, Kenya would not build a pen for her pigs that did not give enough room for the pigs to turn around or pass by each other. However, in analyzing the function \( P \) to decide how to build the pen, Kenya found it useful to consider all the input values that could be the x-dimension of a rectangle. She knew that it didn’t make sense to consider a negative number as the x-dimension for the pen for her pigs, but she asked herself if she could interpret an x-dimension of 0 in any meaningful way. She thought about the formula relating the y-dimension to the x-dimension and decided to include \( x = 0 \). What layout of fencing would correspond to the value \( x = 0 \)? What area would be included inside the fence? Why could the shape created by this fencing layout be called a “degenerate rectangle”?

k. The value \( x = 0 \) is called a **limiting case**. Is there any other limiting case to consider in thinking about values for the x-dimension of the pen for Kenya’s pigs? Explain.

l. Return to your graph of the function \( P \). Adjust your graph to include all the values that could mathematically be the x-dimension of the rectangular pen even though some of these are not reasonable for fencing an area for potbellied pigs.  

(Note: You can plot points corresponding to any limiting cases for the function using a small circle. To include the limiting case, fill in the circle to make a solid dot ●. To not include the limiting case, but just use it to show that the graph does not continue beyond that point, leave the circle open ○.)

5. Use your table, graph, and formula for the function \( P \) to answer questions about the pen for Kenya’s pet pigs.
a. Estimate the area of a pen of with an x-dimension of 10 feet (not yards). Explain your reasoning.

b. Estimate the x-dimension of a rectangle with an area of 30 square yards. Explain your reasoning.

c. What is the domain of the function \( P \)? How do you see the domain in the graph?

d. What point on the graph corresponds to the pen with maximum area?

e. What is the maximum area that the pen for Kenya’s pigs could have? Explain. What do you notice about the shape of the pen?

f. What is the range of the function \( P \)? How can you see the range in the graph?

g. How should you move your finger along the x-axis (right-to-left or left-to-right) so that the x-value increases as your move your finger? If you move your finger along the graph of the function \( P \) in this same direction, do the x-values of the points also increase? As the x-dimension of the pen for Kenya’s pigs increases, sometimes the area increases and sometimes the area decreases. Using your graph, determine the x-values such that the area increase as \( x \) increases. For what x-values does the area decrease as \( x \) increases?

When using tables and formulas, we often look at a function by examining only one or two points at a time, but in high school mathematics, it is important to begin to think about “the whole function,” that is, all of the input-output pairs. We’ve started working on this idea already by using a single letter such as \( f \), \( G \), or \( P \) to refer to the whole collection of input-output pairs. We’ll go further as we proceed with this investigation.
We say that **two functions are equal** (as whole functions) if they have exactly the same input-output pairs. In other words, two functions are equal if they have the **same domain** and the **output values are the same for each input value in the domain**. From a graphical perspective, two functions are equal if their graphs have exactly the same points. Note that the graph of a function consists of all the points which correspond to input-output pairs, but when we draw a graph we often can show only some of the points and indicate the rest. For example, if the graph of the function is a line, we show part of the line and use arrowheads to indicate that the line continues without end.

6. The possibilities for the pen for Kenya’s pet pigs and for Claire’s garden are very similar in some respects but different in others. These two situations involve different **functions**, even though the **formulas** are the same.

   a. If Kenya makes the pen with maximum area, how much more area will the pen for her pet pigs have than Claire’s garden of maximum area? How much area is that in square feet?

   b. What could Claire have done to build her garden with the same area as the maximum area for Kenya’s pen? Do you think this would have been worthwhile?

   c. Consider the situations that led to the functions $G$ and $P$ and review your tables, graphs, formulas related to the two functions. Describe the similarities and differences between Kenya’s pig pen problem and Claire’s garden problem. Your response should include a discussion of **domain** and **range** for the two functions.
From Wonderland to Functionland Learning Task

Consider the following passage from Lewis Carroll’s *Alice’s Adventures in Wonderland*, Chapter VII, “A Mad Tea Party.”

"Then you should say what you mean." the March Hare went on.

"I do," Alice hastily replied; "at least -- at least I mean what I say -- that's the same thing, you know."

"Not the same thing a bit!" said the Hatter, "Why, you might just as well say that 'I see what I eat' is the same thing as 'I eat what I see'!"

"You might just as well say," added the March Hare, "that 'I like what I get' is the same thing as 'I get what I like'!"

"You might just as well say," added the Dormouse, who seemed to be talking in his sleep, "that 'I breathe when I sleep' is the same thing as 'I sleep when I breathe'!"

"It is the same thing with you," said the Hatter, and here the conversation dropped, and the party sat silent for a minute.

Lewis Carroll, the author of *Alice in Wonderland* and *Through the Looking Glass*, was a mathematics teacher who had fun playing around with logic. In this activity, you’ll investigate some basic ideas from logic and perhaps have some fun too.

We need to start with some basic definitions.

A statement is a sentence that is either true or false, but not both.

A conditional statement is a statement that can be expressed in “if … then …” form.

A few examples should help clarify these definitions. The following sentences are statements.

- Atlanta is the capital of Georgia. (*This sentence is true.*)
- Jimmy Carter was the thirty-ninth president of the United States and was born in Plains, Georgia. (*This sentence is true.*)
- The Atlanta Falcons are a professional basketball team. (*This sentence is false.*)
- George Washington had eggs for breakfast on his fifteenth birthday. (*Although it is unlikely that we can find any source that allows us to determine whether this sentence is true or false, it still must either be true or false, and not both, so it is a statement.*)

Here are some sentences that are not statements.
What’s your favorite music video? *(This sentence is a question.)*  
Turn up the volume so I can hear this song. *(This sentence is a command.)*  
This sentence is false. *(This sentence is a very peculiar object called a self-referential sentence. It creates a logical puzzle that bothered logicians in the early twentieth century. If the sentence is true, then it is also false. If the sentence is false, then it is not false and, hence, also true. Logicians finally resolved this puzzling issue by excluding such sentences from the definition of “statement” and requiring that statements must be either true or false, but not both.)*

The last example discussed a sentence that puzzled logicians in the last century. The passage from *Alice in Wonderland* contains several sentences that may have puzzled you the first time you read them. The next part of this activity will allow you to analyze the passage while learning more about conditional statements.

Near the beginning of the passage, the Hatter responds to Alice that she might as well say that “I see what I eat” means the same thing as “I eat what I see.” Let’s express each of the Hatter’s example sentences in “if … then” form.

“I see what I eat” has the same meaning as the conditional statement “If I eat a thing, then I see it.” On the other hand, “I eat what I see” has the same meaning as the conditional statement “If I see a thing, then I eat it.”

1. Express each of the following statements from the Mad Tea Party in “if … then” form.

   a. I like what I get.  
   b. I breathe when I sleep.

2. We use specific vocabulary to refer to the parts of a conditional statement written in “if … then …” form. The **hypothesis** of a conditional statement is the statement that follows the word “if.” So, for the conditional statement “If I eat a thing, then I see it,” the hypothesis is the statement “I eat a thing.” Note that the hypothesis does **not** include the word “if” because the hypothesis is the statement that occurs after the “if.”

   Give the hypothesis for each of the conditionals in item 1, parts a and b.

3. The **conclusion** of a conditional statement is the statement that follows the word “then.” So, for the conditional statement “If I eat a thing, then I see it,” the conclusion is the statement “I see it.” Note that the conclusion does **not** include the word “then” because the conclusion is the statement that occurs after the word “then.”
Give the conclusion for each of the conditionals in item 1, parts a and b.

Now, let’s get back to the discussion at the Mad Tea Party. When we expressed the Hatter’s example conditional statements in “if … then” form, we used the pronoun “it” in the conclusion of each statement rather than repeat the word “thing.” Now, we want to compare the hypotheses (note that the word “hypotheses” is the plural of the word “hypothesis”) and conclusions of the Hatter’s conditionals. To help us see the key relationship between his two conditional statements, we replace the pronoun “it” with the noun “thing.” This replacement doesn’t change the meaning. As far as English prose is concerned, we have a repetitious sentence. However, this repetition helps us analyze the relationship between hypotheses and conclusions.

4. a. List the hypothesis and conclusion for the revised version of each of the Hatter’s conditional statements given below.

<table>
<thead>
<tr>
<th>Hypothesis</th>
<th>Conclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>If I eat a thing, then I see the thing.</td>
<td>__________</td>
</tr>
<tr>
<td>If I see a thing, then I eat the thing.</td>
<td>__________</td>
</tr>
</tbody>
</table>

b. Explain how the Hatter’s two conditional statements are related.

5. There is a term for the new statement obtained by exchanging the hypothesis and conclusion in a conditional statement. This new statement is called the converse of the first.

a. Write the converse of each of the conditional statements in item 1, parts a and b, using “if … then …” form.

b. What happens when you form the converse of each of the conditional statements given as answers for this item part a?

6. The March Hare, Hatter, and Dormouse did not use “if … then” form when they stated their conditionals. Write the converse for each conditional statement below without using “if … then” form.

<table>
<thead>
<tr>
<th>Conditional</th>
<th>Converse</th>
</tr>
</thead>
<tbody>
<tr>
<td>I breathe when I sleep.</td>
<td>___________________</td>
</tr>
<tr>
<td>I like what I get.</td>
<td>___________________</td>
</tr>
<tr>
<td>I see what I eat.</td>
<td>___________________</td>
</tr>
<tr>
<td>I say what I mean.</td>
<td>___________________</td>
</tr>
</tbody>
</table>
7. a. What relationship between breathing and sleeping is expressed by the conditional statement “I breathe when I sleep”? If you make this statement, is it true or false?

c. What is the relationship between breathing and sleeping expressed by the conditional statement “I sleep when I breathe”? If you make this statement, is it true or false?

8. The conversation in passage from Alice in Wonderland ends with the Hatter’s response to the Dormouse "It is the same thing with you." The Hatter was making a joke. Do you get the joke? If you aren’t sure, you may want to learn more about Lewis Carroll’s characterization of the Dormouse in Chapter VII of Alice in Wonderland.

We want to come to some general conclusions about the logical relationship between a conditional statement and its converse. The next steps are to learn more of the vocabulary for discussing such statements and to see more examples.

In English class, you learn about compound sentences. As far as English is concerned, compound sentences consist of two or more independent clauses joined by using a coordinating conjunction such as “and,” “or,” “but,” and so forth, or by using a semicolon.

In logic, the term “compound” is used in a more general sense. A compound statement, or compound proposition, is a new statement formed by putting two or more statements together to form a new statement. There are several specific ways to combine statements to create a compound proposition. Compound statements formed using “and” and “or” are important in the study of probability. In this task, we are focusing on compound propositions created using the “if … then …” form. In order to talk about this type of compound proposition without regard to the particular statements used for the hypothesis and conclusion, we can use variables to represent statements as a whole. This use of variables is demonstrated in the formal definition that follows.
**Definition:** If $p$ and $q$ are statements, then the statement “if $p$, then $q$” is the **conditional statement**, or **implication**, with hypothesis $p$ and conclusion $q$.

We call the variables used above, **statement**, or **propositional, variables**. We seek a general conclusion about the logical relationship between a conditional statement and its converse; we are looking for a relationship that is true no matter what particular statements we substitute for the statement variables $p$ and $q$. That’s why we need to see more examples.

Our earlier discussions of function notation, domain of a function, and range of a function have included conditional statements about inputs and outputs of a function. For the next part of this activity, we consider conditional statements about a particular function, the **absolute value function** $f$, defined as follows:

$$f \text{ is the function with domain all real numbers such that } f(x) = |x|.$$ 

(Note. To give a complete definition of the absolute value function, we must specify the domain and a formula for obtaining the unique output for each input. It is not necessary to specify the range because the domain and the formula determine the set of outputs.)

9. We’ll explore the graph of the absolute value function $f$ and then consider some related conditional statements.

   a. Complete table of values given below.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$0$</th>
<th>$\frac{1}{2}$</th>
<th>$-\frac{1}{2}$</th>
<th>$1$</th>
<th>$-1$</th>
<th>$3.8$</th>
<th>$-3.8$</th>
<th>$-5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x) =</td>
<td>x</td>
<td>$</td>
<td>$0$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$5$</td>
<td>$15$</td>
<td>$\frac{15}{2}$</td>
</tr>
</tbody>
</table>

   d. Most graphing calculators have a standard graphing window which shows the portion of the graph of a function corresponding to $x$-values from $-10$ to $10$ and $y$-values from $-10$ to $10$. On grid paper, set up such a standard viewing window and then use the table of values above to draw the part of the graph of the function $f$ for this viewing window. Does your graph show all of the input/output pairs listed in your table of values? Does your graph show the input/output pairs for those $x$-values from $-10$ to $10$ that were not listed in the table? Explain.
c. Think of your graph as a picture. Do you see a familiar shape? Describe the graph to someone who cannot see it. If you were to extend your graph to include points corresponding to additional values of $x$, say to include $x$-values from $-1000$ to $1000$, would the shape change? What can you do to your sketch to indicate information about the points outside of your “viewing window”? 

e. For what $x$-values shown on your graph of $f$ does the $y$-value increase as $x$ increases, that is, for what $x$-values is it true that, as you move your finger along your graph so that the $x$-values increase, then the $y$-values also increase? (Be sure to give the $x$-values where this happens.) For what $x$-values does the $y$-value decrease as $x$ increases, that is, for what $x$-values is it true that, as you move your finger along your graph so that the $x$-values increase, then the $y$-values decrease? (Be sure to give the $x$-values where this happens.) Considering your answers to the questions in part c, for the whole function $f$, determine (i) those $x$-values such that the $y$-value increases as $x$ increases and (ii) those $x$-values such that the $y$-value decreases as $x$ increases.

10. Evaluate each of the following expressions written in function notation. Be sure to simplify so that there are no absolute value signs in your answers. Use your graph to verify that each of your statements is true.

   a. $f(0) =$ _____ b. $f(-5) =$ _____ c. $f(-\pi) =$ _____ d. $f(3)$

11. The statements in 10, parts a – d, are written using the equals sign. The same ideas can be expressed with conditional statements. Fill in the blanks to form such equivalent statements.

   a. If $x = 0$, then $f(x) =$ ____. b. If $x = -5$, then $f(x) =$ ____. 

   c. If the input for the function $f$ is $-\pi$, then the output for the function $f$ is ____. 

   d. If the input for the function $f$ is $3$, then the output for the function $f$ is ____. 

In the remainder of this task, we will often consider whether a conditional statement is true or false. To say that a conditional is true means that, whenever the hypothesis is true, then the conclusion is also true; and to say that a conditional is false means that the hypothesis is, or can be, true while the conclusion is false.
12. a. Write the converse of each of the **true** conditional statements from item #11. For each converse, use the graph of \( f \) to determine whether the statement is true or false. Organize your work in a table such as the one shown below. For the statements in the table that you classify as false, specify a value of \( x \) that makes the hypothesis true and the conclusion false.

<table>
<thead>
<tr>
<th>Conditional statement</th>
<th>Truth value</th>
<th>Converse statement</th>
<th>Truth value</th>
</tr>
</thead>
<tbody>
<tr>
<td>If ( x = 0 ), then ( f(x) = ___ ).</td>
<td>True</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>True</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>True</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>True</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

b. As indicated by the header line in the table above, whether a statement is true or false is called the **truth value** of the statement. Our goal for this item is to decide whether there is a general relationship between the truth value of a conditional statement and the truth value of its converse. Any particular conditional statement can be true or false, so we need to consider examples for both cases. Add lines to your table from part a for the converses of the following **false** conditional statements. For these statements, and for any converse that you classify as false, give a value of \( x \) that makes the hypothesis true and the conclusion false.

(i) If \( f(x) = 7 \), then \( x = 5 \).  
(ii) If \( f(x) = 2 \), then \( x = 2 \).
c. Complete the following sentence to make a true statement. Explain your reasoning. Is your answer choice consistent with all of the examples of converse in the table above?

*Multiple choice:* The converse of a true conditional statement is ____.

A) always also true  B) always false

C) sometimes true and sometimes false because whether the converse is true or false does not depend on whether the original statement is true or false.

d. Complete the following sentence to make a true statement. Explain your reasoning. Is your answer choice consistent with all of the examples of converse in the table above?

*Multiple choice:* The converse of a false conditional statement is ____.

A) always also false  B) always true

C) sometimes true and sometimes false because whether the converse is true or false does not depend on whether the original statement is true or false.

Whenever we talk about statements in general, without having a particular example in mind, it is useful to talk about the propositional form of the statement. For the propositional form “if \( p \), then \( q \),” the converse propositional form is “if \( q \), then \( p \).”

If two propositional forms result in statements with the same truth value for all possible cases of substituting statements for the propositional variables, we say that the forms are *logically equivalent*. If there exist statements that can be substituted into the propositional forms so that the resulting statements have different truth values, we say that the propositional forms are *not logically equivalent*

e. Consider your answers to parts a and b, and decide how to complete the following statement to make it true. Justify your choice.

The converse propositional form “if \( q \), then \( p \)” is/ is not (choose one) logically equivalent to the conditional statement “if \( p \), then \( q \)”.

f. *Multiple choice:* If you learn a new mathematical fact in the form “if \( p \), then \( q \),” what can you immediately conclude, without any additional information, about the truth value of the converse?

A) no conclusion because the converse is not logically equivalent

B) conclude that the converse is true

C) conclude that the converse is false
g. Look back at the opening of the passage from *Alice in Wonderland*, when Alice hastily replied "I do, at least -- at least I mean what I say -- that's the same thing, you know." What statements did Alice think were logically equivalent? What was the Hatter saying about the equivalence of these statements when he replied to Alice by saying "Not the same thing a bit!"?

There are two other important propositional forms related to any given conditional statement. We introduce these by exploring other inhabitants of the land of functions.

13. Let \( g \) be the function with domain all real numbers such that \( g(x) = |x| + 3 \).

   a. Complete table of values given below.

   \[
   \begin{array}{c|cccccccc}
   x \qquad & 0 & \frac{1}{2} & -\frac{1}{2} & 1 & -1 & 3.8 & -3.8 & -5 \\
   \hline
   g(x) = |x| + 3 & 3 & \frac{7}{2} & \frac{7}{2} & 8 & 21 & 21 & 13
   \end{array}
   \]

   b. On grid paper, draw the portion of the graph of \( g \) for all \( x \)-values such that \(-10 \leq x \leq 10\). To show all of the points corresponding to input/output pairs shown in the table, how much of the \( y \)-axis should your viewing window include? Is the part of the graph that you have drawn representative of the whole graph? Explain.

   c. For what \( x \)-values shown on your graph of \( g \) does the \( y \)-value increase as \( x \) increases? For what \( x \)-values does the \( y \)-value decrease as \( x \) increases? For the whole function \( g \), determine (i) those \( x \)-values such that the \( y \)-value increases as \( x \) increases and (ii) those \( x \)-values such that the \( y \)-value decreases as \( x \) increases.

   d. What is the relationship between the graphs of \( f \) and \( g \)? What in the formulas for \( f(x) \) and \( g(x) \) tells you that the graphs are related in this way?
14. It is clear from the graphs of $f$ and $g$ that, for each input value, the two functions have different output values. For just one example, we see that $f(4) = 4$ but $g(4) = 7$. If we want to emphasize that $g$ is not the absolute value function and that $g(4)$ is different from 4, we could write $g(4) \neq 4$, which is read “$g$ of 4 is not equal to 4.” We now examine some related conditional statements.

a. Complete the following conditional statement to indicate that $g(4) \neq 4$.

If the input of the function $g$ is 4, then the output of the function $g$ is not _____.

b. Consider another true statement about the function $g$; in this case the statement is “$g(-3) \neq 5$.” Use your graph to evaluate $g(-3)$ and verify that “$g(-3) \neq 5$” is a true statement.

Let $p$ represent the statement “The input of the function $g$ is $-3$.”
Let $q$ represent the statement “The output of the function $g$ is not 5.”
What statement is represented by “if $p$, then $q$”? Does this statement express the idea that $g(-3) \neq 5$?

In logic, we form the **negation** of a statement $p$ by forming the statement “It is not true that $p$.” For convenience, we use “not $p$” to refer to the negation of the statement $p$. For a specific choice of statement, when we translate “not $p$” into English, we can usually state the negation in a more direct way. For example:

- when $p$ represents the statement “The input of the function $g$ is $-3$,”
  then “not $p$” represents “The input of the function $g$ is **not** $-3$”, and

- when, as above, $q$ represents the statement “The output of the function $g$ is not 5,”
  then “not $q$” represents “The output of the function $g$ is not **not** 5,” or more simply “The output of the function $g$ is 5.”

c. If $p$ and $q$ are represent the statements indicated in part b:

(i) What statement is represented by “If not $p$, then not $q$”?

(ii) Is this inverse statement true? Explain your reasoning.

(iii) Does this inverse statement tell you that $g(-3) \neq 5$?
A statement of the form “If not \( p \), then not \( q \)” is called the inverse of the conditional statement “if \( p \), then \( q \).” Note that the inverse is formed by negating the hypothesis and conclusion of a conditional statement.

d. The table below includes statements about the functions \( f \) and \( g \). Fill in the blanks in the table. Be sure that your entries for the truth value columns agree with the graphs for \( f \) and \( g \). For the statements in the table that are false, give a value of \( x \) that makes the hypothesis true and the conclusion false.

<table>
<thead>
<tr>
<th>Conditional statement</th>
<th>Truth value</th>
<th>Inverse statement</th>
<th>Truth value</th>
</tr>
</thead>
<tbody>
<tr>
<td>If ( x = 4 ), then ( f(x) \neq 9 ).</td>
<td>True</td>
<td>If ( x \neq 4 ), then ( f(x) = 9 ).</td>
<td></td>
</tr>
<tr>
<td>If ( g(x) \neq 3 ), then ( x \neq 0 ).</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>If ( x = 0 ), then ( g(x) \neq 3 ).</td>
<td></td>
<td>If ( g(x) \neq 6 ), then ( x \neq 3 ).</td>
<td>True</td>
</tr>
</tbody>
</table>

e. Consider the results in the table above, and then decide how to complete the following statement to make it true. Justify your choice.

The inverse propositional form “if not \( p \), then not \( q \)” is/ is not (choose one) logically equivalent to the conditional statement “if \( p \), then \( q \).”

f. Multiple choice: If you learn a new mathematical result in the form “if \( p \), then \( q \)”, what can you immediately conclude, without any additional information, about the truth value of the inverse?

A) no conclusion because the inverse is not logically equivalent
B) conclude that the inverse is true
C) conclude that the inverse is false

15. Let \( h \) be the function with domain all real numbers such that \( h(x) = 2|x| \).

a. On grid paper, draw the portion of the graph of \( h \) for all \( x \)-values such that \(-10 \leq x \leq 10\).

b. For \( x = -7, -2, 0, 5, 8 \), compare \( f(x) \) and \( h(x) \). What is the relationship between the values? Does this relationship hold for every real number? How do the graphs of \( f \) and \( h \) compare? Explain why the graphs have this relationship.
c. **When any** positive real number is used as the input for the absolute value function $f$, then the output is the same as the input.
   For example, $f(1) = 1, f(4) = 4, f(10) = 10$, and so forth. However, the function $h$ does not have this property. For example, $h(10) \neq 10$. Express this inequality in two different ways by completing the following conditional statements.

   (i) If the input of the function $h$ is 10, then the output of the function $h$ is not ________.  

   (ii) If the output of the function $h$ is 10, then the input of the function $h$ ________.

d. For the conditional statement in (i), part c, above, let $p$ denote the hypothesis of the statement, and let $q$ denote the conclusion. Then, express the **propositional form** of the statement (ii), part c, using “not $p$” and “not $q$”.

A statement of the form “If not $q$, then not $p$” is called the **contrapositive** of the conditional statement “if $p$, then $q$.” Note that the contrapositive is formed by both negating and exchanging the hypothesis and conclusion.


e. The table below includes statements about the function $h$. Fill in the blanks in the table. Be sure that your answers for the truth value columns are consistent with the graph of $h$. For the statements in the table classified as false, give a value of $x$ that makes the hypothesis true and the conclusion false.

<table>
<thead>
<tr>
<th>Conditional statement</th>
<th>Truth value</th>
<th>Contrapositive statement</th>
<th>Truth value</th>
</tr>
</thead>
<tbody>
<tr>
<td>If $x = 4$, then $h(x) = 3$.</td>
<td></td>
<td>If $h(x) \neq 3$, then $x \neq 4$.</td>
<td></td>
</tr>
<tr>
<td>If $x = 6$, then $h(x) = 12$.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>If $h(x) = 0$, then $x = 0$.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>If $x &lt; 3$, then $h(x) &lt; 6$.</td>
<td>False, let $x = -4$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

f. On each line of the table, how do the truth value of the conditional statement and its contrapositive compare?
16. a. Suppose that you are given that “if $p$, then $q$” is a true statement for some particular choice of $p$ and $q$. For example, suppose that there is a function $k$ whose domain and range are all real numbers and it is true that, if the input to the function $k$ is 17, then the output of function $k$ is –86. What is the **hypothesis** of the contrapositive statement? What is the **conclusion** of the contrapositive statement? Given that $k(17) = –86$, is the contrapositive true? Explain.

b. Suppose that you are given that the contrapositive statement “if not $q$, then not $p$” is a true statement for some particular choice of $p$ and $q$. For example, suppose that there is a function $k$ whose domain and range are all real numbers and it is true that, if the output of the function $k$ is not 101, then the input to the function $k$ is not 34. If we regard “If the output of the function $k$ is not 101, then the input to the function $k$ is not 34” as the contrapositive statement, what was the original conditional statement? If we are given that the contrapositive statement is true, must the original conditional be true? Explain.

c. Based on your reasoning in parts a and b, if a conditional statement is true, could the contrapositive be false? If a conditional statement is false, could the contrapositive statement be true?

d. Consider your answer to part c, and decide how to complete the following statement to make it true. Justify your choice.

The propositional form “if $p$, then $q$” is **is**/is not (choose one) logically equivalent to its contrapositive “if not $q$, then not $p$.”

e. **Multiple choice:** If you learn a new mathematical result in the form “if $p$, then $q$,” what can you immediately conclude, without any additional information, about the truth value of the contrapositive?

A) no conclusion because the contrapositive is not logically equivalent

B) conclude that the contrapositive is true

C) conclude that the contrapositive is false
17. Summarizing the information about forming related conditional statements, we see that the conditional “if \( p \), then \( q \)” has three related conditional statements:

**converse**: “if \( q \), then \( p \)” (swaps the hypothesis/conclusion)

**inverse**: “if not \( p \), then not \( q \)” (negates the hypothesis/conclusion)

**contrapositive**: “if not \( q \), then not \( p \)” (negates hypothesis/conclusion, then swaps)

Which, if any, of these is logically equivalent to the original conditional statement and always has the same truth value as the original? ________________

The absolute value function \( f \) and the functions \( g \) and \( h \) that you have worked with in this investigation are not linear. However, in your study of functions prior to Mathematics I, you have worked with many linear functions. We conclude this investigation with discussion about converse, inverse, and contrapositive using a linear function.

18. Consider the linear function \( F \) which converts a temperature of \( c \) degrees Celsius to the equivalent temperature of \( F(c) \) degrees Fahrenheit. The formula is given by

\[
F(c) = \frac{9}{5}c + 32, \text{ where } c \text{ is a temperature in degrees Celsius.}
\]

a. What is freezing cold in degrees Celsius? in degrees Fahrenheit? Verify that the formula for \( F \) converts correctly for freezing temperatures.

b. What is boiling hot in degrees Celsius? in degrees Fahrenheit? Verify that the formula for \( F \) converts correctly for boiling hot temperatures.

c. Draw the graph of \( F \) for values of \( c \) such that \(-100 \leq c \leq 400\). What is the shape of the graph you drew? Is this the shape of the whole graph?

d. Verify that “if \( c = 25 \), then \( F(c) = 77 \)” is true. What is the contrapositive of this statement? How do you know that the contrapositive is true without additional verification?

e. What is the converse of “if \( c = 25 \), then \( F(c) = 77 \)”? How can you use the formula for \( F \) to verify that the converse is true? What is the contrapositive of the converse? How do you know that this last statement is true without additional verification?

f. There is a statement that combines a statement and its converse; it’s called a **biconditional**.

**Definition**: If \( p \) and \( q \) are statements, then the statement “\( p \) if and only if \( q \)” is called a **biconditional statement** and is logically equivalent to “if \( q \), then \( p \)” and “if \( p \), then \( q \)”.

Write three true biconditional statements about values of the function \( F \). Explain how you know that the statements are true.
Sequences as Functions Learning Task

In your previous study of functions, you have seen examples that begin with a problem situation. In some of these situations, you needed to extend or generalize a pattern. In the following activities, you will explore patterns as sequences and view sequences as functions.

A sequence is an ordered list of numbers, pictures, letters, geometric figures, or just about any object you like. Each number, figure, or object is called a term in the sequence. For convenience, the terms of sequences are often separated by commas. In Mathematics I, we focus primarily on sequences of numbers and often use geometric figures and diagrams as illustrations and contexts for investigating various number sequences.

1. Some sequences follow predictable patterns, though the pattern might not be immediately apparent. Other sequences have no pattern at all. Explain, when possible, patterns in the following sequences:
   a. 5, 4, 3, 2, 1
   b. 3, 5, 1, 2, 4
   c. 2, 4, 3, 5, 1, 5, 1, 5, 1, 7
   d. S, M, T, W, T, F, S
   e. 31, 28, 31, 30, 31, 30, 31, 31, 30, 31, 30, 31
   f. 1, 2, 3, 4, 5, ..., 999, 1000
   g. 1, -1, 1, -1, 1, -1, ...
   h. 4, 7, 10, 13, 16, ...
   i. 10, 100, 1000, 10000, 100000, ...

The first six sequences above are finite sequences, because they contain a finite number of terms. The last three are infinite sequences because they contain an infinite number of terms. The three dots, called ellipses, indicate that some of the terms are missing. Ellipses are necessary for infinite sequences, but ellipses are also used for large finite sequences. The sixth example consists of the counting numbers from 1 to 1000; using ellipses allows us to indicate the sequence without having to write all of the one thousand numbers in it.
2. In looking for patterns in sequences it is useful to look for a pattern in how each term relates to the previous term. If there is a consistent pattern in how each term relates to the previous one, it is convenient to express this pattern using a recursive definition for the sequence. A recursive definition gives the first term and a formula for how the \( n \)th term relates to the \((n-1)\)th term. For each sequence below, match the sequence with a recursive definition that correctly relates each term to the previous term, and then fill in the blank in the recursive definition.

a. 1, 2, 3, 4, 5, …  
   I. \( t_1 = \ldots \), \( t_n = t_{n-1} + 5 \)

b. 5, 10, 15, 20, 25, …  
   II. \( t_1 = \ldots \), \( t_n = t_{n-1} + 2 \)

c. 1.2, 11.3, 21.4, 31.5, 41.6, 51.7, …  
   III. \( t_1 = \ldots \), \( t_n = t_{n-1} + 1 \)

d. 3 + \( \sqrt{2} \), 5 + \( \sqrt{2} \), 7 + \( \sqrt{2} \), 9 + \( \sqrt{2} \), 11 + \( \sqrt{2} \), …  
   IV. \( t_1 = \ldots \), \( t_n = (-2) \) \( t_{n-1} \)

e. –1, 2, –4, 8, –16, …  
   V. \( t_1 = \ldots \), \( t_n = t_{n-1} + 10.1 \)

3. Some recursive definitions are more complex than those given in item 2. A recursive definition can give two terms at the beginning of the sequence and then provide a formula for the \( n \)th term as an expression involving the two preceding terms, \( n – 1 \) and \( n – 2 \). It can give three terms at the beginning of the sequence and then provide a formula for the \( n \)th term as an expression involving the three preceding terms, \( n – 1 \), \( n – 2 \), and \( n – 3 \); and so forth. The sequence of Fibonacci numbers, 1, 1, 2, 3, 5, 8, 13, 21, …, is a well known sequence with such a recursive definition.

a. What is the recursive definition for the Fibonacci sequence?

b. The French mathematician Edouard Lucas discovered a related sequence that has many interesting relationships to the Fibonacci sequence. The sequence is 2, 1, 3, 4, 7, 11, 18, 29, …, and it is called the Lucas sequence. For important reasons studied in advanced mathematics, the definition of the Lucas numbers starts with the 0th term. Examine the sequence and complete the recursive definition below.

c. What is the 8th term of the Lucas sequence, that is, the term corresponding to \( n = 8 \)?

d. What is the 15th term?
4. Recursive formulas for sequences have many advantages, but they have one disadvantage. If you need to know the 100th term, for example, you first need to find all the terms before it. An alternate way to define a sequence uses a **closed form** definition that indicates how to determine the nth term directly, without the need to calculate other terms. Match each sequence below with a definition in closed form. Verify that the closed form definitions you choose agree with the five terms given for each sequence.

   a. 5, 10, 15, 20, 25, …  
      I. \( t_n = n^3 \), \( for \ n = 1, 2, 3, \ldots \)  
   
   b. 1, 0, 2, 0, 3, 0, 4, …  
      II. \( t_n = 5n \), \( for \ n = 1, 2, 3, \ldots \)  
   
   c. 5, 9, 13, 17, 21, 25, …  
      III. \( t_n = 4n + 1 \), \( for \ n = 1, 2, 3, \ldots \)  
   
   d. 1, 8, 27, 64, 125, …  
      IV. \[
      \begin{align*}
      t_n &= \begin{cases} 
      \frac{n+1}{2}, & \text{if } n = 1, 3, 5, \ldots \\
      0, & \text{if } n = 2, 4, 6, \ldots 
      \end{cases}
      \end{align*}
      \]

Sequences do not need to be specified by formulas; in fact, some sequences are impossible to specify with formulas. When sequences are given by formulas, the two types of formulas most commonly used are the ones explained above: recursive and closed form. In item 4, you probably noticed that a closed form definition for a sequence looks very much like a formula for a function. If you think of the index values as inputs and the terms of the sequence as outputs, then any sequence can be considered to be a function, whether or not you have a formula for the nth term of the sequence.

5. Consider the sequence of dot figures below.

   \[
   \begin{array}{cccccc}
   & \bullet & \bullet & \bullet & \bullet & \bullet \\
   n = 1 & n = 2 & n = 3 & n = 4 & n = 5 \\
   \end{array}
   \]

   a. The sequence continues so that the number of dots in each figure is the next **square number**. Why are these numbers called square numbers?

   b. What is the next figure in the sequence? What is the next square number?

   c. What is the 25th square number? Explain.
d. Is $156$ a square number? Explain.

e. Let $g$ be the function that gives the number of dots in the $n^{\text{th}}$ figure above. Write a formula for this function.

f. Make a table and graph for the function $g$.

g. What is the domain of the function $g$?

h. What is the range of the function $g$?

6. Consider a function $A$ that gives the area of a square of side-length $s$, shown below (and to the right).

a. What is the area of a square of side length 6?

b. Find a formula for the function $A$.

c. Make a table and graph for the function $A$.

d. Use your formula to find the area of squares with the following side lengths: $7$, $\frac{1}{2}$, $4.2$, $\sqrt{3}$, and $200$.

e. What is the domain of $A$ (in the given context)?

f. What are the limiting case(s) in this context? Explain.
g. Change the graph to extend the domain of $A$ to include the limiting case(s). What change did you make in the graph?

7. Compare the functions $g$ and $A$ from items 5 and 6.

a. Explain how similarities and differences in the formulas, tables, and graphs arise from similarities and differences in the contexts.

b. Are these functions equal? Explain.

8. Look back at the finite sequences in 1a and 1b above. Consider the sequence in item 1, part a, to be a function named $h$ and the sequence in item 1, part b to be a function named $k$.

a. Make tables and graphs for the functions $h$ and $k$.

b. How many points are included in each graph?

c. Determine the following:

$$\begin{array}{ll}
\text{domain of } h & \text{range of } h \\
\text{domain of } k & \text{range of } k
\end{array}$$

d. Are these functions equal? Explain.

e. Write a formula for that gives $h(n)$ for each number $n$ in the domain of $h$. What kind of function did you write?

f. Can you write a linear function formula for the function $k$? Explain why or why not?

g. Consider the linear function $d$ given by the formula

$$d(x) = 10 - x,$$

for each real number $x$.

Make a table and graph for the function $d$.

h. Compare the tables, graphs, and formulas for the functions $h$ and $d$. In your comparison, pay attention to domains, ranges, and the shapes of the graphs.
Walking, Falling, and Making Money

In previous mathematics courses, you studied the formula \( \text{distance} = \text{rate} \times \text{time} \), which is usually abbreviated \( d = rt \). If you and your family take a trip and spend 4 hours driving 200 miles, then you can substitute 200 for \( d \), 4 for \( t \), and solve the equation \( 200 = rt \times 4 \) to find that \( r = 50 \). Thus, we say that the average speed for the trip was 50 miles per hour. In this task, we develop the idea of average rate of change of a function, and see that it corresponds to average speed in certain situations.

1. To begin a class discussion of speed, Dwain and Beth want to stage a walking race down the school hallway. After some experimentation with a stop watch, and using the fact that the flooring tiles measure 1 foot by 1 foot, they decide that the distance of the race should be 40 feet and that they will need about 10 seconds to go 40 feet at a walking pace. They decide that the race should end in a tie, so that it will be exciting to watch, and finally they make a table showing how their positions will vary over time. Your job is to help Dwain and Beth make sure that they know how they should walk in order to match their plans as closely as possible.

<table>
<thead>
<tr>
<th>Time (sec.)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dwain’s position (ft.)</td>
<td>0</td>
<td>4</td>
<td>8</td>
<td>12</td>
<td>16</td>
<td>20</td>
<td>24</td>
<td>28</td>
<td>32</td>
<td>36</td>
<td>40</td>
</tr>
<tr>
<td>Beth’s position (ft.)</td>
<td>0</td>
<td>1</td>
<td>3</td>
<td>6</td>
<td>10</td>
<td>15</td>
<td>20</td>
<td>25</td>
<td>30</td>
<td>35</td>
<td>40</td>
</tr>
</tbody>
</table>

a. Draw a graph for this data. Should you connect the dots? Explain.

b. How can you tell that the race is supposed to end in a tie? Provide two explanations.

c. Who is ahead 5 seconds into the race? Provide two explanations.

d. Describe how Dwain should walk in order to match his data. In particular, should Dwain’s speed be constant or changing? Explain how you know, using observations from both the graph and the table.
e. Describe how Beth should walk in order to match his data. In particular, should Beth’s speed be constant or changing? Explain how you know, using observations from both the graph and the table.

f. In your answers above, sometimes you paid attention to the actual data in table. At other times, you looked at how the data change, which involved computing differences between values in the table. Give examples of each. How can you use the graph to distinguish between actual values of the data and differences between data values?

g. Someone asks, “What is Beth’s speed during the race?” Kellee says that this question does not have a specific numeric answer. Explain what she means.

h. Chris says that Beth went 40 feet in 10 seconds, so Beth’s speed is 4 feet per second. But Kellee thinks that it would be better to say that Beth’s average speed is 4 feet per second. Is Chris’s calculation sensible? What does Kellee mean?

i. Taylor explains that to compute average speed over some time interval, you divide the distance during the time interval by the amount of time. Compute Dwain’s average speed over several time intervals (e.g., from 2 to 5 seconds; from 3 to 8 seconds). What do you notice? Explain the result.

j. What can you say about Beth’s speed during the first five seconds of the race? What about the last five seconds? Explain.
k. Trey wants to race alongside Dwain and Beth. He wants to travel at a constant speed during the first five seconds of the race so that he will be tied with Beth after five seconds. At what speed should he walk? Explain how Trey’s walking can provide an interpretation of Beth’s average speed during the first five seconds.

In describing relationships between two variables, it is often useful to talk about the rate of change of one variable with respect to the other. When the rate of change is not constant, we often talk about average rates of change. If the variables are called \( x \) and \( y \), and \( y \) is the dependent variable, then

\[
\text{Average rate of change of } y \text{ with respect to } x = \frac{\text{the change in } y}{\text{the change in } x}.
\]

When \( y \) is distance and \( x \) is time, the average rate of change can be interpreted as an average speed, as we have seen above.

2. Earlier you investigated the distance fallen by a ball dropped from a high place, such as the Tower of Pisa. In that problem, \( y \), measured in meters, is the distance the ball has fallen and \( x \), measured in seconds, is the time since the ball was dropped. You saw that \( y \) is a function of \( x \), and the relationship can be approximated by the formula \( y = f(x) = 5x^2 \). You completed a table like the one below.

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^2 )</td>
<td>0</td>
<td>1</td>
<td>4</td>
<td>9</td>
<td>16</td>
<td>25</td>
<td>36</td>
<td>...</td>
</tr>
<tr>
<td>( y = f(x) = 5x^2 )</td>
<td>0</td>
<td>5</td>
<td>20</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

a. What is the average rate of change of the height of the ball with respect to time during the first second after the ball is dropped? In what units is the average rate of change measured?

b. What is the average rate of change of the height of the ball during the second second after the ball is dropped? after the third, fourth, and fifth? What do these answers say about the speed of the ball?
c. Find the average rate of change over the first five seconds. Is this the same average speed as the average speed for the fifth second during which the ball is falling? Explain.

d. What is the average rate of change over the first two seconds?

e. What is the average rate of change from 2 to 5 seconds after the ball is dropped?

f. What is the average rate of change from \( t = 1.5 \) to \( t = 2 \)?

g. Why is it important to use the phrase “average rate of change” or “average speed” for your calculations in this problem?

Average speed is a common application of the concept of average rate of change but certainly not the only one. There are many applications to analyzing the money a company can make from producing and selling products. The rest of this task explores average rate of change for functions related to the Vee Company and its production and sale of a game called Zingo. The Vee Company is a small privately owned manufacturing company which sells to exclusively to a national chain of toy stores. Zingo games are packaged and sold in cartons holding 24 games each. Due to the size of the Vee Company work force, the maximum number of games per week that can be produced is 6000, which is enough to fill 250 cartons.

3. The table below shows data that the Vee Company has collected about the relationship between the wholesale price per game and the number of cartons of Zingo that the toy store chain will order each week. In the business world, it generally happens that lowering the price of a product increases the number that will be bought; this holds true for the Zingo sales data. Also, it may seem a bit backwards, but in business analysis, price is usually expressed as a function of the number sold, as indicated in the table.

<table>
<thead>
<tr>
<th>no. cartons ordered per week, ( x )</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
<th>60</th>
<th>70</th>
<th>80</th>
<th>90</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>$ price per Zingo</td>
<td>14.50</td>
<td>14.00</td>
<td>13.50</td>
<td>13.00</td>
<td>12.50</td>
<td>12.00</td>
<td>11.50</td>
<td>11.00</td>
<td>10.50</td>
<td>10.00</td>
</tr>
</tbody>
</table>
a. When the number of cartons ordered increases from 10 per week to 20 per week, the price per game changes. If we subtract the 10 cartons-per-week price per game from the 20 cartons-per-week price per game, is the difference positive or negative? What is the difference? What does the positive or negative sign on the number tell you about how the price per game changes as the number of cartons ordered increases from 10 to 20?

b. Calculate the average rate of change of the price per game with respect to the number of cartons ordered as } x \text{ increases from 10 to 20}. What are the units of measure for this average rate of change?

c. Calculate the average rate of change of the price per game with respect to the number of cartons ordered for the increase from 20 to 40 cartons ordered per week, for the increase from 40 to 70 cartons ordered per week, and for the increase from 50 to 100 cartons ordered per week.

d. Based on the data in the table, is the average rate of change of price per game constant for all these increases in the number of cartons ordered, or does it change depending on the particular numbers of cartons ordered?

e. Based on your calculations of average rate of change, determine whether the relationship between } x \text{ and } p(x) \text{ is a linear relationship or a non-linear relationship. Explain.}

f. How much does the Vee Company need to change the price of a Zingo game to sell one more carton per week? Does your answer depend on how many cartons are currently being sold? Explain.
g. The function \( p \) can be viewed as a finite sequence with 250 terms. Explain why and relate this observation to the domain of \( p \). What does the value of the \( n^{\text{th}} \) term mean?

h. Write a formula to calculate \( p(x) \). What information from your answers above do you need in order to find this formula? To what values of \( x \) does the formula apply?

i. Graph the equation \( y = p(x) \) for the domain \( 0 \leq x \leq 250 \). Is this the graph of the function \( p \)? Explain why or why not.

4. In business, the term revenue is used to indicate the money a company receives for sales of its products. In this context, revenue is a function of the number of items sold. The Vee Company’s financial analyst has determined that its revenue, in dollars, for sales of the Zingo game is given by the function \( R \) with the formula

\[
R(x) = 360x - 1.20x^2,
\]

where \( x \) is the number of cartons sold, and \( R(x) \) is revenue measured in dollars.

a. The graph of the function \( R \) is shown at the right. Is the relationship between \( x \) and \( R(x) \) a linear relationship or a non-linear relationship? Explain.

b. Is the average rate change of revenue with respect to number of cartons sold per week constant or changing?

c. Using the formula for the price function in item 3, part h, find the price per Zingo game when 150, 200, and 250 cartons are sold per week. Then, find the price per carton when 150, 200, and 250 cartons are sold per week. Do these values agree with the values of the revenue function when 150, 200, and 250 cartons are sold per week? Explain.
d. When the number of cartons sold per week increases from 50 to 150, is the average rate of change of revenue positive or negative? Explain how to use the graph to find the answer without actually calculating this average rate of change.

e. When the number of cartons sold per week increases from 150 to 250, is the average rate of change of revenue positive or negative? Explain how to use the graph to find the answer without calculating the value.

f. What are the units of measure for the average rate of change of revenue with respect to number of cartons sold per week?

g. Calculate the average rate of change of revenue as \( x \) increases from 50 to 150 and the average rate of change of revenue as \( x \) increases from 150 to 250. What is the relationship between these values? How is this shown in the graph?

h. The average rate of change of revenue as \( x \) increases from 100 to 200 is 0. What feature of the graph causes this to be so? Explain and give other examples of the same phenomenon.
Southern Yards and Garden Learning Task

In this task, you will investigate functions whose formulas involve the algebraic expressions \( \frac{1}{x} \) and \( \sqrt{x} \). You will do so by considering activities of Southern Yard and Garden, a Georgia-owned business that produces grass sod, garden plants, and trees for sale to nurseries and landscaping companies.

Last year, the Research and Development (R&D) Team of the Grass Division at Southern Yard and Garden (SYnG – pronounced “sin gee”) was ready to plant test plots of several new drought resistant grasses. The R&D Team wanted each new grass variety planted in five different test plots at the SYnG experiment farm.

The R&D Team had specific requirements for the plots.
- To assure comparable data, each plot must be the same area, 1200 square feet.
- To assure consideration of a variety of soil and sun/shade conditions, the plots must not be adjacent.
- To prevent any grazing by wildlife in the area, each plot must be surrounded by a special fence.

When the plans were sent to the Chief Financial Officer (CFO) for approval, she concluded that a separate fence for each grass test plot would be a major expense and asked the R&D Team how much fencing would be needed. The R&D Team sent a quick email that the length of fencing needed would depend on the perimeter of the plot, and the CFO replied with a request that they provide a detailed analysis of the possible perimeters for the test plots.

1. Before the CFO’s request for an analysis of possible perimeters, the R&D Team had planned to use rectangular plots 24 feet wide and 50 feet long. They chose these dimensions because they would facilitate cutting of the sod for sale. All SYnG machinery is set to cut sod in 3’ by 1’ rectangular sections for sale to nurseries and landscaping companies.

   a. Verify that a 24’ by 50’ plot has the required area.

   b. How can such a plot be cut into 3’ by 1’ rectangular sections so that there are no leftover pieces of sod? How many of these sections will be produced in a 24’ by 50’ plot?

   c. How much fencing is needed for a 24’ by 50’ grass test plot?
2. The Maintenance Division of SYnG is responsible for maintaining all experimental plots of plants, grass, and tree varieties. The R&D Team decided to consult the Maintenance Division as a first step in developing the analysis for the CFO. The mowing crew cuts the grass in the test plots several times before the sod is established, and the gardeners monitor the use of water, fertilizer, and herbicides. So R&D asked the mowers and gardeners for input. After working with diagrams of mowing patterns for their 60-inch blade mowers, the mowing crew recommended that the plots be by 25’ by 48’ and that the fences have gates in opposite corners so that they could enter in one corner, cut five 60-inch strips and exit at the opposite corner. The gardeners suggested that the plots be 12’ by 100’ to simplify use of existing 12-foot wide equipment currently in use for watering and applying liquid fertilizers.

   a. Use grid paper to represent a 25’ by 48’ rectangle and the mowing crew’s plans for cutting the grass.

   b. Verify that a 25’ by 48’ rectangle has the required area and find its perimeter.

   c. Using the same scale from part a, represent a 12’ by 100’ rectangle on your grid paper.

   d. Verify that a 12’ by 100’ rectangle has the required area and find its perimeter.

3. The R&D Team at SYnG let \( x \) represent one dimension of a rectangle that has area 1200 square feet, let \( y \) represent the other dimension, and explored the relationship between the two variables.

   a. Why is the relationship a functional relationship? Does it matter which dimension you consider to be the input and which to be the output? Explain.

   b. Find a formula to write \( y \) as a function of \( x \).
c. Use your formula to find the value of \( y \) when \( x = 5, 10, 15, 20, 30, 35, 40, 60, 120, 240 \). Organize your results in a table, and add the possible values for \( x \) and \( y \) from item 2, parts b and d.

d. Because the 60-inch mowers must be able to cut the test plots, the smaller dimension of the rectangle must be at least 5’. In this context, what is the domain for the function from part b? Explain your reasoning.

e. **Use technology** to draw a graph of the function from part b. Use the same scale on each axis. The graph has a line of symmetry that is neither horizontal nor vertical. Where is the line?

f. The endpoints of the graph drawn for part e are reflections through the line of symmetry. What do you notice about the relationship between the coordinates of these points? Examine your table of values. Find another pair of points with this relationship between the coordinates. Are these points reflections of each other through the line of symmetry?

g. There is a value of \( x \) such that \( x \) and \( y \) are the same. What is the corresponding point on your graph? What special shape does the rectangle have when \( x = y \)?

h. For what \( x \)-values does the \( y \)-value increase as \( x \)-increases? For what \( x \)-values does the \( y \)-value decrease as \( x \)-increases? Explain your answer based on the graph and the formula.
4. The R&D Team at SYnG used their formula for \( y \) as a function of \( x \) to write a formula for the perimeter in terms of \( x \). Let \( P \) be the perimeter function for this context.

   a. Add a column for perimeter to your table from item 3, part c above.

   b. Using function notation, write a formula to calculate the perimeter for any choice of \( x \)-dimension of the rectangle.

   c. In this context, what is the domain of \( P \)? Explain your reasoning.

   d. Examine your table of values. How do the values of \( P(x) \) change as \( x \) increases? What is the maximum value for the perimeter? What are the dimensions of any rectangle that has the maximum perimeter?

   e. **Use technology** to draw the following on same set of axes: the graph of the function \( P \), the graph of \( y \) as a function of \( x \), and the vertical line through special point noted in item 3, part g. What do you notice about the vertical line?

   f. Examine the graph of \( P \). What value of \( x \) gives the minimum value for \( P(x) \)? What are the dimensions of any rectangle that has the minimum perimeter?
5. The R&D Team provided the CFO with tables and graphs similar to those you have created in responding to items 3 and 4. She estimates that the special fencing to surround the test plots will cost approximately $15 per foot of perimeter to be fenced. Answer the following based on a cost of $15 per foot of perimeter.

a. What are the dimensions for the rectangle that would cost the most to fence? What is the maximum cost?

b. What is the cost of fencing the 24” by 50’ rectangle suggested by the R&D Team?

c. What is the cost of fencing the 25’ by 48’ rectangle suggested by the mowing crew?

d. What is the cost of fencing the 12’ by 100’ rectangle suggested by the gardeners?

e. What are the dimensions of the rectangle that would cost the least to fence? What is the minimum cost?

f. What dimensions do you think that they decided to use and why?

The R&D Team at SYnG is also trying to develop new hardier and more drought tolerant varieties of flowering plants for use in landscaping. The experimental plants are studied for one growing
season and are set in square raised beds with one plant for every square foot of area within the beds. Previously, the number of plats had been restricted to those sizes which allowed for a square arrangement of the plants, similar to the pattern of dots considered in item 5 of the “Sequences as Functions” learning task. Last summer, the Maintenance Division, which is responsible for building the raised beds as well as maintaining them, asked for permission to experiment with a variety of sizes for the beds. They pointed out that as long as the number of plants was the same as the number of square feet enclosed by the raised box, they could find an arrangement that gave each plant a square foot of area, although that area for each plant would not necessarily have a square shape.

6. The R&D Team agreed to allow the Maintenance Division to experiment with the size of the raised beds. Pleased with the chance to experiment, the Maintenance Division decided to build beds for every number of plants from 4 to 400. The length, in feet, of a side of a raised bed is a function of the number of plants that Maintenance will plant in the bed. Name this function \( S \). If \( n \) represents the number of plants in a raised bed, then \( S(n) = \sqrt{n} \).

a. What domain of values does Maintenance plan to use for the function \( S \)?

b. Let \( f \) be the function defined for real all real numbers \( x \geq 0 \) by the formula

\[ f(x) = \sqrt{x}. \]

Make a table of values for \( f \) and draw a graph that includes domain values for \( 0 \leq x \leq 400 \) and indicates the shape of the whole graph.

c. The functions \( f \) and \( S \) both take the square root of the input to find the output. Are their domains the same? Are they equal functions? Explain.

d. How can you use the graph of \( f \) to obtain the graph of \( S \)?

e. The function \( S \) could be considered as a sequence. Do you think it is helpful to think of the function this way? Why or why not?
7. The Maintenance Division uses recycled railroad cross ties to build the raised beds. They arrange the cross ties as shown in the figure at the right.

a. The end view of each cross tie is a square approximately nine inches on a side. What is the length of the side of a square bed needed for 4 plants? What is the length of each cross tie, with length measured in feet, needed to make the box for this bed?

b. Answer the questions from part a for 9 plants, 16 plants, and 23 plants.

c. The Maintenance Division needs a cross-tie length function that inputs the number of plants, $n$, and outputs the length $L(n)$ of the cross ties needed to build the raised bed. Write the formula to calculate $L(n)$ given $n$.

d. How can you use the graph of $f$ from item 6, part b, to draw the graph of $L$, the cross-tie length function?

e. Draw a plan for a raised bed for 24 plants. Specify the lengths of the cross-ties and how you would arrange the plants. Explain how you know that your arrangement gives each plant one square foot of growing area.
Painted Cubes Learning Task

The Vee Company, which produces the Zingo game, is working on a new product: a puzzle invented by one of its employees. The employee, Martin, made a large cube from 1,000 smaller cubes, each having edge length one centimeter, by using temporary adhesive to hold the small cubes together. He painted the faces of the large cube, but when the paint had dried, he separated the large cube into the original 1000 centimeter cubes. The object of his puzzle is to reassemble the cube back together so that no unpainted faces are showing.

The manager responsible for developing Martin’s puzzle into a Vee Company product thought that a 1000 cube puzzle might have too many pieces and decided that he should investigate the puzzle starting with smaller versions.

1. The cube at the right is made of smaller unpainted cubes, each having edge length 1 centimeter. All the faces of the large cube are painted yellow.
   a. How many small cubes were used to make the large cube?
   b. If you could take the large cube apart into the original centimeter cubes, how many cubes would be painted on
      i. three faces?
      ii. two faces?
      iii. one face?
      iv. no faces?

2. Consider large cubes with edge lengths of 3, 4, 5, 6, and 7 centimeters by building and/or sketching models, and answer the same questions that you answered for the large cube of edge length 2. Organize all your answers in a table as shown below.

<table>
<thead>
<tr>
<th>Edge length of large cube</th>
<th>Number of centimeter cubes</th>
<th>Number of small centimeter cubes painted on</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>3 faces</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
3. Study the pattern in the table when the edge length of the large cube is used as input and the
output is the number of centimeter cubes used in constructing the large cube.
   a. Denote this functional relationship \( N \) so that \( x \) represents the edge length of the large cube
      and \( N(x) \) represents the number of centimeter cubes used to make the large cube. Write an
      equation expressing \( N(x) \) in terms of \( x \).

   b. If we want to consider this relationship in general, not limiting ourselves to Martin’s
      puzzle, what should we use for the domain of the function \( N \)?

   c. Let \( f \) be the function such that the formula for \( f(x) \) is the same as the formula for \( N(x) \) but
      the domain is all real numbers. Make a table for values of \( f \). Include some \( x \)-values that are
      in the domain of the function \( N \) and some that are not in the domain of \( N \), especially some
      negative numbers and fractions.

   d. Sketch the graphs of \( f \) and \( N \) on the same axes for \( -10 \leq x \leq 10 \). How does the graph of \( f \)
      help you understand the graph of \( N \)?

4. Study the pattern in the table when edge length of the large cube is used as input and the output
is the number of small centimeter cubes that have 0 faces painted.
   a. Denote this functional relationship \( U \) so that \( x \) represents the edge length of the large cube
      and \( U(x) \) represents the number of unpainted centimeter cubes. Write an equation
      expressing \( U(x) \) in terms of \( x \).

   b. If we want to consider this relationship in general, not limiting ourselves to Martin’s
      puzzle, what should we use for the domain of the function \( U \)?

   c. Let \( g \) be the function such that the formula for \( g(x) \) is the same as the formula for \( U(x) \) but
      the domain is all real numbers. Make a table for values of \( g \). Include some \( x \)-values that
      are in the domain of the function \( U \) and some that are not in the domain of \( U \), especially
      some negative numbers and fractions.
d. Sketch the graphs of \( g \) and \( U \) on the same axes for \(-10 \leq x \leq 10\). How does the graph of \( g \) help you understand the graph of \( U \)? Explain.

5. Consider the context of the functions, \( N \) and \( U \).
   a. For any edge length \( x \) for the large cube, geometrically, what do the numbers \( N(x) \) and \( U(x) \) represent?

   b. Do your formulas for the functions \( N \) and \( U \) show this relationship?

   c. How does this relationship show up in the graphs? Explain the relationship in the graphs in terms of inputs and outputs to the two functions.

   d. Consider the graphs of \( f \) and \( g \). Are they related in the same way that the graphs of \( N \) and \( U \) are related?

6. Use the data in the table from item 2 above to make a table where edge length of the large cube is used as input and the output is the number of small centimeter cubes that have 1 face painted.
   a. Give a geometric explanation for why all of the output numbers are multiples of 6.

   b. Denote this functional relationship \( S \), for single face painted, and again let \( x \) represent the edge length of the large cube. Express each output as a multiple of six, that is, write 0 as \( 6(0) \), write 6 as \( 6(1) \), etc. What pattern do you see? Use this pattern to write an equation expressing \( S(x) \) in terms of \( x \).
c. If we want to consider this relationship in general, not limiting ourselves to Martin’s puzzle, what should we use for the domain of the function \( S \)?

d. Let \( h \) be the function such that the formula for \( h(x) \) is the same as the formula for \( S(x) \) but the domain is all real numbers. Make a table for values of \( h \). Include the values from the function \( S \) and include values of \( x \) not in the domain of \( S \), especially some negative numbers and fractions.

e. Sketch the graphs of \( h \) and \( S \) on the same axes for \(-10 \leq x \leq 14\). How does the graph of \( h \) help you understand the graph of \( S \)? Explain.

7. You have seen functions similar to the function \( h \) before. In an earlier learning task, you considered the function which gives the distance \( y \), in meters, that a ball dropped from a high place will fall in \( x \) seconds; the formula is \( y = 5x^2 \). In another learning task, you considered the function \( A \) for the area of a square given the length of a side, \( s \), where \( A(s) = s^2 \).

Consider the functions specified by the equations below; use the set of all real numbers as the domain for each function.

(i) \( y = x^2 \) 
(ii) \( y = 3x^2 \) 
(iii) \( y = 6x^2 \)
Each of these equations has the form $y = ax^2$, where $a$ is a constant real number. In each of these situations, we say that $y$ varies directly as the square of $x$ and call the value for $a$ the constant of variation.

a. On the same axes, sketch the graphs, with domain all real numbers, for equations (i) – (iii) above. What is the value of the constant of variation, $a$, for each of these equations? How does the value of $a$ affect the shape of the graph?

b. On the same axes, sketch the graphs of $y = x^2$ and $y = -x^2$. What is the effect of changing the sign of $a$? Graph $y = 5x^2$ and $y = -5x^2$ on the same axes. Is the effect the same? Explain why the sign change affects the graph in this way.

c. Choose two values for $a$ such that $0 < a < 1$, and graph $y = x^2$ and $y = ax^2$ for these two values on the same axes. Does the value of $a$ affect these graphs in the same way that you described in part a above?

8. Typical braking distances, $d$, of an automobile for a given speed $s$ are given in the following table. [Note that braking distance is different from stopping distance since there is a reaction time between the time when a driver decides to brake and the time when the brakes are actually applied.]

<table>
<thead>
<tr>
<th>Speed (miles per hour)</th>
<th>$s$</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
<th>60</th>
<th>70</th>
</tr>
</thead>
<tbody>
<tr>
<td>Braking distance (feet)</td>
<td>$d$</td>
<td>5</td>
<td>20</td>
<td>45</td>
<td>80</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
a. The braking distance for an automobile **varies directly** as the square of the speed. What is the general form of the equation relating $d$ and $s$? Use the data in the table to find the specific formula for $d$ as a function of $s$.

b. Fill in the missing values in the table.

c. What should be the domain for braking distance $d$ as a function of speed $s$?

d. How does the graph of this function compare to the graph of the function $y = x^2$ from item 7(i) above?

e. What is the average rate of change of $d$ with respect to $s$ when $s$ changes from 10 to 20? when $s$ changes from 60 to 70? Give the unit of measure and explain the meaning of the numbers.

9. Use the data in the table from #2 to make a table where edge length of the large cube is used as input and the output is the number of small centimeter cubes that have 2 faces painted.

a. Give a geometric explanation for why all of the output numbers are multiples of 12.
b. Denote this functional relationship $T$, for two faces painted, and again let $x$ represent the edge length of the large cube. Write an equation expressing $T(x)$ in terms of $x$.

c. If we want to consider this relationship in general, not limiting ourselves to Martin’s puzzle, what should we use for the domain of the function $T$?

d. Let $k$ be the function such that the formula for $k(x)$ is the same as the formula for $T(x)$ but the domain is all real numbers. Sketch the graphs of $k$ and $T$ on the same axes. What kind of functional relationship does each of these functions possess?

10. Use the data in the table from #2 to make a table where edge length of the large cube is used as input and the output is the number of small centimeter cubes that have 3 faces painted.

a. Give a geometric explanation for why all of the output numbers are the same.

b. Denote this functional relationship $H$, for half of the faces painted, and again let $x$ represent the edge length of the large cube. Write the equation for $H(x)$.

e. If we want to consider this relationship in general, not limiting ourselves to Martin’s puzzle, what should we use for the domain of the function $H$?

c. Let $j$ be the function such that the formula for $j(x)$ is the same as the formula for $H(x)$ but the domain is all real numbers. Sketch the graphs of $j$ and $H$ on the same axes. Why are these called constant functions?

* Adapted from the “Painted Cubes” section of Frogs, Fleas, and Painted Cubes: Quadratic Relationships in the Connected Mathematics 2 series, Pearson Prentice Hall.