Table of Contents

INTRODUCTION: ..................................................................................................... 3
Tiling Learning Task................................................................................................... 7
Tiling Pools Learning Task....................................................................................... 14
I’ve Got Your Number Learning Task ................................................................. 17
Alternate Version for Included Sample Lesson Plan I’ve Got Your Number Learning Task ........................................................................................................... 27
Just Jogging Learning Task ...................................................................................... 31
Ladder Length Learning Task................................................................................... 33
INTRODUCTION:
In this unit students develop skill in adding, subtracting, multiplying, and dividing elementary polynomial, rational, and radical expressions. By using algebraic expressions to represent quantities in context, students understand algebraic rules as general statements about operations on real numbers. The focus of the unit is the development of students’ abilities to read and write the symbolically intensive language of algebra.

ENDURING UNDERSTANDINGS:
• Algebraic equations can be identities that express properties of operations on real numbers.
• Equivalence of algebraic expressions means that the expressions have the same numerical value for all possible values of the variable.
• Equivalent expressions are useful tools in computation and problem solving.
• It takes only one counterexample to show that a general statement is not true.

KEY STANDARDS ADDRESSED:

MM1A2. Students will simplify and operate with radical expressions, polynomials, and rational expressions.
   a. Simplify algebraic and numeric expressions involving square root.
   b. Perform operations with square roots.
   c. Add, subtract, multiply, and divide polynomials.
   d. Add, subtract, multiply, and divide rational expressions.
   e. Factor expressions by greatest common factor, grouping, trial and error, and special products limited to the formulas below.

\[
\begin{align*}
(x + y)^2 &= x^2 + 2xy + y^2 \\
(x - y)^2 &= x^2 - 2xy + y^2 \\
(x + y)(x - y) &= x^2 - y^2 \\
(x + a)(x + b) &= x^2 + (a + b)x + ab \\
(x + y)^3 &= x^3 + 3x^2y + 3xy^2 + y^3 \\
(x - y)^3 &= x^3 - 3x^2y + 3xy^2 - y^3
\end{align*}
\]

f. Use the binomial theorem to expand polynomials.*
   g. Use area and volume models for polynomial arithmetic.
MM1A3. Students will solve simple equations.
   a. Solve quadratic equations in the form $ax^2 + bx + c = 0$, where $a = 1$, by using factorization and finding square roots where applicable.

   * In this unit, use of the binomial theorem limits $n$ to two or three, in future units, the value of $n$ will be increased.

\[(a\pm b)^n = an \pm nc_{n-1} an-1b + nc_{n-2} an-2b^2 \pm \ldots\ldots + nc_1 ab^{n-1} \pm bn\]

RELATED STANDARDS ADDRESSED:
MM1G2. Students will understand and use the language of mathematical argument and justification.
   a. Use conjecture, inductive reasoning, deductive reasoning, counterexamples, and indirect proof as appropriate

MM1P1. Students will solve problems (using appropriate technology).
   a. Build new mathematical knowledge through problem solving.
   b. Solve problems that arise in mathematics and in other contexts.
   c. Apply and adapt a variety of appropriate strategies to solve problems.
   d. Monitor and reflect on the process of mathematical problem solving.

MM1P2. Students will reason and evaluate mathematical arguments.
   a. Recognize reasoning and proof as fundamental aspects of mathematics.
   b. Make and investigate mathematical conjectures.
   c. Develop and evaluate mathematical arguments and proofs.
   d. Select and use various types of reasoning and methods of proof.

MM1P3. Students will communicate mathematically.
   a. Organize and consolidate their mathematical thinking through communication.
   b. Communicate their mathematical thinking coherently and clearly to peers, teachers, and others.
   c. Analyze and evaluate the mathematical thinking and strategies of others.
   d. Use the language of mathematics to express mathematical ideas precisely.

MM1P4. Students will make connections among mathematical ideas and to other disciplines.
   a. Recognize and use connections among mathematical ideas.
   b. Understand how mathematical ideas interconnect and build on one another to produce a coherent whole.
   c. Recognize and apply mathematics in contexts outside of mathematics.

MM1P5. Students will represent mathematics in multiple ways.
   a. Create and use representations to organize, record, and communicate mathematical ideas.
   b. Select, apply, and translate among mathematical representations to solve problems.
   c. Use representations to model and interpret physical, social, and mathematical phenomena.
UNIT OVERVIEW

Prior to this unit, students need to have worked extensively with operations on integers, rational numbers, and square roots of nonnegative integers as indicated in the Grade 6 – 8 standards for Number and Operations.

In the unit students will apply and extend all of the Grade 6 – 8 standards related to writing algebraic expressions, performing operations with algebraic expressions, and working with relationships between variable quantities. Students are assumed to have a deep understanding of linear relationships between variable quantities. Students should understand how to find the area of triangles, rectangles, squares, and trapezoids and know the Pythagorean theorem.

In the unit students will apply the basic function concepts of domain, range, rule of correspondence, and interpreting graphs of functions learned in Unit 1 of Mathematics 1.

The unit begins with intensive work in writing linear and quadratic expressions to represent quantities in a real-world context. The initial focus is developing students’ abilities to read and write meaningful statements using the language of algebra. In the study of operations on polynomials, the special products of standard MM1A2 are studied as product formulas and interpreted extensively through area models of multiplication. Factoring polynomials and solving quadratic equations by factoring are covered in a later unit, but the work with products and introduction of the zero factor property are designed to build a solid foundation for these later topics.

The work with rational and radical expressions is grounded in work with real-world situations that show applications of working with such expressions. Students need to practice computational skills, but extensive applications are needed to demonstrate that the skills have important applicability in modeling and understanding the world around us. The work with rational expressions includes consideration of the application to calculating average speeds.

Students are given many opportunities to use geometric reasoning to justify algebraic equivalence and understand algebraic rules as statements about real number operations. Early in their study of algebra, students may have difficulty grasping the full content of abstract algebraic statements. Many of the questions in the tasks of this unit are intended to guide students to see how a geometric or other relationship from a physical context is represented by an algebraic expression. So, although some of the questions may seem very simple, it is important that they not be skipped. Gaining the ability to see all the information in an abstract algebraic statement takes time and lots of practice.

Algebra is the language that allows us to make general statements about the behavior of the numbers and gaining facility with this language is essential for every educated citizen of the twenty-first century. Throughout this unit, it is important to:
• Require students to explain how their algebraic expressions, formulas, and equations represent the geometric or other physical situation with which they are working.

• Encourage students to come up with many different algebraic expressions for the same quantity, to use tables and graphs to verify expressions are equivalent, and to use algebraic properties to verify algebraic equivalence.

• Make conjectures about relationships between operations on real numbers and then give geometric and algebraic explanations of why the relationship always holds or use a counterexample to show that the conjecture is false.

**TASKS:** The remaining content of this framework consists of student tasks or activities. The first is intended to launch the unit. Activities designed to allow students to build their own algebraic understanding through exploration follow. The last task is designed to demonstrate the type of assessment activities students should be comfortable with by the end of the unit. Thorough Teacher’s Guides which provide solutions, discuss teaching strategy, and give additional mathematical background are available to accompany each task.
Tiling Learning Task

Latasha and Mario are high school juniors who worked as counselors at a day camp last summer. One of the art projects for the campers involved making designs from colored one-square-inch tiles. As the students worked enthusiastically making their designs, Mario noticed one student making a diamond-shaped design and wondered how big a design, with the same pattern, that could be made if all 5000 tiles available were used. Later in the afternoon, as he and Latasha were putting away materials after the children had left, he mentioned the idea to Latasha. She replied that she saw an interesting design too and wondered if he were talking about the same design. At this point, they stopped cleaning up and got out the tiles to show each other the designs they had in mind.

Mario presented the design that interested him as a sequence of figures as follows:

1. To make sure that you understand the design that was of interest to Mario, answer the following questions.

   a) How many rows of tiles are in each of Mario’s figures?

   b) What pattern do you observe that relates the number of rows to the figure number? Explain in a sentence.

   c) Use this pattern to predict the number rows in Figure 12, Figure 47, and Figure 111 if these figures were to be drawn.
d) Write an algebraic expression for the number of rows in Figure $k$. Explain why your pattern will always give the correct number of rows in Figure $k$. Can your expression be simplified? If so, simplify it.

e) What is the total number of tiles in each figure above?

f) What pattern do you observe that relates the total number of tiles to the figure number? Explain in a sentence.

g) Use this pattern to predict the total number of tiles in Figure 12, Figure 47, and Figure 111 if these figures were to be drawn.

h) Write an algebraic expression for the total number of tiles in Figure $k$. Explain why your pattern will always give the correct total number of tiles in Figure $k$. 

When Latasha saw Mario’s figures, she realized that the pattern Mario had in mind was very similar to the one that caught her eye, but not quite the same. Latasha pushed each of Mario’s designs apart and added some tiles in the middle to make the following sequence of figures.

2. Answer the following questions for Latasha’s figures.

   a) How many rows of tiles are in each of the figures above?

   b) What pattern do you observe that relates the number of rows to the figure number? Explain in a sentence.

   c) Use this pattern to predict the number of rows in Figure 12, Figure 47, and Figure 111 if these figures were to be drawn.

   d) Write an algebraic expression for the number of rows in Figure $k$. Explain why your pattern will always give the correct number of rows in Figure $k$. Can your expression be simplified? If so, simplify it.

   e) What is the total number of tiles in each figure above?

   f) What pattern do you observe that relates the total number of tiles to the figure number? Explain in a sentence.
g) Use this pattern to predict the total number of tiles in Figure 12, Figure 47, and Figure 111 if these figures were to be drawn.

h) Write an algebraic expression for the total number of tiles in Figure $k$. Explain why your pattern will always give the correct total number of tiles in Figure $k$.

i) Give a geometric reason why the number of tiles in Figure $k$ is always an even number. Look at the algebraic expression you wrote in part d.

j) Give an algebraic explanation of why this expression always gives an even number. [Hint: If your expression is not a product, use the distributive property to rewrite it as a product.]

3. Mario started the discussion with Latasha wondering whether he could make a version of the diamond pattern that interested him that would use all 5000 tiles that they had in the art supplies. What do you think? Explain your answer. If you can use all 5000 tiles, how many rows will the design have? If a similar design cannot be made, what is the largest design that can be made with the 5000 tiles, that is, how many rows will this design have and how many tiles will be used?

4. What is the largest design in the pattern Latasha liked that can be made with no more than 5000 tiles? How many rows does it have? Does it use all 5000 tiles? Justify your answers.
Let \( M_1, M_2, M_3, M_4 \), and so forth represent the sequence of numbers that give the total number of tiles in Mario’s sequence of figures.

Let \( L_1, L_2, L_3, L_4 \) and so forth represent the sequence of numbers that give the total number of tiles in Latasha’s sequence of figures.

5. Write an equation that expresses each of the following:
   
   a) the relationship between \( L_1 \) and \( M_1 \)
   
   b) the relationship between \( L_2 \) and \( M_2 \)

   c) the relationship between \( L_3 \) and \( M_3 \)

   d) the relationship between \( L_4 \) and \( M_4 \)

   e) the general relationship between \( L_k \) and \( M_k \), where \( k \) can represent any positive integer.

**Triangular numbers** are positive integers such that the given number of dots can be arranged in an equilateral triangle. The first few triangular numbers are as follows.

\[
\begin{array}{cccc}
1 & 3 & 6 & 10 & 15 \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\end{array}
\]

Let \( T_1, T_2, T_3, T_4 \), and so forth represent the sequence of triangular numbers.

6. a) Examine the arrangement of dots for \( T_4 \). How many dots are in row 1? row 2? row 3? row 4?

b) Write \( T_4 \) as a sum of four positive integers.

c) Write each of \( T_2, T_3, \) and \( T_5 \) as a sum 2, 3, and 5 positive integers, respectively.

d) Explain what sum you would need to compute to find \( T_{12} \). Compute \( T_{12} \) doing the addition yourself.

e) Explain what sum you would need to compute to find \( T_{47} \). Use technology to find \( T_{47} \).
7. The triangular numbers could also be represented by arrangements of square tiles instead of dots.
   a) Draw the first few figures to represent triangular numbers using square tiles.

   b) Compare these figures to Latasha’s diamond-shaped figures. Write a sentence comparing the triangular-number figures and Latasha’s figures.

   c) Write an equation using \( L_k \) and \( T_k \) to express the relationship between the number tiles in the \( k \)-th one of Latasha’s figures and the \( k \)-th triangular number figure.

   d) Check your equation by comparing the numbers \( L_1 \) and \( T_1 \), \( L_2 \) and \( T_2 \), \( L_3 \) and \( T_3 \), \( L_4 \) and \( T_4 \), \( L_5 \) and \( T_5 \), \( L_{12} \) and \( T_{12} \), and \( L_{47} \) and \( T_{47} \).

   e) Write a formula to calculate \( T_k \), the \( k \)-th triangular number, without summing the first \( k \) positive integers.

   f) Check your formula by using it to calculate \( T_1 \), \( T_2 \), \( T_3 \), \( T_4 \), \( T_5 \), \( T_{12} \), and \( T_{47} \).

   g) Give a geometric reason why your formula always gives a whole number.

   h) Give an algebraic reason why the formula must always give a whole number.

   i) What sum does your formula calculate?
8. a) Compare the triangular-number figures to Mario’s figures. Write a sentence comparing the triangular-number figures and Mario’s figures.

b) Write an equation using $M_k$ and $T_k$ to express the relationship between the number tiles in the $k$-th one of Mario’s figures and the $k$-th triangular number figure.

c) What is another name for the sequence $M_1, M_2, M_3, M_4, \ldots$?

d) Write a sentence expressing the relationship expressed in part b); use the familiar name for the numbers in the sequence $M_1, M_2, M_3, M_4, \ldots$. 
Tiling Pools Learning Task

In this task, you will continue to explore how different ways of reasoning about a situation can lead to algebraic expressions that are different but equivalent to each other. We will use swimming pools as the context throughout this task.

In the figures below there are diagrams of swimming pools that have been divided into two sections. Swimming pools are often divided so that different sections are used for different purposes such as swimming laps, diving, area for small children, etc.

(a) For each pool, write two different but equivalent expressions for the total area.
(b) Explain how these diagrams and expressions illustrate the Distributive Property.

In-ground pools are usually surrounded by a waterproof surface such as concrete. Many homeowners have tile borders installed around the outside edges of their pools to make their pool area more attractive. Superior Pools specializes in custom pools for residential customers and often gets orders for square pools of different sizes. The diagram at the right shows a pool that is 8 feet on each side and is surrounded by two rows of square tiles. Superior Pools uses square tiles that are one foot on each side for all of its tile borders.

The manager at Superior Pools is responsible for telling the installation crew how many border tiles they need for each job and needs an equation for calculating the number of tiles needed for a square pool depending on the size of the pool. Let \( N \) represent the total number of tiles needed when the length of a side of the square pool is \( s \) feet and the border is two tiles wide.

3. Write a formula in terms of the variable \( s \) that can be used to calculate \( N \).

4. Write a different but equivalent formula that can be used to calculate \( N \).
5. Give a geometric explanation of why the two different expressions in your formulas for the number of border tiles are equivalent expressions. Include diagrams.

6. Use the Commutative, Associative, and/or Distributive properties to show that your expressions for the number of border tiles are equivalent.

Some customers who have pools installed by Superior Pools want larger pools and choose a rectangular shape that is not a square. Many of these customers also choose to have tile borders that are 2 tiles wide.

7. How many 1-foot square border tiles are needed to put a two-tile-wide border around a pool that is 12 feet wide and 30 feet long?

8. Write an equation for finding the number $N$ of border tiles needed to put a two-tile-wide border around a pool that is $L$ feet long and $W$ feet wide. Explain, with diagrams, how you found your expression.

9. Explain why the area $A$ of the tile border (in square feet) is the same number as the number of tiles that are needed for the border. Write an equation for finding the area $A$ of the tile border using an expression that is different from but equivalent to the expression used in the equation for $N$ given in answering question 8. Use algebraic properties to show that your expressions for $A$ and $N$ are equivalent.
A company that sells hot tubs creates a tile border for its products by placing 1-foot-square tiles along the edges of the tub and triangular tiles at the corners as shown. The triangular tiles are made by cutting the square tiles in half along a diagonal.

10. Suppose a hot tub has sides of length 6 feet. How many square tiles are needed for the border?

11. Write an equation for the number of square tiles \( N \) needed to create such a border on a hot tub that has sides that are \( s \) feet long.

12. Write a different but equivalent expression for the number of border tiles \( N \). Explain why this expression is equivalent to the one given in your answer to question 11.

13. Below are three expressions that some students wrote for the number of tiles needed for the border of a square hot tub with sides \( s \) feet long.

\[
(i) \quad 4 \left( \frac{s + s + 1}{2} \right) \\
(ii) \quad 4 \left( \frac{s + s}{4} \right) + 2 \\
(iii) \quad (s + 2)^2 - 4 \left( \frac{1}{2} \right) - s^2
\]

(a) Use each expression to find the number of border tiles if \( s = 0 \).

(b) Do you think that the expressions are equivalent? Explain.

(c) Use each expression to find the number of tiles if \( s = 10 \). Does this result agree with your answer to part (b)? Explain.

(d) What can you say about testing specific values as a method for determining whether two different expressions are equivalent?

(e) Use algebraic properties to show the equivalence of those expressions in 11, 12, and 13 which are equivalent.

* Adapted from the “Equivalent Expressions” section of Say It With Symbols: Making Sense of Symbols in the Connected Mathematics 2 series.
I’ve Got Your Number Learning Task

Equivalent algebraic expressions, also called algebraic identities, give us a way to express results with numbers that always work a certain way. In this task you will explore several “number tricks” that work because of basic algebra rules. It is recommended that you do this task with a partner.

Think of a number and call this number \( x \).

Now think of two other numbers, one that is 2 more than your original number and a second that is 3 more than your original number. No matter what your choice of original number, these two additional numbers are represented by \( x + 2 \) and \( x + 3 \).

Multiply your \( x + 2 \) and \( x + 3 \) and record it as Answer 1 ________ .

Find the square of your original number, \( x^2 \) , five times your original number, \( 5x \) , and the sum of these two numbers plus six, \( x^2 + 5x + 6 \), and record it as Answer 2 ________ .

Compare Answers 1 and 2. Are they the same number? They should be. If they are not, look for a mistake in your calculations.

**Question:** How did the writer of this task know that your Answers 1 and 2 should be the same even though the writer had no way of knowing what number you would choose for \( x \)?

**Answer:** Algebra proves it has to be this way.

To get Answer 1, we multiply the number \( x + 2 \) by the number \( x + 3 \): \((x + 2)(x + 3)\)

We can use the distributive property several times to write a different but equivalent expression.

First treat \((x + 2)\) as a single number but think of \( x + 3 \) as the sum of the numbers \( x \) and 3, and apply the distributive property to obtain: \((x + 2) \cdot x + (x + 2) \cdot 3\)

Now, change your point of view and think of \( x + 2 \) as the sum of the numbers \( x \) and 2, and apply the distributive property to each of the expressions containing \( x + 2 \) as a factor to obtain: \(x \cdot x + 2 \cdot x + x \cdot 3 + 2 \cdot 3\)

Using our agreements about algebra notation, rewrite as: \( x^2 + 2x + 3x + 6\)

Add the like terms \( 2x \) and \( 3x \) by using the distributive property in the other direction: \( 2x + 3x = (2 + 3)x = 5x \)
The final expression equivalent to \((x + 2)(x + 3)\) is: 
\[x^2 + 5x + 6\]

The last expression is Answer 2, so we have shown that \((x + 2)(x + 3) = x^2 + 5x + 6\) no matter what number you choose for \(x\). Notice that 5 is the sum of 2 and 3 and 6 is the product of 2 and 3.

The above calculations are just an example of the following equivalence of algebraic expressions.

\[(x + a)(x + b) = x^2 + (a + b)x + ab\]  \hspace{1cm} \text{(Pattern 1)}

For the remainder of this task, we’ll refer to the above equivalence as Pattern 1. Note that we used \(a = 2\) and \(b = 3\) is our example, but your task right now is to show, geometrically, that \(a\) could represent any real number and so could \(b\).

1. (a) Each of the diagrams below illustrates Pattern 1. Match each diagram with one of the following cases for \(a\) and \(b\).

   \begin{itemize}
   \item \textit{Case 1}: \(a\) positive, \(b\) positive \hspace{1cm} \text{Figure A}.
   \item \textit{Case 2}: \(a\) positive, \(b\) negative \hspace{1cm} \text{Figure B}.
   \item \textit{Case 3}: \(a\) negative, \(b\) positive \hspace{1cm} \text{Figure C}.
   \item \textit{Case 4}: \(a\) negative, \(b\) negative \hspace{1cm} \text{Figure D}.
   \end{itemize}
(b) Thinking of the case it represents, for each diagram above, find the rectangle whose area is 
\((x + a)(x + b)\) and use a pencil to put diagonal stripes on this rectangle. Then explain how the diagram illustrates the pattern. Note that when \(a\) is a negative number, \(a = -|a|\), and when \(a\) is a positive number, then \(a = |a|\); and similarly for \(b\). Tell two of your explanations to another student and let that student explain the other two to you.

(c) What happens to the pattern and the diagrams if \(a\) is 0? if \(b\) is 0? if both \(a\) and \(b\) are 0?

2. Pattern 1 can be used to give an alternate way to multiply two digit numbers that have the same digit in the ten’s place.
For example, \((31)(37)\) can be thought of as \((30 + 1)(30 + 7)\) so we let \(x = 30\), \(a = 1\), and \(b = 7\). Since \(30^2 = 900\), \(1 + 7 = 8\) and \(1 \cdot 7 = 7\),
\[(31)(37) = (30 + 1)(30 + 7) = 900 + (8)(30) + 7 = 900 + 240 + 7 = 1147.\]

Use Pattern 1 to calculate each of the following products.

(a) \((52)(57) = \)

(b) \((16)(13) = \)

(c) \((48)(42) = \)

(d) \((72)(75) = \)
3. Look at your result for 2(c). You can get the answer another way by doing the following.
Take 4, the common ten’s digit, and multiply it by the next integer, 5, to get 20 as the number of hundreds in the answer. Then multiply the units digits 4 x 6 to get 24 for the last two digits.

\[
\begin{array}{c}
4 & 8 \\
\times & 4 \ 2 \\
\hline
2 & 0 & 1 & 6 \\
\end{array}
\]

\[
4 \cdot 5 = 20 \quad \quad 8 \cdot 2 = 16
\]

Use this scheme to calculate the products below and verify the answers using Pattern 1.

(a) \((34)(36) = \) _____.
(b) \((63)(67) = \) _____.
(c) \((81)(89) = \) _____.
(d) \((95)(95) = \) _____.

(e) In each of the products immediately above, look at the pairs of units digits: 8 and 2 in the example, 4 and 6 for part (a), 3 and 7 in part (b), 1 and 9 in part (c), and 5 and 5 in part (d). What sum do each of the pairs have?

(f) Now let’s use Pattern 1 to see why this scheme works for these products with the property you noted in part (e). To represent two-digit numbers with the same ten’s digit, start by using \(n\) to represent the ten’s digit. So, \(n\) is 4 for the example and 3 for part (a). What is \(n\) for parts (b), (c), and (d)?

(g) Next, represent the first two-digit number as 10 \(n + a\) and the second one as 10 \(n + b\).
In part (a):
\[
(32)(38) = (30 + 2)(30 + 8) = (10n + a)(10n + b)
\]
for \(n = 3, a = 2,\) and \(b = 8.\)
List \(n\) _____, \(a\) _____, and \(b\) _____ for part (b).
List \(n\) _____, \(a\) _____, and \(b\) _____ f for part (c).
List \(n\) _____, \(a\) _____, and \(b\) _____ f for part (d)
(h) Use Pattern 1 to multiply \((10n + a)(10n + b)\) where we have numbers like the example and parts (a), (b), (c), and (d). We are assuming that each of \(n, a,\) and \(b\) is a single digit number. What are we assuming about the sum \(a + b?\) \(a + b = \_\_\_.\)

Write your result in the form \(100k + ab\) where \(k\) is an expression containing the variable \(n\) and numbers.

Write the expression for \(k\).

Explain why \(k\) represents the product of two consecutive integers.

(i) Create three other multiplication exercises that can be done with this scheme and exchange your exercises with a classmate. When you both are done, check each other’s answers.

\[
\begin{array}{ccc}
(i) & (ii) & (iii) \\
\_\_\_\_\_ & \_\_\_\_\_ & \_\_\_\_\_ \\
\end{array}
\]

(j) Explain the quick way to calculate each of the following:

\((15)(15)\)

\((45)(45)\)

\((85)(85)\)

Why do each of these products fit the pattern of this question?

(k) Summarize in your own words what you’ve learned in the parts of question 3.
4. The parts of question 3 explored one special case of Pattern 1. Now, let’s consider another special case to see what happens when \( a \) and \( b \) in Pattern 1 are the same number. Start by considering the square below created by adding 4 to the length of each side of a square with side length \( x \).

\[ \begin{array}{c}
\text{x} \\
\text{4}
\end{array} \]

\[ \begin{array}{c}
\text{x} \\
\text{4}
\end{array} \]

(a) What is the area of the square with side length \( x \)?

(b) The square with side length \( x + 4 \) has greater area. Use Pattern 1 to calculate its total area. When you use Pattern 1, what are \( a \), \( b \), \( a + b \)?

(c) How much greater is the area of the square with side length \( x + 4 \)? Use the figure to show this additional area. Where is the square with area 16 square units?

(d) How would your answers to parts (b) and (c) change if the larger square had been created to have side length \( x + y \), that is, if both \( a \) and \( b \) are both the same number \( y \)?

(e) At the right, draw a figure to illustrate the area of a square with side length \( x + y \) assuming that \( x \) and \( y \) are positive numbers. Use your figure to explain the pattern below.

\[
(x + y)^2 = x^2 + 2xy + y^2 \quad \text{(Pattern 2 - the Square of a Sum)}
\]

5. This pattern gives a rule for squaring a sum. Use it to calculate each of the following by making convenient choices for \( x \) and \( y \).

(a) \( 30^2 = \)

(b) \( 54^2 = \)

(c) \( 65^2 = \)

(d) \( 2.1^2 = \)
(e) Look back at question 3, (k). Why will the rule for squaring a sum also work on those exercises? Can the method of Question 3 be used for the square of any sum?

6. We can extend the ideas of questions 4 and 5 to cubes.

(a) What is the volume of a cube with side length 4? _____

(b) What is the volume of a cube with side length $x$? _____

(c) Now determine the volume of a cube with side length $x + 4$. First, use the rule for squaring a sum to find the area of the base of the cube. __________

Now use the distributive property several times to multiply the area of the base by the height, $x + 4$. Simplify your answer.

(d) Repeat parts (b) and (c) for a cube with side length $x + y$. Write your result as a rule for the cube of a sum.

area of the base of the cube __________

area of base multiplied by the height, $x + y$:

_________________________ – (Pattern 3 – the Cube of a Sum)

(e) Making convenient choices for $x$ and $y$, use Pattern 3 to find the following cubes.

$11^3 =$

$23^3 =$

$101^3 =$

Use the rule for cubing a sum to cube $2 = 1 + 1$. Do you get the same number as $(2)(2)(2)$?

(f) Use the cube of the sum pattern to simplify the following expressions.

$(t + 5)^3 =$

$(w + 2)^3 =$
7. (a) Let $y$ represent any positive number. Go back to Pattern 1 and substitute $-y$ for $a$ and for $b$ to get the following rule for squaring a difference.

\[(x - y)^2 = x^2 - 2xy + y^2\]  

(Pattern 4 - The Square of a Difference)

(b) In the diagram below find the square with side $x$, the square with side $y$, and two different rectangles with area $xy$. Now, use the diagram to give a geometric explanation of the rule for the Square of a Difference.

(c) By making a convenient choices for $x$ and $y$, use the Square of a Difference pattern to find the following squares. Note that $99 = 100 - 1$, $38 = 40 - 2$, and $17 = 20 - 3$.

\[99^2\]
\[38^2\]
\[17^2\]

8. (a) Find a rule for the cube of a difference.

(b) Check your rule for the Cube of a Difference by using it to calculate:
the cube of 1 using $1 = 2 - 1$ and the cube of 2 using $2 = 5 - 3$.

\[1^3\]
\[2^3\]

9. Now let’s consider what happens in Pattern 1 if $a$ and $b$ are opposite real numbers. Use Pattern 1 to calculate each of the following. Substitute other variables for $x$ as necessary.

(a) Calculate $(x + 8)(x - 8)$. Remember that $x - 8$ can also be expressed as $x + (-8)$.

\[(x + 8)(x - 8)\]

(b) Calculate $(x - 6)(x + 6)$. Remember that $x - 6$ can also be expressed as $x + (-6)$.

\[(x - 6)(x + 6)\]

(c) Calculate $(z + 12)(z - 12)$. Remember that $z - 12$ can also be expressed as $z + (-12)$.

\[(z + 12)(z - 12)\]
(d) Calculate \((w + 3)(w - 3)\). Remember that \(w - 3\) can also be expressed as \(w + (-3)\).
\[(w + 3)(w - 3)\]

(e) Calculate \((t - 7)(t + 7)\). Remember that \(t - 7\) can also be expressed as \(t + (-7)\).
\[(t - 7)(t + 7)\]

(f) Substitute \(y\) for \(a\) and \(-y\) for \(b\) in Pattern 1 to find an pattern for the product \((x + y)(x - y)\).
\[(x + y)(x - y) \quad \text{(Pattern 6)}\]

10. Make appropriate choices for \(x\) and \(y\) to use Pattern 6 to calculate each of the following.

(a) \((101)(99)\)

(b) \((22)(18)\)

(c) \((45)(35)\)

(d) \((63)(57)\)

(e) \((6.3)(5.7)\)

(f) \[
\begin{pmatrix}
\frac{3}{2} & \frac{1}{2}
\end{pmatrix}
\]

11. 
(a) In Question 10, you computed several products of the form \((x + y)(x - y)\) verifying that the product is always of the form \(x^2 - y^2\). Thus, if we choose values for \(x\) and \(y\) so that \(x = y\), then the product \((x + y)(x - y)\) will equal 0. If \(x = y\), what is \(x - y\)?

(b) Is there any other way to choose numbers to substitute \(x\) and \(y\) so that the product \((x + y)(x - y)\) will equal 0? If so, what is \(x + y\)?

(c) In general, if the product of two numbers is zero, what must be true about one of them?
(d) Consider Pattern 2 for the Square of a Sum: \((x + y)^2 = x^2 + 2xy + y^2\). Is there a way to choose numbers to substitute for \(x\) and \(y\) so that the product \(xy\) equals 0?

(e) Is it ever possible that \((x + y)^2\) could equal \(x^2 + y^2\)? Explain your answer.

(f) Could \((x – y)^2\) ever equal \(x^2 + y^2\)? Could \((x – y)^2\) ever equal \(x^2 – y^2\)? Explain your answer.
Alternate Version for Included Sample Lesson Plan I’ve Got Your Number

Learning Task

Equivalent algebraic expressions, also called algebraic identities, give us a way to express results with numbers that always work a certain way. In this task you will explore several “number tricks” that work because of basic algebra rules. It is recommended that you do this task with a partner.

Your teacher will give you a “number trick” card(s). Work with a partner to complete the trick. Be prepared to share your thoughts as you describe your solution(s). Can you generalize what you have found?

The same concepts you explored through your cards to give an alternate way to multiply two digit numbers that have the same digit in the ten’s place. For example, (31)(37) can be thought of as (30 + 1)(30 + 7).

\[
\begin{array}{ccc}
30 & 1 \\
30 & 900 & 30 \\
7 & 210 & 7 \\
\end{array}
\]

or \[900 + 210 + 30 + 7 = 1147\]

Rewrite these similarly and use the area models to calculate each of the following products.

(a) \((52)(57)\)

(b) \((16)(13)\)

(c) \((48)(42)\)

(d) \((72)(75)\)

All of the previous products involved addition, how do you think it would be different if they also included subtraction? What if the products involved both addition and subtraction? Make a conjecture before you test it using the next set of cards from your teacher.
Your teacher will give you a “number trick” card(s). Work with a partner to complete the trick. Be prepared to share your thoughts as you describe your solution(s). Can you generalize what you have found?

We can use the same concepts you explored through your cards to give an alternate way to multiply two digit numbers that have the same digit in the ten’s place. For example, $(27)(37)$ can be thought of as $(30 - 3)(30 + 7)$.

Can you use an area model to show this multiplication? Compare your model with the model above.

1. Use both addition and subtraction to rewrite these similarly and use the area models to calculate each of the following products.
   
   (a) $(46)(57)$
   
   (b) $(16)(25)$
   
   (c) $(38)(42)$
   
   (d) $(62)(75)$

Looking at part c, is there anything special about the binomials that you wrote and the answer that you got?

2. With a partner compose three other multiplication questions that use the same idea. Explain your thinking. What must be always be true for this special situation to work?

3. Make appropriate choices for $x$ and $y$ to use Identity 6 to calculate each of the following.
   
   (a) $(101)(99)$
   
   (b) $(22)(18)$
   
   (c) $(45)(35)$
   
   (d) $(63)(57)$
   
   (e) $(6.3)(5.7)$
   
   (f) $\begin{pmatrix} 3 \\ 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$
4. (a) In Question 3, you computed several products of the form \((x + y)(x - y)\) verifying that the product is always of the form \(x^2 - y^2\). Thus, if we choose values for \(x\) and \(y\) so that \(x = y\), what will the product be?

(b) Is there any other way to choose numbers to substitute \(x\) and \(y\) so that the product \((x + y)(x - y)\) will equal 0?

(c) In general, if the product of two numbers is zero, what must be true about one of them?

(d) These products are called conjugates. Give two examples of other conjugates.

(e) So, \((x + y)(x - y) = \ldots\)

5. Now, let’s consider another special case to see what happens when the numbers are the same. Start by considering the square below created by adding 4 to the length of each side of a square with side length \(x\).

(a) What is the area of the square with side length \(x\)?

(b) The square with side length \(x + 4\) has greater area. Use Identity 1 to calculate its total area. When you use Identity 1, what are \(a\), \(b\), \(a + b\)?

(c) How much greater is the area of the square with side length \(x + 4\)? Use the figure to show this additional area. Where is the square with area 16 square units?

(d) How would your answers to parts (b) and (c) change if the larger square had been created to have side length \(x + y\), that is, if both \(a\) and \(b\) are the same? If both \(a\) and \(b\) are both the same number \(y\)?

(e) At the right, draw a figure to illustrate the area of a square with side length \(x + y\) assuming that \(x\) and \(y\) are positive numbers. Use your figure to explain the identity below.

\[
(x + y)^2 = x^2 + 2xy + y^2
\]
5. This identity gives a rule for squaring a sum. Use it to calculate each of the following by making convenient choices for \( x \) and \( y \).

(a) \( 302^2 \)
(b) \( 54^2 \)
(c) \( 65^2 \)
(d) \( 2.1^2 \)
(e) Is it ever possible that \( (x + y)^2 \) could equal \( x^2 + y^2 \)? Explain your answer.
(f) Could \( (x - y)^2 \) ever equal \( x^2 - y^2 \)? Explain your answer.

6. We can extend the ideas of questions 4 and 5 to cubes.

(a) What is the volume of a cube with side length 4?_______
(b) What is the volume of a cube with side length \( x \)?_______
(c) Now determine the volume of a cube with side length \( x + 4 \). First, use the rule for squaring a sum to find the area of the base of the cube.

Now use the distributive property several times to multiply the area of the base by the height, \( x + 4 \). Simplify your answer.
(d) Repeat parts (b) and (c) for a cube with side length \( x + y \). Write your result as a rule for the cube of a sum.

(area of the base of the cube \______________
area of base multiplied by the height, \( x + y \): \______________
(e) Making convenient choices for \( x \) and \( y \), use Identity 3 to find the following cubes.

\( 11^3 \)
\( 23^3 \)
\( 101^3 \)

Use the rule for cubing a sum to cube \( 2 = 1 + 1 \). Do you get the same number as \( (2)(2)(2) \)?
(f) Use the cube of the sum to simplify the following expressions.

\( (t + 5)^3 \)
\( (w + 2)^3 \)
Just Jogging Learning Task

For distances of 12 miles or less, a certain jogger can maintain an average speed of 6 miles per hour while running on level ground.

1. If this jogger runs around a level track at an average speed of 6 mph, how long in hours will the jogger take to run each of the following distances? [Express your answers as fractions of an hour in simplest form.]
   (a) 3 miles    (b) 9 miles    (c) 1 mile    (d) \(\frac{1}{2}\) mile    (e) \(\frac{1}{10}\) mile

2. Analyze your work in Question 1. Each answer can be found by using the number of miles, a single operation, and the number 6. What operation should be used? Write an algebraic expression for the time it takes in hours for this jogger to run \(x\) miles on level ground at an average speed of 6 miles per hour.

3. Each day this jogger warms up with stretching exercises for 15 minutes, jogs for a while, and then cools down for 15 minutes. How long would this exercise routine take, in hours, if the jogger ran for 5 miles? [Express your answer as a fraction in simplest form.]

4. Let \(T\) represent the total time in hours it takes for this workout routine when the jogger runs for \(x\) miles. Write a formula for calculating \(T\) given \(x\), where, as in Question 2, \(x\) is number of miles the jogger runs. Express the formula for \(T\) as a single algebraic fraction.

5. If the jogger skipped the warm-up and cool-down period and used this additional time to jog, how many more miles would be covered? Does this answer have any connection to the answer to question 4 above?
Suppose this same jogger decides to go to a local park and use one of the paths there for the workout routine one day each week. This path is a gently sloping one that winds its way to the top of a hill.

6. If the jogger can run at an average speed of 5.5 miles per hour up the slope and 6.5 miles per hour going down the slope, how long, in hours, will it take for the jogger to cover 2 miles by going uphill for 1 mile and then returning 1 mile back down the hill? Give an exact answer expressed as a fraction in simplest terms and then give a decimal approximation correct to three decimal places.

7. If the jogger can run at an average speed of 5.3 miles per hour up the slope and 6.7 miles per hour going down the slope, how long, in hours, will it take for the jogger to cover 2 miles by going uphill for 1 mile and then returning 1 mile back down the hill? Give an exact answer expressed as a fraction in simplest terms and then give a decimal approximation correct to three decimal places.

8. Write an algebraic expression for the total time, in hours, that it takes the jogger to cover 2 miles by going uphill for 1 mile and then returning 1 mile back down the hill if the jogger runs uphill at an average speed that is \( c \) miles per hour slower than the level-ground speed of 6 miles per hour and runs downhill at an average speed that is \( c \) miles per hour faster than the level-ground speed of 6 miles per hour. Simplify your answer to a single algebraic fraction. Verify that your expression gives the correct answers for Questions 6 and 7.

9. The average speed in miles per hour is defined to be the distance in miles divided by the time in hours spent covering the distance.

(a) What is the jogger’s average speed for a two mile trip on level ground?

(b) What is the jogger’s average speed for the two mile trip in question 6?

(c) What is the jogger’s average speed for the two mile trip in question 7?

(d) Write an expression for the jogger’s average speed over the two-mile trip (one mile up and one mile down) when the average speed uphill is \( c \) miles per hour slower than the level-ground speed of 6 miles per hour and the average speed downhill at an average speed that is \( c \) miles per hour faster than the level-ground speed of 6 miles per hour. Express your answer as a simplified algebraic fraction.

(e) Use the expression in part (d) to recalculate your answers for parts (b) and (c)? What value of \( c \) should you use in each part?

10. For what value of \( c \) would the jogger’s average speed for the two-mile trip (one mile up and one mile down) be 4.5 miles per hour? For this value of \( c \), what would be the jogger’s average rate uphill and downhill?
Firefighters are important members of the community who not only fight fires but also participate in a variety of rescue activities. A ladder truck is an important tool that firefighters use to get water to and rescue people from heights above the ground floor. This activity explores relationships between the building floor firefighters need to reach and the length to which they need to extend the ladder mounted on the truck.

One metropolitan fire department has a truck with a 100 ft extension ladder. The ladder is mounted on the top of the truck. When the ladder is in use, the base of the ladder is 10 feet above the ground.

Suppose that a ladder truck is parked so that the base of the ladder is 20 feet from the side of an apartment building. Because there are laundry and storage rooms in the basement of the building, the base of the windows in first floor apartments are 10 feet above the ground, and there is a distance of 10 feet between the base of the windows on adjacent floors. See the diagram.

1. Find the length to which the ladder needs to be extended to reach the base of a window on the second floor. What is the exact answer? Would an approximation be more meaningful in this situation? Make an approximation to the nearest tenth of a foot and check your answer.
2. Find the length to which the ladder needs to be extended to reach the base of a window on the third floor.

3. Find the length to which the ladder needs to be extended to reach the base of a window on the fourth floor.

4. Let $n$ represent the number of the floor that the ladder needs to reach. Write a function where the input is $n$ and the output is the height $h$, in feet above the ground, of the base of windows on that floor. Does this function express a linear relationship?

5. Write a function where the input is $n$ and the output is the length $L$ of the ladder when it is extended to reach the base of a window on floor $n$. If possible, simplify this expression. Does this function express a linear relationship?

6. In this situation, what is the highest floor that the ladder could reach?