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INTRODUCTION:
In this unit students will explore the properties of circles and use these properties to solve problems involving arcs, angles, sectors, chords, tangent lines and secant lines. Students will continue their study of measurement geometry in a study of the surface area and volume of a sphere.

ENDURING UNDERSTANDINGS:
- Properties of circles are connected and appropriate for describing many aspects of our world.
- Properties of angles, triangles, quadrilaterals, and polygons are connected and appropriate for describing many aspects of our world.
- Geometric ideas are useful in all areas of mathematics including algebra, trigonometry, and analysis.
- Concepts and measurements of surface area and volume of spheres are useful.
- Relationships between change in length of radius or diameter, surface area, and volume exist.

KEY STANDARDS ADDRESSED:

MA1G4. Students will understand the properties of circles.
   a. Understand and use properties of chords, tangents, and secants as an application of triangle similarity.
   b. Understand and use properties of central, inscribed, and related angles.
   c. Use the properties of circles to solve problems involving the length of an arc and the area of a sector.
   d. Justify measurements and relationships in circles using geometric and algebraic properties.

MA1G5. Students will find and compare the measures of spheres.
   a. Use and apply surface area and volume of a sphere.
   b. Determine the effect on surface area and volume of changing the radius or diameter of a sphere.
RELATED STANDARDS ADDRESSED:

MA1P1. Students will solve problems (using appropriate technology).
   a. Build new mathematical knowledge through problem solving.
   b. Solve problems that arise in mathematics and in other contexts.
   c. Apply and adapt a variety of appropriate strategies to solve problems.
   d. Monitor and reflect on the process of mathematical problem solving.

MA1P2. Students will reason and evaluate mathematical arguments.
   a. Recognize reasoning and proof as fundamental aspects of mathematics.
   b. Make and investigate mathematical conjectures.
   c. Develop and evaluate mathematical arguments and proofs.
   d. Select and use various types of reasoning and methods of proof.

MA1P3. Students will communicate mathematically.
   a. Organize and consolidate their mathematical thinking through communication.
   b. Communicate their mathematical thinking coherently and clearly to peers, teachers, and others.
   c. Analyze and evaluate the mathematical thinking and strategies of others.
   d. Use the language of mathematics to express mathematical ideas precisely.

MA1P4. Students will make connections among mathematical ideas and to other disciplines.
   a. Recognize and use connections among mathematical ideas.
   b. Understand how mathematical ideas interconnect and build on one another to produce a coherent whole.
   c. Recognize and apply mathematics in contexts outside of mathematics.

MA1P5. Students will represent mathematics in multiple ways.
   a. Create and use representations to organize, record, and communicate mathematical ideas.
   b. Select, apply, and translate among mathematical representations to solve problems.
   c. Use representations to model and interpret physical, social, and mathematical phenomena.

UNIT OVERVIEW:

Unit 3 begins with representations of the fundamental figures used in the study of lines, line segments, angles, and their relationships to the circle(s) they intersect. The tasks are focused on investigating properties and relationships that occur among circles, lines, and angles formed by circles and lines and on generalizing the observations to theorems with justifications. By the completion of the unit, students have discovered and proven theorems and properties about circles, lines, and angles formed by circles and lines. Since a great deal of geometry has been studied in middle school, the definitions, theorems, postulates, and corollaries are listed for reference purposes. These concepts are not an inclusive list and should not be taught in isolation. However, due to evidence of frequent difficulty and
misunderstanding associated with these concepts, instructors should pay particular attention to them and to how their students are able to explain and apply them. Because students learned many of the basis constructions in seventh grade, this unit relies heavily on the use of computer produced figures and the ability to “drag” the individual parts of these figures to see the results of the movement in the measurements related to these figures. These types of investigations allow the student to “see” the results of changes quickly with the assurance that the measurements are correct. Since the Geogebra Website is free, teachers are encouraged to use it extensively in this unit.

**PREVIOUSLY LEARNED Terms and definitions:**
A **circle** is the set of all points in a plane that are equidistant (the length of the radius) from a given point, the **center**, of the circle.

A **chord** is a segment in the interior of a circle whose endpoints are on the circle.

A **diameter** is a segment between two points on a circle, which passes through the center of the circle. The diameter is the longest chord of the circle.

**Terms and Definitions**

An **arc** is a connected section of the circumference of a circle. An arc has a linear measurement, which is the portion of the circumference, and an arc has a degree measurement, which is a portion of the 360 degree circle.

If a circle is divided into two unequal arcs, the shorter arc is called the **minor arc** and the longer arc is called the **major arc**.

If a circle is divided into two equal arcs, each arc is called a **semicircle**.

A **secant line** is a line that intersects a circle at two points on the circle.

A **tangent line** is a line that intersects the circle at exactly one point.

A **central angle** of a circle is an angle whose vertex is the center of the circle.

An **inscribed angle** is an angle in a circle, whose vertex is on the circle and whose sides contain chords of the circle.

A **sector** of a circle is a region in the interior of the circle bounded by two radii and an arc of the circle.

**Properties, theorems, and corollaries:**

- An inscribed angle is equal to half its intercepted arc.
- If two inscribed angles intercept the same arc, then they are congruent.
- If a quadrilateral is inscribed in a circle then opposite angles are supplementary.
- An angle formed by a chord and a tangent is equal to half the intercepted arc.
- An angle inscribed in a semicircle is a right angle.
- If a line is tangent to a circle, then the line is perpendicular to the radius drawn to the tangent line at the point of tangency.
- If a line in the plane of a circle is perpendicular to a radius at its endpoint on the circle then the line is tangent to the circle.
- In the same circle or in equal circles:
  - Equal minor arcs have equal central angles.
Georgia Department of Education

Accelerated Mathematics I

Unit 3  2nd Edition

- Equal chords have equal arcs.
- Equal arcs have equal chords.
- Equal chords are equally distant from the center.
- Chords equally distant from the center are equal.

- A diameter that is perpendicular to a chord bisects the chord and its arc.
- If two inscribed angles intercept the same arc, then the angles are equal.
- If a quadrilateral is inscribed in a circle, then opposite angles are supplementary.
- If the scale factor of two similar solids is $a:b$, then
  - The ratio of corresponding lengths is $a:b$,
  - The ratio of corresponding areas is $a^2:b^2$; and
  - The ratio of the volumes is $a^3:b^3$.

- The lengths of arcs are proportional to the sizes of the central angles, but lengths of chords are neither proportional to the sizes of the central angles nor to the arcs subtended by the central angles.
- If the radius of a circle is perpendicular to a chord, then the radius bisects the chord and bisects the intercepted arc of the central angle.
- The relationships between the radius of a circle ($r$) and the distance between a line and the circle explain the position of the line and the circle.
- Intersecting chord theorem
- Intersecting secant theorem
- The sum of opposite angles of an inscribed quadrilateral in a circle is always $180^\circ$.
- The measure of an angle inscribed in a semicircle is $90^\circ$.
- Chords equidistant from the center of a circle are congruent.
- Congruent chords are equidistant from the center.
- The perpendicular bisector of a chord passes through the center of a circle.
- A line through the center of a circle that is perpendicular to a chord, bisects the chord.
- A line through the center of the circle bisecting a chord is perpendicular to the chord.
  Note: *(These last three are equivalent statements stemming from the uniqueness of the perpendicular bisector.)*
- If a central angle and an inscribed angle of a circle are subtended by the same chord and on the same side of the chord, then the central angle is twice the inscribed angle.
- If two angles are subtended by the same chord and on the same side of the chord, then the sum of their measures totals $180$ degrees.
- If two angles are subtended by the same chord and on opposite sides of the chord, then the sum of their measures totals $180$ degrees.
- The surface area of a sphere is equal to $4 \pi r^2$.
- The volume of a sphere is equal to $\frac{4}{3} \pi r^3$.
- Arc length is $(\frac{n}{360})(2\pi r)$
- Area of a sector is $(\frac{n}{360})(\pi r^2)$
Tasks:
The remaining content of this framework consists of student learning tasks designed to allow students to learn by investigating situations with a real-world context. The first learning task is intended to launch the unit. Its primary focus is to explore the relationships between lines, angles, and circles leading to an in-depth study of their relationships in the next few tasks. As students become familiar with the ways in which the figures and their measurements relate, they will begin to make conjectures and establish generalizations of the properties and theorems of the unit. The use of interactive geometry applets through GeoGebra allows students to thoroughly investigate the relationships of angles, arcs, and their measures. The last task is designed to demonstrate the type of assessment activities students should be comfortable with by the end of the unit and gives students a chance to apply many of the ideas they have learned throughout the unit.
Sunrise on the First Day of a New Year Learning Task

Figure 1: Sunrise

In some countries in Asia, many people visit the seashore on the east side of their countries the first day of every New Year. While watching the gorgeous scene of the sun rising up from the horizon over the ocean, the visitors wish good luck on their new year.

As the sun rises above the horizon, the horizon cuts the sun at different positions. By simplifying this scene, we can mathematically think of the relationships between lines and circles and the angles formed by these lines and parts of the circle. We can use a circle to represent the sun and a line to represent the horizon.

3) A circle is not a perfect representation of the sun. Why not?
4) Using the simplified diagram above, describe the different types of intersections the sun and horizon may have. Illustrate the intersections you described and explain how they differ.

5) A tangent line is a line that intersects a circle at exactly one point, while a secant line intersects a circle in two points. Do any of your drawings in #2 have a tangent or a secant? If so, identify them. Is it possible for a line to intersect a circle in 3 points? 4 points? Explain why or why not.

6) When a secant line intersects a circle in two points, it creates a chord. A chord is a segment whose endpoints lie on the circle.

   a. How does a chord differ from a secant line?

   b. How many chords can be in a circle?

   c. What is the longest chord in a circle? Explain how you know?

   d. Describe the relationship between the distance chords are from the center of a circle and the length of the chords.

   e. Mary made the following conjecture: If two chords are the same distance from the center of the circle, the chords are congruent. Do you agree or disagree? Support your answer mathematically. State the converse of this conjecture and explain whether or not it is true.

   f. Ralph was looking at the figure to the right. He made the following conjecture: A radius perpendicular to a chord bisects the chord. Prove his conjecture is true. Remember, if we can prove something is always true it can be named a theorem.

   g. Is the converse of the statement in part f true? How would you prove that the converse is true?
5. Think back to the sunrise. As the sun rises you see a portion of its outer circumference. A portion of circle’s circumference is called an arc. An arc is a curve that has two endpoints that lie on the circle.

a. Describe what happens to the visible arc of the circumference of the sun as the sun rises. Describe the similarities and differences between the arcs of a sunrise and the arcs of a sunset.

b. If a circle is divided into two unequal arcs, the shorter arc is called the minor arc and the longer arc is called the major arc. If a circle is divided into two equal arcs, each arc is called a semicircle. Use these words to describe the arcs of the sunrise.

c. What must be true for an arc to be a semicircle?

![Figure 2. The radius and the distance between the center of a circle and a line]

6. a. For what lines is \( d \) less than \( r \)? Specifically, given a circle and lines in a plane, determine what length is greater than the other for each case. Refer to the above picture.
Use one of the notations of <, = or > between \( d \) and \( r \) in the following:

i) \( d (_,) r \) for a secant line,

ii) \( d (_,) r \) for a tangent line, and

iii) \( d (_,) r \) for the others.

b. For i) above, how can you be sure that your answer is correct? Prove it by using the Pythagorean Theorem for a right triangle. Use the picture below.
Figure 3. The distance between a secant line and the center of a circle, using the radius of the circle

In the picture, \( OA \) is the radius \( (r) \) of the circle, and \( OH \) is the distance \( (d) \) between the center of the circle and the secant line.

Apply the Pythagorean Theorem to the right \( \triangle OHA \).

\[
OH^2 + HA^2 = OA^2
\]

Which segment is longer: \( OH(d) \) or \( OA(r) \)?

Why? Discuss this with others in your group.

\( OA(r) > OH(d) \).
Is it Shorter Around or Across Learning Task

1. What conditions determine a unique circle on a plane? In other words, given a point on a plane and the length of the radius from that point, how many different circles can we draw?

2. A sector of a circle is a wedge of the circle with its point at the center and whose two sides are radii which subtend an arc of the circle.

![Figure 2](image)

a. The length of arc AB is \( l \) and the length of arc AC is \( 3l \). Using a protractor, measure \( \angle AOB \) and \( \angle AOC \). Compare the measures of these two angles.

b. Are the measures of \( \angle AOB \) and \( \angle AOC \) proportional to the lengths of their corresponding arcs?

c. Measure and compare the lengths of chords \( \overline{AB} \) and \( \overline{BC} \). Are the lengths of these segments proportional to the measures of their corresponding central angles?
3. If radius $OD$ is the perpendicular to chord $AB$, then $OD$ bisects chord $AB$ and $OD$ bisects arc $AB$, so that the measure of arc $AD = $ the measure of arc $BD$. Prove or disprove these statements.

4. Do any relationships exist among the center of the circle and the bisectors of chords? Discuss this with others in your group.

5. The diagram below shows a swing set. If $AE$ is 6.5 ft and represents the swing hanging straight down and a child can swing 12 feet total from D to F, what is the measure of the arc that a child swings? ($AB$ and $AC$ represent the poles of the swing set.)
Angles of a Circle Learning Task

In this learning task, a set of investigations using GeoGebra is recommended.

The link to the list of items is:

At this link are investigations for every type of angle as well as handouts that can be printed.
Central and Inscribed Angles (by hand)

Using a compass and straight edge construct a circle of any size on paper. Mark the center of the circle Q.

Place two points, A and B, on the circle and construct the two segments (radii), \( QA \) and \( QB \).

Using a protractor, measure \( \angle AQB \) and record that measurement. \( \angle AQB = \)

Next place point C on the circle opposite A and B.

Now construct the two segments (chords), \( CA \) and \( CB \).

Measure \( \angle ACB \) and record that measurement. \( \angle ACB = \)

The arc formed between A and B is called a \( \underline{\text{___}} \) arc.

Use your protractor and measure \( \overparen{AB} \). \( AB = \overparen{AQB} \)

The arc from A to B that passes through C is called a \( \underline{\text{___}} \) arc.

Now that we know what \( AB \) measures, what does \( ACB \) measure? \( ACB = \underline{\phantom{0}} AB \)

How did you figure this?

What do you notice about the measurements you made?

Discuss with a neighbor and see if they discovered the same relationship on their circle.

The angle with vertex at the center of the circle is called a \( \underline{\text{___}} \) angle and its measure is \( \underline{\phantom{0}} \) intercepted arc.

The angle with vertex on the circle is an \( \underline{\phantom{0}} \) angle and its measure is equal to \( \underline{\phantom{0}} \) its intercepted arc.

An inscribed angle’s measure is equal to \( \underline{\phantom{0}} \) the central angle with the same intercepted arc.

Now on the same circle construct another central angle \( \angle DQE \) with the measure equal to the measure of \( \angle AQB \).

Pick another point \( P \) on the circle and construct the inscribed angle that intercepts the same arc as \( \angle DQE \).

Measure these two angles and see if your conjecture about their measurements still holds.

Use your straight edge to connect \( AB \) and \( DE \).

Measure these lengths in centimeters to the nearest tenth.

Are they the same length?

In the same circle or in equal circles:
1) equal chords have equal \( \underline{\phantom{0}} \);  
2) equal arcs have equal \( \underline{\phantom{0}} \);  
3) congruent chords determine two central angles that are \( \underline{\phantom{0}} \).
Lines and Line Segments of a Circle Learning Task

**Finding the Center Again**

Show the students a broken plate or some circular object that has only part showing. Ask them if they have any ideas about how to find the entire circle. Anthropologists find artifacts that are only parts of the complete item and must work to discover what the item might be – depending on its size. Do astronomers see an entire crate on the moon or must they use mathematics to determine the size of the crater?

After the discussion the following activity can be done with MIRAs.
Have the students construct a circle on a sheet of paper and construct two nonparallel and noncongruent chords.
Using a MIRA construct the perpendicular bisector of each chord.
Do these two perpendicular bisectors intersect?
Do you notice anything about the intersection?
Compare with your neighbors and try to fill in the blanks of the following statements.
The perpendicular bisectors of chords of a circle _________________.
The perpendicular from the center of a circle to a chord is the ________________ of the chord. (And the __________ of the arc.)
If two chords of a circle are congruent then they determine two central angles of the circle that are ________________.

The following link investigates the length of intersecting chords:

The following link investigates the length of secant segments and tangent segments:
Figure 2
**Another Way to Find the Center and Facts about Tangents Learning Task**

1. Draw a circle using a compass and mark the center O.
2. Put a pencil point on the circle and then put the straight edge up next to the pencil so that it touches the circle in one and only one point.
3. Label that point P (point of tangency).
4. Use the straight edge to draw the tangent line through P and mark point A on the line.
5. Construct the radius $\overline{OP}$.
6. Measure angle $\angle APO$ using a protractor.
7. Repeat the above activity using the same circle.
8. Put another point S on the circle.
9. This time construct a line through the radius $\overline{OS}$ and either construct a perpendicular at S with compass and straight edge or use a MIRA to construct the perpendicular at $S \perp \overline{OS}$.
10. Mark point B on the perpendicular line.
11. Are both $\angle APO$ and $\angle BSO$ both right angles?

Check with your neighbors to see if they got the same results and complete the following statement.

A tangent to a circle ______________ to the radius drawn to the point of tangency.

![Diagram](image.png)

$\overline{BA} \perp \overline{DB}$ and $\overline{AC} \perp \overline{DC}$ With this information what can you conclude about two tangents from the same point outside the circle?
What do these tangent conjectures have to do with space travel?
**Notes on Summary of Angles and Segments:**

<table>
<thead>
<tr>
<th>Location of Vertex</th>
<th>Relationship</th>
</tr>
</thead>
<tbody>
<tr>
<td>On the center of circle</td>
<td></td>
</tr>
<tr>
<td>On the circle</td>
<td></td>
</tr>
<tr>
<td>Outside the circle</td>
<td></td>
</tr>
<tr>
<td>Inside the circle but not the center of the circle</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Segments</th>
<th>Length Relationship</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intersecting chords</td>
<td></td>
</tr>
<tr>
<td>Chord and Tangent at pt. of Tangency</td>
<td></td>
</tr>
<tr>
<td>Two intersecting secants</td>
<td></td>
</tr>
<tr>
<td>Two intersecting tangents</td>
<td></td>
</tr>
<tr>
<td>Intersecting secant and tangent</td>
<td></td>
</tr>
</tbody>
</table>
Arc Lengths and Areas of Sectors Learning Task

Now that arcs have been discovered, a review of the area and circumference of a circle is in order. An understanding of arc lengths as a portion of the total circumference and the area of a sector as a partial area of the circle’s total area develops in this task.

The length of an arc equals the circumference times the measure of the central angle divided by 360°. The area of a sector equals the area of the circle times the measure of the central angle divided by 360°. See circle below and use proportions to find the area of the sector and the length of the arc.

\[
C = 20\pi \\
A = 10^2 \pi = 100\pi \\
\frac{60^\circ}{360^\circ} = \frac{x}{20\pi} \\
\frac{60^\circ}{360^\circ} = \frac{x}{100\pi}
\]

Students made a pie chart using percentages in a previous task. In the game show *Wheel of Fortune*, three contestants compete to earn money and prizes for spinning a wheel and solving a word puzzle. The game requires some understanding of probability and the use of the English language. Make a spinner to use in the *Wheel of Fortune* game.

Have students create a playing wheel that has eight spaces (sectors) marked $3000, $750, $900, $400, Bankrupt, $600, $450, and Lose a Turn.

Source for this is the link below from *The World’s Largest Math Event 8*.

http://my.nctm.org/eresources/view_article.asp?article_id=6221& page=12
**Sphere Learning Task**

Using the materials you have been given, slice the orange through the great circle. Place one-half of the orange cut-side down in the center of the paper plate. Trace the great circle onto your paper plate. Using a radius of $r$, what is the area of this circle? Now, peel the orange making sure to keep every piece of the peeling. After the orange is peeled, place the pieces of the peel on the circle on your paper plate, breaking the peel to make it fit exactly. As you fill the circle, discard the pieces that you have already used, count the number of times the peel would completely fill the great circle. Using the area of the circle that you found earlier, what is the total area that the orange peel can cover? Considering that this is the area the orange peel covers and that it originally covered the entire orange or sphere. What would you conjecture is the relationship of the surface area of the orange and the radius of the great circle? How does the radius of the great circle and the radius of the sphere compare? Will this always be the case? Why or why not?

To find the formula for the volume of a sphere, you will need a sphere and a cylinder that have a relationship as shown below:

![Diagram of sphere and cylinder](image)

That is, one where the diameter of the base of the cylinder is the same as the diameter of the sphere and the height of the cylinder is the same as the diameter of the sphere.

Using the following GeoGebra link:


have the students follow the instructions to use the formulas that they found above to calculate the volume and surface area of spheres with five different radii lengths. Make a chart of the values and find the relationship between the change in radii to the change in surface area to the change in volume.