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INTRODUCTION:

In unit 3, properties of exponents are study with great detail and the logarithmic functions are introduced as an inverse of the function. Since Accelerated Mathematics II, we have studied a variety of functions and have learned to solve them with a plethora of methods. This unit gives us a chance to bring it together and solve equations with known strategies and how to solve when those are not enough. A comprehensive understanding of work done in Accelerated Mathematics I will be necessary. Weaknesses of equation solving will be made evident in this unit, so there will need to be a readiness to address them.

ENDURING UNDERSTANDINGS:

- Solve exponential and logarithmic equations analytically and graphically
- Use characteristics of functions to understand graphic solutions
- Find and interpret solutions of higher order polynomials
- Importance of exponential and logarithmic models to interpret real phenomena.

KEY STANDARDS ADDRESSED:

MA2A4. Students will explore logarithmic functions as inverses of exponential functions.

d. Understand and use properties of logarithms by extending laws of exponents.
g. Explore real phenomena related to exponential and logarithmic functions including half-life and doubling time.

MA2A5. Students will solve a variety of equations and inequalities.

a. Find real and complex roots of higher degree polynomial equations using the factor theorem, remainder theorem, rational root theorem, and fundamental theorem of algebra, incorporating complex and radical conjugates.
b. Solve polynomial, exponential, and logarithmic equations analytically, graphically, and using appropriate technology.
c. Solve polynomial, exponential, and logarithmic inequalities analytically, graphically, and using appropriate technology. Represent solution sets of inequalities using interval notation.
d. Solve a variety of types of equations by appropriate means choosing among mental calculation, pencil and paper, or appropriate technology.

**RELATED STANDARDS ADDRESSED:**

**MA2A3. Students will analyze graphs of polynomial functions of higher degree.**
   d. Investigate and explain characteristics of polynomial functions, including domain and range, intercepts, zeros, relative and absolute extrema, intervals of increase and decrease, and end behavior.

**MAP1. Students will solve problems (using appropriate technology).**
   a. Build new mathematical knowledge through problem solving.
   b. Solve problems that arise in mathematics and in other contexts.
   c. Apply and adapt a variety of appropriate strategies to solve problems.
   d. Monitor and reflect on the process of mathematical problem solving.

**MA2P2. Students will reason and evaluate mathematical arguments.**
   a. Recognize reasoning and proof as fundamental aspects of mathematics.
   b. Make and investigate mathematical conjectures.
   c. Develop and evaluate mathematical arguments and proofs.
   d. Select and use various types of reasoning and methods of proof.

**MA2P3. Students will communicate mathematically.**
   a. Organize and consolidate their mathematical thinking through communication.
   b. Communicate their mathematical thinking coherently and clearly to peers, teachers, and others.
   c. Analyze and evaluate the mathematical thinking and strategies of others.
   d. Use the language of mathematics to express mathematical ideas precisely.

**MA2P4. Students will make connections among mathematical ideas and to other disciplines.**
   a. Recognize and use connections among mathematical ideas.
   b. Understand how mathematical ideas interconnect and build on one another to produce a coherent whole.
   c. Recognize and apply mathematics in contexts outside of mathematics.

**MM2P5. Students will represent mathematics in multiple ways.**
   a. Create and use representations to organize, record, and communicate mathematical ideas.
   b. Select, apply, and translate among mathematical representations to solve problems.
   c. Use representations to model and interpret physical, social, and mathematical phenomena.
Unit Overview:
Since Unit 3 addressed the properties of exponentials and logarithms as functions, Unit 5 will start with the properties of logarithms as an extension of rules of exponents (MA2A4d). After looking at the properties, we plan to solve application problems using our properties of logarithms and exponents. Since we will be finding roots when we solve our application problems, we will have the opportunity to discuss finding roots for other types of functions they have already studied, namely higher-degree polynomials. This will allow for the introduction and investigation of theorems for finding complex roots of polynomials. The plan is to weave inequalities into the unit in each sub-section. This unit could almost be broken into two parts, solving logs and exponentials, and then solving polynomials. One connection that we have talked about making is the logarithmic transformation of data to determine if a scatterplot represents exponential data or power function data. This seems to be a natural bridge between the two topics.

Historical Background: Discover logarithmic properties: A calculator driven exploration task, and eventually lead into an activity which asks: If $2^1 = 2$ and $2^2 = 4$, what power do we need for $2^x = 3$?

Potato Lab: Newton’s Law of Cooling: Heat the potato in the microwave, take temp afterwards, place potato in refrigerator, record readings in fridge and of potato, calculate the cooling coefficient and derive Newton’s formula.

Is it Safe to Eat: With this lab, students will also use inequalities with an emphasis on the food service industry. For example, at what temperature would the potato be too hot to serve in a restaurant? How long could the potato sit at room temperature before it cooled too much to be served?

Polynomial Root Task: In this task, students will revisit concepts of solving quadratics from Math 2 while taking the opportunity to extend their strategies with the concepts of synthetic division, the remainder theorem, and the rational root theorem. This task is intended to be used as guided instruction in the classroom.

Suitcase Design Task: In this culminating activity, the students will need to use their knowledge of polynomials to design a scale model suitcase that fits within certain criteria. In doing so, they will make use of techniques to solve equations and inequalities that have been emphasized in this unit.
Vocabulary and formulas:

Common logarithm: A logarithm with a base of 10. A common logarithm is the power, $a$, such that $10^a = b$. The common logarithm of $x$ is written $\log x$. For example, $\log 100 = 2$ because $10^2 = 100$.

Exponential functions: A function of the form $y = a \cdot b^x$ where $a > 0$ and either $0 < b < 1$ or $b > 1$.

Logarithmic functions: A function of the form $y = \log_b x$, with $b \neq 1$ and $b$ and $x$ both positive. A logarithmic function is the inverse of an exponential function. The inverse of $y = b^x$ is $y = \log_b x$.

Higher order polynomials: A polynomial is considered to be higher order if the degree of the polynomial is greater than 2.

Rational Root Theorem: A theorem that provides a complete list of possible rational roots of the polynomial equation $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 = 0$ where all coefficients are integers.

Synthetic Division: Synthetic division is a shortcut method for dividing a polynomial by a linear factor of the form $(x - a)$. It can be used in place of the standard long division algorithm.

Remainder Theorem: An application of polynomial long division. It states that the remainder of a polynomial $f(x)$ divided by a linear divisor $(x - a)$ is equal to $f(a)$.
LAUNCHING TASK: HISTORICAL RELEVANCE AND OVERVIEW OF PROPERTIES

Historical Background of Logarithms:

John Napier, a Scottish mathematician, thought of numbers as being one continuous series. This idea seems to be commonly held, but chances are that we are thinking about whole numbers. This thought of a continuous series is actually extended to any number. Every number that can be conceived is the result of a base number being raised to a power. Two is a comfortable number with which to work. The powers of two are commonly used and relatively easy to figure if they are not committed to memory.

1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, 4096…

Napier’s father was a tax collector, and John often helped his father with computations. His work prompted the development of a convention that would allow him to shorten his process. Multiple multiplications can be done quickly with the use of exponents that have been studied in class earlier this year. Now with the thought of the multiplication and the understanding of all values as the result of a number raised to a power, the following question is posed.

If \(2^2 = 4\) and \(2^3 = 8\) then what power would 2 need to be raised to get 6 as a result?

Without referring to your calculator, discuss a possible answer to this question with your neighbor. Be prepared to share your rationale with the class.

Let’s collect the responses from each pair in the class and list on the board or on your own paper. Find an average of all the responses in the class. Does the average of the responses seem appropriate?

Use a calculator to evaluate your response. How would you alter your original answer seeing this result?

Consider a similar situation with 10. Knowing that \(10^2 = 100\) and \(10^3 = 1000\), 10 to what power would equal 500?

Now, let’s look at these familiar values from a different perspective. Considering \(10^2 = 100\) and \(10^3 = 1000\), find the arithmetic mean of 2 and 3. Now find the square root of the product of 100 and 1000. What is the relationship of 2.5 to 316.227?
**Investigating the Properties of Logarithms**

**For the purpose of this activity we will be using Common Logs.**

**PART I**

1. Complete the following table using your calculator. Round answers to four decimal places.

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>log 5</td>
<td>0.6990</td>
</tr>
<tr>
<td>log 10 – log 2</td>
<td></td>
</tr>
<tr>
<td>log 3</td>
<td></td>
</tr>
<tr>
<td>log 18 – log 6</td>
<td></td>
</tr>
<tr>
<td>log 7</td>
<td></td>
</tr>
<tr>
<td>log 28 – log 4</td>
<td></td>
</tr>
<tr>
<td>log ½</td>
<td></td>
</tr>
<tr>
<td>Log 3 – log 6</td>
<td></td>
</tr>
<tr>
<td>log 2</td>
<td></td>
</tr>
<tr>
<td>Log 8 – log 4</td>
<td></td>
</tr>
</tbody>
</table>

2. Using any patterns you see in the results above, what generalizations could be made?

3. How could we find the value of log 3 if the “3” button is missing from our calculator? Explain.

**PART II**

4. Complete the following table using your calculator. Round answers to four decimal places.

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>log 12</td>
<td></td>
</tr>
<tr>
<td>log 6 + log 2</td>
<td></td>
</tr>
<tr>
<td>log 18</td>
<td></td>
</tr>
<tr>
<td>log 3 + log 6</td>
<td></td>
</tr>
<tr>
<td>log 9</td>
<td></td>
</tr>
<tr>
<td>log 3 + log 3</td>
<td></td>
</tr>
<tr>
<td>log 20</td>
<td></td>
</tr>
<tr>
<td>log 4 + log 5</td>
<td></td>
</tr>
<tr>
<td>log 26</td>
<td></td>
</tr>
<tr>
<td>log 2 + log 13</td>
<td></td>
</tr>
</tbody>
</table>

5. Using any patterns you see in the results above, what generalizations could be made?
6. Have you noticed a similar result before? Where?

7. How could we find the value of log 30 if the “3” button is missing from our calculator? Explain.

PART III

8. Complete the following table.

<table>
<thead>
<tr>
<th>Number</th>
<th>Equivalent value with a different base</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>$2^4, 4^2$</td>
</tr>
<tr>
<td>64</td>
<td>$2^6, 4^3, 8^2$</td>
</tr>
<tr>
<td>81</td>
<td></td>
</tr>
<tr>
<td>49</td>
<td></td>
</tr>
<tr>
<td>25</td>
<td></td>
</tr>
</tbody>
</table>

Complete the following table using your calculator. Round values to four decimal places.

<table>
<thead>
<tr>
<th>Number</th>
<th>Equivalent value</th>
</tr>
</thead>
<tbody>
<tr>
<td>4log 2</td>
<td>1.2041</td>
</tr>
<tr>
<td>log 16</td>
<td></td>
</tr>
<tr>
<td>2log 5</td>
<td></td>
</tr>
<tr>
<td>log 25</td>
<td></td>
</tr>
<tr>
<td>3log 4</td>
<td></td>
</tr>
<tr>
<td>6log 2</td>
<td></td>
</tr>
<tr>
<td>log 64</td>
<td></td>
</tr>
<tr>
<td>2log 7</td>
<td></td>
</tr>
<tr>
<td>log 49</td>
<td></td>
</tr>
</tbody>
</table>

9. Using the two previous tables, what generalization(s) can be made? Can these generalizations be linked to your previous knowledge of exponents? How?

10. A student noticed that log ½ gave the same value as –log 2. How is this possible?

Summarize all the properties of logarithms you know. Compare your results with others in the class.
Potato Lab: Cooling Effect

Using the given data, you are going to investigate the cooling rate of a small baked potato. This data could be collected during a class, but instead of the potato cooling in room temperature, the potato could cool in a refrigerator. This would expedite the cooling process to fit inside of one class period.

1. Plot the data given below as a scatterplot:

<table>
<thead>
<tr>
<th>Time</th>
<th>Temp °C</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>90.7</td>
</tr>
<tr>
<td>2</td>
<td>86.6</td>
</tr>
<tr>
<td>5</td>
<td>80.2</td>
</tr>
<tr>
<td>6</td>
<td>77.3</td>
</tr>
<tr>
<td>10</td>
<td>67.3</td>
</tr>
<tr>
<td>11</td>
<td>65.1</td>
</tr>
<tr>
<td>15</td>
<td>57.6</td>
</tr>
<tr>
<td>16</td>
<td>56.0</td>
</tr>
<tr>
<td>20</td>
<td>50.4</td>
</tr>
<tr>
<td>21</td>
<td>49.2</td>
</tr>
<tr>
<td>25</td>
<td>44.9</td>
</tr>
<tr>
<td>26</td>
<td>43.9</td>
</tr>
<tr>
<td>30</td>
<td>40.5</td>
</tr>
<tr>
<td>31</td>
<td>39.8</td>
</tr>
<tr>
<td>35</td>
<td>37.1</td>
</tr>
<tr>
<td>36</td>
<td>36.5</td>
</tr>
<tr>
<td>40</td>
<td>34.2</td>
</tr>
<tr>
<td>41</td>
<td>33.7</td>
</tr>
</tbody>
</table>

A. What type of curve do you think the data represents?

B. How could you be certain?
C. Linearize the data given below by taking the natural logarithm of all the temperature values.

<table>
<thead>
<tr>
<th>Time</th>
<th>Temp</th>
<th>ln (temp)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>90.7</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>86.6</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>80.2</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>77.3</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>67.3</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>65.1</td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>57.6</td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>56.0</td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>50.4</td>
<td></td>
</tr>
<tr>
<td>21</td>
<td>49.2</td>
<td></td>
</tr>
<tr>
<td>25</td>
<td>44.9</td>
<td></td>
</tr>
<tr>
<td>26</td>
<td>43.9</td>
<td></td>
</tr>
<tr>
<td>30</td>
<td>40.5</td>
<td></td>
</tr>
<tr>
<td>31</td>
<td>39.8</td>
<td></td>
</tr>
<tr>
<td>35</td>
<td>37.1</td>
<td></td>
</tr>
<tr>
<td>36</td>
<td>36.5</td>
<td></td>
</tr>
<tr>
<td>40</td>
<td>34.2</td>
<td></td>
</tr>
<tr>
<td>41</td>
<td>33.7</td>
<td></td>
</tr>
</tbody>
</table>

D. Create a scatterplot using your time values and the natural log values. How linear does your data look?

E. Use your calculator to find the linear regression equation of your data. What is the linear equation? What is your correlation coefficient? What does this tell you about your equation?

F. Even though you have a linear equation, what formula did you actually find?

\[ \ln (y) = _____ x + _____ \]

How do you transform this formula into an exponential equation?
G. Analyze the exponential equation you have derived from the linearized scatterplot data. Give its domain, range, intercepts (if any), extremum (if any), asymptotes, end behavior, and any other important information that you can determine.

H. Graph your exponential equation along with your original scatterplot of the time v. temperature data. How well does the function fit the data?

I. According to your equation, at what time will the temperature of the potato reach 20° C? If the room temperature is 23.5° C, how is it possible that the potato will cool to a temperature lower than 23.5° C?

J. Suppose the initial temperature of the hot potato was approximately 92.3° C and the room temperature at which the potato is cooling is 23.5° C. What problems do you see between the collected data and your exponential equation?

K. What is the significance of the room temperature in this problem? How should this be reflected in the graph of your exponential equation?

L. What changes could you make to your original data to correct the problems you found in part J?

M. Create a scatterplot with your new data. Linearize the data once again, find your linear regression line, and then find your exponential model based on the linear regression. Graph your exponential equation and compare to your scatterplot in part L and to the original scatterplot. What final change would you make to fit your equation to the original data? Write your equation below.
N. What relation is there between the coefficient of your exponential function and the initial temperature and/or room temperature?

O. Summarize your findings into a generic equation. Use $T_i$ for the initial temperature, $T_f$ for the final temperature, and $T_s$ for the temperature of the surroundings.

P. Depending on the size of the potato, what other value might change in your equation? Let $k$ represent this value and rewrite your equation.

Congratulations, you have just found the formula for Newton’s Law of Cooling!
**Is it Safe to Eat?**

The USDA recommends keeping prepared food out of the “danger zone” for bacteria growth of 60°C to 5°C. Specifically, *Escherichia coli* bacteria (*E. coli*) has a doubling rate of 20 minutes when introduced into an optimum growing environment. An optimum environment for *E. coli* would consist of plenty of glucose (think starchy food) and a temperature between 27°C and 37°C.

A. Over what time interval after its initial “baking” would your potato enter into the USDA “danger zone” for bacteria growth? (use the potato data and equation from the original problem)

When would the potato be in the optimum growing temperature window for *E. coli* bacteria?

B. Find an equation to model the exponential growth rate of *E. coli* bacteria in an optimum environment.

C. Using parts A and B, determine how many bacterium of *E. coli* could be present in your potato if one hundred bacteria were introduced into the potato environment during the optimum growing window for *E. coli* bacteria.

D. Suppose the temperature of the potato remained constantly in the optimum environment for *E. coli* growth. After how many hours will there be more than 1000 bacteria present?

E. Suppose the potato stayed in the optimum environment for *E. coli* growth for a 24 hour period. Consider that the mass of an *E. coli* bacterium is roughly $10^{-12}$ grams. How many grams of bacteria would be present at the end of the 24-hour period.

F. Is this possible? What might happen to prevent this unchecked growth?
Notes on Polynomial Root Task:

Finding Roots of Higher Order Polynomials:
Solving polynomials that have a degree greater than those solved in GPS Algebra is going to require the use of skills that were developed when we solved those quadratics last year. Let’s begin by taking a look at some second degree polynomials and the strategies used to solve them. These equations have the form $ax^2 + bx + c = 0$, and when they are graphed the result is a parabola.

Factoring is used to solve quadratics of the form $ax^2 + bx + c = 0$ when the roots are rational.

1. Find the roots of the following quadratic functions:
   a. $f(x) = x^2 - 5x - 14$
   b. $f(x) = x^2 - 64$
   c. $f(x) = 6x^2 + 7x - 3$
   d. $f(x) = 3x^2 + x - 2$

Another option for solving a quadratic whether it is factorable but particularly when it is not is to use the quadratic formula. Remember that we developed this concept during Math II.

$$ax^2 + bx + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Remember that $b^2 - 4ac$ is the discriminant and gives us the ability to determine the nature of the roots.

$$b^2 - 4ac$$
- $> 0$ \hspace{1cm} 2 real roots
- $= 0$ \hspace{1cm} 1 real root
- $< 0$ \hspace{1cm} real roots (imaginary)

2. Find the number and nature of the roots, and the roots for each of the following.
   a. $f(x) = 4x^2 - 2x + 9$
   b. $f(x) = 3x^2 + 4x - 8$
   c. $f(x) = x^2 - 5x + 9$

Let’s take a look at the situation of a polynomial that is one degree greater. When the polynomial is a third degree, will there be any similarities when we solve?
Suppose we want to find the roots of \( f(x) = x^3 + 2x^2 - 5x - 6 \). By inspecting the graph of the function, we can see that one of the roots is distinctly 2.
Since we know that \( x = 2 \) is a solution to \( f(x) \), we also know that \((x - 2)\) is a factor of the expression \( x^3 + 2x^2 - 5x - 6 \). This means that if we divide \( x^3 + 2x^2 - 5x - 6 \) by \((x - 2)\) there will be a remainder of zero.

First, let’s think about something we learned in elementary school, long division. Can you use the same process that you used to solve \( 46 \overline{)3768} \)?

What did you think about to start the division problem? Try to complete the entire long division problem.

Now, we are going to use the same idea to divide polynomials. Specifically,

\[
x - 2 \overline{x^3 + 2x^2 - 5x - 6}
\]

Your teacher will give you several of these to practice.

This can be quite tedious, let us consider another way to show this division called synthetic division.

However, when the divisor is linear, there is a short cut. The next part of this task will explore how it works and why it only works when there is a linear divisor.
The following excerpt is taken from:

**9.04 SYNTHETIC DIVISION AND SYNTHETIC SUBSTITUTION**

The labor of dividing a polynomial by \( x - t \) can be reduced considerably by eliminating the symbols that occur repetitiously in the procedure. Let us consider the following division:

\[
\begin{array}{c|cccc}
 & 4x^3 + 5x^2 + 3x + 2 \\
\hline
x - 2 | 4x^4 - 3x^3 - 7x^2 - 4x - 9 \\
\end{array}
\]

\[
\begin{array}{c|cccc}
 & 4x^4 - 8x^3 \\
\hline
 & 5x^3 - 7x^2 \\
 & 5x^3 - 10x^2 \\
\hline
 & 3x^2 - 4x \\
 & 3x^2 - 6x \\
\hline
 & 2x - 9 \\
 & 2x - 4 \\
\hline
 & -5
\end{array}
\]

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We may streamline this division, as follows, leaving out the various powers of \( x \) but maintaining the coefficients in their proper places.

\[
\begin{array}{cccc}
4 & 5 & 3 & 2 \\
\hline
-2 | 4 & -3 & -7 & -4 & -9 \\
\hline
 & -8 \\
 & 5 \\
\hline
 & -10 \\
 & 3 \\
\hline
 & -6 \\
 & 2 \\
\hline
 & -4 \\
 & -5
\end{array}
\]

The above arrangement may be “collapsed” to give the following:

\[
\begin{array}{cccc}
-2 | 4 & -3 & -7 & -4 & -9 \\
\hline
 & -8 & 10 & -6 & -4 \\
\hline
 & 4 & 5 & 3 & 2 & -5
\end{array}
\]
Let's practice synthetic division before we tackle how to solve cubic polynomials in general.

3. Do the following division problems synthetically.
   a. \[
   \frac{10x^3 - 17x^2 - 7x + 2}{x - 2}
   \]
   b. \[
   \frac{x^3 + 3x^2 - 10x - 24}{x + 4}
   \]

Note that: \[-8 = 4(-2)\]
\[-10 = 5(-2)\]
\[-6 = 3(-2)\]
\[-4 = 2(-2)\]

Since it is generally easier to add than to subtract, we shall replace \(-2\) by \(2\) and add, rather than subtract, in each column beginning with the second from the left. Hence we have the final streamlined division known as synthetic division:

\[
\begin{array}{c|cccc}
2 & 4 & -3 & 7 & -4 \\
\hline & & 8 & 10 & 6 & 4 \\
& & 4 & 5 & 3 & 2 & -5 \\
\end{array}
\]

There are several points to be noted in connection with this procedure:

1. The number in the upper left-hand corner is \("t}\", if we are dividing by \(x - t\).
2. The top row consists of coefficients of terms of the dividend polynomial in order of descending degree. Any missing term in the sequence must be indicated by a zero coefficient. For example, we shall treat \(5x^4 + 3x\) as \(5x^4 + 0x^3 + 0x^2 + 3x + 0\).
3. The left-hand coefficient in the top row is merely "brought down" to the third row.
4. The procedure is then one of "multiply by \(t\) and add."
5. The third row, except for the right-hand number, consists of the coefficients of powers of \(x\) in the quotient polynomial, in order of descending degree.
6. The right-hand number in the third row is the remainder, when the divisor is \(x - t\), which, by the Remainder Theorem, also represents the value of the dividend polynomial at \(x = t\).
7. In view of the Remainder Theorem the process is known equally well as synthetic substitution.
The main thing to notice about solving cubic polynomials (or higher degree polynomials) is that a polynomial that is divisible by \((x - k)\) has a root at \(k\). Synthetic division applied to a polynomial and a factor result in a zero for the remainder. This leads us to the Factor Theorem, which states A polynomial \(f(x)\) has a factor \((x - k)\) if and only if \(f(k) = 0\).

Solving cubic polynomials can be tricky business sometimes. A graphing utility can be a helpful tool to identify some roots, but in general there is no easy formula for solving cubic polynomials like the quadratic formula aids us in solving quadratics.

There is however a tool that we can use for helping us to identify Rational Roots of the polynomial in question.

The Rational Root Theorem states that any rational solutions to a polynomial will be in the form of \(\frac{p}{q}\) where \(p\) is a factor of the constant term of the polynomial (the term that does not show a variable) and \(q\) is a factor of the leading coefficient. This is actually much simpler than it appears at first glance.

Let us consider the polynomial \(f(x) = x^3 - 5x^2 - 4x + 20\)

Identify \(p\) (all the factors of 20):

Identify \(q\) (all the factors of the lead coefficient, 1):

Identify all possible combinations of \(\frac{p}{q}\):

If \(f(x) = x^3 - 5x^2 - 4x + 20\) is going to factor, then one of these combinations is going to “work”, that is, the polynomial will divide evenly. So the best thing to do is employ a little trial and error. Let’s start with the smaller numbers, they will be easier to evaluate. Substitute for \(x\): 1, -1, 2, -2, 4, -4 …20, -20.
Why would substituting these values in for x be a useful strategy?

Why do we not have to use synthetic division on every one?

Let us define what the Remainder Theorem states and how it helps us.

Hopefully, you did not get all the way to -20 before you found one that works. Actually, 2 should have worked. Once there is one value that works, we can go from there.

Use the factor $(x - 2)$ to divide $f(x)$. This should yield:

$$f(x) = x^3 - 5x^2 - 4x + 20 = (x - 2)(x^2 - 3x - 10)$$

By factoring the result we can find all the factors:

$$f(x) = x^3 - 5x^2 - 4x + 20 = (x - 2)(x + 2)(x - 5)$$

Therefore the roots are 2, -2, and 5.

What could be done if this portion was not factorable?

**Use the Quadratic Formula**

4. For each of the following find each of the roots, classify them and show the factors.

   a. $f(x) = x^3 - 5x^2 - 4x + 20$

   Possible rational roots:

   Show work for Synthetic Division and Quadratic Formula (or Factoring):

   Complete Factorization: ________________________________
b. $f(x) = x^3 + 2x^2 - 5x - 6$

Possible rational roots:

Show work for Synthetic Division and Quadratic Formula (or Factoring):

Complete Factorization: ____________________________________

Roots and Classification

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c. $f(x) = 4x^3 - 7x + 3$

Possible rational roots:

Show work for Synthetic Division and Quadratic Formula (or Factoring):

Complete Factorization: ____________________________________

Roots and Classification

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What happens when we come to a function that is a 4th degree?
Well, just like the cubic there is no formula to do the job for us, but extending our strategies that we used on the cubics we can tackle any quartic function.

A. Develop your possible roots using the \( \frac{p}{q} \) method.
B. Use synthetic division with your possible roots to find an actual root. If you started with a 4th degree, that makes the dividend a cubic polynomial.
C. Continue the synthetic division trial process with the resulting cubic. Don’t forget that roots can be used more than once.
D. Once you get to a quadratic, use factoring techniques or the quadratic formula to get to the other two roots.

5. For each of the following find each of the roots, classify them and show the factors.

a. \( f(x) = x^4 + 2x^3 - 9x^2 - 2x + 8 \)
   
   Possible rational roots:

   Show work for Synthetic Division and Quadratic Formula (or Factoring):

   Complete Factorization: ___________________________________________________________________

   Roots and Classification
   
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b. \( f(x) = x^4 - 11x^3 - 13x^2 + 11x + 12 \)
   
   Possible rational roots:

   Show work for Synthetic Division and Quadratic Formula (or Factoring):
Complete Factorization: ____________________________________

Roots and Classification

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\[ f(x) = x^5 - 12x^4 + 49x^3 - 90x^2 + 76x - 24 \]

Possible rational roots:

Show work for Synthetic Division and Quadratic Formula (or Factoring):

Complete Factorization: ____________________________________

Roots and Classification

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\[ f(x) = x^5 - 5x^4 + 8x^3 - 8x^2 + 16x - 16 \]

Possible rational roots:

Show work for Synthetic Division and Quadratic Formula (or Factoring):

Complete Factorization: ____________________________________
Let’s consider a scenario where the roots are imaginary.

Suppose that you were asked to find the roots of \( f(x) = x^4 - x^3 + 3x^2 - 4x - 4 \).
There are only 6 possible roots: \( \pm 1, \pm 2, \pm 4 \). In the light of this fact, let’s take a look at the graph of this function.

It should be apparent that none of these possible solutions are roots of the function. And without a little help at this point we are absolutely stuck. None of the strategies we have discussed so far help us at this point.

But consider that we are given that one of the roots of the function is \( 2i \). Because roots come in pairs (think for a minute about the quadratic formula); an additional root should be \(-2i\). So, let’s take these values and use them for synthetic division.

Though the values may not be very clean, this process should work just as it did earlier. Take a moment and apply what you have been doing to this function.