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INTRODUCTION:

Conic sections are presented from both an algebraic and a geometric point of view. Students address equations in standard and general forms. Graphing is done by hand and using graphing technology. Parabolas have been studied in previous courses as quadratic functions, but in this unit they are addressed as a type of conic section and the two presentations are connected. Many of the applications of conic sections depend on their reflective properties.

ENDURING UNDERSTANDINGS:

Solving systems of equations using substitution.
Finding quadratic equations to model graphs given a set of points.
Solving systems of equations using matrices.
Graphing quadratic functions in the form \( y = a(x-h)^2+k \).
Writing equations of a line given the slope and a point on the line.
Finding slopes of perpendicular lines.
Graphing functions and adjusting graphing windows on function graphing technology.

KEY STANDARDS ADDRESSED:

MA2G3. Students will investigate the relationships between lines and circles.

a. Find equations of circles.
b. Graph a circle given an equation in general form.
c. Find the equation of a tangent line to a circle at a given point.
d. Solve a system of equations involving a circle and a line.
e. Solve a system of equations involving two circles.

MA2G4. Students will recognize, analyze, and graph the equations of the conic sections (parabolas, circles, ellipses, and hyperbolas).

a. Convert equations of conics by completing the square.
b. Graph conic sections, identifying fundamental characteristics.
c. Write equations of conic sections given appropriate information.

MA2G5. Students will investigate planes and spheres.

c. Recognize and understand equations of planes and spheres.
RELATED STANDARDS ADDRESSED:

MA2P1. Students will solve problems (using appropriate technology).

a. Build new mathematical knowledge through problem solving.
b. Solve problems that arise in mathematics and in other contexts.
c. Apply and adapt a variety of appropriate strategies to solve problems.
d. Monitor and reflect on the process of mathematical problem solving.

MA2P2. Students will reason and evaluate mathematical arguments.

a. Recognize reasoning and proof as fundamental aspects of mathematics.
b. Make and investigate mathematical conjectures.
c. Develop and evaluate mathematical arguments and proofs.
d. Select and use various types of reasoning and methods of proof.

MA2P3. Students will communicate mathematically.

a. Organize and consolidate their mathematical thinking through communication.
b. Communicate their mathematical thinking coherently and clearly to peers, teachers, and others.
c. Analyze and evaluate the mathematical thinking and strategies of others.
d. Use the language of mathematics to express mathematical ideas precisely.

MA2P4. Students will make connections among mathematical ideas and to other disciplines.

a. Recognize and use connections among mathematical ideas.
b. Understand how mathematical ideas interconnect and build on one another to produce a coherent whole.
c. Recognize and apply mathematics in contexts outside of mathematics.

MA2P5. Students will represent mathematics in multiple ways.

a. Create and use representations to organize, record, and communicate mathematical ideas.
b. Select, apply, and translate among mathematical representations to solve problems.
c. Use representations to model and interpret physical, social, and mathematical phenomena.
Unit 7 Conic Section Tasks:

Getting to Know Conic Sections.

Part 1. Slicing a Cone. Students cut paper models of cones to produce the different conic sections.

Part 2. Conic Equations. A TI-83/84 program is provided to graph of conic sections entered in general form, $Ax^2 + Cy^2 + Dx + Ey + F = 0$. Students investigate how different values of $A$ and $C$ determine the shape of the graph.

Circles.

Part 1. Graphing Circles. Beginning with the geometric locus definition of a circle, the Pythagorean Theorem is used to develop the algebraic equations. Circles are studied in both general and standard forms. Completing the square is introduced and general form equations are rewritten in standard graphing form. Students graph circles; write equations of circles given graphs; and graph circles using function graphing technology.

Part 2. Radio Stations. Systems of a circle and a line and systems of two circles are investigated. Equations of lines tangent to circles with known centers and points of tangency are introduced. Overlapping circular broadcast regions of Georgia FM radio stations provide the context for writing equations of circles and solve systems of equations. Solutions are verified using technology.

Part 3: Equations of Spheres. The understanding of the equation of a circle will be used to find the equation of a sphere. Given the equation of a sphere, students will be able to identify the center and the radius.

Parabolas. Students have previously studied parabolas as functions and have experience with transformations applied to parabolas. Known graphing techniques are expanded to include graphing from different algebraic forms. Reflection properties of parabolas based on the position of the focus and directrix are the determining factors in different real world applications. Paper clip parabolas illustrate the support structure of parabolic cables in suspension bridges.

Is It Really an Ellipse? Beginning with the geometric locus definition of an ellipse, the general and standard forms of the algebraic equations of ellipses are developed. Students graph ellipses; write equations of ellipses given graphs and various information about vertices, foci, major and minor axes; and graph ellipses using technology. Building an elliptical pool table provides the background for a task where students must determine whether or not an unmarked figure is a true ellipse or just a oval shape. Students use patty paper, string, thumbtacks, measurements, and technology to justify their solution.
Hyperbolas. Beginning with the geometric locus definition of a hyperbola, the general and standard forms of the algebraic equations of hyperbolas are developed. Students graph hyperbolas; write equations of hyperbolas given graphs and various information about vertices, foci, transverse axes and conjugate axes; and graph ellipses using technology. LORAN is introduced to provide practice in writing hyperbola equations to solve problems.

Let's Go Fishing. Conic sections and a variety of functions are used to graph a cartoon style fish. Working with a specific graphic, students must write equations to produce the fish on a function graphing technology. Various regression methods, systems of equations, and knowledge of conic sections are necessary to find the equations. Since the figure must be graphed using a technology format, all equations need to be written as functions and domains must be limited so the graphs are drawn over the proper intervals. This task is very versatile and can be adjusted to fit a variety of instructional needs.
Getting to Know Conic Sections Learning Task:
A Greek mathematician, Menachmus, a tutor to Alexander the Great, is credited with the discovery of conic sections sometime between 360 - 350 BC. He formed the figures by slicing a plane through double napped cones. Later another Greek mathematician, Apollonius, wrote an eight volume study, *Conics*, during the period 262 - 190 BC. Apollonius gave names, parabola, ellipse, and hyperbola to the figures. It is rumored that one reason he studied conic sections was to create a weapon to launch projectiles to keep ships from entering the harbor of his city.

Part 1. Slicing a Cone

Materials: 5 cone shaped paper cups
scissors
play dough
1 strand of linguini

1. Cut the first cone across perpendicular to the central axis and parallel to the base. What shape is formed by the cut? Draw a picture to show the shape of your cut section.

2. Cut the second cone diagonally so that the cut is not parallel to the base or to an outside edge, which is the slant height. Describe the resulting shape. Draw a picture to show the shape of your cut section.

3. Cut the third cone parallel to the outside edge or slant height. Describe the shape. Draw a picture to show the shape of your cut section.

4. Using the fourth and fifth cones, pack the apexes with some play dough and using a strand of linguini line the cones up like an hour glass. Cut sections on both cones perpendicular to the bases and parallel to the central axis now represented by the linguini. Describe the shapes. Draw a picture to show the shape of your cut section.
5. Use your cones to fill in the blanks in the below.

   If a plane intersects a cone parallel to the base of the cone, their intersection forms a(n) ___ __________.

   If a plane is not parallel to the base of a cone and the plane does not intersect the base of the cone, the intersection of the plane and the cone forms a(n) __ _____.

   If a plane intersects a cone perpendicular to the base of the cone, their intersection forms a(n) __ _____________.

   If a plane intersects a cone parallel to a line extending from the base to the vertex of the cone and running along the surface of the cone, their intersection forms a(n) ___ _______. 
Part 2. Conic Equations

All conic sections have equations which can be written in the form

\[ Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \]

which is known as the general quadratic equation. Equations in this form are difficult to graph. Most graphing calculators and computer programs only graph equations in function form and in order to use them we must change the general quadratic equation by solving for \( y \) in terms of \( x \).

Use the program to graph each equation. Sketch each graph and name the conic represented by each equation. You may need to adjust the window for some of the graphs to get a complete picture of the graph. If you use a window other than the standard \([-10, 10]\) by \([-10, 10]\) window, name the window on your sketch of the graph.

After completing the graphs, group the equations by type by writing them in the proper section in the table below. Write each equation in general form before entering the coefficients into the program.
Write each equation in \( Ax^2 + Cy^2 + Dx + Ey + F = 0 \) form when you put it in the table below.

<table>
<thead>
<tr>
<th>CIRCLE</th>
<th>ELLIPSE</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>PARABOLA</th>
<th>HYPERBOLA</th>
</tr>
</thead>
<tbody>
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<td></td>
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</tr>
</tbody>
</table>

1. \(-2x^2 + y^2 + 4x + 6y + 3 = 0\)  
2. \(2x^2 + 2y^2 - 4x + 16y + 2 = 0\)
3. \(x^2 + 16y^2 - 64y = 0\)  
4. \(3x^2 + 3y^2 = 36\)
5. \(x^2 + y^2 = -4x + 6y + 3\)  
6. \(16x^2 - 25y^2 - 32x + 100y - 484 = 0\)
7. \(16x^2 + 4y^2 + 32x - 8y = 44\)  
8. \(x - 2 = y^2 - 10y\)
9. \(y = x^2 + 2x - 4\)  
10. \(9x^2 - 3 = 18x + 4y\)
11. \(x^2 + 4y^2 + 6x - 8y = 3\)  
12. \(7x^2 - 5y^2 = 48 - 20y - 14x\)

All of the equations in this task are of the form \( Ax^2 + Cy^2 + Dx + Ey + F = 0 \). Are there any generalizations you can make about the equations in this form based on the graphs you have just seen? If so, list your observations in the table below.

<table>
<thead>
<tr>
<th>CIRCLE</th>
<th>ELLIPSE</th>
<th>PARABOLA</th>
<th>HYPERBOLA</th>
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Circles Learning Task:

By definition a circle is the set of all points on a plane equidistant (radius) from a given point (center).

If this circle is drawn on an axis system, with the center located at (0, 0) with radius \( r \), it is possible to write an algebraic equation for the circle.

Suppose the center point is located at the origin \((0, 0)\). Choosing one point \((x, y)\) allows us to form a right triangle with radius \( r \) as the hypotenuse. One leg is the perpendicular segment from \((x, y)\) to the x-axis at point \((x, 0)\). The second leg is the segment from the point \((x, 0)\) back to the origin. Use the Pythagorean Theorem to write an equation for \( r \),

\[
x^2 + y^2 = r^2.
\]

Because this equation is true for every point on the circle, it can be given as the equation of the circle itself.
Now suppose the center point is located away from the origin at point \((h, k)\).
Following the procedure used with a circle located with its center at the origin, pick a point \((x, y)\) on the circle and form a right triangle with the other vertices at \((x, k)\) and \((h, k)\). The hypotenuse is \(r\) and the legs are \(y - k\) and \(x - h\).

According to the Pythagorean Theorem,
\[
r^2 = (x - h)^2 + (y - k)^2.
\]

By expanding the binomial terms this equation can be written as
\[
r^2 = x^2 - 2hx + h^2 + y^2 - 2ky + k^2 \quad \text{or} \quad x^2 + y^2 - 2hx - 2ky + h^2 + k^2 = r^2.
\]

\((x - h)^2 + (y - k)^2 = r^2\) is called the standard form equation for a circle with a center at \((h, k)\) and radius \(r\). By multiplying and collecting terms, the standard form equation can be written as
\[
x^2 + y^2 - 2hx - 2ky + h^2 + k^2 = r^2 \quad \text{which then fits into the general conic equation} \quad Ax^2 + Cy^2 + Dx + Ey + F = 0. \quad \text{In order to be a circle} \quad A = C \quad \text{in the general conic equation.}
\]

**Part 1. Graphing Circles**

1. Write equations for the following circle graphs in both standard form and general form.

   a. 

   ![Diagram of a circle with center at \((0,0)\) and radius \(r\)]
b.

To change from general form to standard form, it is necessary to complete the square for x and y. **Completing the square** is an algebraic tool used to change equations of conic sections given in general form, \( Ax^2 + Cy^2 + Dx + Ey + F = 0 \), to standard form, \( (x - h)^2 + (y - k)^2 = r^2 \). Standard form is the form used to graph conic sections.

Perfect squares are numbers or expressions which have exactly two identical factors.

\[
(2)(2) = 4 \quad (-5)(-5) = 25 \quad (3x)(3x) = 9x^2\quad (-6y)(-6y) = 36y^2 \quad (x + 2)(x + 2) = x^2 + 4x + 4
\]

Consider the following geometric area models of three perfect squares. The area is given as both factors and as a quadratic expression.

\[
\text{area} = (x + 2)(x + 2) = x^2 + 4x + 4
\]

\[
\text{area} = (x - 2)(x - 2) = x^2 - 4x + 4
\]

\[
\text{area} = (x - 4)(x - 4) = x^2 - 8x + 16
\]
2. Find the products of the following expressions.

a. \((x + 1)^2 = (x + 1)(x + 1) = \)

b. \((x - 3)^2 = (x - 3)(x - 3) = \)

c. \((x - 5)^2 = (x - 5)(x - 5) = \)

d. \((x + 7)^2 = (x + 7)(x + 7) = \)

e. \((x + n)^2 = (x + n)(x + n) = \)

3. Each of the products in #2. is a perfect square. Use the results of 2. to complete each of the squares and show their factored forms. Include Geometric diagrams to illustrate the perfect squares.

a. \(x^2 + 20x + ____ = (x + ____)^2\)

b. \(x^2 - 12x + ____ = (x - ____)^2\)

c. \(x^2 + 18x + ____ = (x + ____)^2\)

d. \(x^2 - 7x + ____ = (x - ____)^2\)

e. \(x^2 + 2nx + ____ = (x + ____)^2\)

In order to graph a circle given in general form, it is necessary to change to standard form. In order to rewrite \(x^2 + y^2 + 2x - 4y - 11 = 0\) in standard form to facilitate graphing, it is necessary to complete the square for both \(x\) and \(y\).

\[
x^2 + y^2 + 2x - 4y - 11 = 0
\]
\[
(x^2 + 2x) + (y^2 - 4y) = 11
\]
\[
(x^2 + 2x + 1) + (y^2 - 4y + 4) = 11 + 1 + 4
\]
\[
(x + 1)^2 + (y - 2)^2 = 16
\]

circle with center at (-1, 2) and radius 4
To change $x^2 + y^2 + 2x - 4y - 11 = 0$ to standard form, it is necessary to remove a factor of 2 before completing the square for both $x$ and $y$.

$2x^2 + 2y^2 - 4x + 6y - 4 = 0$

$(x^2 - 2x + 1) + (y^2 + 3y + 9/4) = 2 + 1 + 9/4$

balance the equation by adding 1 and $9/4$ to both sides of the equation factor

$(x - 1)^2 + (y + 1.5)^2 = 5.25$

circle with center at (1, -1.5) and radius

4. Change the following equations to standard form. Graph the circles; identify the centers and the radii.

a. $x^2 + y^2 + 2x + 4y - 20 = 0$

b. $x^2 + y^2 - 4y = 0$

c. $x^2 + y^2 - 6x - 10y = 2$
To graph the circle $x^2 + y^2 + 2x - 4y - 11 = 0$ using a TI83/TI84 it is necessary to solve for $y$ after changing the equation to standard form.

\[
(x + 1)^2 + (y - 2)^2 = 16
\]
\[
(y - 2)^2 = 16 - (x + 1)^2
\]
\[
\sqrt{(y - 2)^2} = \pm \sqrt{16 - (x + 1)^2}
\]
\[
y - 2 = \pm \sqrt{16 - (x + 1)^2}
\]
\[
y = 2 \pm \sqrt{16 - (x + 1)^2}
\]

Enter this result as two functions $y_1 = 2 + \sqrt{16 - (x + 1)^2}$ and $y_2 = 2 - \sqrt{16 - (x + 1)^2}$. In order to minimize the distortion caused by the rectangular screen of the graphing calculator, use a window with a x to y ratio of 3 to 2. Otherwise circles appear as ellipses.

5. Write the equations from #4. as you would enter them in a graphing calculator and list an appropriate graphing window to show the entire circle graph.

a. $x^2 + y^2 + 2x + 4y - 20 = 0$

b. $x^2 + y^2 - 4y = 0$

c. $x^2 + y^2 - 6x - 10y = 2$
Part 2 Systems of Equations Containing a Circle and a Line or Two Circles

First, consider systems of equations containing a circle and a line. Since these systems contain equations of two different degrees we solve them using graphing and substitution.

1. Sketch all possible graphing configurations for a line and a circle.

Given a system of equations such as \[
\begin{align*}
   y &= -x + 1 \\
   x^2 + (-x + 1)^2 &= 9 \\
   x^2 + x^2 - 2x + 1 &= 9 \\
   2x^2 - 2x - 8 &= 0 \\
   x^2 - x - 4 &= 0
\end{align*}
\] Solve y in terms of x and substitute into the circle equation.

Using the Quadratic Formula \(x = \frac{1 \pm \sqrt{17}}{2}\), which gives \(x = 2.56\) and \(x = -1.56\). Substituting these values into the linear equation yields \(y = -1.56\) and \(y = 2.56\) respectively. Therefore the line intersects the circle in two points \((2.56, -1.56)\) and \((-1.56, 2.56)\).

Solving this system by graphing by hand is best done with a ruler and a compass to give the most accurate results. Using a graphing calculator and implementing Zoom features gives good results.

Enter the linear equation and both parts of the circle; choose a friendly window; and, graph the system.
2. Solve the following systems of equations algebraically and then check the solutions using graphing.

a. \[
\begin{align*}
  x^2 + y^2 &= 34 \\
  x - y &= 2
\end{align*}
\]

b. \[
\begin{align*}
  x^2 + y^2 &= 9 \\
  2y &= x + 8
\end{align*}
\]

c. \[
\begin{align*}
  x^2 + y^2 &= 25 \\
  2x + y &= 10
\end{align*}
\]

d. \[
\begin{align*}
  x^2 + y^2 &= 25 \\
  -3x + 4y &= 25
\end{align*}
\]

e. \[
\begin{align*}
  x^2 + y^2 &= 10 \\
  x + 3y &= 10
\end{align*}
\]

3. Write equations of a line and a circle which satisfy the following conditions: (In all three cases, justify your solutions by solving for the points of intersection and by graphing your equations.)

a. the line and circle have no points in common,

b. the line and circle are tangent, and,

c. the line and circle have two points of intersection

In the systems of equations we have just completed, some of the problems concerned lines tangent to circles. It is possible to find the equation of a line tangent to a given circle if you know the point of tangency. From our study of circles in Geometry, we know that a tangent line intersects a circle in exactly one point called the point of tangency. Also recall that a radius drawn to the point of tangency is perpendicular to the tangent.
In the following diagram, the **point of tangency** is \((a, b)\) and the **radius** is \(r\). The equation of this circle is \(x^2 + y^2 = r^2\) and by substituting \((a, b)\) for \((x, y)\) we could get \(a^2 + b^2 = r^2\).

In order to write the equation of a line, we need a point on the line and the slope. In this case, we know the slope of the radius is \(\frac{b}{a}\). Since the tangent line is perpendicular to the radius, the slope of the tangent line must be \(-\frac{a}{b}\). \((a, b)\) is a point on both the radius and the tangent line. Substituting \((a, b)\) and \(m = -\frac{a}{b}\) into the point-slope formula gives

\[
y - b = -\frac{a}{b} (x - a)
\]

\[
b(y - b) = -a(x - a)
\]

\[
ax + by = r^2
\]

the formula of the line tangent to a circle, with center \((0, 0)\) and radius \(r\), at point \((a, b)\).

4a. Find an equation of the tangent line to a circle with the equation \(x^2 + y^2 = 9\) with the point of tangency at \((1, \sqrt{8})\).

4b. Write the equation of a circle with the center at the origin tangent to the line \(2x + 3y = 13\).

Now we will solve systems of equations consisting of two circles. These systems can be solved using elimination, substitution, and graphing.

5. How many different solutions are possible when solving a system of two circles? Use diagrams to explain your answer.

Consider the system of equations

\[
\begin{align*}
x^2 + (y - 2)^2 &= 9 \\
(x - 2)^2 + (y - 3)^2 &= 1
\end{align*}
\]

In order to solve this system, first expand the binomials to get

\[
\begin{align*}
x^2 + y^2 - 4y + 4 &= 9 \\
x^2 + y^2 - 4x - 6y + 13 &= 1
\end{align*}
\]

then collect like terms and simplify:

\[
\begin{align*}
x^2 + y^2 - 4y - 5 &= 0 \\
x^2 + y^2 - 4x - 6y + 12 &= 0
\end{align*}
\]
Subtract and to get $4x + 2y - 17 = 0$. Note that this result is the equation of a line. Any solutions of this system must be points on this line. Solve for $y$ in terms of $x$ to get $y = -2x + 8.5$. Substitute $-2x + 8.5$ in place of $y$ in one of the original equations to get

\[
x^2 + (-2x + 8.5 - 2)^2 = 9
\]
\[
x^2 + (-2x + 6.5)^2 = 9
\]
\[
x^2 + 4x^2 - 26x + 42.25 - 9 = 0
\]
\[
5x^2 - 26x + 33.25 = 0
\]

Solve for $x$ using the Quadratic Formula and get $x = \frac{26 \pm \sqrt{11}}{10}$ or $x = 2.9$ or $x = 2.2$. Solving for $y$ gives the solutions $(2.9, 2.7)$ and $(2.2, 4.1)$.

If you solve the system by graphing using a graphing calculator, the problem could be done as follows. First solve each circle for $y$ in terms of $x$ and separate the results into two functions.

\[
x^2 + (y - 2)^2 = 9\]
\[
(x - 2)^2 + (y - 3)^2 = 1
\]

The graph does not give much information as it is so use the Zoom features and take a closer look at the points of intersection. Zoom features do not produce a friendly window which distorts the graph. But, you can still use Trace and find the points of intersection.

$(2.2, 4.1)$

$(2.9, 2.7)$
Application Task 1  RADIO STATIONS.
1. Radio signals emitted from a transmitter form a pattern of concentric circles. Write equations for three concentric circles.

2. Randy listens to radio station WYAY from Atlanta. Randy's home is located 24 miles east and 32 miles south of the radio station's transmitter. His house is located on the edge of WYAY's maximum broadcast range.
   a. When a radio signal reaches Randy's house, how far has it traveled? Sketch WYAY's listening area of the partial map of Georgia given. On the map let Atlanta's WYAY have coordinates (0, 0) and use the scale as 100 miles = 60 mm.

   b. Find an equation which represents the station's maximum listening area.

   c. Determine four additional locations on the edge of WYAY's listening area, give coordinates correct to tenths.
3. Randy likes to listen to country music. Several of his friends have suggested that in addition to WYAY, he try station WXAG in Athens and WDEN in Macon. WYAY, WXAG, and WDEN are FM stations which normally have an average broadcast range of 40 miles. Use the map included with the indicated measures to answer the following questions.

a. Given the location of Randy's can he expect to pick up radio signals from WXAG and WDEN? Explain how you know.

b. What are the coordinates of the intersections of the broadcast areas of station WYAY and station WDEN? Does it matter whether you find the intersections using miles or mms? How do you know?

Solve the system

\[
\begin{align*}
x^2 + y^2 &= 1600 \\
(x - 40)^2 + (y + 63.3)^2 &= 1600
\end{align*}
\]
Application Task 2 CROP CIRCLES. Crop circles, geometric patterns formed by flattening grain crops, have been documented since the 17th century in English woodcuts. Early crop circles were simple circular formations, but more recent formations have increased in complexity and now include figures other than circles. Many crop circles are extremely large and are best viewed from the air. Their size and precision of patterns, added to the fact that many appear mysteriously overnight have fueled interest in this phenomenon. A large number of crop circles began appearing in the English countryside during the 1970s and during the 1980s were reported in Australia and the United States. Many crop circles are known to be man-made, but a large number have unexplained origins. For more information on crop circles visit, http://en.wikipedia.org/wiki/Crop_circle

More recent crop circles are very intricate and appear similar to very beautiful computer graphics. Suppose your class decides to duplicate a simple crop circle design as part of a project logo. Choose one of the crop circle pictures shown (or a picture approved by your teacher) below and develop equations to generate similar designs on your calculator.
Part 3 Equation of a Sphere

Just as we have looked at the equation of a circle: the set of all points, \((x, y)\), a given distance, \(r\), from a given point \((h, k)\), we can extend that definition from lying in a plane to three dimensions. A sphere is the set of all points, \((x, y, z)\), a given distance, \(r\), from a given point, \((h, k, j)\). Can you conjecture how to extend the equation of a circle and the distance formula in two dimensions to the equation of a sphere in three dimensions? Show how you know.
Parabolas Learning Task:

Parabolas were studied in previous courses as quadratic functions where the equations were based on the position of the vertex and additional points were found using values of x on either side of the axis of symmetry. Equations were in the vertex form, \( y = a(x-h)^2 + k \), or in general quadratic form \( y = ax^2 + bx + c \). We will use these forms and expand our study by including the geometric locus definition of the parabola.

The locus definition of the parabola is given in terms of a fixed point, the focus, and a fixed line, the directrix, on a plane. Each point on the parabola has a distance to the focus point which equals the distance from that point to the directrix. On the diagram below, point \( P \) is located at \( (x, y) \), the focus point is located at \( (0, 3) \), and the directrix has the equation \( y = -3 \). Note that any point on the directrix line can be written in the form \( (x, -3) \). The distances shown on the diagram are \( PF \), from the parabola to the focus, and \( PD \), from the parabola to the directrix, are both equal to 4. Similar distances from each point on the parabola have the same relationship, \( PF = PD \), but are different lengths. Draw segments from several of the points on the parabola to the focus and to the directrix to convince yourself that the relationship \( PF = PD \) holds for all points.
Using the relationship PF = PD, distance formula, the points P(x, y), F(0, 3), and D(x, -3), we can find an algebraic equation for the parabola. As you work through the following procedures, describe what is being done in each step.

\[ PF = \sqrt{(x - 0)^2 + (y - 3)^2} \text{ and } PD = \sqrt{(x - x)^2 + (y + 3)^2} \]

apply the distance formula

\[ \sqrt{(x - 0)^2 + (y - 3)^2} = \sqrt{(x - x)^2 + (y + 3)^2} \]

substitution

\[ \sqrt{x^2 + (y - 3)^2} = \sqrt{(y + 3)^2} \]

simplify

\[ x^2 + (y - 3)^2 = (y + 3)^2 \]

square both sides of the equation

\[ x^2 = (y + 3)^2 - (y - 3)^2 \]

subtract \((y - 3)^2\) from both sides

\[ x^2 = y^2 + 6y + 9 - (y^2 - 6y + 9) \]

expand the binomial terms

\[ x^2 = 12y \]

simplify

\[ y = \frac{1}{12} x^2 \]

division

\[ y = \frac{1}{12} x^2 \] can be written as \( y = \frac{1}{12} (x - 0)^2 + 0 \), the vertex form, where the vertex is \((0, 0)\) and \( a = \frac{1}{12} \), a positive value, meaning the parabola opens upward. \( y = \frac{1}{12} x^2 \) also fits the general form with \( a = \frac{1}{12} \), \( b = 0 \) and \( c = 0 \). These observations all agree with the parabola shown in the starting diagram.

Adding the focus and the directrix gives more information about the parabola. The distance from the focus point at \((0, 3)\) to the vertex at \((0, 0)\) is 3 units and the distance from the vertex to the directrix is also 3 units. These distances are generalized to give the \textbf{position of the focus as} \((0, p)\), the \textbf{equation of the directrix as} \( y = -p \), and the equation of the parabola as \( y = \frac{1}{p} x^2 \).

The parabola equation \( y = \frac{1}{12} x^2 \) is \( y = \frac{1}{4(3)} x^2 \). Parabolas can open horizontally as well as vertically and the information on parabolas with vertices at \((0, 0)\) are summarized in the following table.
Summary of Parabola Information

\[ Ax^2 + Cy^2 + Dx + Ey + F = 0 \]

where \( A = 0 \) or \( C = 0 \) but not both \( = 0 \)

vertex at \((h, k)\)

**Horizontal Directrix**

\[ y - k = \frac{1}{4p} (x - h)^2 \]

- \( p > 0 \) opens up
- \( p < 0 \) opens down
- focus \((h, k + p)\)
- directrix \(y = k - p\)
- axis of symmetry \(x = h\)

**Vertical Directrix**

\[ x - h = \frac{1}{4p} (y - k)^2 \]

- \( p > 0 \) opens right
- \( p < 0 \) opens left
- focus \((h + p, k)\)
- directrix \(x = h - p\)
- axis of symmetry \(y = k\)
Parabolas have special reflective properties which make them useful shapes for many items including flashlights, car headlights, suspension bridges such as the Golden Gate Bridge, solar cookers, and satellite dishes. Two properties are especially important when considering applications of parabolas. First, all rays in the interior of a parabola parallel to the axis of symmetry are reflected toward the focus. And, all rays emitted from the focus are reflected so that each reflected ray runs parallel to the axis of symmetry and perpendicular to the directrix.

1. In each of the following problems, list the coordinates of the vertex, the coordinates of the focus, the equation of the directrix, and graph the parabola.
   a. \((y - 3)^2 = -12(x + 2)\)
   b. \(-\frac{1}{8}(x - 1)^2 = y\)
c. \( x = (y - 4)^2 \)

d. \( (x + 1)^2 = 2(y + 3) \)

2. Approximate the location of the focus point and the directrix on the following parabola. Show how rays emitted from the focus would travel.

\[
\bullet
\]

To change parabolas given in general form to standard form to facilitate graphing, it is necessary to complete the square. Given the parabola \( 2x^2 - 4x + y + 4 = 0 \), first rewrite the equation with x terms and y terms on different sides of the equation.

\[
2x^2 - 4x + y + 4 = 0
\]

\[
y + 4 = -2x^2 + 4x \quad \text{separate x terms and y terms}
\]

\[
y + 4 = -2(x^2 - 2x) \quad \text{prepare to complete the square by factoring out -2}
\]

\[
y + 4 - 2 = -2(x^2 - 2x + 1) \quad \text{complete the square; add -2 to both sides}
\]

\[
y + 2 = -2(x - 1)^2 \quad \text{factor}
\]

3. Write each of the following equations in standard form. List the vertex, coordinates of the focus and equation of the directrix. Graph the parabola.
a. \( y^2 - 8y + 8x + 8 = 0 \)

b. \( x^2 - 6x + 12y + 21 = 0 \)

4. **Parabolas and Suspension Bridges.**

Suspension bridges depend on parabolically curved cables to support the weight of the road bed of the bridge. The weight of the bridge is evenly distributed among the support cables which run parallel to the axis of symmetry, similar to the paths of rays reflected off the surface of parabolic reflectors.

**Work in groups of 2 or 3 with the following materials.**

Materials: box of #1 smooth paper clips
3 bulletin board pens
1 inch per square grid paper
piece of cardboard at least 10" x 10"
marker pen

1. Link 16 paper clips together to form a chain
2. Hang 2 linked paper clips at each joint of the
3. Attach the 1 inch grid paper to the piece of cardboard
4. Draw a horizontal line across the top of the grid paper about 1" from the top edge
5. Find the midpoint of the line and draw a perpendicular line at least 10" long
6. Use 2 of the pins to hang your paper clip chain so that each end is on the horizontal line 4" from the vertical line.
7. Your weighted chain should form a parabola; use the 3rd pin to mark the vertex and fix the position of your vertex.
8. Mark points at each paperclip joint; try to mark as close to the center of each joint as possible.
9. Sketch your figure.
10. Choose an appropriate ordered pair for the vertex point of your parabola and mark an x-axis and a y-axis. Using the points you have, write an equation in \( y = a(x - h)^2 + k \) form for your weighted paper clip chain.
The paper clip chain with no weight attached is a catenary curve. When weight is attached, the curve becomes a parabola. Catenary curves and parabolas are often mistaken for each other. The Golden Gate bridge is a suspension bridge in San Francisco, California. The towers are 1280 meters apart and rise 160 meters above the road. The cable just touches the sides of the road midway between the towers. What is the height of the cable 200 meters from a tower?
Is It Really an Ellipse? Learning Task:

An ellipse is the set of all points P in a plane where the sum of the distances from two fixed points (foci) remains constant. In the drawing shown, A and B are the fixed focus points and P is any point on the ellipse where |PA + PB| = 10. Using the two sets of equally spaced circles centered at A and B, check the distances of each of the points marked from foci A and B, are the sums |PA + PB| all equal to 10? Yes.

The midpoint of AB is the center of the ellipse. For the ellipse shown, the center is (0, 0). The vertices occur at the points (5, 0), (-5, 0), (0, 4), and (0, -4). The segments joining the vertices lying along the x-axis is 10 units long; and, the axis joining the vertices along the y-axis is 8 units long. The longer axis is called the major axis and the shorter axis is called the minor axis. Using the distance formula and the geometric definition of a ellipse, |PA + PB| = 10, makes it possible to find to write an equation of the ellipse in both general form and standard form. Focus point A is located at (3, 0) and B is located at (-3, 0). The distance formula gives the following distances. As you work through the following procedures, describe what is being done in each step.
\[ PA = \sqrt{(x - 3)^2 + (y - 0)^2} \quad \text{and} \quad PB = \sqrt{(x + 3)^2 + (y - 0)^2} \]

\[ PA + PB = 10 \]

\[ \sqrt{(x - 3)^2 + (y - 0)^2} + \sqrt{(x + 3)^2 + (y - 0)^2} = 10 \]

\[ \sqrt{(x - 3)^2 + (y - 0)^2} = 10 - \sqrt{(x + 3)^2 + (y - 0)^2} \]

\[ (\sqrt{(x - 3)^2 + (y - 0)^2})^2 = (10 - \sqrt{(x + 3)^2 + (y - 0)^2})^2 \]

\[ (x - 3)^2 + y^2 = 100 - 20 \sqrt{(x + 3)^2 + y^2} + (x + 3)^2 + y^2 \]

\[ x^2 - 6x + 9 + y^2 = 100 - 20 \sqrt{(x + 3)^2 + y^2} + x^2 + 6x + 9 + y^2 \]

\[ -12x - 100 = -20 \sqrt{(x + 3)^2 + y^2} \]

\[ 3x + 25 = 5 \sqrt{(x + 3)^2 + y^2} \]

\[ (3x + 25)^2 = (5 \sqrt{(x + 3)^2 + y^2})^2 \]

\[ 9x^2 + 150x + 625 = 25((x + 3)^2 + y^2) \]

\[ 9x^2 + 150x + 625 = 25(x^2 + 6x + 9 + y^2) \]

\[ 9x^2 + 150x + 625 = 25x^2 + 150x + 225 + 25y^2 \]

\[ 16x^2 + 25y^2 = 400 \quad \text{[general form for an ellipse]} \]

and \[ \frac{16}{400} x^2 + \frac{25}{400} y^2 = 1 \] which can be written as

\[ \frac{x^2}{25} + \frac{y^2}{16} = 1 \quad \text{[standard form for an ellipse]} \]
Summary of Ellipse Information

\[ Ax^2 + Cy^2 + Dx + Ey + F = 0 \]

where \( A \neq C \) and \( AC > 0 \)

\[
\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1
\]

center: \( (h, k) \)

\[ a^2 > b^2 \quad \text{b}^2 > a^2 \]

major axis: horizontal \quad \text{major axis: vertical}

length major axis: 2a \quad \text{length major axis: 2b}

length minor axis: 2b \quad \text{length minor axis: 2a}

\[ c^2 = a^2 - b^2 \quad c^2 = b^2 - a^2 \]

foci: \( (h \pm c, k) \) \quad \text{foci:} \( (h, k \pm c) \)
In the equation, \( \frac{x^2}{25} + \frac{y^2}{16} = 1 \), \( a^2 = 25 \) so \( a = \pm 5 \) and two of the ellipse vertices are located horizontally 5 units right and left of the center at \((5, 0)\) and \((-5, 0)\). \( b^2 = 16 \) so \( b = \pm 4 \) and the remaining two vertices are located vertically 4 units above and below the center at \((0, 4)\) and \((0, -4)\). Since \( a^2 > b^2 \), the major axis lies horizontally with a length of 10 and the minor axis lies vertically with a length of 8. The vertices on the minor axis are referred to as co-vertices.

1. Write an equation in both standard and general forms for each ellipse.

a. 

```

```

b. 

```
```

c. 

```

```
2. Graph each ellipse. Identify the vertices and co-vertices. Find the lengths of the major and minor axes.

a. \( \frac{x^2}{9} + \frac{y^2}{16} = 1 \)

b. \( 4x^2 + 9y^2 = 36 \)

c. \( 4(x - 3)^2 + 9(y + 2)^2 = 36 \)
3. Use completing the square to change each general form ellipse to standard form. Next change the standard form equation to equations ready to be graphed using function graphing technology. Identify an appropriate graphing window.
   a. \(9x^2 + 4y^2 + 8y - 32 = 0\)

   b. \(x^2 + 4y^2 + 6x - 8y - 3 = 0\)

Ellipses have special reflective properties. Energy waves emanating from one focus point will bounce off the ellipse and travel to the other focus. "Whispering galleries" are elliptical domes where any sound waves produced at one focus can be heard at the other focus and no where in between. Statuary Hall in the U.S. Capital building is elliptical. John Quincy Adams, while a member of the House of Representatives, discovered this acoustical phenomenon. He situated his desk at a focal point of the elliptical ceiling, and could easily hear conversations being held at the opposing party leader's desk located near the other focal point. The Mormon Tabernacle is another example of a whispering gallery.

![Diagram of an ellipse with reflected waves]

A nonsurgical treatment for kidney stones, Extracorporeal Shock Wave Lithotripsy, uses the same reflection property of the ellipse to break up the stones without damaging any tissue around them. A patient is positioned in an elliptical chamber so that the kidney stone is at one focus and high energy sound waves are created at the other focus. The sound waves reflect off the wall of the chamber and converge break up the kidney stone.

Elliptical pool tables also depend on the reflection qualities of ellipses. A ball moving across one focus point and hitting the wall of the table will travel to a pocket located at the other focus point. Suppose you wanted to build an elliptical pool table.

4. In order to build an elliptical pool table you must first be able to construct a perfect ellipse. The figure below looks like an ellipse, but is it a true ellipse? And if the figure is an ellipse, where would the pockets need to be located to ensure the billiard balls follow correct paths around the table?
a. Locate the vertical and horizontal axes of the figure and draw them on the diagram.

b. Write an equation of the ellipse which has axes with the measures you found on this drawing.

c. Use your equation to solve for eight additional points which should be on the diagram if it is a true ellipse. Plot the points you found. Are the points on the figure?

d. If you believe the figure in the diagram is an ellipse, locate the position of the focus points on the diagram.
e. One of your classmates has heard of a method for constructing ellipses using string and thumbtacks. She illustrates by placing two thumbtacks on a line 6 inches apart. She ties the string to form a loop with a circumference of 16 inches. Next, she loops the string around both thumbtacks and holding a pencil in the string so that the string always forms a triangle, traces a figure. She says her figure is an ellipse. Is she right? Justify your answer.

f. Consider the following ellipse equations. Compare the graphs. How are they similar? How do they differ?

\[
\frac{x^2}{2.25} + \frac{y^2}{6.25} = 1 \quad \frac{x^2}{9} + \frac{y^2}{25} = 1 \quad \frac{x^2}{900} + \frac{y^2}{2500} = 1
\]

g. Write a plan for building an elliptical pool table. Include a scale drawing showing the measurements you would use and the placement of the pocket and the other focus.
Hyperbolas Learning Tasks:
A hyperbola is the set of all points $P$ in a plane where the absolute value of the differences of the distances from two fixed points (foci) remains constant. In the drawing shown, $A$ and $B$ are the fixed focus points and $P$ is any point on a branch of the hyperbola where $|PA - PB| = 3$. Using the equally spaced circles centered at $A$ and $B$, check the distances of each of the points marked from foci $A$ and $B$, are the differences $|PA - PB|$ all equal to 3?

The midpoint of $AB$ is the center of the hyperbola. For the hyperbola shown, the center is $(0, 0)$. The vertices occur at the points of intersection of the branches and $AB$. The segment joining the vertices is the transverse axis and the conjugate axis lies on the perpendicular bisector of the transverse axis.

Using the distance formula and the geometric definition of a hyperbola makes it possible to find to write an equation of the hyperbola in both general form and standard form.
Focus point \( A \) is located at \((2, 0)\) and \( B \) is located at \((-2, 0)\). The distance formula gives the following distances. As you work through the following procedures, describe what is being done in each step.

\[
PA = \sqrt{(x-2)^2 + (y-0)^2} \quad \text{and} \quad PB = \sqrt{(x+2)^2 + (y-0)^2}
\]

apply the distance formula

then \(|PA - PB| = 3\) and \(PA - PB = \pm 3\)

definition of absolute value

\[
\sqrt{(x-2)^2 + (y-0)^2} - \sqrt{(x+2)^2 + (y-0)^2} = \pm 3
\]

substitution

\[
\sqrt{(x-2)^2 + (y-0)^2} = \pm 3 + \sqrt{(x+2)^2 + (y-0)^2}
\]

separate the radical expressions

\[
\left(\sqrt{(x-2)^2 + (y-0)^2}\right)^2 = (\pm 3 + \sqrt{(x+2)^2 + (y-0)^2})^2
\]

square both sides of the equation

\[
(x-2)^2 + y^2 = 9 \pm 6\sqrt{(x+2)^2 + y^2} + (x+2)^2 + y^2
\]

begin to expand the right side

\[
x^2 - 4x + 4 + y^2 = 9 \pm 6\sqrt{(x+2)^2 + y^2} + x^2 + 4x + 4 + y^2
\]

expand the binomials

\[-8x - 9 = \pm 6\sqrt{(x+2)^2 + y^2}
\]

simplify

\[8x + 9 = \pm 6\sqrt{(x+2)^2 + y^2}
\]

multiply both sides by -1

\[(8x + 9)^2 = (\pm 6\sqrt{(x+2)^2 + y^2})^2
\]

square again to remove radical

\[64x^2 + 144x + 81 = 36((x+2)^2 + y^2)
\]

expand the binomials

\[64x^2 + 144x + 81 = 36(x^2 + 4x + 4 + y^2)
\]

continue to expand the binomials

\[64x^2 + 144x + 81 = 36x^2 + 144x + 144 + 36y^2
\]

multiply by 36

\[28x^2 - 36y^2 = 63\] [general form for a hyperbola]

combine like terms

and \(\frac{28}{63}x^2 - \frac{36}{63}y^2 = 1\) which can be written as

divide by 63
The equation of the hyperbola is given as:
\[
\frac{x^2}{\frac{63}{28}} - \frac{y^2}{\frac{63}{36}} = 1
\]

This is the standard form for a hyperbola.

### Summary of Hyperbola Information

The equation of a hyperbola in standard form is:
\[
Ax^2 - Cy^2 + Dx + Ey + F = 0
\]
where \(A \neq C\) and \(AC < 0\)

- Center: \((h, k)\)
- Transverse axis: horizontal
- Length transverse axis: 2a
- Length conjugate axis: 2b
- Asymptotes: \(y = \pm \frac{b}{a} x\)
- Foci: \((h \pm c, k)\)

For a hyperbola with the equation:
\[
\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1
\]
- Center: \((h, k)\)
- Transverse axis: vertical
- Length transverse axis: 2a
- Length conjugate axis: 2b
- Asymptotes: \(y = \pm \frac{a}{b} x\)
- Foci: \((h, k \pm c)\)

\[c^2 = a^2 + b^2\]
Using the standard form equation we found, \( \frac{x^2}{28} - \frac{y^2}{36} = 1 \), it is possible to name values for the hyperbola. The equation can be rewritten in the form \( \frac{x^2}{2.25} - \frac{y^2}{1.75} = 1 \). The coordinates of the vertices are \((1.5, 0)\) and \((-1.5, 0)\). The length of the transverse axis is 3. The length of the conjugate axis is approximately 2.6. And the equations of the asymptotes are 
\[ y = \pm \frac{13}{15} x \] or 
\[ y = \pm \frac{13}{15} x \]. Because we initially considered the geometric definition, we know the position of the vertices to be \((2, 0)\) and \((-2, 0)\). But, if we used the focus relationship, \( c^2 = a^2 + b^2 \), the equation gives \( c^2 = 2.25 + 1.75 = 4 \) which confirms our results.

Hyperbolic shapes are used for horns, street lamps, space heaters, and cooling towers for nuclear reactors. Rays emanating from one focus point, \( A \), reflect off point \( P \) on the hyperbola as if they had emanated from the other focus point \( B \). This has the effect of spreading out any waves coming from a focus point.
1. For each of the following hyperbolas, find the equation in standard form and draw the graph.
   a. vertices at (-3, 1) and (3, 1) and asymptotes of \( y = \pm \frac{4}{3} x \)
b. vertices at (0, 1) and (0, -1) and asymptotes \( y = \pm \frac{1}{3} x \)

2. Write the standard equation for each hyperbola, give the coordinates of the center, vertices, and foci. What direction does the transverse axis lie?

a. \( 4x^2 - 9y^2 - 8x + 54y - 113 = 0 \)

\[
\frac{(x - 1)^2}{9} - \frac{(y - 3)^2}{4} = 1
\]

b. \( y^2 - 9x^2 - 6y - 36x - 36 = 0 \)

\[
\frac{(y - 3)^2}{9} - \frac{(x + 2)^2}{1} = 1
\]

3. For each of the following hyperbolas, give the coordinates of the center, and the vertices. Write the equations of the asymptotes. Sketch the graph and include all of the values you found as well as identifying the transverse and conjugate axes.

a. \( 4x^2 - 25y^2 = 100 \)
b. \[
\frac{(y-1)^2}{4} - \frac{(x+2)^2}{9} = 1
\]

4. For the hyperbola graph, identify the vertices, the asymptotes, the length of the transverse axis and the length of the conjugate axis. Write an equation in standard form for the graph.

5. You are writing a report for science about LORAN (LOng RAnge Navigation) radio navigation systems and would like to include accurate graphs. The program you have available only graphs functions and you need to graph hyperbolas.
   a. Will a program designed to graph functions graph a hyperbola? Explain.
b. Suppose you wished to graph a hyperbola with the equation \( \frac{x^2}{16} - \frac{y^2}{25} = 1 \). Show how you would go about graphing this equation using a TI83/TI84. Include a possible graphing window.

c. Show how you would go about graphing the equation \( \frac{(y-8)^2}{4} - \frac{(x+6)^2}{9} = 1 \) using a TI83/TI84. Include a possible graphing window.

d. Your report includes this problem to illustrate a LORAN application. Two radio stations located at A and B transmit simultaneously to a ship located at P. The onboard computer converts the time difference |PA – PB| between the time the ship receives a signal from each station and this locates the ship on one branch of a hyperbola. Suppose the ship receives the signal from the station located at A 1200 microseconds before it receives the signal from station at B. Station A is located 400 miles due east of station B. Assuming that radio signals travel at a speed of 980 feet per microsecond, find the equation of the hyperbola where the ship lies.