Accelerated Mathematics III
Frameworks
Student Edition

Unit 7
Extended Trigonometry
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INTRODUCTION:

In this unit, students are given the opportunity to delve more deeply into trigonometric relationships. The tasks in Unit 7 offer contextual situations in which students will solve trigonometric equations both graphically and algebraically. Students will apply the law of sines and the law of cosines and use technology when appropriate. There are also tasks included which focus on the verification and application of \( A = \frac{1}{2}ab \sin C \) to find the area of a triangle.

Students will learn to represent vectors algebraically and geometrically. One task will focus on the conversion of vectors expressed using rectangular coordinates and vectors expressed using magnitude and direction. The tasks will provide realistic problem situations that require the use of vectors in the solution process.

KEY STANDARDS ADDRESSED:

MA3A6. Students will solve trigonometric equations both graphically and algebraically.
   a. Solve trigonometric equations over a variety of domains, using technology as appropriate.
   b. Use the coordinates of a point on the terminal side of an angle to express \( x \) as \( r \cos \theta \) and \( y \) as \( r \sin \theta \).
   d. Apply the law of sines and the law of cosines.

MA3A7. Students will verify and apply \( \frac{1}{2}ab \sin C \) to find the area of a triangle.

MM4A10. Students will understand and use vectors.
   a. Represent vectors algebraically and geometrically.
   b. Convert between vectors expressed using rectangular coordinates and vectors expressed using magnitude and direction.
   c. Add, subtract, and compute scalar multiples of vectors.
   d. Use vectors to solve realistic problems.

MA3A11. Students will use complex numbers in trigonometric form.
   a. Represent complex numbers in trigonometric form.
   b. Find products, quotients, powers, and roots of complex numbers in trigonometric form.

MA3A12. Students will explore parametric representations of plane curves.
   a. Convert between Cartesian and parametric form.
   b. Graph equations in parametric form showing direction and beginning and ending points where appropriate.
MA3A13. Students will explore polar equations.
Express coordinates of points in rectangular and polar form.
Graph and identify characteristics of simple polar equations including lines, circles, cardioids, limacons, and roses.

RELATED STANDARDS ADDRESSED:

MA3P1. Students will solve problems (using appropriate technology).
   a. Build new mathematical knowledge through problem solving.
   b. Solve problems that arise in mathematics and in other contexts.
   c. Apply and adapt a variety of appropriate strategies to solve problems.
   d. Monitor and reflect on the process of mathematical problem solving.

MA3P2. Students will reason and evaluate mathematical arguments.
   a. Recognize reasoning and proof as fundamental aspects of mathematics.
   b. Make and investigate mathematical conjectures.
   c. Develop and evaluate mathematical arguments and proofs.
   d. Select and use various types of reasoning and methods of proof.

MA3P3. Students will communicate mathematically.
   a. Organize and consolidate their mathematical thinking through communication.
   b. Communicate their mathematical thinking coherently and clearly to peers, teachers, and others.
   c. Analyze and evaluate the mathematical thinking and strategies of others.
   d. Use the language of mathematics to express mathematical ideas precisely.

MA3P4. Students will make connections among mathematical ideas and to other disciplines.
   a. Recognize and use connections among mathematical ideas.
   b. Understand how mathematical ideas interconnect and build on one another to produce a coherent whole.
   c. Recognize and apply mathematics in contexts outside of mathematics.

MA3P5. Students will represent mathematics in multiple ways.
   a. Create and use representations to organize, record, and communicate mathematical ideas.
   b. Select, apply, and translate among mathematical representations to solve problems.
   c. Use representations to model and interpret physical, social, and mathematical phenomena.
Methods for Solving Trigonometric Equations

Trigonometric functions are used to model real world data so being able to solve these equations is an important skill. Solving trigonometric equations is similar to solving algebraic equations but there are various things you have learned about trigonometric functions that you will need to remember to solve trigonometric equation.

Trigonometric functions are periodic functions which mean there are times an equation may have an infinite number of solutions. There are other times when solutions will be restricted to limited domains. Let’s start with a familiar problem situation.

Sydney is riding a Ferris wheel that has a diameter of 40 meters. The wheel revolves at a rate of 1.5 revolutions per minute. Sydney’s height about the ground, h, after t minutes can be modeled be the equation \( h = 21 - 30 \cos 3\pi t \). How long after she starts riding will she be 31 meters above the ground?

We need to solve the equation \( h = 21 - 30 \cos 3\pi t \) for \( h=31 \).

Solving this equation begins with basic algebra.

\[
\begin{align*}
h &= 21 - 30 \cos 3\pi t \\
31 &= 21 - 30 \cos 3\pi t \\
10 &= -30 \cos 3\pi t \\
-\frac{1}{3} &= \cos 3\pi t
\end{align*}
\]

1. Replace \( 3\pi t \) with \( \theta \).

2. Think back to the unit circle. When is \( \cos \theta = -\frac{1}{3} \)?

3. Set these values equal to \( 3\pi t \) to solve for \( t \).

4. You should get \( t = \frac{2}{9} + \frac{2}{3}k \) or \( t = \frac{4}{9} + \frac{2}{3}k \) for your solution. Why are there multiple solutions to this problem? Does it fit the situation to have more than one answer?

5. Go back to the problem.

How long after she starts riding will her she be 31 meters above the ground?

Use the solution for \( t \) in problem #4 (or your solution) to answer this question.

6. At what other times was Sydney 31 meters above the ground?
7. If Sydney rides for about 3 minutes, how many times will she be exactly 31 meters above the ground?

8. Solving these problems require a method similar to what you just did. Pay close attention to the indicated domain for your solutions.

| a.   | \(2\sqrt{3} \sin \theta = 3\) for principal values of \(\theta\) |
| b.   | \(2 = \sqrt{2} \sec \theta\) for \(0 \leq x \leq 2\pi\) |
| c.   | \(\cos 3x + 1 = 0\) for \(0^\circ \leq x \leq 360^\circ\) |
| d.   | \(\cot 2x - \sqrt{3} = 0\) for all real values of \(x\) |

Other types of trigonometric equations arise that can be solved looking at the unit circle.

9. Solve \(2 \cos \theta + 1 < 0\) for \(0 \leq \theta \leq 2\pi\) (Use a unit circle when solving.)
The following problems require use of trigonometric identities and relationship.

Example: Solve \( \sec^2(x) + \cos^2(x) = 2 \) for \( 0^\circ \leq x \leq 360^\circ \)

\[
\sec^2(x) + \cos^2(x) = 2
\]
Replace \( \sec^2(x) \) with \( 1 + \tan^2(x) \) and replace \( \cos^2(x) \) with \( 1 - \sin^2(x) \)

\[
1 + \tan^2(x) + 1 - \sin^2(x) = 2
\]
Simplify sides

\[
2 + \tan^2(x) - \sin^2(x) = 2
\]
Subtract 2 from each side

\[
\tan^2(x) - \sin^2(x) = 0
\]
Add \( \sin^2(x) \) to both sides

\[
\tan^2(x) = \sin^2(x)
\]
Use definition \( \tan = \sin/\cos \)

\[
\frac{\sin^2(x)}{\cos^2(x)} = \sin^2(x)
\]
Simplify

\[
\cos(x) = \pm 1
\]
What angle(s) has a cos of +1 or -1

\[
x = 0^\circ, 90^\circ
\]

This problem requires the use of identities and algebraic factoring.

Solve: \( 2\cos^2(x) + 3\sin(x) = 3 \) for \( 0 < x < \pi \).

\[
2\cos^2(x) + 3\sin(x) = 3
\]
Subtract 3 to set equal to zero.

\[
2\cos^2(x) + 3\sin(x) - 3 = 0
\]
We need to get everything in terms of \( \sin(x) \) or everything in terms of \( \cos(x) \). Since we know that \( \cos^2(x) = 1 - \sin^2(x) \):

\[
2(1 - \sin^2(x)) + 3\sin(x) - 3 = 0
\]
Distribute 2

\[
2 - 2\sin^2(x) + 3\sin(x) - 3 = 0
\]
Simplify

\[
-2\sin^2(x) + 3\sin(x) - 1 = 0
\]
Multiply through by -1 so first term is positive

\[
2\sin^2(x) - 3\sin(x) + 1 = 0
\]
Factor

\[
(2\sin(x) - 1)(\sin(x) - 1) = 0
\]
Set factors equal to zero

\[
\sin x = \frac{1}{2} \text{ or } \sin x = 1
\]
Think back to the unit circle.

\[
x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{\pi}{2}
\]

10. Solve \( \sin(x)\cos(x) - \frac{1}{2} \cos x = 0 \) for principal values of \( x \).

11. Solve \( 2 \sec^2(x) - \tan^4 = -1 \) for all real values of \( x \)
Some equations are difficult to solve with only algebraic methods. Calculators can be valuable tools in solving these equations.

12. What are some strategies that could be used to solve \( \sin x = 4 \cos x \) for \( 0 < x < 2\pi \)?

13. Solve \( \tan 2x = \sin x \) for \( 0 < x < \pi \).
   a. How many solutions are there?
   b. What method did you use to find the solutions?
   c. Explain 3 ways you can use your calculator to find the solutions.

Let’s finish with one more application problem. The range of a projectile that is launched with an initial velocity \( v \) at an angle of \( \theta \) with the horizon is given by \( R = \frac{v^2}{g} \sin 2\theta \), where \( g \) is the acceleration due to gravity or 9.8 meters per second squared.

14. If a projectile is launched with an initial velocity of 25 meters/second what angle is required to reach a range of 30 meters?

15. What if the initial velocity drops to 15 meters/second? At what angle will the range be 30 meters?

16. What if the initial velocity increases to 50 meters/second? At what angle will the range be 25 meters?

17. For what initial velocity is the maximum range 30 meters?
Law of Sines and Cosines Learning Task

During a baseball game an outfielder caught a ball hit to dead center field, 400 feet from home plate. If the distance from home base to first base is 90 feet, how far does the outfielder have to throw the ball to get it to first base?

1. Draw a picture of the problem above.

2. What information do we know? What do we not know?

Typically, you have solved triangles that are right triangles. This is a case where we do not have a right triangle to solve. We know two sides and one included angle. The Law of Cosines can be applied in cases where the triangle is not a right triangle.

Consider the triangle to the right. In our problem we know b, a, and the angle at C. Follow these steps to derive a way to solve for c knowing just that much information.

3. Use the Pythagorean Theorem to find \( c^2 \) in terms of \( h, a \) and \( x \). Multiply as needed to get rid of any parentheses.
4. Solve for \( b^2 \). Look for an expression in #3 that is equivalent to \( b^2 \) and replace it with \( b^2 \).

5. Find \( \cos c \) and solve for \( x \).

6. Replace \( x \) in your equation from #4 with your expression from #5 and simplify completely.

Your answer to #6 is one of three formulas that make up the **Law of Cosines**. Each of the formulas can be derived in the same way you derived this one by working with each vertex and the other heights of the triangle.

**Law of Cosines**

Let \( a, b, \) and \( c \) be the lengths of the legs of a triangle opposite angles \( A, B, \) and \( C \). Then,

\[
a^2 = b^2 + c^2 - 2bccosA \\
b^2 = a^2 + c^2 - 2accosB \\
c^2 = a^2 + b^2 - 2abcosC
\]

These formulas can be used to solve for unknown lengths and angles in a triangle.

7. Solve the baseball problem at the beginning of this task.

8. Two airplanes leave an airport, and the angle between their flight paths is 40º. An hour later, one plane has traveled 300 miles while the other has traveled 200 miles. How far apart are the planes at this time?

9. A triangle has sides of 8 and 7 and the angle between these sides is 35º. Solve the triangle. (Find all missing angles and sides.)

10. Three soccer players are practicing on a field. The triangle they create has side lengths of 18, 14, and 15 feet. At what angles are they standing from each other?
11. Is it possible to know two sides of a triangle and the included angle and not be able to solve for the third side?

When using the Law of Cosines it is necessary to know the angle included between two sides. However, there are times the angle that is known is not the angle included between two known sides. And there are other times where we might know two angles and only one side. In both of these cases the Law of Cosines cannot be used. In those cases you can use the Law of Sines.

Law of Sines
Let a, b, and c be the lengths of the legs of a triangle opposite angles A, B, and C. Then,

\[
\begin{align*}
\frac{a}{\sin A} &= \frac{b}{\sin B} &= \frac{c}{\sin C}
\end{align*}
\]

In the triangle to the right the measurements for angle B and A are missing and the length of AC. Follow the steps below to solve for the measure using the Law of Sines.

12. This is the information that we know so far.

\[
\frac{6.7}{\sin 33^\circ} = \frac{5.4}{\sin A} = \frac{b}{\sin B}
\]

Solve the first two ratios to find A.

13. Find angle B.

14. Use either the first or second ratio with the last ratio to solve for side b.

15. Draw a picture and solve the following problem.

A surveyor is near a river and wants to calculate the distance across the river. He measures the angle between his observations of two points on the shore, one on his side and one on the other side, to be 28°. The distance between him and the point on his side of the river can be measured and is 300 feet. The angle formed by him, the point on his side of the river, and the point directly on the opposite side of the river is 128°. What is the distance across the river?
From Accelerated Mathematics I, you know that the measures of two sides and a non-included angle will not necessarily work together to create a triangle. The Hinge Theorem is a geometric theorem that focuses on this idea. You may have explored this idea in GPS Geometry when you studied congruent triangles.

Consider the two triangles below. Given sides of 7 cm and 4.2 cm with a non-included angle of $30^\circ$, there are two triangles that can be created. This is why angle-side-side is not a congruency theorem for triangles.

In trigonometry we consider this to be the ambiguous case for solving triangles.

It is also possible to have only one solution. What if side $a$ is 3.5 cm?

Now we have a right triangle. In this case there is only one solution.

16. What would happen if side $a$ is less than 3.5 cm?

17. Order the sides of the triangle to the right from longest to shortest. (The figure is NOT drawn to scale.)

18. How did you decide the order of the sides?
19. Sketch the information given and decide if it is possible to create a triangle with the given information. Explain your answer.

Triangle 1: \( A = 40^\circ, a = 8 \text{ cm and } b = 5 \text{ cm.} \)

Triangle 2: \( A = 150^\circ, a = 5 \text{ cm and } b = 8\text{cm} \)

Be sure to take time to analyze the data you are given when you are solving a triangle to determine if you might have a set of data that will not yield a triangle. This could save you a lot of work!

Remember, if the triangle has 2 solutions, that means there are 2 possible angles for \( b \) when you take the inverse sin.

20. Solve the following triangle. Determine if there is no solution, one solution or two solutions.

Measure of angle \( a = 35^\circ, a = 10 \text{ cm and } b = 16 \text{ cm} \)

21. A ship traveled 60 miles due east and then adjusted its course 15 degrees northward. After traveling for a while the ship turned back towards port. The ship arrived in port 139 miles later. How far did the ship travel on the second leg of the journey? What angle did the ship turn through when it headed back to port?
Finding a New Area Formula for Triangles Learning Task

A developer needs to find the area of some plots of land he is interested in buying. Each plot is owned by a different person and neither owner knows the actual area of the land. The diagram below illustrates the plots he wants to buy but he wants to know the area before buying it. Fortunately, he knows a great math student named Valerie and he hopes she will be able to help him out.

Here are the steps Valerie used to find the area. Work through the steps and see what you think about her method.

1. The usual formula for the area of a triangle is \( \frac{1}{2} bh \) where the height is perpendicular to the base. But there are no perpendicular segments in the diagram to be used as the height.

   Looking at the diagram, she decided to drop her heights from point C and point D. Sketch these heights on the diagram to the right.

2. Knowing basic trigonometry came in handy at this point and she quickly determined the lengths of her heights. Calculate the heights now.

3. Now that she had a base and a height she quickly calculated the areas. Use your calculated heights to find the area of the land.
4. After working on this problem, she started wondering if it would be possible to use this method on any triangle. She noticed that she had started with a triangle in which she knew two sides and the included angle. She drew the diagram below to represent a general triangle.

Working as if she know the measure of angle B and sides c and a, she sketched in the altitude from angle C.

Sketch the altitude of the triangle from angle C and calculate its length in terms of a and B.

Using the calculated height and the two known sides, Valerie concluded that she had a formula that would find the area of any triangle given two sides and the included angle.

Valerie’s formula: \[ \text{Area} = \frac{1}{2}ac\sin B \]

Does your work agree with her final answer? Explain.

5. Wanda worked as if she knew angle A and sides b and c. She drew her altitude from C. Her formula was different from Valerie’s formula. Determine if Wanda’s method works. How is her formula similar to Valerie’s?

6. Would it be possible to develop a third formula for the area? If so, find it. If not, explain why not.

Optional Problem:

7. There is another formula that is based on knowing the angles in a triangle and one side measure. Use the Law of Sines to derive the formula.
Vectors in the City Learning Task

Amy is spending some time in a city that is laid out in square blocks. The blocks make it very easy to get around so most directions are given in terms of the number of blocks you need to walk and the direction you need to go. Amy had a series of places she wanted to go to when she left the hotel on Monday morning. The concierge had helped her map out the directions. The directions looked like this:

<table>
<thead>
<tr>
<th>Stop 1</th>
<th>3 East</th>
<th>5 North</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stop 2</td>
<td>5 East</td>
<td>2 North</td>
</tr>
<tr>
<td>Stop 3</td>
<td>2 East</td>
<td>1 North</td>
</tr>
</tbody>
</table>

1. Use a piece of graph paper and record Amy’s trip. Place her hotel at the origin.

2. How many total blocks East did Amy walk?

3. How many total blocks North did Amy walk?

4. Add directions to the chart for Amy to walk so she can get back to the hotel in only one turn.

5. A very tired Amy got back to the hotel and wished she could have walked directly back to the hotel from her last stop. How much shorter would it have been if Amy could have walked directly back?

6. The next morning she was ready to go again. She had even more stops this time. On Tuesday her directions looked like this.

<table>
<thead>
<tr>
<th>Stop 1</th>
<th>2 West</th>
<th>0 North</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stop 2</td>
<td>0 West</td>
<td>4 South</td>
</tr>
<tr>
<td>Stop 3</td>
<td>3 West</td>
<td>1 South</td>
</tr>
<tr>
<td>Stop 4</td>
<td>5 West</td>
<td>1 South</td>
</tr>
<tr>
<td>Stop 5</td>
<td>0 West</td>
<td>2 South</td>
</tr>
</tbody>
</table>
7. Amy walked back to the hotel only making one turn. She was surprised that it took her the same amount of time to walk back as it had on Monday. Add her last set of directions to return to hotel to the chart above.

Is there a reason that it took her the same amount of time to walk back?

8. Was her shortest distance back to the hotel the same on Monday and Tuesday?

9. On Wednesday her path was going to be taking her all over the place. Map out her directions on a grid.

<table>
<thead>
<tr>
<th>Stop 1</th>
<th>2 East</th>
<th>3 North</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stop 2</td>
<td>4 West</td>
<td>2 North</td>
</tr>
<tr>
<td>Stop 3</td>
<td>3 East</td>
<td>1 South</td>
</tr>
<tr>
<td>Stop 4</td>
<td>5 SOUTHWEST*</td>
<td></td>
</tr>
<tr>
<td>Stop 5</td>
<td>3 East</td>
<td>1 South</td>
</tr>
</tbody>
</table>

*go further west than south

10. Amy was amazed to find herself back at the hotel when she finished on Wednesday. Did you end up back at the hotel on your grid? If not, go back and check your SouthWest direction. Think back to the Pythagorean Theorem to determine where she ends up in 5 blocks.

11. Look back at the directions for Monday and Tuesday. What do you notice about the sums of the blocks as she made her stops and the directions you added to get her back to the hotel in just one turn.

12. What happens if you add the blocks she walked on Wednesday? Do you get zero? What if you replace the 5 blocks southwest with 3 blocks south and 4 blocks west? What do you need to do about the signs of the numbers to get it to sum to zero?

13. How are the directions North, South, East and West related to the coordinate plane?


15. What would NNE mean? What would it look like on your coordinate plane? Can you identify at least two coordinates that lie on a line pointed in the NNE direction?
If Amy were not walking around a city where buildings did not block her path, it would be a shorter walk for her if she could walk directly to the position indicated by the two directions given to her at the hotel.

16. Determine a single set of directions that could get Amy to a point that is 3 blocks East and 4 blocks North of her hotel. Note that her direction is not exactly NE. Determine another way to define her direction from the hotel.

17. Did you determine a distance and a direction for Amy to walk? How did you describe her direction?

In mathematics we use directed line segments, or vectors, to indicate a magnitude (length or distance) and a direction.

In Amy’s situation, using the Pythagorean Theorem helps us find the magnitude of 5. To describe the direction, we use the angle the vector makes with the x-axis. The angles are measured in the same way angles are measured on the unit circle: counter-clockwise is positive, counter clockwise is negative.

18. The specific direction Amy traveled can be found using \( \theta = \tan^{-1} \frac{4}{3} \). Is that what you did? If not, do it now and determine the direction she was traveling.

The length of the directed line segment is used to represent the magnitude of a vector. The direction the segment points in is indicative of the direction of the vector.

19. Look at the vectors below. Describe them in your own words.

![Vectors](image)

The directions given to Amy were the horizontal and vertical components of a vector. It is possible to determine the magnitude and direction of the vector using the components of the vector.

20. A pilot traveled due East for 3 miles and then turned due North for 8 miles. Write a single vector to describe his distance and direction from his base.
At other times we want to take a single vector apart and look at its components. This is possible using geometry and trigonometry.

21. A ship leaves port and sails 58 miles in a direction 48° North of due East. Find the magnitude of the vertical and horizontal components. (The drawing should help.)

22. A plane flies 150 miles 38° south of due West. Draw a diagram of the flight and determine the horizontal and vertical components of the flight.

Often, there are two vectors that combine to describe a third vector. The sum of two or more vectors is called the resultant vector. There are several ways to find the resultant vector.

23. Draw a diagram to represent the following problem. A ship leaves port and travels 49 miles at a direction of 30°. The ship then turns an additional 40° north of due east and travels for 89 miles. At that point the ship drops anchor. A helicopter needs to join the ship. If the helicopter is leaving the same port, what vector should be reported to the pilot that describes a direct path to the plane? (Hint: Draw a diagram. Create two right triangles to find the total horizontal and vertical distances.)

24. Vectors are also used to represent forces in physics. Two forces can act together or against each other. Either way, one force will have a definite effect when combined with another force.
   a. Imagine two people pulling on your arms with the same amount of force, both in a northerly direction (picture yourself facing north). What effect would they have?
   b. What if one person pulls towards the north while the other pulls towards the south?
   c. What if the person pulling towards the north is stronger than the one pulling south? How would that effect you?
When two or more vectors are added together, the resultant vector can be found by summing the horizontal and vertical components of each vector.

25. You jump into a river intending to swim straight across to the other side. When you started swimming you realized the current was stronger than you expected. In fact, the current was pushing you directly south at 4 miles/hour. You were swimming directly East at 1 mile/hour an hour. If you do not change your direction, where will you land when you touch the other side 15 minutes later?

26. Since you do not want to land so far downriver you decide to swim at a direction of 35° north of due east. How far downstream will you now land?

27. A plane traveling at 400 mph is flying with a bearing of 40°. There is a wind of 50 mph from the South. If no correction is made for the wind, what are the final bearing and ground speed of the plane?

28. Vectors can also be represented algebraically using ordered pairs. If vector \( \mathbf{w} \) starts at the origin and ends at the point \(-4, 6\), determine the magnitude and direction of the vector. Vector \( \mathbf{w} \) is written as \( \mathbf{w} = (-4, 6) \).

29. If adding vectors written algebraically, add each of the corresponding terms to get the resultant vector. Similarly, if you are to subtract the vector, subtract the corresponding terms. Multiplying a vector is called scalar multiplication. Scalar multiplication multiplies each part of the vector.

Let \( \mathbf{w} = (-2, 3), \mathbf{a} = (4, 10), \mathbf{m} = (-4.2, -8), \mathbf{k} = (0.5, 22) \)

\[
\mathbf{w} + \mathbf{k} = (-2 + 0.5, 3 + 22) = (-1.5, 25) \\
\mathbf{a} - \mathbf{m} = (4 - (-4.2), 10 - (-8)) = (0.2, 18) \\
3\mathbf{w} + 2\mathbf{a} = 3(-2, 3) + 2(4, 10) = (-6, 9) + (8, 20) = (2, 29)
\]

Complete the following:

\[
4\mathbf{a} - 5\mathbf{k} + \mathbf{k} = (-14, -48)
\]

An interactive website to use with vectors is: