Mathematics III
Frameworks
Student Edition

Unit 1
Matrices

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INTRODUCTION:

In previous mathematics courses students have used continuous functions to deal with a wide variety of data. In this unit the approach to data is discrete. Matrices allow students to store and retrieve data easily. Data arranged in matrices can be manipulated as a single entity while still being maintained as individual values. Students usually find matrix algebra operations to be very appealing since most operations can be done with a variety of calculators and/or computer programs. The tasks in this unit are designed to introduce matrix algebra and to provide practical applications for matrix transposes, determinants, inverses, and powers.

ENDURING UNDERSTANDINGS:

- Matrices are used to store and operate with data.
- Properties of matrices are used when operating with data.
- Matrices are used to represent and solve problems.

KEY STANDARDS ADDRESSED:

MM3A4. Students will perform basic operations with matrices.

a. Add, subtract, multiply, and invert matrices, when possible, choosing appropriate methods, including technology.
b. Find the inverses of two-by-two matrices using pencil and paper, and find inverses of larger matrices using technology.
c. Examine the properties of matrices, contrasting them with properties of real numbers.

MM3A5. Students will use matrices to formulate and solve problems.

a. Represent a system of linear equations as a matrix equation.
b. Solve matrix equations using inverse matrices.
c. Represent and solve realistic problems using systems of linear equations.

MM3A6. Students will solve linear programming problems in two variables.

a. Solve systems of inequalities in two variables, showing the solutions graphically.
b. Represent and solve realistic problems using linear programming.

MM3A7. Students will understand and apply matrix representations of vertex-edge graphs.

a. Use graphs to represent realistic situations.
b. Use matrices to represent graphs, and solve problems that can be represented by graphs.
RELATED STANDARDS ADDRESSED:

**MM3P1. Students will solve problems (using appropriate technology).**

a. Build new mathematical knowledge through problem solving.
b. Solve problems that arise in mathematics and in other contexts.
c. Apply and adapt a variety of appropriate strategies to solve problems.
d. Monitor and reflect on the process of mathematical problem solving.

**MM3P2. Students will reason and evaluate mathematical arguments.**

a. Recognize reasoning and proof as fundamental aspects of mathematics.
b. Make and investigate mathematical conjectures.
c. Develop and evaluate mathematical arguments and proofs.
d. Select and use various types of reasoning and methods of proof.

**MM3P3. Students will communicate mathematically.**

a. Organize and consolidate their mathematical thinking through communication.
b. Communicate their mathematical thinking coherently and clearly to peers, teachers, and others.
c. Analyze and evaluate the mathematical thinking and strategies of others.
d. Use the language of mathematics to express mathematical ideas precisely.

**MM3P4. Students will make connections among mathematical ideas and to other disciplines.**

a. Recognize and use connections among mathematical ideas.
b. Understand how mathematical ideas interconnect and build on one another to produce a coherent whole.
c. Recognize and apply mathematics in contexts outside of mathematics.

**MM3P5. Students will represent mathematics in multiple ways.**

a. Create and use representations to organize, record, and communicate mathematical ideas.
b. Select, apply, and translate among mathematical representations to solve problems.
c. Use representations to model and interpret physical, social, and mathematical phenomena.
**Unit Overview:**

A matrix is an organized rectangular array of numbers and a determinant is a value associated with a square matrix. The earliest known example of numbers being placed in an array in order to aid analysis is clay tablets from Babylon of the fourth century BC. Matrices are tools for organizing and storing information. Management of information such as cost of materials, time to produce each item, available inventory by month, needed inventory, and so forth provide material to create a variety of matrices, develop definitions, and basic matrix operations. Dimensions and dimension labels are used to provide rationale for addition and multiplication procedures. Emphasis is placed on interpreting entries as matrices are written, added, multiplied (scalar and regular) and transposed. Properties of real numbers are examined to determine which properties are true in matrix operations. Students find inverses and determinants by hand and by using technology. Determinants are used to find areas of triangles and test for collinear points. Systems of equations are written as matrix equations and inverses of coefficient matrices are used to solve the systems. Students also learn how to find equations of lines passing through two given points by solving for coefficients in system of two equations in the form $y = ax + b$ and equations of parabolas passing through three given points by solving for coefficients in a system of three equations written in the form $y = ax^2 + bx + c$. This task includes a lab where students solve a system of three equations to determine what is hidden in a sealed lunch bag. Lastly, the unit task uses vertex-edge digraphs to study various group interactions including food webs and gossiping. Predator prey graphs and powers of related adjacency matrices are interpreted to determine the effects of environmental changes on a food web.
Central High School Booster Club Learning Task:

In order to raise money for the school, the Central High School Booster Club offered spirit items prepared by members for sale at the school store and at games. They sold stuffed teddy bears dressed in school colors, tote bags and tee shirts with specially sewn and decorated school insignias. The teddy bears, tote bags, and tee shirts were purchased from wholesale suppliers and decorations were cut, sewn and painted, and attached to the items by booster club parents. The wholesale cost for each teddy bear was $4.00, each tote bag was $3.50 and each tee shirt was $3.25. Materials for the decorations cost $1.25 for the bears, $0.90 for the tote bags and $1.05 for the tee shirts. Parents estimated the time necessary to complete a bear was 15 minutes to cut out the clothes, 20 minutes to sew the outfits, and 5 minutes to dress the bears. A tote bag required 10 minutes to cut the materials, 15 minutes to sew and 10 minutes to glue the designs on the bag. Tee shirts were made using computer generated transfer designs for each sport which took 5 minutes to print out, 6 minutes to iron on the shirts, and 20 minutes to paint on extra detailing.

The booster club parents made spirit items at three different work meetings and produced 30 bears, 30 tote bags, and 45 tee shirts at the first session. Fifteen bears, 25 tote bags, and 30 tee shirts were made during the second meeting; and, 30 bears, 35 tote bags and 75 tee shirts were made at the third session. They sold the bears for $12.00 each, the tote bags for $10.00 each and the tee shirts for $10.00 each. In the first month of school, 10 bears, 15 tote bags, and 50 tee shirts were sold at the bookstore. During the same time period, Booster Club members sold 50 bears, 20 tote bags, and 100 tee shirts at the games.
The following is a **matrix**, a rectangular array of values, showing the wholesale cost of each item as well as the cost of decorations. "wholesale" and "decorations" are labels for the matrix **rows** and "bears", "totes", and "shirts" are labels for the matrix **columns**. The **dimensions of this matrix** called \( A \) are 2 rows and 3 columns and matrix \( A \) is referred to as a \([2 \times 3]\) **matrix**. Each number in the matrix is called an **entry**.

\[
A = \begin{bmatrix}
4.00 & 3.50 & 3.25 \\
1.25 & .90 & 1.05
\end{bmatrix}
\]

It is sometimes convenient to write matrices (plural of matrix) in a simplified format without labels for the rows and columns. Matrix \( A \) can be written as an array.

\[
A = \begin{bmatrix}
4.00 & 3.50 & 3.25 \\
1.25 & .90 & 1.05
\end{bmatrix}
\]

where the values can be identified as \( A = \begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23}
\end{bmatrix} \). In this system, the entry \( a_{22} = .90 \), which is the cost of decorations for tote bags.

1. Write and label matrices for the information given on the Central High School Booster Club's spirit project.

   a. Let matrix \( B \) show the information given on the time necessary to complete each task for each item. Labels for making the items should be **cut/print**, **sew/iron**, and **dress/decorate**.

   b. Find matrix \( C \) to show the numbers of bears, totes, and shirts produced at each of the three meetings.

   c. Matrix \( D \) should contain the information on items sold at the bookstore and at the game.

   d. Let matrix \( E \) show the sales prices of the three items.

2. Matrices are called **square matrices** when the number of **rows** = the number of **columns**. A matrix with only one row or only one column is called a **row matrix** or a **column matrix**. Are any of the matrices from 1. **square matrices** or **row matrices** or **column matrices**? If so, identify them.

   Since matrices are arrays containing sets of discrete data with dimensions, they have a particular set of rules, or algebra, governing operations such as addition, subtraction, and multiplication. In order to **add two matrices**, the matrices must have the same dimensions. And, if the matrices have row and column labels, these labels must also match. Consider the following problem and matrices.
Several local companies wish to donate spirit items which can be sold along with the items made by the Booster Club at games help raise money for Central High School. J J's Sporting Goods store donates 100 caps and 100 pennants in September and 125 caps and 75 pennants in October. Friendly Fred's Food store donates 105 caps and 125 pennants in September and 110 caps and 100 pennants in October. How many items are available each month from both sources?

To add two matrices, add corresponding entries. Let

\[
J = \begin{bmatrix}
caps & 100 \\
pennants & 100 
\end{bmatrix}, \quad F = \begin{bmatrix}
caps & 125 \\
pennants & 75 
\end{bmatrix}
\]

then

\[
J + F = \begin{bmatrix}
caps & 100+105 \\
pennants & 100+125 
\end{bmatrix}
\]

Subtraction is handled like addition by subtracting corresponding entries.

3. Construct a matrix G with dimensions \([1 \times 3]\) corresponding to production cost per item. Use this new matrix G and matrix E from #1 to find matrix P, the profit the Booster Club can expect from the sale of each bear, tote bag, and tee shirt.

Another type of matrix operation is known as scalar multiplication. A scalar is a single number such as 3 and matrix scalar multiplication is done by multiplying each entry in a matrix by the same scalar.

Let

\[
M = \begin{bmatrix}
-2 & 0 & 5 \\ 1 & -3 & 4 
\end{bmatrix}, \quad M = \begin{bmatrix}
-6 & 0 & 15 \\ 3 & -9 & 12 
\end{bmatrix}
\]

4. Use scalar multiplication to change matrix B (problem #1) from minutes required per item to hours required per item.
Matrices can also be multiplied together. Since each matrix represents an array of data, rules for multiplying them together depends on the position of each entry. Consider the following example.

At the beginning of November a stomach virus hits Central High School. Students in the Freshman and Sophomore classes are either well, a little sick, or really sick. The following tables show Freshmen and Sophomores according to their levels of sickness and their gender.

<table>
<thead>
<tr>
<th>Student Population</th>
<th>% of Sick Students</th>
</tr>
</thead>
<tbody>
<tr>
<td>Categories</td>
<td>Male</td>
</tr>
<tr>
<td>Freshmen</td>
<td>250</td>
</tr>
<tr>
<td>Sophomores</td>
<td>200</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Suppose school personnel needed to prepare a report and include the total numbers of well and sick male Freshmen and Sophomores in the school.

\[
\text{well Freshmen males + well Sophomore males = well males} \\
(.2)(250) + (.25)(200) = 100
\]

\[
\text{a little sick Freshmen males + a little sick Sophomore males = a little sick males} \\
(.5)(250) + (.4)(200) = 205
\]

\[
\text{really sick Freshmen males + really sick Sophomore males = really sick males} \\
(.3)(250) + (.35)(200) = 145
\]

Notice the positions of the values in these products. We are multiplying rows by columns to get the information we want. Translating the tables to matrices and using the rows by columns pattern of multiplication we get the following result.

\[
\begin{bmatrix}
F \\
S
\end{bmatrix}
\begin{bmatrix}
M & F \\
250 & 300 \\
200 & 275
\end{bmatrix}
= \\
\begin{bmatrix}
.2 & .25 \\
.5 & .4 \\
.3 & .35
\end{bmatrix}
\begin{bmatrix}
F \\
M
\end{bmatrix}
= \\
\begin{bmatrix}
.2(250)+(.25)(200) \\
.5(250)+(.4)(200) \\
.3(250)+(.35)(200)
\end{bmatrix} + \\
\begin{bmatrix}
.2(300) + (.25)(275) \\
.5(300) + (.4)(275) \\
.3(300)+(.35)(275)
\end{bmatrix}
= \\
\begin{bmatrix}
100 & 128.75 \\
205 & 260 \\
145 & 186.25
\end{bmatrix}
\]
[level of sickness x class] * [class x gender] = [level of sickness x gender]

\[ [3 \times 2] \times [2 \times 2] = [3 \times 2] \]

This procedure illustrates the multiplication of two matrices. In order to multiply two matrices, the number of columns of the matrix on the left must equal the number of rows of the matrix on the right. Also the labels of the columns of the left matrix must be the same as the labels of the rows of the right matrix. **If the dimensions of two matrices are not appropriately matched, it is not possible to multiply them.**

5. Given the following matrices, find their products if possible.

\[
L = \begin{bmatrix} 1 & 3 \\ -5 & 4 \end{bmatrix} \quad M = \begin{bmatrix} -1 & 2 & 7 & -1 \\ 5 & 4 & 3 & 2 \end{bmatrix} \quad N = \begin{bmatrix} 3 & 0 \\ -2 & 1 \\ 5 & 5 \\ -1 & 2 \\ 6 & 3 \end{bmatrix} \quad T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

\[
S = \begin{bmatrix} well & 60\% & 70\% \\ sick & 40\% & 30\% \end{bmatrix} \quad C = \begin{bmatrix} Jr & 150 & 210 \\ Sr & 100 & 50 \end{bmatrix}
\]

a. LM
b. LN
c. LT
d. MN
e. SC  [Sometimes it is necessary to exchange the rows and columns of a matrix in order to make it possible to multiply. This process is called finding the **transpose** of a matrix and is most useful with labeled matrices.]
6. Using the matrices you wrote in problems 1. and 3. and matrix multiplication, find matrices to show the

a. the amount of profit made at the bookstore and at the games; and,

b. the amount of time (in minutes) it took to perform each task at the three work sessions.

REFERENCES:


Walk Like a Mathematician Learning Task:

Matrices allow us to perform many useful mathematical tasks which ordinarily require large number of computations. Some types of problems which can be done efficiently with matrices include solving systems of equations, finding the area of triangles given the coordinates of the vertices, finding equations for graphs given sets of ordered pairs, and determining information contained in vertex edge graphs. In order to address these types of problems, it is necessary to understand more about matrix operations and properties; and, to use technology to perform some of the computations.

Matrix operations have many of the same properties as real numbers. There are more restrictions on matrices than on real numbers, however, because of the rules governing matrix addition, subtraction, and multiplication. Some of the real number properties which are more useful when considering matrix properties are listed below.

| COMMUTATIVE | a + b = b + a | ab = ba |
| ASSOCIATIVE | (a + b) + c = a + (b + c) | (ab)c = a(bc) |
| IDENTITY | There exists a unique real number zero, 0, such that a + 0 = 0 + a = a | There exists a unique real number one, 1, such that a * 1 = 1 * a = a |
| INVERSE | For each real number a, there is a unique real number - a such that a + (-a) = (-a) + a = 0 | For each nonzero real number a, there is a unique real number \( \frac{1}{a} \) such that \( a\left(\frac{1}{a}\right) = \left(\frac{1}{a}\right)a = 1 \) |
The following is a set of matrices without row and column labels. Use these matrices to complete the problems.

\[
D = \begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix} \quad E = \begin{bmatrix} 2 & 4 & 1 \\ 0 & -1 & 3 \\ 1 & 0 & 2 \end{bmatrix} \quad F = \begin{bmatrix} 5 \\ -2 \end{bmatrix} \quad G = \begin{bmatrix} 0 & -1 \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} \quad H = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad J = \begin{bmatrix} 4 & 2 \\ -1 & 3 \end{bmatrix} \quad K = \begin{bmatrix} 3 & 4 & -1 \\ 0 & -2 & 3 \\ 5 & 1 & 2 \end{bmatrix} \quad L = \begin{bmatrix} 5 & -2 \\ -1 & 0 \end{bmatrix}
\]

1. Are matrix addition and matrix multiplication commutative? What would matrices have to look like in order to be commutative under addition and multiplication?

2. Are matrix addition and matrix multiplication associative?

3. Is there a zero or identity, 0, for addition in matrices? If so, what does a zero matrix look like? Is a zero matrix unique?

4. Do matrices have a one or an identity, I, for multiplication? If so what does an identity matrix look like; is it unique; and, does it satisfy the property \( a * I = I * a = a \)?

5. Consider matrices D and G from the original set of matrices. D and G are inverse matrices. In order for a matrix to have an inverse, it must satisfy two conditions.

   1. The matrix must be a square matrix.
   2. No row of the matrix can be a multiple of any other row.

   Both D and G are 2x2 matrices; and, the rows in D are not multiples of each other. The same is true of G.

   The notation normally used for a matrix and its inverse is \( D \) and \( D^{-1} \) or \( G \) and \( G^{-1} \).

   The product of two inverse matrices should be the identity matrix, I.

   Find \( D * G \) and \( G * D \).
6. The following formula can be used to find the inverse of a 2x2 matrix. Given matrix \( A \) where the rows of \( A \) are not multiples of each other:

\[
A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{then} \quad A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}
\]

For higher order matrices, we will use technology to find inverses.

Find the inverse of matrix \( J \) from the matrices listed above. Verify that \( J \) and \( J^{-1} \) are inverses.

7. Another type of matrix operation is finding the determinant. Only square matrices have determinants. To find determinants of 2x2 matrices by hand use the following procedure.

\[
\text{determinant}(A) = \text{det}(A) = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc
\]

To find determinants of 3x3 matrices use the following procedure: given matrix \( B = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \), rewrite the matrix and repeat columns 1 and 2 to get \( \begin{bmatrix} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{bmatrix} \). Now multiply and combine products according to the following patterns.

\[
\begin{vmatrix} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{vmatrix}
\]

and

\[
\begin{vmatrix} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{vmatrix}
\]

to give \( \text{det}(B) = aei + bfg + cdh - ceg - afh - bdi \).

We will also use expansion by minors and technology to find determinants of matrices.
The determinant of a matrix can be used to find the area of a triangle. If \((x_1, y_1), (x_2, y_2),\) and \((x_3, y_3)\) are vertices of a triangle, the area of the triangle is

\[
\text{Area} = \pm \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.
\]

a. Given a triangle with vertices \((-1, 0), (1, 3),\) and \((5, 0)\), find the area using the determinant formula. Verify that area you found is correct using geometric formulas.

b. Suppose you are finding the area of a triangle with vertices \((-1, -1), (4, 7),\) and \((9, -6)\). You find the area of the triangle to be \(-52.5\) and your partner works the same problem and gets \(+52.5\). After checking both solutions, you each have done your work correctly. How can you explain this discrepancy?

c. Suppose another triangle with vertices \((1, 1), (4, 2),\) and \((7, 3)\) gives an area of \(0\). What do you know about the triangle and the points?

d. A gardener is trying to find a triangular area behind his house that encloses 1750 square feet. He has placed the first two fence posts at \((0, 50)\) and at \((40, 0)\). The final fence post is on the property line at \(y = 100\). Find the point where the gardener can place the final fence post.
Candy? What Candy? Do We Get to Eat It? Learning Task:

Suppose you walked into class one day and found a big stack of sealed lunch bags full of candy on a table just waiting for you to rip them open and devour their chocolaty contents. But, you could not even touch them until you figured out how many pieces of each brand of candy was contained in each bag. Well, today is the day. Each group of three gets one bag which must remain unopened until you can tell how many pieces of each type of candy W, X, Y, or Z. Each bag holds 3 different types of candy and a total of 9 pieces of candy.

Your task is to determine exactly what is in your bag by writing a system of equations and solving that system using matrices.

A system of equations such as \( \begin{cases} ax + by = c \\ dx + ey = f \end{cases} \) can be written as a matrix equation where

\[
\begin{bmatrix} a & b \\ d & e \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c \\ f \end{bmatrix} \quad \text{or} \quad CV = A
\]

where \( C \) is the coefficient matrix, \( V \) is the variables matrix, and \( A \) is the answers matrix. The variables matrix can be isolated by multiplying each side of the equation by the inverse of \( C \).

\[
CV = A \\
C^{-1}CV = C^{-1}A \\
V = C^{-1}A
\]

To solve a system of equations such as \( \begin{cases} 5x - y = 7 \\ 2x + 3y = -1 \end{cases} \) write the matrix equation.

\[
\begin{bmatrix} 5 & -1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \end{bmatrix}
\]

\[
\frac{1}{17} \begin{bmatrix} 3 & 5 & -1 \\ -2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{17} \begin{bmatrix} 3 & 1 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 7 \\ -1 \end{bmatrix}
\]

\[
\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{17} \begin{bmatrix} 3 & 1 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 7 \\ -1 \end{bmatrix}
\]

\[
x = \frac{20}{17} \quad \text{and} \quad y = -\frac{19}{17}
\]
Solving systems of equations of higher order can be accomplished using a similar format and a TI-84 graphing calculator to find the inverse of the coefficient matrix and find the necessary products.

\[
\begin{align*}
\begin{cases}
 x - 2y + 3z &= 3 \\
 2x + y + 5z &= 8 \\
 3x - y - 3z &= -22
\end{cases}
\end{align*}
\]

To solve the system write the coefficient matrix \( C = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & 5 \\ 3 & -1 & -3 \end{bmatrix} \) and the answers matrix \( A = \begin{bmatrix} 3 \\ 8 \\ -22 \end{bmatrix} \). Enter \( C \) and \( A \) into the TI-84.

\[
[C] = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & 5 \\ 3 & -1 & -3 \end{bmatrix} \quad [H] = \begin{bmatrix} 3 \\ 8 \\ -22 \end{bmatrix} \quad [C]^{-1} \ast [H] = \begin{bmatrix} -4 \\ 1 \\ 3 \end{bmatrix}
\]

1. For the following systems of equations, write the matrix equation and solve for the variables.

   a. \( 2x + 3y = 2 \) \\
   \( 4x - 9y = -1 \)

   b. \( 9x - 7y = 5 \) \\
   \( 10x + 3y = -16 \)

   c. \( 5x - 4y + 3z = 15 \) \\
   \( 6x + 2y + 9z = 13 \) \\
   \( 7x + 6y - 6z = 6 \)

2. Systems of equations can be used to write an equation for a graph given coordinates of points on the graph. Suppose you are asked to find an equation of a line passing through the points (2, -5) and (-1, -4). Knowing the graphing form of the equation of a line is \( y = mx + b \), you can form a system of equations \( \begin{cases}
 -5 = 2m + b \\
 -4 = -1m + b
\end{cases} \). Solving this system gives \( m = -\frac{1}{3} \) and \( b = -4 \frac{1}{3} \) and an equation of \( y = -\frac{1}{3}x - 4 \frac{1}{3} \).
Consider the following graph. The parabola passes through the points (-3, 5), (1, 1), and (2, 10). Write a system of equations for this graph. Solve the system and write an equation for the parabola. Justify your answer.

Choose values from the Nutrition Chart and the totals given on the card attached to your bag and write a system of equations that describes the information regarding the snacks in your group’s paper bag. List your equations below. Solve this system of equations using matrices. Explain how you determined how many pieces of each type of candy are in your bag.

1. _____________________________
2. _____________________________
3. _____________________________

Suppose you are working with a younger group of students and want to do this activity using two types of candy instead of three. Explain how you could use the nutrition information and matrix multiplication to label your bags of candy for other students.
AN OKEFENOKEE FOOD WEB  Learning Task:

Recent weather conditions have caused a dramatic increase in the insect population of the Okfenokee Swamp area. The insects are annoying to people and animals and health officials are concerned there will be an increase in disease. Local authorities want to use an insecticide that would literally wipe out the entire insect population of the area. You, as an employee of the Environmental Protection Agency, must determine how detrimental this would be to the environment. Specifically, you are concerned on the effects on the food web of six animals known to populate the swamp.

Consider the following digraph of a food web for the six animals and the insects that are causing the problem.

A **digraph** is a directed vertex edge graph. Here each vertex represents an animal or insects. The direction of the edges indicates whether an animal preys on the linked animal. For example, raccoons eat fish. (Note: the food web shown is simplified. Initial producers of nutrients, plants, have not been included.)
Adjacency matrices can be used in conjunction with digraphs. If we consider just the relationships between raccoons, fish, and frogs in the food web shown, an adjacency matrix would be

\[
\begin{pmatrix}
R & FT & F \\
R & [0 & 1 & 1] \\
FT & 0 & 0 & 0 \\
F & 0 & 1 & 1
\end{pmatrix}
\]

1. Construct the associated matrix F to represent this web. What does a row containing a single one indicate? What does a column of zeros indicate?

2. Which animals have the most direct sources of food? How can this be determined from the matrix? Find the number of direct food sources for each animal.

3. The insect column has the most ones. What does this suggest about the food web?

4. The matrix F^2 denotes indirect (through one intermediary) sources of food. For example, the fish relies on insects for food, and the bear relies on the fish for food, so the insect is an indirect source of food for the bear. Find F^2. Notice that insect column contains all nonzero numbers. What does this indicate?

5. Compute additional powers of the food web matrix to represent the number of direct and indirect sources of food for each animal. Which animal has the most food sources?

6. Construct a new matrix G to represent the food web with no insects. What effect does this have on the overall animal population? What has happened to the row sums? Compare these with those of the original matrix. What does a row sum of zero indicate?

7. Will all the animals be affected by the insecticide? Which animal(s) will be least affected?

8. Organize and summarize your findings in a brief report to the health officials. Take and support a position on whether using an insecticide to destroy the insect population is harmful to the environment.
REFERENCES:


The Skateboard Learning Task

SK8MAN, Inc., manufactures and sells skateboards. A skateboard is made of a deck, two trucks that hold the wheels (see Figure 1.2.1), four wheels, and a piece of grip tape. SK8MAN manufactures the decks of skateboards in its own factory and purchases the rest of the components. Currently, SK8MAN manufactures two types of skateboards: Sporty (Figure 1.2.2, top) and Fancy (Figure 1.2.2, bottom).

G. F. Hurley, the production manager at SK8MAN, needs to decide the production rate for each type of skateboard in order to make the most profit. Each Sporty board earns $15 profit, and each Fancy board earns $35 profit. However, Mr. Hurley might not be able to produce as many boards of either style as he would like, because some of the necessary resources are limited. We say that the production rates are constrained by the availability of the resources.

To produce a skateboard deck, the wood must be glued and pressed, then shaped. After a deck has been produced, the trucks and wheels are added to the deck to complete a skateboard.

Skateboard decks are made of either North American maple or Chinese maple. A large piece of maple wood is peeled into very thin layers called veneers. A total of seven veneers are glued at a gluing machine and then placed in a hydraulic press for a period of time (See Figure 1.2.3). After the glued veneers are removed from the press, eight holes are drilled for the truck mounts. Then the new deck goes into a series of shaping, sanding, and painting processes. Figure 1.2.4 shows a deck during the shaping process.

The Sporty board is a less expensive product, because its quality is not as good as the Fancy board. Chinese maple is used in the manufacture of Sporty decks. North American maple is used for
Fancy decks. Because Chinese maple is soft, it is easier to shape. On average, it takes a worker 5 minutes to shape a Sporty board. However, a Fancy board requires 15 minutes to shape. SK8MAN, Inc., is open for 8 hours a day, 5 days a week.

To make decisions in order to maximize the company’s weekly profit, operations researchers use a technique known as *linear programming*. Answering the following questions will help you understand this technique.

One way to approach the problem is to make some guesses and test the profit generated by each guess. Suppose SK8MAN decides the company should make 200 of each model per week.

1. How much profit would be generated?

2. Is there enough available time to shape that number of each model?

3. Answer the same two questions if SK8MAN decides to make:
   a. 50 Sporty and 350 Fancy
   b. 350 Sporty and 50 Fancy

4. Can you find a production mix for which there is enough shaping time?

5. How much profit do the production rates you found generate for the company?

The first step in the formulation of a linear programming problem is to define the *decision variables* in the problem. The decision variables are then used to define the *objective function*. This function captures the goal in the problem. In the SK8MAN problem, the goal is to maximize the company’s profits per week. Therefore, the objective function should represent the weekly profit. That profit comes from the sale of the two different styles of skateboards, so we may begin the problem formulation with:

Let: $x_1$ = the weekly production rate of Sporty boards,
$x_2$ = the weekly production rate of Fancy boards, and
$z$ = the amount of profit SK8MAN, Inc earns per week.

Now, since we know the profit for each style of skateboard, we can write the objective function by expressing $z$ in terms of $x_1$ and $x_2$ is?

The last step in the formulation of the problem is to represent any constraints in terms of the decision variables. In this case, Mr. Hurley cannot just decide to make as many boards as he wants,
because the number made is constrained by the available shaping time. Eight hours per day, five days per week is a 40-hour workweek. However, the information about shaping time is expressed in minutes, so we convert 40 hours to 2,400 minutes. Now, if SK8MAN makes $x_1$ Sportys and $x_2$ Fancys per week, that uses __________ minutes of shaping time. Thus, the shaping time constraint is ______________.

There are also two not-so-obvious, but completely logical constraints. Since the production rate cannot be a negative number for either type of skateboard, $x_1 \geq 0$ and $x_2 \geq 0$. We call these last two inequalities the non-negativity constraints.

Finally, the problem formulation written out all together looks like this:

Let: $x_1 =$ the weekly production rate of Sporty boards,  
$x_2 =$ the weekly production rate of Fancy boards, and  
$z =$ the amount of profit SK8MAN, Inc earns per week.

Maximize: $z = 15x_1 + 35x_2$,  
subject to: $5x_1 + 15x_2 \leq 2400$ and $x_1, x_2 \geq 0$.

Now that the problem has been formulated, we will solve it graphically. To do so, we set up a coordinate plane with $x_1$ as the horizontal axis and $x_2$ as the vertical axis. Then, we graph the constraints, as shown in Figure 1.2.5.

Every point in the shaded region satisfies all three constraints. Thus, every point in the shaded region is a solution to the system of linear inequalities, and there are an infinite number of points. This shaded region is called the feasible region. Each ordered pair in the feasible region represents a combination of Fancys and Sportys that SK8MAN, Inc., could produce without violating any of the constraints. The solution to our problem of maximizing profit requires us to pick the one mix of products that will generate the most profit, our objective function.
If $5x_1 + 15x_2 = 2400$, then $x = \frac{2400 - 5x_1}{15} = 160 - \frac{1}{3}x_1$ is an equivalent equation written in slope intercept form, according to how we graphed the constraints. Keep in mind that our decision variables are the weekly production rates for both types of skateboards. Thus, the profit generated by a particular mix of products is not a variable.

Let us suppose SK8MAN, Inc., generates $3,500 of profit each week. $3500 = 15x_1 + 35x_2$ represents this situation algebraically, and the graph in Figure 1.2.6 represents it geometrically. If $0$ of profit were made, then the equation would be $0 = 15x_1 + 35x_2$. A $7,000 weekly profit would mean $7000 = 15x_1 + 35x_2$. As you can see in the graph in Figure 1.2.7, all these lines are parallel; only their vertical intercepts are changing. Each equation represents a line of constant profit. For example, $(0, 100), (120, 49), \text{and } (233.33, 10)$ are collinear, and each coordinate pair generates a profit of $3,500 when substituted into the objective function.

Figure 1.2.6: Feasible region with line of constant profit of $3,500

Figure 1.2.7: Feasible region with several lines of constant profit

The coordinates of every point on a particular line of constant profit represent a mix of products that generates that amount of profit. When the weekly profit is assumed to be $3,500, you can see there is an infinite number of points on that line that lie within the feasible region. If we were to continue to graph lines with increasingly large values for profit, we would eventually happen upon a line that intersects the feasible region at just one point. If the value of $z$ were made any larger, none of points on the new line would intersect the feasible region.
So, how many Sporty and how many Fancy skateboards will optimize profit for SK8MAN, Inc.?

Adapted from MINDSET project materials found at http://www.mindsetproject.org/