Georgia
Standards of Excellence
Transition Curriculum
Frameworks

Mathematics

GSE Coordinate Algebra ⇨ GSE Geometry

Polynomials, Radicals, and Quadratics

Georgia Department of Education

Richard Woods, Georgia's School Superintendent

"Educating Georgia's Future"

<u>Georgia Standards of Excellence Transition Frameworks</u>

Table of Contents

Contents

GSE Transition Frameworks

GSE Coordinate Algebra ⇨ GSE Geometry: Polynomials, Radicals, and Quadratics

OVERVIEW

Students who are transitioning from GSE Coordinate Algebra to GSE Geometry will not have been exposed to the content from GSE Algebra I which addresses polynomials, radicals, and quadratics. This transition unit further develops student understanding of radicals, introduces quadratics, and introduces operations with polynomials. This critical content is provided in both web-based and print formats.

In this unit students will:

- Understand similarities between the system of polynomials and the system of integers.
- Understand that the basic properties of numbers continue to hold with polynomials.
- Draw on analogies between polynomial arithmetic and base–ten computation, focusing on properties of operations, particularly the distributive property.
- Operate with polynomials with an emphasis on expressions that simplify to linear or quadratic forms.
- Rewrite (simplifying and operations of addition, subtraction, and multiplication) expressions involving radicals.
- Use and explain properties of rational and irrational numbers
- Explain why the sum or product of rational numbers is rational; why the sum of a rational number and an irrational number is irrational; and why the product of a nonzero rational number and an irrational number is irrational.
- focus on quadratic functions, equations, and applications
- explore variable rate of change
- learn to factor general quadratic expressions completely over the integers and to solve general quadratic equations by factoring by working with quadratic functions that model the behavior of objects that are thrown in the air and allowed to fall subject to the force of gravity
- learn to find the vertex of the graph of any polynomial function and to convert the formula for a quadratic function from standard to vertex form.
- apply the vertex form of a quadratic function to find real solutions of quadratic equations that cannot be solved by factoring
- develop the concept of discriminant of a quadratic equation
- learn the quadratic formula
- explain why the graph of every quadratic function is a translation of the graph of the basic function $f(x) = x^2$

STANDARDS ADDRESSED IN THIS UNIT

Students who are transitioning from GSE Coordinate Algebra to GSE Geometry will not have been exposed to the following standards.

Extend the properties of exponents to rational exponents.

MGSE9-12.N.RN.2 Rewrite expressions involving radicals ~~and rational exponents using the properties of exponents.~~ (i.e., simplify and/or use the operations of addition, subtraction, and multiplication, with radicals within expressions limited to square roots).

Use properties of rational and irrational numbers.

MGSE9-12.N.RN.3 Explain why the sum or product of rational numbers is rational; why the sum of a rational number and an irrational number is irrational; and why the product of a nonzero rational number and an irrational number is irrational.

Interpret the structure of expressions

MGSE9–12.A.SSE.1 Interpret expressions that represent a quantity in terms of its context. *(Focus on quadratic expressions; compare with linear and exponential functions studied in Coordinate Algebra.)*

MGSE9–12.A.SSE.1a Interpret parts of an expression, such as terms, factors, and coefficients, in context. *(Focus on quadratic expressions; compare with linear and exponential functions studied in Coordinate Algebra.)*

MGSE9–12.A.SSE.1b Given situations which utilize formulas or expressions with multiple terms and/or factors, interpret the meaning (in context) of individual terms or factors. *(Focus on quadratic expressions; compare with linear and exponential functions studied in Coordinate Algebra.)*

MGSE9-12.A.SSE.2 Use the structure of an expression to rewrite it in different equivalent forms. For example, see $x^4 – y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 – y^2) (x^2 + y^2)$.

Write expressions in equivalent forms to solve problems

MGSE9-12.A.SSE.3 Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.

MGSE9-12.A.SSE.3a Factor any quadratic expression to reveal the zeros of the function defined by the expression.

MGSE9-12.A.SSE.3b Complete the square in a quadratic expression to reveal the maximum and minimum value of the function defined by the expression.

Perform arithmetic operations on polynomials

MGSE9–12.A.APR.1 Add, subtract, and multiply polynomials; understand that polynomials form a system analogous to the integers in that they are closed under these operations.

Reasoning with Equations and Inequalities
MGSE9-12.A.REI.4 Solve quadratic equations in one variable.

MGSE9-12.A.REI.4a Use the method of completing the square to transform any quadratic equation in x into an equation of the form $(x – p)^2 = q$ that has the same solutions. Derive the quadratic formula from $ax^2 + bx + c = 0$.

MGSE9-12.A.REI.4b Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, factoring, completing the square, and the quadratic formula, as appropriate to the initial form of the equation (limit to real number solutions).

Interpret functions that arise in applications in terms of the context

MGSE9-12.F.IF.4 Using tables, graphs, and verbal descriptions, interpret the key characteristics of a function which models the relationship between two quantities. Sketch a graph showing key features including: intercepts; interval where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; ~~and periodicity.~~

MGSE9-12.F.IF.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*

MGSE9-12.F.IF.6 Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.

Analyze functions using different representations

MGSE9-12.F.IF.7 Graph functions expressed algebraically and show key features of the graph both by hand and by using technology.

MGSE9-12.F.IF.7a Graph linear and quadratic functions and show intercepts, maxima, and minima (as determined by the function or by context).

MGSE9-12.F.IF.8 Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.

MGSE9-12.F.IF.8a Use the process of factoring and completing the square in a quadratic function to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context. *For example, compare and contrast quadratic functions in standard, vertex, and intercept forms.*

MGSE9-12.F.IF.9 Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one function and an algebraic expression for another, say which has the larger maximum.*

Build new functions from existing functions

MGSE9-12.F.BF.3 ~~Identify the effect on the graph of replacing f(x) by f(x) + k, k f(x), f(kx), and f(x + k) for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology.~~ Include recognizing even and odd functions from their graphs and algebraic expressions for them.

Construct and Compare Linear, Quadratic, and Exponential Models and solve problems

MGSE9-12.F.LE.3 Observe using graphs and tables that a quantity increasing exponentially eventually exceeds a quantity increasing ~~linearly~~, quadratically, or (more generally) as a polynomial function.

WEB-BASED CONTENT

Web-based content available at: http://www.gavirtuallearning.org/Resources/SharedMath1(MathCopy).aspx

Content specific to the GPS Transition Frameworks can be accessed through the following links:

Simplifying Radical Expressions Module
Factor Polynomials Completely
Graphing Quadratics
Graphing Quadratics in Standard Form
Average Rate of Change
Solving Quadratics by Completing the Square
Using the Quadratic Formula and Discriminant
Quadratic Module Assessments

POLYNOMIAL PATTERNS

GEORGIA STANDARDS OF EXCELLENCE

Perform arithmetic operations on polynomials

MGSE9–12.A.APR.1 Add, subtract, and multiply polynomials; understand that polynomials form a system analogous to the integers in that they are closed under these operations. *(Focus on polynomial expressions that simplify to forms that are linear or quadratic in a positive integer power of x.)*

Interpret the structure of expressions

MGSE9–12.A.SSE.1a Interpret parts of an expression, such as terms, factors, and coefficients, in context.

STANDARDS FOR MATHEMATICAL PRACTICE

2. Reason abstractly and quantitatively by requiring students to make sense of quantities and their relationships to one another in problem situations.

7. Look for and make use of structure by expecting students to apply rules, look for patterns and analyze structure.

Common Student Misconceptions

1. Some students will apply the distributive property inappropriately. Emphasize that it is the distributive property of multiplication over addition. For example, the distributive property can be used to rewrite $2(x + y)$ as $2x + 2y$, because in this product the second factor is a sum (i.e., involving addition). But in the product $2(xy)$, the second factor, (xy), is itself a product, not a sum.

2. Some students will still struggle with the arithmetic of negative numbers. Consider the expression $(-3)\bullet(2 + (-2))$. On the one hand, $(-3)\bullet(2 + (-2)) = (-3)\bullet(0) = 0$. But using the distributive property, $(-3)\bullet(2 + (-2)) = (-3)\bullet(2) + (-3)\bullet(-2)$. Because the first calculation gave 0, the two terms on the right in the second calculation must be opposite in sign. Thus, if we agree that $(-3)\bullet(2) = -6$, then it must follow that $(-3)\bullet(-2) = 6$.

3. Students often forget to distribute the subtraction to terms other than the first one. For example, students will write $(4x + 3) - (2x + 1) = 4x + 3 - 2x + 1 = 2x + 4$ rather than $4x + 3 - 2x - 1 = 2x + 2$.

4. Students will change the degree of the variable when adding/subtracting like terms. For example, $2x + 3x = 5x^2$ rather than $5x$.

5. Students may not distribute the multiplication of polynomials correctly and only multiply like terms. For example, they will write $(x + 3)(x - 2) = x^2 - 6$ rather than $x^2 - 2x + 3x - 6$.

The following activity is a modification from NCTM's Illuminations Polynomial Puzzler: http://illuminations.nctm.org/LessonDetail.aspx?id=L798

Comments:

When students have completed their puzzlers, allow them to share their answers and thinking with the class. Here are some ideas to help you structure this:

- ***Do not simply put up the answer key. Have students write their solutions to the puzzlers on the board or fill them in on an overhead copy of the activity sheet. As they fill in the spaces, ask them to explain verbally or in writing how they approached the puzzle.***
- ***If students worked in pairs, allow them to present the solutions in pairs.***
- ***As students are reflecting, you may wish to ask them questions such as***

❖ ***Did you use a traditional method to expand and factor, such as FOIL, or did you develop your own strategies as you worked?***
❖ ***Were there certain paths to solving the polynomial puzzlers that were easier than others? Why?***

Assessment can be made by creating tables using the templates below. The question marks can be completed by the instructor, and then, the students will complete the tables.

	?	
?		***?***
	?	

		?
?		***?***
		?

1. Can you find the pattern to the number puzzle below? Explain the pattern.

2	–6	–12
4	1	4
8	–6	–48

2. Now, use the pattern to complete this table.

3	***–5***	–15
8	–2	–16
24	***10***	240

HINT: Start with the question marks.

3. This can be expanded to multiplication with polynomials by solving the following:

1	$x + 3$	***$x + 3$***
$–2x + 5$	2	***$–4x + 10$***
$–2x + 5$	***$2x + 6$***	***$–4x^2 –2x + 30$***

4. What about this one?

–5	***$–2x + 3$***	$10x – 15$
$3x – 2$	***4***	***$12x – 8$***
$–15x + 10$	$8x + 12$	***$120x^2 –260x + 120$***

5. Work the following on your own for 10 minutes, and then complete the tables with a partner.

a.

1	$x + 7$	***$x + 7$***
$-2x + 5$	2	***$-4x + 10$***
$-2x + 5$	***$2x + 14$***	***$-4x^2 - 18x + 70$***

b.

-2	$x - 3$	***$-2x + 6$***
3	$-5x + 1$	***$-15x + 3$***
-6	***$-5x^2 + 16x - 3$***	$30x^2 - 96x + 18$

c.

-4	***2***	-8
$x + 3$	***$x - 3$***	***$x^2 - 9$***
$-4x - 12$	$2x - 6$	$-8x^2 + 72$

d.

$x + 3$	***3***	***$3x + 9$***
2	***$4x$***	$8x$
$2x + 6$	$12x$	***$24x^2 + 72x$***

e.

2	***$x + 5$***	$2x + 10$
$x + 3$	7	***$7x + 21$***
$2x + 6$	***$7x + 35$***	***$14x^2 + 112x + 210$***

f.

6	***$2x + 2$***	***$12x + 12$***
3	$x + 3$	***$3x + 9$***
18	***$2x^2 + 8x + 6$***	$36x^2 + 144x + 108$

POLYNOMIAL PATTERNS

The following activity is a modification from NCTM's Illuminations Polynomial Puzzler http://illuminations.nctm.org/LessonDetail.aspx?id=L798

1. Can you find the pattern to the number puzzle below? Explain the pattern.

2	–6	–12
4	1	4
8	–6	–48

2. Now, use the pattern to complete this table.

3	?	–15
	–2	?
		240

HINT: Start with the question marks.

3. This can be expanded to multiplication with polynomials by solving the following:

1	$x + 3$	
$-2x + 5$	2	

4. What about this one?

–5		$10x - 15$
$3x - 2$		
	$-8x + 12$	

5. Work the following on your own for 10 minutes, and then complete the tables with a partner.

a.

1	$x + 7$	
$-2x + 5$	2	

b.

	$x - 3$	
3	$-5x + 1$	
		$30x^2 - 96x + 18$

c.

-4		-8
	$2x - 6$	$-8x^2 + 72$

d.

$x + 3$		
2		$8x$
	$12x$	

e.

		$2x + 10$
$x + 3$	7	
$2x + 6$		

f.

6		
	$x + 3$	
18		$36x^2 + 144x + 108$

MODELING

GEORGIA STANDARDS OF EXCELLENCE

MGSE9–12.A.APR.1 Add, subtract, and multiply polynomials; understand that polynomials form a system analogous to the integers in that they are closed under these operations.

MGSE9–12.A.SSE.1a Interpret parts of an expression, such as terms, factors, and coefficients, in context.

MGSE9–12.A.SSE.1b Given situations which utilize formulas or expressions with multiple terms and/or factors, interpret the meaning (in context) of individual terms or factors.

MGSE9–12.N.Q.1 Use units of measure (linear, area, capacity, rates, and time) as a way to understand problems:

a. Identify, use, and record appropriate units of measure within context, within data displays, and on graphs;
b. Convert units and rates using dimensional analysis (English–to–English and Metric–to–Metric without conversion factor provided and between English and Metric with conversion factor);
c. Use units within multi–step problems and formulas; interpret units of input and resulting units of output.

STANDARDS FOR MATHEMATICAL PRACTICE

2. Reason abstractly and quantitatively by requiring students to make sense of quantities and their relationships to one another in problem situations.

7. Look for and make use of structure by expecting students to apply rules, look for patterns and analyze structure.

Common Student Misconceptions

1. Some students will apply the distributive property inappropriately. Emphasize that it is the distributive property of multiplication over addition. For example, the distributive property can be used to rewrite $2(x + y)$ as $2x + 2y$, because in this product the second factor is a sum (i.e., involving addition). But in the product $2(xy)$, the second factor,(xy), is itself a product, not a sum.

2. Some students will still struggle with the arithmetic of negative numbers. Consider the expression $(-3)\bullet(2 + (-2))$. On the one hand, $(-3) \bullet(2 + (-2)) = (-3) \bullet(0) = 0$. But using the distributive property, $(-3)\bullet(2 + (-2)) = (-3)\bullet(2) + (-3)\bullet(-2)$. Because the first calculation gave 0, the two terms on the right in the second calculation must be opposite in sign. Thus, if we agree that $(-3)\bullet(2) = -6$, then it must follow that $(-3)\bullet(-2) = 6$.

3. Students often forget to distribute the subtraction to terms other than the first one. For example, students will write $(4x+ 3) –(2x+ 1) = 4x + 3 – 2x + 1 = 2x + 4$ rather than $4x + 3 – 2x –1 = 2x + 2$.

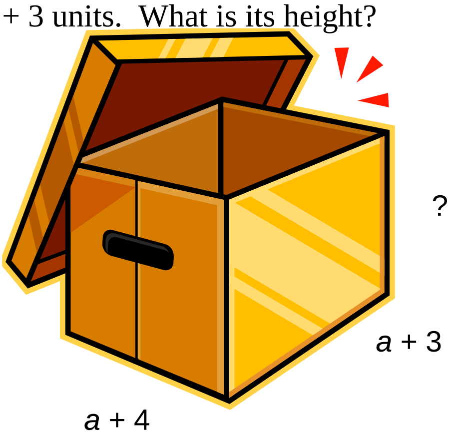
4. Students will change the degree of the variable when adding/subtracting like terms. For example, $2x+ 3x= 5x^2$ rather than $5x$.

5. Students may not distribute the multiplication of polynomials correctly and only multiply like terms. For example, they will write $(x + 3)(x – 2) = x^2 – 6$ rather than $x^2 – 2x + 3x –6$

The problems below will be placed on the walls around the room with large sheets of paper under each. Students will work in teams of four people to travel around the room and write their solutions

Problem A (extension problem)

The volume in cubic units of the box is $a^3 + 8a^2 + 19a + 12$. Its length is $a + 4$ units and its width is $a + 3$ units. What is its height?



answer: a + 1

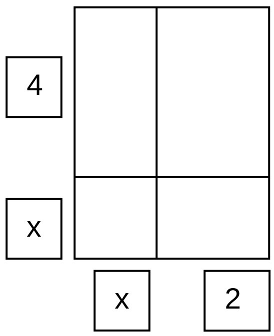
under the papers. Each team should be given a letter name that corresponds to their starting problem. After each team is given about 3–4 minutes on a problem, the teacher should call time, and the teams move to the next station.
Review and practice: http://www.wiziq.com/tutorial/111186–7–5–Multiplying–a–Polynomial–by–a–Monomial

Having each time write in a different color can be beneficial as well as designating roles for each team member such as scribe, director, checker, and presenter. After all teams have rotated through all the problems, the teams can travel back through to check for differences in answers. This can lead to a discussion on which problems are correct or a discussion on the different methods used to arrive at the same answer.

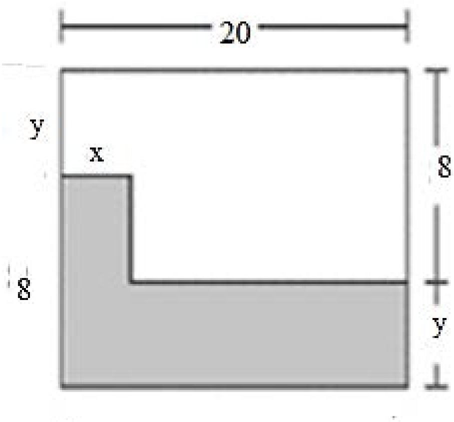
Problem B

What is an illustration of $(x + 2)(x + 4)$?

Possible answer:



Problem C: This rectangle shows the floor plan of an office. The shaded part of the plan is an area that is getting new tile. Write an algebraic expression that represents the area of the office that is getting new tile.

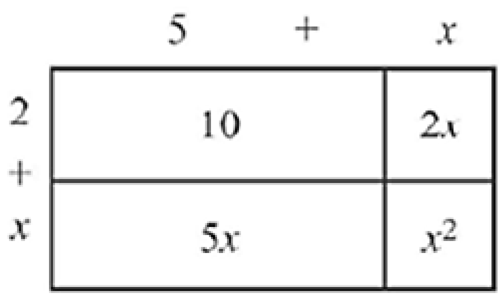


Possible Answer

8x –xy + 20y

Problem D

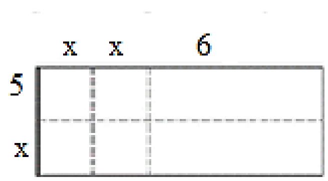
What is the rectangle modeling?



***Answer:* $(x + 5)(x + 2)$**

Problem E

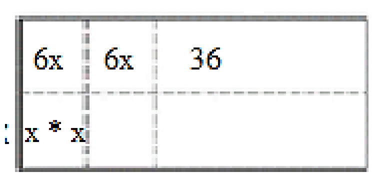
What is the product of the expression represented by the model below?



***Answer:* $2x^2 + 16x + 30$**

Problem F

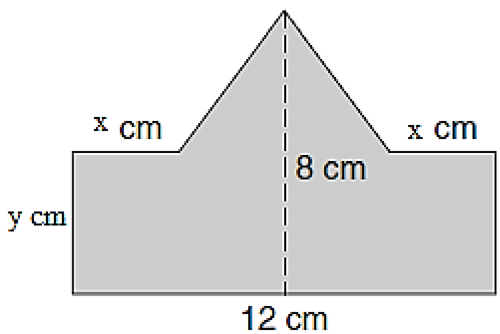
Write the dimensions for the rectangle below.



***Answer:* $(x + 6)$ by $(x + x + 6)$ or $(x + 6)$ by $(2x + 6)$**

Problem G

Find the area, including units, of the shape below.

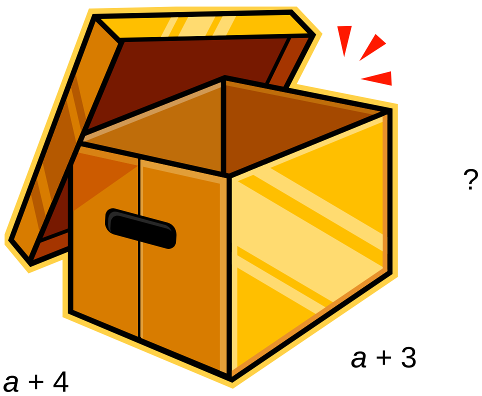


Answer: ***–8x + xy + 6y + 48***

MODELING

Problem A

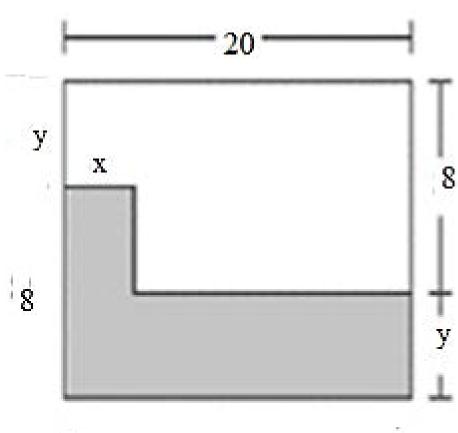
The volume in cubic units of the box is $a^3 + 8a^2 + 19a + 12$. Its length is $a + 4$ units and its width is $a + 3$ units. What is its height?



Problem B

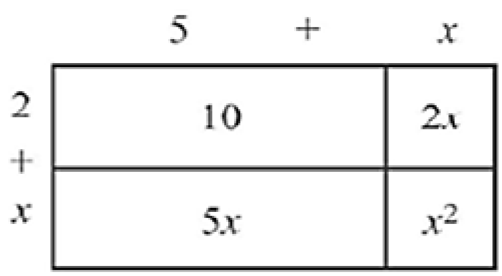
What is an illustration of $(x + 2)(x + 4)$?

Problem C: This rectangle shows the floor plan of an office. The shaded part of the plan is an area that is getting new tile. Write an algebraic expression that represents the area of the office that is getting new tile.



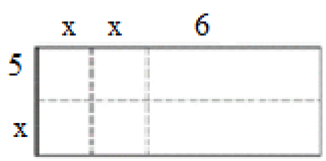
Problem D

What is the rectangle modeling?



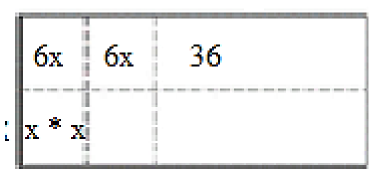
Problem E

What is the product of the expression represented by the model below?



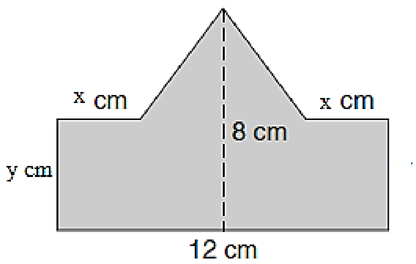
Problem F

Write the dimensions for the rectangle below.



Problem G

Find the area, including units, of the shape below.



Visualizing Square Roots (Learning Task)

Adapted from http://www.mathedpage.org/geometry-labs/gl/gl-9.pdf

Mathematical Goals:

- To build the ideas of square and square root on their geometric interpretation.
- To justify simplification of radicals using geometric representations.

Essential Questions:

- How do I represent radicals visually?
- What is the relationship between the radicand and the area of a square?
- How do I justify simplification of radicals using geometric representations?

GEORGIA STANDARDS OF EXCELLENCE

MGSE9-12.N.RN.2 Rewrite expressions involving radicals ~~and rational exponents using the properties of exponents.~~ (i.e., simplify and/or use the operations of addition, subtraction, and multiplication, with radicals within expressions limited to square roots).

STANDARDS FOR MATHEMATICAL PRACTICE

1. Make sense of problems and persevere in solving them
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically
6. Attend to precision
7. Look for and make use of structure

Materials Needed:

- Dot Paper , Graph Paper, or Geoboard

Grouping:

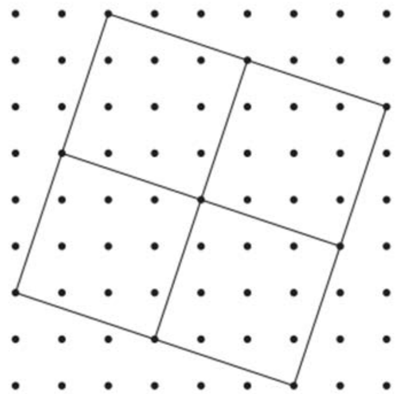
- Individual/Partner

Time Needed:

- 50-60 minutes

Visualizing Square Roots (Learning Task)– Teacher Notes

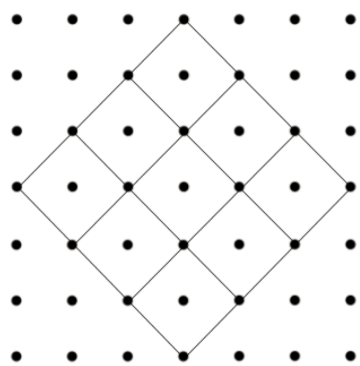
Adapted from http://www.mathedpage.org/geometry-labs/gl/gl-9.pdf



1a. In the above figure, what are the following measures?

i. The area of one of the small squares — 10 units²
ii. The side of one of the small squares — $\sqrt{10}$ units
iii. The area of the large square — 40 units²
iv. The side of the large square — $\sqrt{40}$ units

1b. Explain, using the answers to Problem 1a, why $\sqrt{40} = 2\sqrt{10}$.
Both values represent the length of the side of the large square.



2a. In the above figure, what are the following measures?

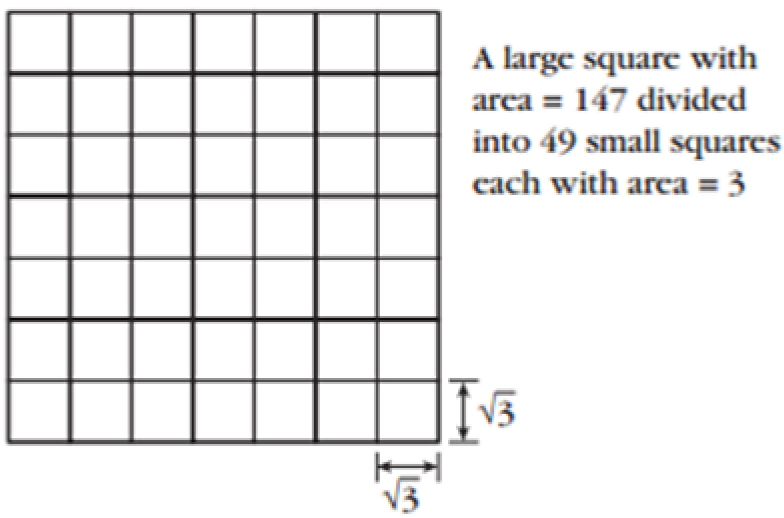
i. The area of one of the small squares — 2 units²
ii. The side of one of the small squares — $\sqrt{2}$ units
iii. The area of the large square — 18 units²
iv. The side of the large square — $\sqrt{18}$ units

2b. Explain, using the answers to problem 2a, why $\sqrt{18} = 3\sqrt{2}$
Both values represent the length of the side of the large square.

3. On dot paper, create a figure to show that $\sqrt{8} = 2\sqrt{2}$, $\sqrt{18} = 3\sqrt{2}, \sqrt{32} = 4\sqrt{2}, and\ \sqrt{50} = 5\sqrt{2}$.

4. On dot paper, create a figure to show that $\sqrt{20} = 2\sqrt{5}$ and $\sqrt{45} = 3\sqrt{5}$.

In the figure on the previous page, and in the figures you made in Problems 3 and 4, a larger square is divided up into *a square number of squares.* This is the basic idea for writing square roots in *simple radical form.* The figure need not be made on dot paper. For example, consider $\sqrt{147}$. Since 147 = 3 ·49, and since 49 is a square number, we can divide a square of area 147 units ^{2} into 49 squares, each of area 3 untis ^{2}:



You will notice that the side of the larger square is $\sqrt{147} = 7\sqrt{3}$

5. Write the following in simple radical form.

i.	$\sqrt{12}$	$2\sqrt{3}$
ii.	$\sqrt{45}$	$3\sqrt{5}$
iii.	$\sqrt{24}$	$2\sqrt{6}$
iv.	$\sqrt{32}$	$4\sqrt{2}$
v.	$\sqrt{75}$	$5\sqrt{3}$
vi.	$\sqrt{98}$	$7\sqrt{2}$

Visualizing Square Roots (Learning Task)

Name________________________________ **Date**_________

Adapted from http://www.mathedpage.org/geometry-labs/gl/gl-9.pdf

Mathematical Goals:

- To build the ideas of square and square root on their geometric interpretation.
- To justify simplification of radicals using geometric representations.

Essential Questions:

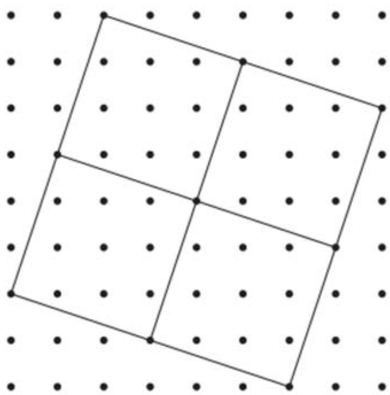
- How do I represent radicals visually?
- What is the relationship between the radicand and the area of a square?
- How do I justify simplification of radicals using geometric representations?

GEORGIA STANDARDS OF EXCELLENCE

MGSE9-12.N.RN.2 Rewrite expressions involving radicals ~~and rational exponents using the properties of exponents.~~ (i.e., simplify and/or use the operations of addition, subtraction, and multiplication, with radicals within expressions limited to square roots).

STANDARDS FOR MATHEMATICAL PRACTICE

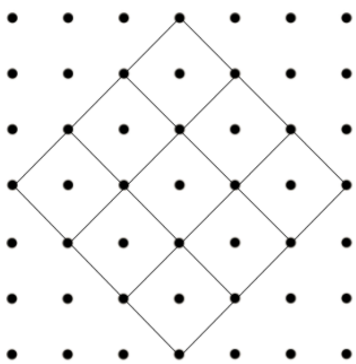
1. Make sense of problems and persevere in solving them
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically
6. Attend to precision
7. Look for and make use of structure



1a. In the above figure, what are the following measures?

v. The area of one of the small squares
vi. The side of one of the small squares
vii. The area of the large square
viii. The side of the large square

1b. Explain, using the answers to Problem 1a, why $\sqrt{40} = 2\sqrt{10}$.



2a. In the above figure, what are the following measures?

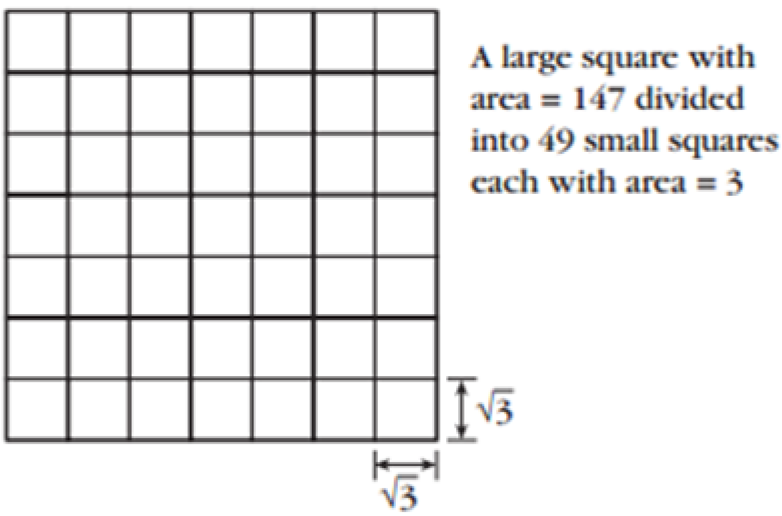
v. The area of one of the small squares
vi. The side of one of the small squares
vii. The area of the large square
viii. The side of the large square

2b. Explain, using the answers to problem 2a, why $\sqrt{18} = 3\sqrt{2}$

3. On dot paper, create a figure to show that $\sqrt{8} = 2\sqrt{2}$, $\sqrt{18} = 3\sqrt{2}$, $\sqrt{32} = 4\sqrt{2}$, *and* $\sqrt{50} = 5\sqrt{2}$.

4. On dot paper, create a figure to show that $\sqrt{20} = 2\sqrt{5}$ and $\sqrt{45} = 3\sqrt{5}$.

In the figure on the previous page, and in the figures you made in Problems 3 and 4, a larger square is divided up into *a square number of squares.* This is the basic idea for writing square roots in *simple radical form.* The figure need not be made on dot paper. For example, consider $\sqrt{147}$. Since 147 = 3 ·49, and since 49 is a square number, we can divide a square of area 147 units ^{2} into 49 squares, each of area 3 untis ^{2}:



You will notice that the side of the larger square is $\sqrt{147} = 7\sqrt{3}$

5. Write the following in simple radical form.

vii. $\sqrt{12}$
viii. $\sqrt{45}$
ix. $\sqrt{24}$
x. $\sqrt{32}$
xi. $\sqrt{75}$
xii. $\sqrt{98}$

FORMATIVE ASSESSMENT LESSON: RATIONAL & IRRATIONAL NUMBERS 1

Source: Formative Assessment Lesson Materials from Mathematics Assessment Project

http://map.mathshell.org/materials/download.php?fileid=1245

ESSENTIAL QUESTIONS:

How do you classify numbers as rational or irrational?

How do you move between different representations of rational and irrational numbers?

TASK COMMENTS:

Tasks and lessons from the Mathematics Assessment Project are specifically designed to help teachers effectively formatively assess their students. The way the tasks and lessons are designed gives the teacher a clear understanding of what the students are able to do and not do. Within the lesson, teachers will find suggestions and question prompts that will help guide students towards understanding. For more information access the MAP website:

http://www.map.mathshell.org/materials/background.php?subpage=formative

The task, *Rational & Irrational Numbers – 1*, is a Formative Assessment Lesson (FAL) that can be found at the website: http://map.mathshell.org/materials/lessons.php?taskid=424&subpage=concept

The FAL document provides a clear lesson design, from the opening of the lesson to the closing of the lesson.

The PDF version of the task can be found at the link below:

http://map.mathshell.org/materials/download.php?fileid=1245

GEORGIA STANDARDS OF EXCELLENCE

Extend the properties of exponents to rational exponents.

MGSE9-12.N.RN.2 Rewrite expressions involving radicals ~~and rational exponents using the properties of exponents.~~ (i.e., simplify and/or use the operations of addition, subtraction, and multiplication, with radicals within expressions limited to square roots).

Use properties of rational and irrational numbers.

MGSE9-12.N.RN.3 Explain why the sum or product of rational numbers is rational; why the sum of a rational number and an irrational number is irrational; and why the product of a nonzero rational number and an irrational number is irrational.

Interpret the structure of expressions

MGSE9–12.A.SSE.1a Interpret parts of an expression, such as terms, factors, and coefficients, in context.

STANDARDS FOR MATHEMATICAL PRACTICE

3. Construct viable arguments and critique the reasoning of others by engaging students on discussion of why they agree or disagree with responses, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

FORMATIVE ASSESSMENT LESSON: RATIONAL & IRRATIONAL NUMBERS 2

Source: Formative Assessment Lesson Materials from Mathematics Assessment Project

http://map.mathshell.org/materials/download.php?fileid=1267

ESSENTIAL QUESTIONS:

How do you find irrational and rational numbers to exemplify general statements?

How do you reason with properties of rational and irrational numbers?

TASK COMMENTS:

Tasks and lessons from the Mathematics Assessment Project are specifically designed to help teachers effectively formatively assess their students. The way the tasks and lessons are designed gives the teacher a clear understanding of what the students are able to do and not do. Within the lesson, teachers will find suggestions and question prompts that will help guide students towards understanding. For more information access the MAP website:

http://www.map.mathshell.org/materials/background.php?subpage=formative

The task, *Rational & Irrational Numbers – 2*, is a Formative Assessment Lesson (FAL) that can be found at the website: http://map.mathshell.org/materials/lessons.php?taskid=434&subpage=concept

The FAL document provides a clear lesson design, from the opening of the lesson to the closing of the lesson.
The PDF version of the task can be found at the link below:

http://map.mathshell.org/materials/download.php?fileid=1267

GEORGIA STANDARDS OF EXCELLENCE

Extend the properties of exponents to rational exponents.

MGSE9-12.N.RN.2 Rewrite expressions involving radicals ~~and rational exponents using the properties of exponents.~~ (i.e., simplify and/or use the operations of addition, subtraction, and multiplication, with radicals within expressions limited to square roots).

Use properties of rational and irrational numbers.

MGSE9-12.N.RN.3 Explain why the sum or product of rational numbers is rational; why the sum of a rational number and an irrational number is irrational; and why the product of a nonzero rational number and an irrational number is irrational.

Interpret the structure of expressions

MGSE9–12.A.SSE.1a Interpret parts of an expression, such as terms, factors, and coefficients, in context.

MGSE9–12.A.SSE.1b Given situations which utilize formulas or expressions with multiple terms and/or factors, interpret the meaning (in context) of individual terms or factors.

STANDARDS FOR MATHEMATICAL PRACTICE

3. Construct viable arguments and critique the reasoning of others by engaging students on discussion of why they agree or disagree with responses, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

CULMINATING TASK: AMUSEMENT PARK PROBLEM

GEORGIA STANDARDS OF EXCELLENCE

Extend the properties of exponents to rational exponents.

MGSE9-12.N.RN.2 Rewrite expressions involving radicals ~~and rational exponents using the properties of exponents.~~ (i.e., simplify and/or use the operations of addition, subtraction, and multiplication, with radicals within expressions limited to square roots).

Use properties of rational and irrational numbers.

MGSE9-12.N.RN.3 Explain why the sum or product of rational numbers is rational; why the sum of a rational number and an irrational number is irrational; and why the product of a nonzero rational number and an irrational number is irrational.

Perform arithmetic operations on polynomials

MGSE9–12.A.APR.1 Add, subtract, and multiply polynomials; understand that polynomials form a system analogous to the integers in that they are closed under these operations.

Interpret the structure of expressions

MGSE9–12.A.SSE.1a Interpret parts of an expression, such as terms, factors, and coefficients, in context.

MGSE9–12.A.SSE.1b Given situations which utilize formulas or expressions with multiple terms and/or factors, interpret the meaning (in context) of individual terms or factors.

Reason quantitatively and use units to solve problems.

MGSE9–12.N.Q.1 Use units of measure (linear, area, capacity, rates, and time) as a way to understand problems:

a. Identify, use, and record appropriate units of measure within context, within data displays, and on graphs;
b. Convert units and rates using dimensional analysis (English–to–English and Metric–to–Metric without conversion factor provided and between English and Metric with conversion factor);
c. Use units within multi–step problems and formulas; interpret units of input and resulting units of output.

MGSE9–12.N.Q.2 Define appropriate quantities for the purpose of descriptive modeling. Given a situation, context, or problem, students will determine, identify, and use appropriate quantities for representing the situation.

MGSE9–12.N.Q.3 Choose a level of accuracy appropriate to limitations on measurement when reporting quantities. *For example, money situations are generally reported to the nearest cent (hundredth). Also, an answers' precision is limited to the precision of the data given.*

STANDARDS FOR MATHEMATICAL PRACTICE

2. Reason abstractly and quantitatively by requiring students to make sense of quantities and their relationships to one another in problem situations.

3. Construct viable arguments and critique the reasoning of others by engaging students on discussion of why they agree or disagree with responses, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

6. Attend to precision by requiring students to calculate efficiently and accurately; and to communicate precisely with others by using clear mathematical language to discuss their reasoning.

7. Look for and make use of structure by expecting students to apply rules, look for patterns and analyze structure.

8. Look for and express regularity in repeated reasoning by expecting students to understand broader applications and look for structure and general methods in similar situations.

The Radical World of Math is reviewing the master plan of a proposed amusement park coming to your area. Your help is needed with the land space and with park signage.

First, the planners need help in designing the land space. The parameters are as follows:

- 15 rows of parking are required
- The rows will be the same length as the park
- The park size will be square with a length of X so that expansion is possible.
- "Green Space" for planting, sitting, or picnicking is a must.
- Parking will be adjacent to only two sides of the park

Your task is to choose 3 possible configurations of land use with 15 rows of parking. Find the area of the picnic (green space) for each configuration. There is more than one way to solve the problem. For your maximum picnic space, write an equation for the total AREA of the park.

As long as all the parameters are met, student designs will be correct. Maximum spaces are the closest to square designs. $(x + 7)(x + 8)$

Extension:

The park is expected to be successful and the planners decide to expand the parking lot by adding 11 more rows. Assume the new plan will add not only 11 rows of parking but will also triple the maximum original green space (approximately). Choose 1 of your park configurations (your best) to complete this section and redraw your park configuration. What is the percentage increase in area that was created by expanding to 26 rows of parking?

Second, signs have to be designed for the park. For one of the areas called "Radical Happenings", the signs must show conversions between radical expressions and exponential expressions. There must be at least 10 signs in all that reflect square roots, cube roots, and fourth roots. Create 10 unique signs for use in the park. Would there be appropriate areas for these values to be placed?

The signs can have values 1 through 10 and be used to designate ten different park areas.

You must create problems demonstrating each concept and have another team try to "get through" your obstacles.

Extension: Using fractals, create advertising for this park. This link may help: http://mathworld.wolfram.com/Fractal.html

CULMINATING TASK: AMUSEMENT PARK PROBLEM

The Radical World of Math is reviewing the master plan of a proposed amusement park coming to your area. Your help is needed with the land space and with park signage.

First, the planners need help in designing the land space. The parameters are as follows:

- 15 rows of parking are required
- The rows will be the same length as the park
- The park size will be square with a length of X so that expansion is possible.
- “Green Space” for planting, sitting, or picnicking is a must.
- Parking will be adjacent to only two sides of the park

Your task is to choose 3 possible configurations of land use with 15 rows of parking. Find the area of the picnic (green space) for each configuration. There is more than one way to solve the problem. For your maximum picnic space, write an equation for the total AREA of the park.

Extension:
The park is expected to be successful and the planners decide to expand the parking lot by adding 11 more rows. Assume the new plan will add not only 11 rows of parking but will also triple the maximum original green space (approximately). Choose 1 of your park configurations (your best) to complete this section and redraw your park configuration. What is the percentage increase in area that was created by expanding to 26 rows of parking?

Second, signs have to be designed for the park. For one of the areas called “Radical Happenings”, the signs must show conversions between radical expressions and exponential expressions. There must be at least 10 signs in all that reflect square roots, cube roots, and fourth roots. Create 10 unique signs for use in the park. Would there be appropriate areas for these values to be placed?
You must create problems demonstrating each concept and have another team try to “get through” your obstacles.

Extension: Using fractals, create advertising for this park. This link may help: http://mathworld.wolfram.com/Fractal.html

PAULA'S PEACHES

GEORGIA STANDARDS OF EXCELLENCE

MGSE9-12.A.SSE.3 Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.

> **MGSE9-12.A.SSE.3a** Factor any quadratic expression to reveal the zeros of the function defined by the expression.

MGSE9-12.A.CED.2 Create linear, quadratic, and exponential equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. (The phrase "in two or more variables" refers to formulas like the compound interest formula, in which $A = P(1 + r/n)^{nt}$ has multiple variables.)

MGSE9-12.F.IF.4 Using tables, graphs, and verbal descriptions, interpret the key characteristics of a function which models the relationship between two quantities. Sketch a graph showing key features including: intercepts; interval where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; ~~and periodicity.~~

MGSE9-12.F.IF.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*

MGSE9-12.F.IF.6 Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.

MGSE9-12.F.IF.7 Graph functions expressed algebraically and show key features of the graph both by hand and by using technology.

> **MGSE9-12.F.IF.7a** Graph linear and quadratic functions and show intercepts, maxima, and minima (as determined by the function or by context).

MGSE9-12.F.IF.8 Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.

MGSE9-12.F.IF.8a Use the process of factoring and completing the square in a quadratic function to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context. *For example, compare and contrast quadratic functions in standard, vertex, and intercept forms.*

MGSE9-12.F.IF.9 Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one function and an algebraic expression for another, say which has the larger maximum.*

STANDARDS FOR MATHEMATICAL PRACTICE

1. Make sense of problems and persevere in solving them by requiring students to interpret and make meaning of a problem and find a logical starting point, and to monitor their progress and change their approach to solving the problem, if necessary.
2. Reason abstractly and quantitatively by requiring students to make sense of quantities and their relationships to one another in problem situations.
4. Model with mathematics by expecting students to apply the mathematics concepts they know in order to solve problems arising in everyday situations, and reflect on whether the results are sensible for the given scenario.
5. Use appropriate tools strategically by expecting students to consider available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a compass, a calculator, software, etc.
6. Attend to precision by requiring students to calculate efficiently and accurately; and to communicate precisely with others by using clear mathematical language to discuss their reasoning.
7. Look for and make use of structure by expecting students to apply rules, look for patterns and analyze structure.
8. Look for and express regularity in repeated reasoning by expecting students to understand broader applications and look for structure and general methods in similar situations.

Common Student Misconceptions
1. Some students may believe that factoring and completing the square are isolated techniques within a unit of quadratic equations. Teachers should help students to see the value of these skills in the context of solving higher degree equations and examining different families of functions.
2. Students may think that the minimum (the vertex) of the graph of $y = (x + 5)^2$ is shifted to the right of the minimum (the vertex) of the graph $y = x^2$ due to the addition sign. Students should explore expels both analytically and graphically to overcome this misconception.
3. Some students may believe that the minimum of the graph of a quadratic function always occur at the y–intercept.
4. Students may believe that equations of linear, quadratic and other functions are abstract and exist on "in a math book," without seeing the usefulness of these functions as modeling real–world phenomena.
5. Additionally, they believe that the labels and scales on a graph are not important and can be assumed by a reader and that it is always necessary to use the entire graph of a function when solving a problem that uses that function as its model.
6. Students may interchange slope and y–intercept when creating equation. For example, a taxi cab cost \$4 for a dropped flag and charges \$2 per mile. Students may fail to see that \$2 is a rate of change and is slope while the \$4 is the starting cost and incorrectly write the equation as $y = 4x + 2$ instead of $y = 2x + 4$.
7. Given a graph of a line, students use the x–intercept for b instead of the y–intercept.
8. Given a graph, students incorrectly compute slope as run over rise rather than rise over run. For example, they will compute slope with the change in x over the change in y.

9. Students do not know when to include the "or equal to" bar when translating the graph of an inequality.
10. Students do not correctly identify whether a situation should be represented by a linear, quadratic, or exponential function.
11. Students often do not understand what the variables represent. For example, if the height h in feel of a piece of lava t seconds after it is ejected from a volcano is given by $h(t) = -16t^2 + 64t + 936$ and the student is asked to find the time it takes for the piece of lava to hit the ground, the student will have difficulties understanding that $h = 0$ at the ground and that they need to solve for t.
12. Students may believe that it is reasonable to input any x–value into a function, so they will need to examine multiple situations in which there are various limitations to the domains.
13. Students may also believe that the slope of a linear function is merely a number used to sketch the graph of the line. In reality, slopes have real–world meaning, and the idea of a rate of change is fundamental to understanding major concepts from geometry to calculus.
14. Students may believe that each family of functions (e.g., quadratic, square root, etc.) is independent of the others, so they may not recognize commonalities among all functions and their graphs.
15. Students may also believe that skills such as factoring a trinomial or completing the square are isolated within a unit on polynomials, and that they will come to understand the usefulness of these skills in the context of examining characteristics of functions.
16. Additionally, student may believe that the process of rewriting equations into various forms is simply an algebra symbol manipulation exercise, rather than serving a purpose of allowing different features of the function to be exhibited.

Teacher Notes
This task introduces students to solving a new type of equation, a quadratic equation of the form $x^2 + bx + c = 0$***. The method uses factorization of the quadratic; hence, the task introduces the factoring formula*** $x^2 + bx + c = (x+m)(x+n)$ ***where*** $b = m + n$ ***and*** $c = m \cdot n$***. The context involves peach production at orchards in central Georgia. Questions about peach production lead to solving quadratic equations, and the need to solve quadratic equations motivates the need to study factoring. While the context is realistic, unlike most of the contexts in other tasks, there is no claim that the numbers are realistic. The numbers where chosen to provide quadratic equations that (i) are equivalent to a quadratic in the form*** $x^2 + bx + c = 0$***, (ii) can be solved by factoring, and (iii) involve values of c for which finding factors m and n such that*** $m + n = b$ ***is not so difficult as to distract students from learning the process of solving these equations.***

In this unit, students will revisit and extend many topics from previous units. This task opens with a simple linear relationship. Understanding this relationship is necessary for writing the formula for the function that provides the continuing context of the task. However, this opening serves two other important purposes. First, since the task opens with a familiar subject, students should find it easy to get started on the task. Second, analyzing this linear relationship requires students to recall important topics from previous units that they will need to apply as they progress through the task.

Paula is a peach grower in central Georgia and wants to expand her peach orchard. In her current orchard, there are 30 trees per acre and the average yield per tree is 600 peaches. Data from the local agricultural experiment station indicates that if Paula plants more than 30 trees per acre, once the trees are in production, the average yield of 600 peaches per tree will decrease by 12 peaches for each tree over 30. She needs to decide how many trees to plant in the new section of the orchard.

Throughout this task assume that, for all peach growers in this area, the average yield is 600 peaches per tree when 30 trees per acre are planted and that this yield will decrease by 12 peaches per tree for each additional tree per acre.

1. Paula believes that algebra can help her determine the best plan for the new section of orchard and begins by developing a mathematical model of the relationship between the number of trees per acre and the average yield in **peaches per tree.**

a. Is this relationship linear or nonlinear? Explain your reasoning.

linear The average rate of change of the function is constant; average yield decreases by 12 peaches per tree each time there is one additional tree per acre.

b. If Paula plants 6 more trees per acre, what will be the average yield in peaches per tree? What is the **average** yield in peaches per tree if she plants 42 trees per acre?

There is a decrease in average yield per tree of 12(6) = 72 peaches.
average yield: 600 – 72 = 528 peaches per tree

42 trees per acre is an increase of 12 trees per acre over the current 30 trees per acre, so the average yield decreases by 12(12) = 144 peaches per tree
average yield: 600 – 144 = 456 peaches per tree

c. Let T be the function for which the input x is the number of trees planted on each acre and $T(x)$ is the average yield in peaches per tree. Write a formula for $T(x)$ in terms of x and express it in simplest form. Explain how you know that your formula is correct.

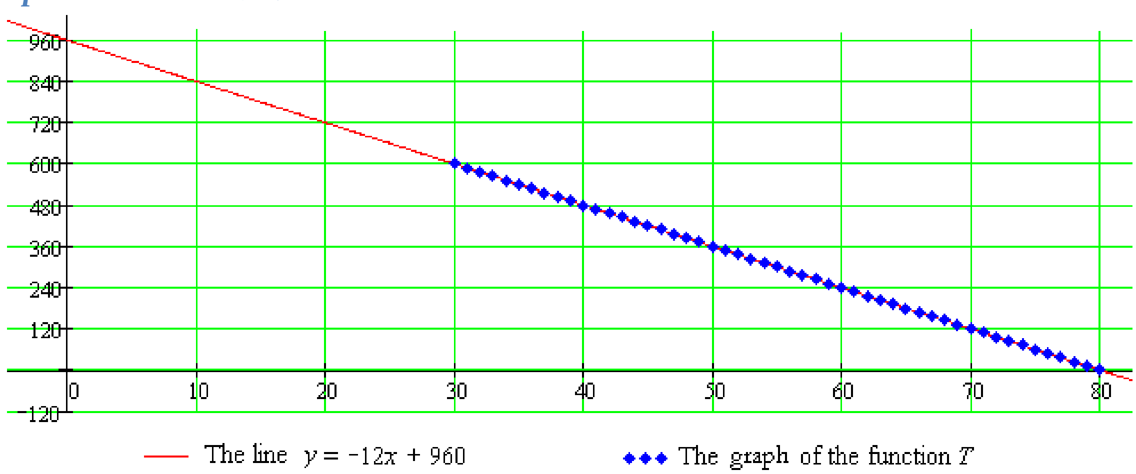
Comments:
Some students will realize that x – 30 is the number of additional trees per acre, the number in excess of 30, see that 12(x – 30) is the total decrease in peaches per tree, and write the formula for T(x) as 600 – 12(x – 30) = 960 – 12x. Other students will likely take a more geometric approach. They may use one of the points (30, 600), (36, 528), or (42, 456) on the graph of the function T and the slope of – 12 to obtain the formula for T(x) using the point-slope formula: y – 600 = – 12(x – 30) ⇒ y = 960 – 12x ⇒ T(x) = 960 – 12x .
Others may use the slope to count backward to a y-intercept. In the table below we decrease x by 5 and increase y by 12(5) = 60 until we get to x = 0. Other students may simply say that a decrease of 30 for x would result in an increase of 12(30) = 360 for y so y = 600 + 360 = 960 when x = 30 – 30 = 0. Of course, the given information applies only to 30 or more trees per acre. However, from the extensive work in previous tasks, students should know that formula will be the same as the formula for the whole line through the point (30, 600) with slope – 12 , so any tools they have for writing that equation can be applied here.

x	30	25	20	15	10	5	0
y	600	660	720	780	840	900	960

Solution:
x – 30 = additional trees per acre;   12(x – 30) = decrease in average yield
T(x) = 600 – 12(x – 30) = 600 – 12x + 360 = 960 – 12x
Sample reason for why the expression is correct: T is a linear function; thus, the graph is a straight line. Since two points are sufficient to determine a line, if the formula agrees with two known points, it must be correct. The expression gives 600 peaches per tree when there are 30 trees per acre, 528 peaches per tree when there are 36 trees per acre, and 456 peaches per tree when there are 42 trees per acre.

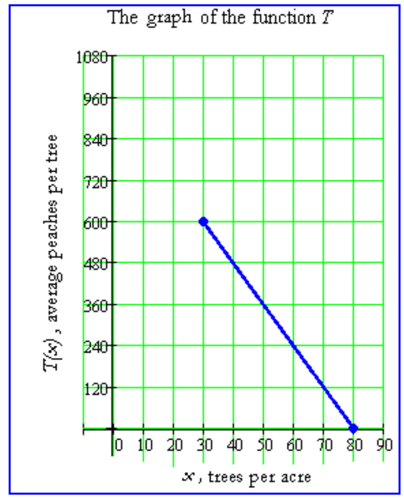
d. Draw a graph of the function *T*. Given that the information from the agricultural experiment station applies only to increasing the number of trees per acre, what is an appropriate domain for the function *T*?

Comments:
Students should realize that the graph of the function consists of points on the graph of the line $y = -12x + 960$ ***whose first coordinates are integers greater than or equal to 30. They should also realize that the output of the function cannot be negative and stop the graph at the x-intercept point, (80, 0). The first graph shown below uses the technique of showing the graph of T as a collection of discrete points on the graph of the function with same formula but domain all real numbers. The second graph is typical of what students may draw. On a smaller scale, the space between the discrete points is not visible.***



<u>Solution:</u>

<u>Domain</u> of T: The set of all positive integers from 30 through 80, that is, $\{30, 31, 32, 33, \ldots, 79, 80\}$.



<u>Comments</u>: Students should realize that the domain is restricted to integer values. As indicated above, if they use a small scale, the discrete nature of the domain may not show up on the graph. Thus, it is important that they answer the question about domain and show that they know the domain contains only positive integers.

2. Since her income from peaches depends on the total number of peaches she produces, Paula realized that she needed to take a next step and consider the total number of peaches that she can produce **per acre**.
    a. With the current 30 trees per acre, what is the yield in total peaches per acre? If Paula plants 36 trees per acre, what will be the yield in total peaches per acre? 42 trees per acre?

(600 peaches per tree)·(30 trees per acre) = 18000 peaches per acre

With 36 trees per acre, there will be 528 peaches per tree.
(528 peaches per tree)·(36 trees per acre) = 19008 peaches per acre
With 42 trees per acre, there will be 456 peaches per tree.
(456 peaches per tree)·(42 trees per acre) = 19152 peaches per acre

<u>Comments</u>: These questions are designed to help students shift their thinking to the issue of total yield of peaches per acre. Students see that for these values, while the number of peaches per tree goes down, the total yield increases.

b. Find the average rate of change of peaches per acre with respect to number of trees per acre when the number of trees per acre increases from 30 to 36. Write a sentence to explain what this number means.

<u>Comments:</u>
The unit of measure for this rate of change is peaches per acre per trees per acre – not an easy unit with which to work. Hence, the question asks for a sentence explanation of the number 168. This question and the next one are intentional review questions but also make sure that students are prepared to answer the question in part g.

Solution:
change in peaches per acre: 19008 – 18000 = 1008
change in trees per acre: 36 – 30 = 6
average rate of change: $\frac{1008}{6} = 168$
On the average, while increasing from 30 to 36 trees per acre, each additional tree per acre produces a total yield of 168 more peaches per acre.

c. Find the average rate of change of peaches per acre with respect to the number of trees per acre when the number of trees per acre increases from 36 to 42. Write a sentence to explain the meaning of this number.

change in peaches per acre: 19152 – 19008 = 144
change in trees per acre: 42 – 36 = 6
average rate of change: $\frac{144}{6} = 24$
On the average, while increasing from 36 to 42 trees per acre, each additional tree per acre produces a total yield of 24 more peaches per acre.

d. Is the relationship between number of trees per acre and yield in peaches per acre linear? Explain your reasoning.

No, the relationship is not linear. We know the relationship is not linear because the average rate of change is not constant. Linear relationships have a constant average rate of change.

e. Let Y be the function that expresses this relationship, that is, the function for which the input x is the number of trees planted on each acre and the output $Y(x)$ is the total yield in peaches per acre. Write a formula for $Y(x)$ in terms of x and express your answer in expanded form.

$Y(x) = x(960 - 12x)$ ***or*** $Y(x) = 960x - 12x^2$
The number of trees per acre must be multiplied by the number of peaches per tree to get the total number of peaches per acre.

f. Calculate $Y(30)$, $Y(36)$, and $Y(42)$. What is the meaning of these values? How are they related to your answers to parts a through c?

Comments:
This question is designed to let students verify that they have written a correct formula; however, the prompt is not stated as such because students will not learn until a higher mathematics course that three points determine a parabola. So, a more accurate description of this item is that it allows students to verify that they do not have a wrong formula.

Solution:

Y(30) = 30(960 – 360) = 30(600) = 18000 or

$Y(30) = 960(30) - 12(30)^2 = 28800 - 12(900) = 28800 - 10800 = 18000$

Y(36) = 36(960 – 432) = 36(528) = 19008 or

$Y(36) = 960(36) - 12(36)^2 = 34560 - 12(1296) = 34560 - 15552 = 19008$

Y(42) = 42(960 – 504) = 42(456) = 19152 or

$Y(42) = 960(42) - 12(42)^2 = 40320 - 12(1764) = 40320 - 21168 = 19152$

Y(30) = 18000 means that there are 18000 peaches per acre when there are 30 trees per acre.
Y(36) = 19008 means that there are 19008 peaches per acre when there are 36 trees per acre.
Y(42) = 19152 means that there are 19152 peaches per acre when there are 42 trees per acre.
The three values are the answers to parts a through c, respectively, because they express the same idea in function notation.

g. What is the relationship between the domain for the function *T* and the domain for the function *Y*? Explain.

The domains are the same. The formula for the function Y applies only to possible numbers of peach trees per acre. These are the values listed in the domain of T.

3. Paula wants to know whether there is a different number of trees per acre that will give the same yield per acre as the yield when she plants 30 trees per acre.
   a. Write an equation that expresses the requirement that *x* trees per acre yields the same total number of peaches per acre as planting 30 trees per acre.

Equation: $960x - 12x^2 = 18000$ or $x(960 - 12x) = 18000$

   b. Use the algebraic rules for creating equivalent equations to obtain an equivalent equation with an expression in *x* on one side of the equation and 0 on the other.

$x(960 - 12x) = 18000$

$960x - 12x^2 = 18000$

$-12x^2 + 960x - 18000 = 0$

   c. Multiply this equation by an appropriate rational number so that the new equation is of the form $x^2 + bx + c = 0$. Explain why this new equation has the same solution set as the equations from parts a and b.

$-\frac{1}{12}(-12x^2 + 960x - 18000) = -\frac{1}{12}(0)$

$x^2 - 80x + 1500 = 0$

Each of the equations shown in parts b and c is obtained from the previous equation by application of the Addition Property of Equality or the Multiplication Property of Equality. Equations obtained by the use of the properties have the same solution set as the original equation.

d. When the equation is in the form $x^2 + bx + c = 0$, what are the values of *b* and *c*

b = – 80 and c = 1500

e. Find integers *m* and *n* such that *m·n* = *c* and *m* + *n* = *b*.

(– 30)(– 50) = 1500 and (– 30) + (– 50) = – 80, so m = – 30 and n = – 50.

f. Using the values of *m* and *n* found in part e, form the algebraic expression $(x+m)(x+n)$ and simplify it.

(x – 30)(x – 50) = $x^2 - 80x + 1500$

g. Combining parts d through f, rewrite the equation from part c in the form $(x+m)(x+n)=0$.

$x^2 - 80x + 1500 = 0$ ***from part c is equivalent to the equation (x – 30)(x – 50) = 0.***

h. This equation expresses the idea that the product of two numbers, $x+m$ and $x+n$, is equal to 0. We know from the discussion in Unit 2 that, when the product of two numbers is 0, one of the numbers has to be 0. This property is called the ***Zero Product Property***. For these particular values of *m* and *n*, what value of *x* makes $x+m=0$ and what value of *x* makes $x+n=0$?

If x – 30 = 0, then x = 30. ***If x – 50 = 0, then x = 50.***

i. Verify that the answers to part h are solutions to the equation written in part a. It is appropriate to use a calculator for the arithmetic.

We substitute 30 for x in the left-hand side of the equation $960x - 12x^2 = 18000$ ***to obtain*** $960(30) - 12(30)^2 = 28800 - 12(900) = 28800 - 10800 = 18000$ ***to show that 30 satisfies the equation.***

We substitute 50 for x in the left-hand side of the equation $960x - 12x^2 = 18000$ ***to obtain*** $960(50) - 12(50)^2 = 48000 - 12(2500) = 48000 - 30000 = 18000$ ***to show that 50 satisfies the equation.***

j. Write a sentence to explain the meaning of your solutions to the equation in relation to planting peach trees.

If Paula plants 30 trees per acre or 50 trees per acre, the yield in total peaches per acre is 18000.

4. Paula saw another peach grower, Sam, from a neighboring county at a farm equipment auction and began talking to him about the possibilities for the new section of her orchard. Sam was surprised to learn about the agricultural research and said that it probably explained the drop in yield for an orchard near him. This peach farm has more than 30 trees per acre and is getting an average total yield of 14,400 peaches per acre. *(Remember: Throughout this task assume that, for all peach growers in this area, the average yield is 600 peaches per tree when 30 trees per acre are planted and that this yield will decrease by 12 peaches per tree for each additional tree per acre.)*

a. Write an equation that expresses the situation that x trees per acre results in a total yield per acre of 14,400 peaches per acre.

Equation: $960x - 12x^2 = 14400$ ***or*** $x(960 - 12x) = 14400$

b. Use the algebraic rules for creating equivalent equations to obtain an equivalent equation with an expression in x on one side of the equation and 0 on the other.

$$x(960 - 12x) = 14400$$
$$960x - 12x^2 = 14400$$
$$-12x^2 + 960x - 14400 = 0$$

c. Multiply this equation by an appropriate rational number so that the new equation is of the form $x^2 + bx + c = 0$. Explain why this new equation has the same solution set as the equations from parts a and b.

$$-\frac{1}{12}(-12x^2 + 960x - 18000 = -\frac{1}{12}(0)$$
$$x^2 - 80x + 1500 = 0$$

d. When the equation is in the form $x^2 + bx + c = 0$, what is value of b and what is the value of c?

$$b = -80 \text{ and } c = 1200$$

e. Find integers m and n such that $m \cdot n = c$ and $m + n = b$.

$$(-20)(-60) = 1200 \text{ and } (-20) + (-60) = -80, \text{ so } m = -20 \text{ and } n = -60.$$

a. Using the values of m and n found in part e, form the algebraic expression $(x+m)(x+n)$ and simplify $(x+m)(x+n)$.

$(x – 20)(x – 60) = x^2 – 80x + 1200$

b. Combining parts d through f, rewrite the equation from part d in the form $(x+m)(x+n)=0$.

$x^2 – 80x + 1200$ from part c is equivalent to the equation $(x – 20)(x – 60) = 0$.

c. This equation expresses the idea that the product of two numbers, $x+m$ and $x+n$, is equal to 0. We know from the discussion in Unit 2 that, when the product of two numbers is 0, one of the numbers has to be 0. What value of x makes $x+m=0$? What value of x makes $x+n=0$?

If $x – 20 = 0$, then $x = 20$. ***If $x – 60 = 0$, then $x = 60$.***

d. Verify that the answers to part h are solutions to the equation written in part a. It is appropriate to use a calculator for the arithmetic.

We substitute 20 for x in the left–hand side of the equation $960x – 12x^2 = 14400$ to obtain $960(20) – 12(20)^2 = 19200 – 12(400) = 19200 – 4800 = 14400$ to show that 20 satisfies the equation.

We substitute 60 for x in the left–hand side of the equation $960x – 12x^2 = 14400$ to obtain $960(60) – 12(60)^2 = 57600 – 12(3600) = 57600 – 43200 = 14400$ to show that 60 satisfies the equation.

e. Which of the solutions verified in part i is (are) in the domain of the function *Y*? How many peach trees per acre are planted at the peach orchard getting 14400 peaches per acre?

Comments:
This question provides the opportunity for discussing the distinction between solving an equation and solving a problem in context. This distinction is often critical in using a quadratic equation to solve an applied problem. Solving an equation in one variable requires that we find all the real numbers which satisfy the given equation. However, the context may put additional restrictions on the solution that are not implied by the equation. In this context the number of trees per acre must be between 30 and 80, inclusive. In other situations, the desired quantity is a measurement that must be positive. In other situations, the problem situation requires solutions that are whole numbers. Thus, the problem solver must eliminate those solutions to the equation that do not satisfy conditions of the context. Since this issue often arises in working with quadratic equations, the task introduces the idea early in the discussion of solving such equations.

Solution:
The solution x = 60 is in the domain of the function Y. The solution x = 20 is not. Thus, there must be 60 trees per acre at the orchard getting a yield 14400 peaches per acre.

The steps in items 3 and 4 outline a method of solving equations of the form $x^2 + bx + c$. These equations are called ***quadratic equations*** and an expression of the form $x^2 + bx + c$ is called a ***quadratic expression***. In general, quadratic expressions may have any nonzero coefficient on the x^2 term. An important part of this method for solving quadratic expressions with coefficient 1 on the x^2term. An important part of this method for solving quadratic equations is the process of rewriting an expression of the form $x^2 + bx + c$ in the form $(x + m)(x + n)$. The identity tells us that the product of the numbers m and n must equal c and that the sum of m and n must equal b.

5. Since the whole expression $(x + m)(x + n)$ is a product, we call the expressions $x + m$ and $x + n$ the **factors** of this product. For the following expressions in the form $x^2 + bx + c$, rewrite the expression as a product of factors of the form $x + m$ and $x + n$. Verify each answer by drawing a rectangle with sides of length $x + m$ and $x + n$, respectively, and showing geometrically that the area of the rectangle is $x^2 + bx + c$.

a. $x^2 + 3x + 2$
(x + 1)(x + 2)

b. $x^2 + 6x + 5$
(x + 1)(x + 5)

c. $x^2 + 5x + 6$
(x + 2)(x + 3)

e. $x^2 + 8x + 12$
(x + 2)(x + 6)

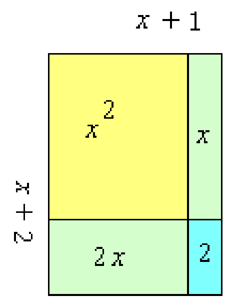
f. $x^2 + 13x + 36$
(x + 4)(x + 9)

g. $x^2 + 13x + 12$
(x + 1)(x + 12)

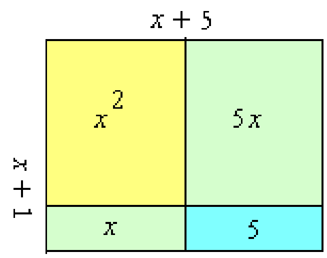
d. $x^2 + 7x + 12$
(x + 3)(x + 4)

The rectangles are drawn below.

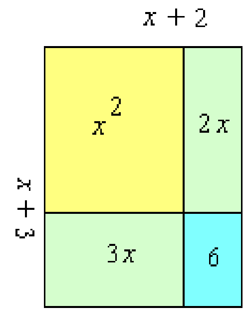
<u>Comments</u>:
The verification asks students to revisit the area model of multiplication and see the factors x + m and x + n as the lengths of sides of a rectangle whose area is the given quadratic expression. It is important to revisit geometric models so that students have concrete meaning for the factoring process. Students need to see the pattern in factoring, but they need to do more than manipulate patterns.



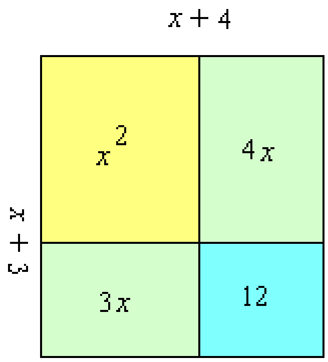
area = $x^2 + 3x + 2$



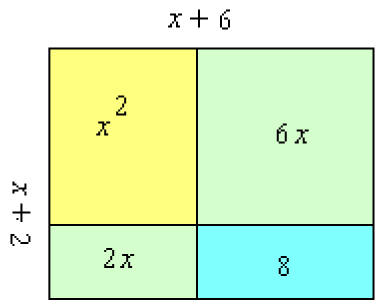
area = $x^2 + 6x + 5$



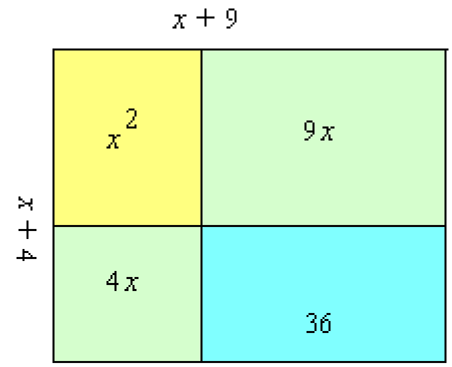
area = $x^2 + 5x + 6$



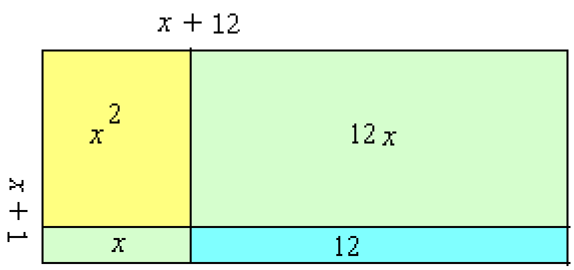
area = $x^2 + 7x + 12$



area = $x^2 + 8x + 12$



area = $x^2 + 13x + 36$



area = $x^2 + 13x + 12$

6. In item 5, the values of b and c were positive. Now use Identity 1 in reverse to factor each of the following quadratic expressions of the form $x^2 + bx + c$ where c is positive but b is negative. Verify each answer by multiplying the factored form to obtain the original expression.

a. $x^2 - 8x + 7$
(x – 1)(x – 7)

b. $x^2 - 9x + 18$
(x – 3)(x – 6)

c. $x^2 - 4x + 4$
(x – 2)(x – 2)

d. $x^2 - 8x + 15$
(x – 3)(x – 5)

e. $x^2 - 11x + 24$
(x – 3)(x – 8)

f. $x^2 - 11x + 18$
(x – 2)(x – 9)

g. $x^2 - 12x + 27$
(x – 3)(x – 9)

The checks by multiplication are shown below.

a. $(x - 1)(x - 7) = x(x - 7) + (-1)(x - 7) = x^2 - 7x - x + 7 = x^2 - 8x + 7$

b. $(x - 3)(x - 6) = x(x - 6) + (-3)(x - 6) = x^2 - 6x - 3x + 18 = x^2 - 9x + 18$

c. $(x - 2)(x - 2) = x(x - 2) + (-2)(x - 2) = x^2 - 2x - 2x + 4 = x^2 - 4x + 4$

d. $(x - 3)(x - 5) = x(x - 5) + (-3)(x - 5) = x^2 - 5x - 3x + 15 = x^2 - 8x + 15$

e. $(x - 3)(x - 8) = x(x - 8) + (-3)(x - 8) = x^2 - 8x - 3x + 24 = x^2 - 11x + 24$

f. $(x - 2)(x - 9) = x(x - 9) + (-2)(x - 9) = x^2 - 9x - 2x + 18 = x^2 - 11x + 18$

g. $(x - 3)(x - 9) = x(x - 9) + (-3)(x - 9) = x^2 - 9x - 3x + 27 = x^2 - 12x + 27$

Paula's Peaches Continued!

7. Now we return to the peach growers in central Georgia. How many peach trees per acre would result in only 8400 peaches per acre? Which answer makes sense in this particular context?

$Y(x) = 960x - 12x^2$ We want to find x such that $Y(x) = 8400$, so we need to solve the equation $960x - 12x^2 = 8400$. We put the equation in the standard form $x^2 + bx + c = 0$ and solve.

$$960x - 12x^2 = 8400$$
$$-12x^2 + 960x - 8400 = 0$$
$$-\frac{1}{12}(-12x^2 + 960x - 8400) = -\frac{1}{12}(0)$$
$$x^2 - 80x + 700 = 0$$
$$(x - 10)(x - 70) = 0$$
$$x - 10 = 0 \quad \text{or} \quad x - 70 = 0$$
$$x = 10 \quad \text{or} \quad x = 70$$

Both 10 and 70 solve the equation, but the expression for total peaches does not apply to values of x less than 30, so 70 is the only solution that meets the requirements of the context. Thus, we can conclude that it would take 70 trees per acre to have a yield of only 8400 peaches per acre.

8. If there are no peach trees on a property, then the yield is zero peaches per acre. Write an equation to express the idea that the yield is zero peaches per acre with x trees planted per acre, where x is number greater than 30. Is there a solution to this equation? Explain why only one of the solutions makes sense in this context.

<u>Comments</u>:

The situation with the solution $x = 0$ is a little bit complicated. The number 0 solves the equation. It is true that 0 trees per acre will result in a total yield of 0 peaches per acre, but this is just common sense. We cannot interpret the solution $x = 0$ as telling us that 0 trees per acre will result in a total yield of 0 peaches per acre because the equation is meaningful only for integer values of x from 30 through 80.

<u>Solution</u>:

We want to find x such that $Y(x) = 0$, so the equation is: $960x - 12x^2= 0$.. We put the equation in the standard form $x^2 + bx + c = 0$ and solve.

$$-12x^2 + 960x = 0$$
$$-\frac{1}{12}(-12x^2 + 960x) = -\frac{1}{12}(0)$$
$$x^2 - 80x = 0$$
$$x(x - 80) = 0$$
$$x = 0 \quad \text{or} \quad x - 80 = 0$$
$$x = 0 \quad \text{or} \quad x = 80$$

The equation expressing the idea that the yield is 0 peaches per acre has a solution for more than 30 trees per acre; the answer is 80 trees per acre. This is consistent with the original information. In item 1, we saw that 80 trees per acre would reduce the yield to 0 peaches per tree. Thus, the total yield per acre is 0 peaches.

9. At the same auction where Paula heard about the peach grower who was getting a low yield, she talked to the owner of a major farm supply store in the area. Paula began telling the store owner about her plans to expand her orchard, and the store owner responded by telling her about a local grower that gets 19,200 peaches per acre. Is this number of peaches per acre possible? If so, how many trees were planted?
To answer this question we need to solve the equation $960x - 12x^2 = 19200$.

$$-12x^2 + 960x - 19200 = 0$$

$$-\frac{1}{12}(-12x^2 + 960x - 19200) = -\frac{1}{12}(0)$$

$$x^2 - 80x + 1600 = 0$$

$$(x - 40)(x - 40) = 0$$

$$x - 40 = 0$$

$$x = 40$$

Yes, with 40 trees per acre, the total yield in 19200 peaches per acre.

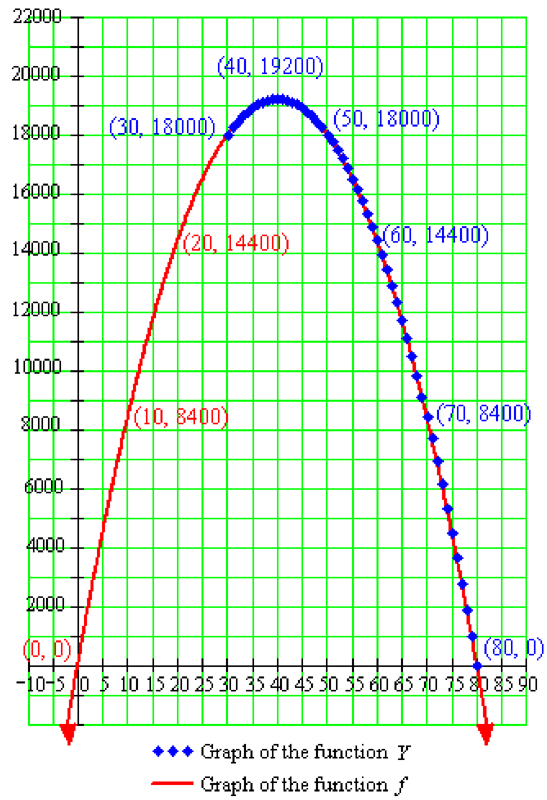
10. Using graph paper, explore the graph of Y as a function of x.

a. What points on the graph correspond to the answers for part j from questions 3 and 4?
(30, 18000), (50, 18000), (60, 14400)

b. What points on the graph correspond to the answers to items 7, 8, and 9?
(70, 8400), (80, 0), (40, 19200)

c. What is the relationship of the graph of the function Y to the graph of the function f, where the formula for $f(x)$ is the same as the formula for $Y(x)$ but the domain for f is all real numbers?

The graph of the function Y consists of those points on the graph of the function f such that x is in the domain for Y = {30, 31, 32, …, 79, 80}.



d. Items 4, 7, and 8 give information about points that are on the graph of f but not on the graph of Y. What points are these?

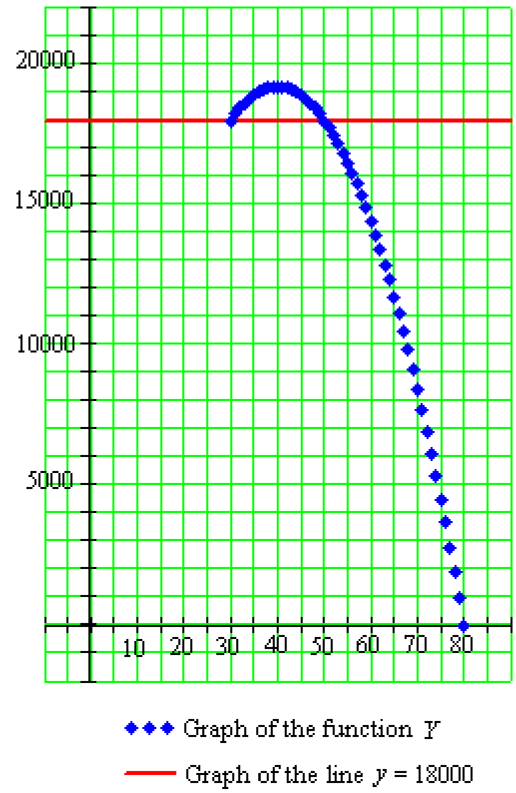
(20, 14400), (10, 8400), (0, 0)

e. Graph the functions f and Y on the same axes. How does your graph show that the domain of f is all real numbers? How is the domain of Y shown on your graph?

<u>Solution</u>:
The graphs of f and Y are shown at the right. The graph of f shows that the domain is all real numbers by indicating that the graph continues beyond the grid area shown. The graph of Y shows that domain is the set of integers from 30 through 80 by showing distinct points on the graph of f at integer values. Note that, for part of the domain, the discrete points are close enough together that drawing them with dots fills in the line.

<u>Comments</u>:
The graph at the right was drawn using technology to plot all 51 points on the graph of Y accurately. It is provided to show that even on a small scale, the points on the graph of Y show up as discrete points. It is unreasonable to expect students to calculate and graph all 51 points accurately. However, they can draw the graph of f and then indicate integer spaced points on this graph.

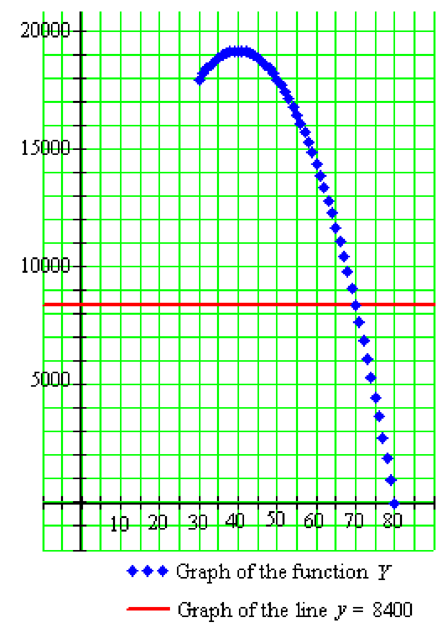
f. Draw the line $y = 18000$ on the graph drawn for item 10, part e. This line is the graph of the function with constant value 18000. Where does this line intersect the graph of the function Y? Based on the graph, how many trees per acre give a yield of more than 18000 peaches per acre?



Comments:
This question shows this part of the solution to the equation as the intersection of the line and the graph of Y gives more information than the algebraic solution of the equation.

Solution:
The line is shown on the graph at the right. The line intersects the graph of the function Y at the points (30. 18000) and (50, 18000). Based on the graph, when the number of trees is 31, 32, …, 49, that is, when the number of trees is strictly between 30 and 50, the yield is more than 18000 peaches per acre.

g. Draw the line $y = 8400$ on your graph. Where does this line intersect the graph of the function Y? Based on the graph, how many trees per acre give a yield of fewer than 8400 peaches per acre?



The line is shown on the graph at the right. The line intersects the graph of the function Y at the point (70, 8400). Based on the graph, when the number of trees is 71, 72, …, 80, that is, when the number of trees is more than 70, the yield is fewer than 8400 peaches per acre.

h. Use a graphing utility and this intersection method to find the number of trees per acre that give a total yield closest to the following numbers of peaches per acre:
(i) 10000 (ii) 15000
(iii) 20000

Using the trace feature on the graph at the right we find that (i) 68 trees per acre gives the yield closest to 10000 peaches per acre, (ii) 59 trees per acre gives the yield closest to 15000 peaches per acre, and (iii) 40 trees per acre gives the yield closest to 20000 peaches per acre.

i. Find the value of the function Y for the number of trees given in answering (i) – (iii) in part c above.

Y(68) = 68(960 – 816) = 68(144) = 9792
Y(59) = 59(960 – 708) = 59(252) = 14868
Y(40) = 40(960 – 480) = 40(480) = 19200

11. In items 5 and 6, we used factoring as part of a process to solve equations that are equivalent to equations of the form $x^2 + bx + c = 0$ where b and c are integers. Look back at the steps you did in items 3 and 4, and describe the process for solving an equation of the form $x^2 + bx + c = 0$. Use this process to solve each of the following equations, that is, to find all of the numbers that satisfy the original equation. Verify your work by checking each solution in the original equation.

Step 1: Factor the expression $x^2 + bx + c$ and write an equivalent equation containing the factored form.
Step 2: By the Zero Product Property, since the product is zero, one of the factors must equal zero. Set each factor equal to zero.
Step 3: Solve each equation from Step 2. Each of the solutions is a solution for the quadratic equation.

a. $x^2 - 6x + 8 = 0$

(x – 2)(x – 4) = 0
x – 2 = 0 or x – 4 = 0
x = 2 or x = 4
Solutions: 2, 4

Checks:
$(2)^2 - 6(2) + 8 = 4 - 12 + 8 = 0$
$(4)^2 - 6(4) + 8 = 16 - 24 + 8 = 0$

b. $x^2 - 15x + 36 = 0$

(x – 3)(x – 12) = 0
x – 3 = 0 or x – 12 = 0
x = 3 or x = 12
Solutions: 3, 12

Checks:
$(3)^2 - 15(3) + 36 = 9 - 45 + 36 = 0$
$(12)^2 - 15(12) + 36 = 144 - 180 + 36 = 0$

c. $x^2 + 28x + 27 = 0$

(x + 1)(x + 27) = 0
x + 1 = 0 or x + 27 = 0
x = –1 or x = –27
Solutions: –1, –27

Checks:
$(-1)^2 + 28(-1) + 27 = 1 - 28 + 27 = 0$
$(-27)^2 + 28(-28) + 27 = 729 - 756 + 27 = 0$

d. $x^2 - 3x - 10 = 0$

(x + 2)(x – 5) = 0
x + 2 = 0 or x – 5 = 0
x = –2 or x = 5

Checks:
$(-2)^2 - 3(-2) - 10 - 4 + 6 - 10 = 0$
$(5)^2 - 3(5) - 10 = 25 - 15 - 10 = 0$

Solutions: –2, 5

e. $x^2 + 2x - 15 = 0$

(x – 3)(x + 5) = 0
x – 3 = 0 or x + 5 = 0
x = 3 or x = –5
Solutions: 3, –5

Checks:
$(3)^2$ + 2(3) – 15 = 9 + 6 – 15 = 0
$(–5)^2$ + 2(–5) – 15 = 25 – 10 – 25 = 0

f. $x^2 - 4x - 21 = 0$

(x + 3)(x – 7) = 0
x + 3 = 0 or x – 7 = 0
x = –3 or x = 7
Solutions: –3, 7

Checks:
$(–3)^2$ – 4(–3) – 21 = 9 + 12 – 21 = 0
$(7)^2$ – 4(7) – 21 = 49 – 28 – 21 = 0

g. $x^2 - 7x = 0$

(x)(x – 7) = 0
x = 0 or x – 7 = 0
x = 0 or x = 7
Solutions: 0, 7

Checks:
$(0)^2$ – 7(0) = 0 – 0 = 0
$(7)^2$ – 7(7) = 49 – 49 = 0

h. $x^2 + 13x = 0$

(x)(x + 13) = 0
x = 0 or x + 13 = 0
x = 0 or x = –13
Solutions: 0, –13

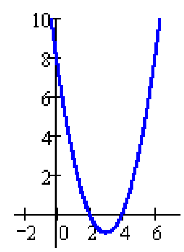
Checks:
$(0)^2$ + 13(0) = 0 + 0 = 0
$(–13)^2$ + (13)(–13) = 169 – 169 = 0

12. For each of the equations solved in question 11, do the following:

a. Use technology to graph a function whose formula is given by the left–hand side of the equation.
b. Find the points on the graph which correspond to the solutions found in item 8.
c. How is each of these results an example of the intersection method explored above?

a. The graph of y = x^2 – 6x + 8 is shown at the right.

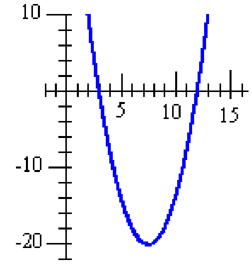
b. The numbers 2 and 4 are solutions to the quadratic equation x^2 – 6x + 8= 0. The corresponding points are (2, 0) and (4, 0).



---

a. The graph of $y = x^2 – 15x + 36$ is shown at the right.

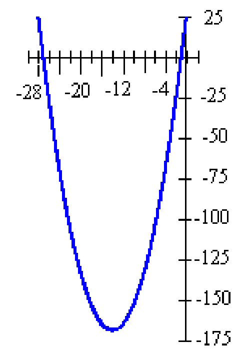
b. The numbers 3 and 12 are solutions to the quadratic equation $x^2 – 15x + 36 = 0$. The corresponding points are (3, 0) and (12, 0).



---

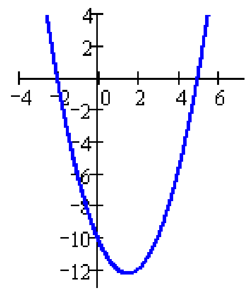
a. The graph of $y = x^2 + 28x + 27$ is shown at the right.

b. The numbers –1 and –27 are solutions to the quadratic equation $x^2 + 28x + 27 = 0$. The corresponding points are (–1, 0) and (–27, 0).



a. The graph of $y = x^2 - 3x – 10$ is shown at the right.

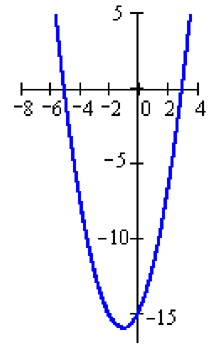
b. The numbers 5 and –2 are solutions to the quadratic equation $x^2 – 3x – 10 = 0$. The corresponding points are (5, 0) and (–2, 0).



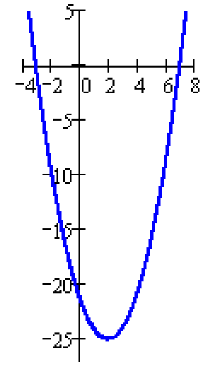
---

a. The graph of $y = x^2 + 2x – 15$ is shown at the right.

b. The numbers 3 and –5 are solutions to the quadratic equation $x^2 + 2x – 15 = 0$. The corresponding points are (3, 0) and (–5, 0).

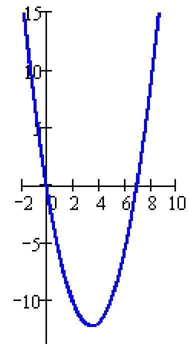


a. The graph of $y = x^2 - 4x - 21$ is shown at the right.



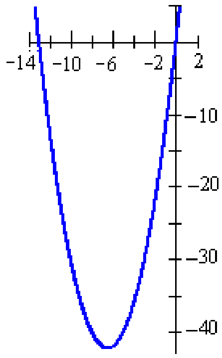
b. The numbers 7 and –3 are solutions to the quadratic equation $x^2 - 4x - 21 = 0$. The corresponding points are (7, 0) and (–3, 0).

a. The graph of $y = x^2 - 7x$ is shown at the right.



b. The numbers 0 and 7 are solutions to the quadratic equation $x^2 - 7x = 0$. The corresponding points are (0, 0) and (7, 0).

a. The graph of $y = x^2 + 13x$ is shown at the right.



b. The numbers 0 and –13 are solutions to the quadratic equation $x^2 + 13x = 0$. The corresponding points are (0, 0) and (–13, 0).

c. For each graph, the points that correspond to the solutions of the equation are the x–intercepts of the graph. The x–axis is the line $y = 0$. So the solutions of the original equation are the x–coordinates of the intersection points of two functions, one given by an equation of the form $y = x^2 + bx + c$ and the other given by the equation $y = 0$.

Paula's Peaches

Paula is a peach grower in central Georgia and wants to expand her peach orchard. In her current orchard, there are 30 trees per acre and the average yield per tree is 600 peaches. Data from the local agricultural experiment station indicates that if Paula plants more than 30 trees per acre, once the trees are in production, the average yield of 600 peaches per tree will decrease by 12 peaches for each tree over 30. She needs to decide how many trees to plant in the new section of the orchard.

1. Paula believes that algebra can help her determine the best plan for the new section of orchard and begins by developing a mathematical model of the relationship between the number of trees per acre and the average yield in **peaches per tree.**
   a. Is this relationship linear or nonlinear? Explain your reasoning.

   b. If Paula plants 6 more trees per acre, what will be the **average** yield in peaches per tree? What is the **average** yield in peaches per tree if she plants 42 trees per acre?

   c. Let T be the function for which the input x is the number of trees planted on each acre and $T(x)$ is the average yield in peaches per tree. Write a formula for $T(x)$ in terms of x and express it in simplest form. Explain how you know that your formula is correct.

   d. Draw a graph of the function T. Given that the information from the agricultural experiment station applies only to increasing the number of trees per acre, what is an appropriate domain for the function T?

2. Since her income from peaches depends on the total number of peaches she produces, Paula realized that she needed to take a next step and consider the total number of peaches that she can produce **per acre**.

   a. With the current 30 trees per acre, what is the yield in total peaches per acre? If Paula plants 36 trees per acre, what will be the yield in total peaches per acre? 42 trees per acre?

   b. Find the average rate of change of peaches per acre with respect to number of trees per acre when the number of trees per acre increases from 30 to 36. Write a sentence to explain what this number means.

   c. Find the average rate of change of peaches per acre with respect to the number of trees per acre when the number of trees per acre increases from 36 to 42. Write a sentence to explain the meaning of this number.

d. Is the relationship between number of trees per acre and yield in peaches per acre linear? Explain your reasoning.

e. Let Y be the function that expresses this relationship; that is, the function for which the input x is the number of trees planted on each acre and the output $Y(x)$ is the total yield in peaches per acre. Write a formula for $Y(x)$ in terms of x and express your answer in expanded form.

f. Calculate $Y(30)$, $Y(36)$, and $Y(42)$. What is the meaning of these values? How are they related to your answers to parts a through c?

g. What is the relationship between the domain for the function T and the domain for the function Y? Explain.

3. Paula wants to know whether there is a different number of trees per acre that will give the same yield per acre as the yield when she plants 30 trees per acre.

a. Write an equation that expresses the requirement that x trees per acre yields the same total number of peaches per acre as planting 30 trees per acre.

b. Use the algebraic rules for creating equivalent equations to obtain an equivalent equation with an expression in x on one side of the equation and 0 on the other.

c. Multiply this equation by an appropriate rational number so that the new equation is of the form $x^2 + bx + c = 0$. Explain why this new equation has the same solution set as the equations from parts a and b.

d. When the equation is in the form $x^2 + bx + c = 0$, what are the values of b and c

e. Find integers m and n such that $m \cdot n = c$ and $m + n = b$.

f. Using the values of m and n found in part e, form the algebraic expression $\left(x+m\right)\left(x+n\right)$ and simplify it.

g. Combining parts d through f, rewrite the equation from part c in the form $\left(x+m\right)\left(x+n\right)=0$.

h. This equation expresses the idea that the product of two numbers, $x+m$ and $x+n$, is equal to 0. We know from the discussion in Unit 2 that, when the product of two numbers is 0, one of the numbers has to be 0. This property is called the ***Zero Product Property***. For these particular values of *m* and *n*, what value of *x* makes $x+m=0$ and what value of *x* makes $x+n=0$?

i. Verify that the answers to part h are solutions to the equation written in part a. It is appropriate to use a calculator for the arithmetic.

j. Write a sentence to explain the meaning of your solutions to the equation in relation to planting peach trees.

4. Paula saw another peach grower, Sam, from a neighboring county at a farm equipment auction and began talking to him about the possibilities for the new section of her orchard. Sam was surprised to learn about the agricultural research and said that it probably explained the drop in yield for a orchard near him. This peach farm has more than 30 trees per acre and is getting an average total yield of 14,400 peaches per acre. *(Remember: Throughout this task assume that, for all peach growers in this area, the average yield is 600 peaches per tree when 30 trees per acre are planted and that this yield will decrease by 12 peaches per tree for each additional tree per acre.)*

a. Write an equation that expresses the situation that *x* trees per acre results in a total yield per acre of 14,400 peaches per acre.

b. Use the algebraic rules for creating equivalent equations to obtain an equivalent equation with an expression in *x* on one side of the equation and 0 on the other.

c. Multiply this equation by an appropriate rational number so that the new equation is of the form $x^2+bx+c=0$. Explain why this new equation has the same solution set as the equations from parts and b.

d. When the equation is in the form $x^2+bx+c=0$, what is value of *b* and what is the value of *c*?

e. Find integers *m* and *n* such that $m \cdot n = c$ and $m + n = b$.

f. Using the values of *m* and *n* found in part e, form the algebraic expression $(x+m)(x+n)$ and simplify it.

g. Combining parts d through f, rewrite the equation from part d in the form $(x+m)(x+n)=0$.

h. This equation expresses the idea that the product of two numbers, $x + m$ and $x + n$, is equal to 0. We know from the discussion in Unit 2 that, when the product of two numbers is 0, one of the numbers has to be 0. What value of x makes $x + m = 0$? What value of x makes $x + n = 0$?

i. Verify that the answers to part h are solutions to the equation written in part a. It is appropriate to use a calculator for the arithmetic.

j. Which of the solutions verified in part i is (are) in the domain of the function Y? How many peach trees per acre are planted at the peach orchard getting 14400 peaches per acre?

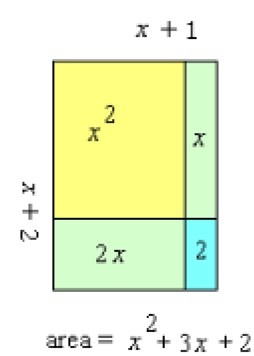
The steps in items 3 and 4 outline a method of solving equations of the form $x^2 + bx + c$. These equations are called ***quadratic equations*** and an expression of the form $x^2 + bx + c$ is called a ***quadratic expression***. In general, quadratic expressions may have any nonzero coefficient on the x^2 term. An important part of this method for solving quadratic expressions with coefficient 1 on the x^2term. An

important part of this method for solving quadratic equations is the process of rewriting an expression of the form $x^2 + bx + c$ in the form $(x + m)(x + n)$. The identity tells us that the product of the numbers m and n must equal c and that the sum of m and n must equal b.

5. Since the whole expression $(x + m)(x + n)$ is a product, we call the expressions $x + m$ and $x + n$ the **factors** of this product. For the following expressions in the form $x^2 + bx + c$, rewrite the expression as a product of factors of the form $x + m$ and $x + n$. Verify each answer by drawing a rectangle with sides of length $x + m$ and $x + n$, respectively, and showing geometrically that the area of the rectangle is $x^2 + bx + c$.

On a separate sheet of paper:

Example: $x^2 + 3x + 2$
Solution: (x + 1)*(x + 2)



a. $x^2 + 6x + 5$

b. $x^2 + 5x + 6$

c. $x^2 + 7x + 12$

d. $x^2 + 8x + 12$

e. $x^2 + 13x + 36$

f. $x^2 + 13x + 12$

6. In item 5, the values of b and c were positive. Now use Identity 1 in reverse to factor each of the following quadratic expressions of the form $x^2 + bx + c$ where c is positive but b is negative. Verify each answer by multiplying the factored form to obtain the original expression.

On a separate sheet of paper:

a. $x^2 - 8x + 7$

b. $x^2 - 9x + 18$

c. $x^2 - 4x + 4$

d. $x^2 - 8x + 15$

e. $x^2 - 11x + 24$

f. $x^2 - 11x + 18$

g. $x^2 - 12x + 27$

Paula's Peaches Continued!

7. Now we return to the peach growers in central Georgia. How many peach trees per acre would result in only 8400 peaches per acre? Which answer makes sense in this particular context?

8. If there are no peach trees on a property, then the yield is zero peaches per acre. Write an equation to express the idea that the yield is zero peaches per acre with x trees planted per acre, where x is number greater than 30. Is there a solution to this equation? Explain why only one of the solutions makes sense in this context.

9. At the same auction where Paula heard about the peach grower who was getting a low yield, she talked to the owner of a major farm supply store in the area. Paula began telling the store owner about her plans to expand her orchard, and the store owner responded by telling her about a local grower that gets 19,200 peaches per acre. Is this number of peaches per acre possible? If so, how many trees were planted?

10. Using graph paper, explore the graph of *Y* as a function of *x*.

   a. What points on the graph correspond to the answers for part j from questions 3 and 4?

   b. What points on the graph correspond to the answers to questions 7, 8, and 9?

   c. What is the relationship of the graph of the function *Y* to the graph of the function *f* , where the formula for *f*(*x*) is the same as the formula for *Y*(*x*) but the domain for *f* is all real numbers?

   d. Questions 4, 7, and 8 give information about points that are on the graph of *f* but not on the graph of *Y*. What points are these?

   e. Graph the functions *f* and *Y* on the same axes. How does your graph show that the domain of *f* is all real numbers? How is the domain of *Y* shown on your graph?

   f. Draw the line $y = 18000$ on the graph drawn for item 10, part e. This line is the graph of the function with constant value 18000. Where does this line intersect the graph of the function *Y*? Based on the graph, how many trees per acre give a yield of more than 18000 peaches per acre?

   g. Draw the line $y = 8400$ on your graph. Where does this line intersect the graph of the function *Y*? Based on the graph, how many trees per acre give a yield of <u>fewer than</u> 8400 peaches per acre?

   h. Use a graphing utility and this intersection method to find the number of trees per acre that give a total yield **closest** to the following numbers of peaches per acre:
   (i) 10000 (ii) 15000 (iii) 20000

   i. Find the value of the function *Y* for the number of trees given in answering (i) – (iii) in part c above.

11. In items 5 and 6, we used factoring as part of a process to solve equations that are equivalent to equations of the form $x^2 + bx + c = 0$ where *b* and *c* are integers. Look back at the steps you did in items 3 and 4, and describe the process for solving an equation of the form $x^2 + bx + c = 0$. Use this process to solve each of the following equations, that is, to find all of the numbers that satisfy the original equation. Verify your work by checking each solution in the original equation.

a. $x^2 - 6x + 8 = 0$

b. $x^2 - 15x + 36 = 0$

c. $x^2 + 28x + 27 = 0$

d. $x^2 - 3x - 10 = 0$

e. $x^2 + 2x - 15 = 0$

f. $x^2 - 4x - 21 = 0$

g. $x^2 - 7x = 0$

h. $x^2 + 13x = 0$

12. For each of the equations solved in question 11, do the following.

  a. Use technology to graph a function whose formula is given by the left–hand side of the equation.
  b. Find the points on the graph which correspond to the solutions found in question 8.
  c. How is each of these results an example of the intersection method explored above?

COMPLETING THE SQUARE AND DERIVING THE QUADRATIC FORMULA

GEORGIA STANDARDS OF EXCELLENCE

MGSE9-12.A.REI.4a Use the method of completing the square to transform any quadratic equation in x into an equation of the form $(x - p)^2 = q$ that has the same solutions. Derive the quadratic formula from $ax^2 + bx + c = 0$.

MGSE9-12.A.SSE.3b Complete the square in a quadratic expression to reveal the maximum and minimum value of the function defined by the expression.

MGSE9-12.F.IF.7 Graph functions expressed algebraically and show key features of the graph both by hand and by using technology.

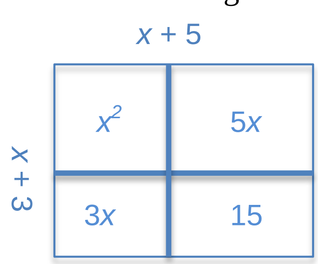
STANDARDS FOR MATHEMATICAL PRACTICE

7. Look for and make use of structure by expecting students to apply rules, look for patterns and analyze structure.
8. Look for and express regularity in repeated reasoning by expecting students to understand broader applications and look for structure and general methods in similar situations.

The following set of tasks and activities is best done one section at a time, with students having time to work on each set and then the teacher facilitating a class discussion about the general ideas in each section before moving to the next. If multiplying binomials and factoring trinomials has been previously introduced with area models and/or algebra tiles, then the first two sections can be skipped or very briefly reviewed. Not all parts of all problems need to be completed. Instead, the students should work enough parts to understand the patterns. In Section 2, it is important for students to do numbers 8 and 9.

Section 1: Area models for multiplication

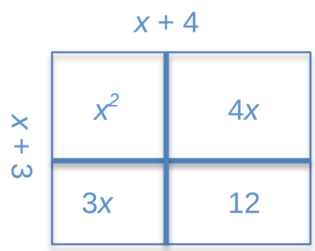
1. If the sides of a rectangle have lengths $x + 3$ and $x + 5$, what is an expression for the area of the rectangle? Draw the rectangle, label its sides, and indicate each part of the area.



The area of the rectangle is x^2 +8x +15.

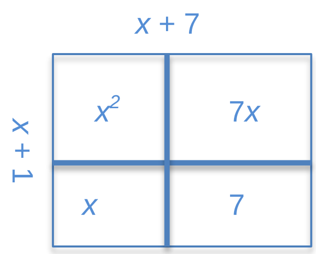
2. For each of the following, draw a rectangle with side lengths corresponding to the factors given. Label the sides and the area of the rectangle:

   a. $(x + 3)(x + 4)$



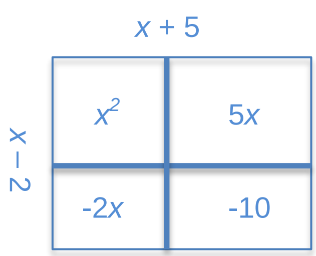
The area of the rectangle is $x^2 + 7x + 12$.

   b. $(x + 1)(x + 7)$



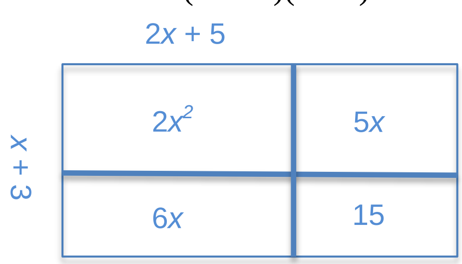
The area of the rectangle is $x^2 + 8x + 7$.

   c. $(x – 2)(x + 5)$



The area of the rectangle is $x^2 + 3x – 10$.

   d. $(2x + 1)(x + 3)$

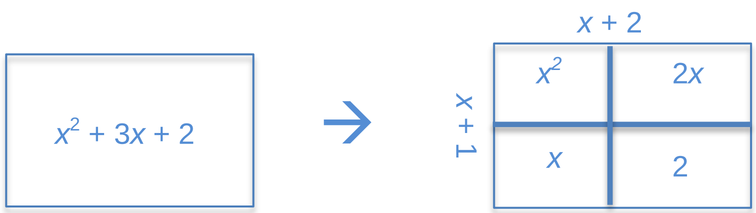


The area of the rectangle is $2x^2 + 11x + 15$.

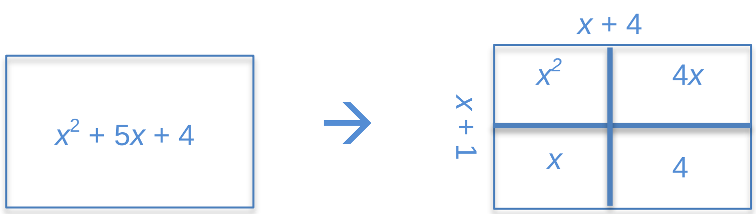
Section 2: Factoring by thinking about area and linear quantities ***(It would be best to have algebra tiles available for use in this section.)***

For each of the following, draw a rectangle with the indicated area. Find appropriate factors to label the sides of the rectangle.

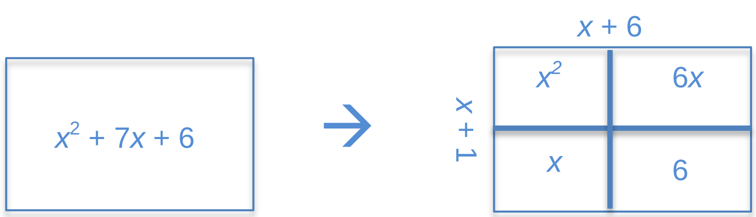
1. $x^2 + 3x + 2 = \mathbf{(x + 1)(x + 2)}$



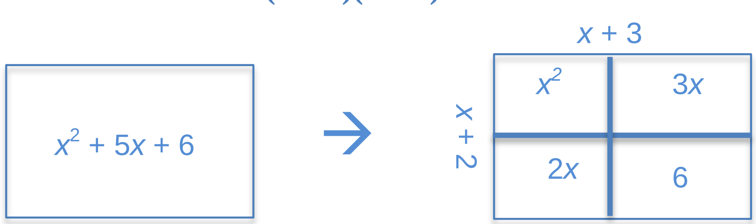
2. $x^2 + 5x + 4 = \mathbf{(x + 1)(x + 4)}$



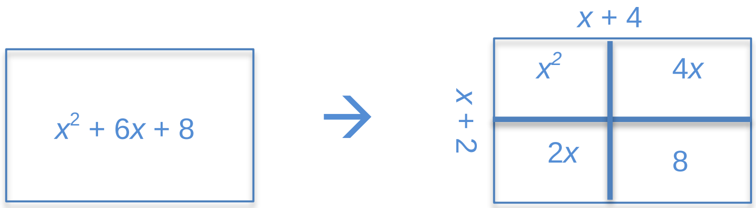
3. $x^2 + 7x + 6 = \mathbf{(x + 1)(x + 6)}$



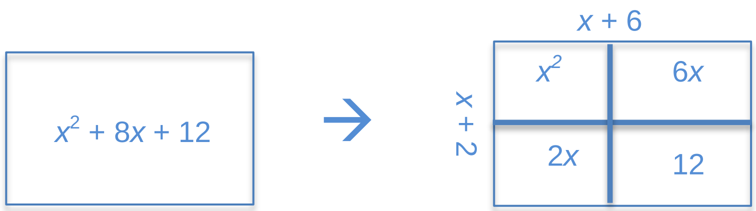
4. $x^2 + 5x + 6 = \mathbf{(x + 3)(x + 2)}$



5. $x^2 + 6x + 8 = (x + 4)(x + 2)$



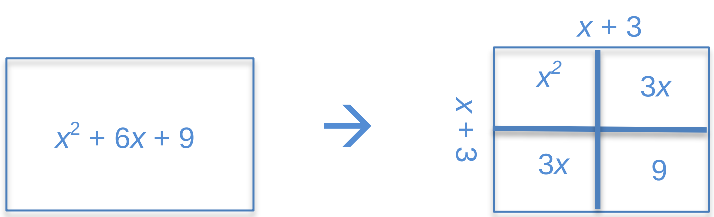
6. $x^2 + 8x + 12 = (x + 6)(x + 2)$



7. $x^2 + 7x + 12 = (x + 3)(x + 4)$

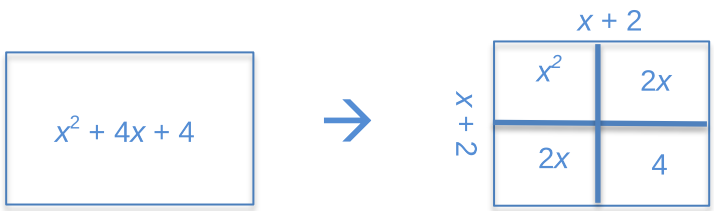
8. $x^2 + 6x + 9 = (x + 3)(x + 3) = (x + 3)^2$

Note: It is important for students to notice that this forms a square – geometrically, and it is the square of a binomial.



9. $x^2 + 4x + 4 = (x + 2)(x + 2) = (x + 2)^2$

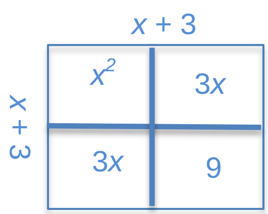
Note: It is important for students to notice that this forms a square – geometrically, and it is the square of a binomial.



Section 3: Completing the square

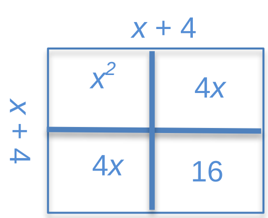
1. What number can you fill in the following blank so that $x^2 + 6x +$ ____ will have two equal factors? What are the factors? Draw the area and label the sides. What shape do you have?

9; $(x + 3)(x + 3) = (x + 3)^2$ It is a square.



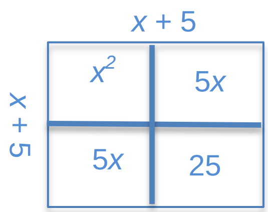
2. What number can you fill in the following blank so that $x^2 + 8x +$ ____ will have two equal factors? What are the factors? Draw the area and label the sides. What shape do you have?

16; $(x + 4)(x + 4) = (x + 4)^2$ It is a square.



3. What number can you fill in the following blank so that $x^2 + 10x +$ ____ will have two equal factors? What are the factors? Draw the area and label the sides. What shape do you have?

25; (x + 5)(x + 5) = (x + 5)² It is a square.



4. What would you have to add to $x^2 + 12x$ in order to make a square? ***36*** What could you add to $x^2 + 20x$ to make a square? ***100*** What about $x^2 + 50x$? ***625*** What if you had $x^2 +$ ***b****x*? ***Add*** $\left(\frac{b}{2}\right)^2$

Section 4: Solving equations by completing the square

1. Solve $x^2 = 9$ without factoring. How many solutions do you have? What are your solutions? ***There are two solutions.*** $x = \pm 3$
2. Use the same method as in question 5 to solve $(x + 1)^2 = 9$. How many solutions do you have? What are your solutions? ***There are two solutions.*** $(x+1) = \pm 3$***, so x = -4 or 2***
3. In general, we can solve any equation of this form $(x + h)^2 = k$ by taking the square root of both sides and then solving the two equations that we get. Solve each of the following:
   a. $(x + 3)^2 = 16$

$(x+3) = \pm 4$**, so *x = -7 or 1***

   b. $(x + 2)^2 = 5$

$(x+2) = \pm\sqrt{5}$**, so** $x = -2 \pm \sqrt{5}$

   c. $(x - 3)^2 = 4$

$(x-3) = \pm 2$**, so *x = 1 or 5***

   d. $(x - 4)^2 = 3$

$(x-4) = \pm\sqrt{3}$**, so** $x = 4 \pm \sqrt{3}$

4. Now, if we notice that we have the right combination of numbers, we can actually solve other equations by first putting them into this, using what we noticed in questions 1 – 4. Notice that if we have $x^2 + 6x + 9 = 25$, the left side is a square, that is, $x^2 + 6x + 9 = (x + 3)^2$. So, we can rewrite $x^2 + 6x + 9 = 25$ as $(x + 3)^2 = 25$, and then solve it just like we did the problems in question 7. (What do you get?) $(x+3) = \pm 5$**, so *x* = *-8 or 2***

5. Sometimes, though, the problem is not written quite in the right form. That's okay. We can apply what we already know about solving equations to write it in the right form, and then we can solve it. This is called completing the square. Let's say we have $x^2 + 6x = 7$. The left side of this equation is not a square, but we know what to add to it. If we add 9 to both sides of the equation, we get $x^2 + 6x + 9 = 16$. Now we can solve it just like the ones above. What is the solution? $(x+3)^2 = 16$***, so*** $(x+3) = \pm 4$***, and x = -7 or 1***

6. Try these:
   a. $x^2 + 10x = -9$

$$x^2 + 10x + 25 = -9 + 25$$
$$(x+5)^2 = 16$$
$$(x+5) = \pm 4$$

x = -1 or -9

   b. $x^2 + 8x = 20$

$$x^2 + 8x + 16 = 20 + 16$$
$$(x+4)^2 = 36$$
$$(x+4) = \pm 6$$

x = -10 or 2

   c. $x^2 + 2x = 5$

$$x^2 + 2x + 1 = 5 + 1$$
$$(x+1)^2 = 6$$
$$(x+1) = \pm\sqrt{6}$$
$$x = -1 \pm \sqrt{6}$$

   d. $x^2 + 6x – 7 = 0$

$$x^2 + 6x = 7$$
$$x^2 + 6x + 9 = 7 + 9$$
$$(x+3)^2 = 16$$
$$(x+3) = \pm 4$$

x = -7 or 1

e. $2x^2 + 8x = -6$

$$\frac{2}{2}x^2 + \frac{8}{2}x = \frac{-6}{2}$$
$$x^2 + 4x = -3$$
$$x^2 + 4x + 4 = -3 + 4$$
$$(x + 2)^2 = 1$$
$$(x + 2) = \pm 1$$
$$x = -3 \text{ or } -1$$

Section 5: Deriving the quadratic formula by completing the square

If you can complete the square for a general quadratic equation, you will derive a formula you can use to solve any quadratic equation. Start with $ax^2 + bx + c = 0$, and follow the steps you used in Section 4.

$$ax^2 + bx + c = 0$$
$$ax^2 + bx = -c$$
$$x^2 + \frac{b}{a}x = \frac{-c}{a}$$
$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \frac{-c}{a} + \left(\frac{b}{2a}\right)^2$$
$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{4ac}{4a^2}$$
$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$
$$\left(x + \frac{b}{2a}\right) = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$
$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Thus, the quadratic formula is: if $ax^2 + bx + c = 0$, then $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

COMPLETING THE SQUARE AND DERIVING THE QUADRAIC FORMULA

Section 1: Area models for multiplication

1. If the sides of a rectangle have lengths $x + 3$ and $x + 5$, what is an expression for the area of the rectangle? Draw the rectangle, label its sides, and indicate each part of the area.

2. For each of the following, draw a rectangle with side lengths corresponding to the factors given. Label the sides and the area of the rectangle:
   a. $(x + 3)(x + 4)$

   b. $(x + 1)(x + 7)$

   c. $(x - 2)(x + 5)$

   d. $(2x + 1)(x + 3)$

Section 2: Factoring by thinking about area and linear quantities

For each of the following, draw a rectangle with the indicated area. Find appropriate factors to label the sides of the rectangle.

1. $x^2 + 3x + 2$

2. $x^2 + 5x + 4$

3. $x^2 + 7x + 6$

4. $x^2 + 5x + 6$

5. $x^2 + 6x + 8$

6. $x^2 + 8x + 12$

7. $x^2 + 7x + 12$

8. $x^2 + 6x + 9$

9. $x^2 + 4x + 4$

Section 3: Completing the square

1. What number can you fill in the following blank so that $x^2 + 6x +$ ____ will have two equal factors? What are the factors? Draw the area and label the sides. What shape do you have?

2. What number can you fill in the following blank so that $x^2 + 8x +$ ____ will have two equal factors? What are the factors? Draw the area and label the sides. What shape do you have?

3. What number can you fill in the following blank so that $x^2 + 4x +$ ____ will have two equal factors? What are the factors? Draw the area and label the sides. What shape do you have?

4. What would you have to add to $x^2 + 10x$ in order to make a square? What could you add to $x^2 + 20x$ to make a square? What about $x^2 + 50x$? What if you had $x^2 + \boldsymbol{b}x$?

Section 4: Solving equations by completing the square

1. Solve $x^2 = 9$ without factoring. How many solutions do you have? What are your solutions?

2. Use the same method as in question 5 to solve $(x + 1)^2 = 9$. How many solutions do you have? What are your solutions?

3. In general, we can solve any equation of this form $(x + h)^2 = k$ by taking the square root of both sides and then solving the two equations that we get. Solve each of the following:
   a. $(x + 3)^2 = 16$

   b. $(x + 2)^2 = 5$

   c. $(x - 3)^2 = 4$

   d. $(x - 4)^2 = 3$

4. Now, if we notice that we have the right combination of numbers, we can actually solve other equations by first putting them into this, using what we noticed in questions 1 – 4. Notice that if we have $x^2 + 6x + 9 = 25$, the left side is a square, that is, $x^2 + 6x + 9 = (x + 3)^2$. So, we can rewrite $x^2 + 6x + 9 = 25$ as $(x + 3)^2 = 25$, and then solve it just like we did the problems in question 7. (What do you get?)

5. Sometimes, though, the problem is not written quite in the right form. That's okay. We can apply what we already know about solving equations to write it in the right form, and then we can solve it. This is called completing the square. Let's say we have $x^2 + 6x = 7$. The left side of this equation is not a square, but we know what to add to it. If we add 9 to both sides of the equation, we get $x^2 + 6x + 9 = 16$. Now we can solve it just like the ones above. What is the solution?

6. Try these:
   a. $x^2 + 10x = -9$

   b. $x^2 + 8x = 20$

   c. $x^2 + 2x = 5$

   d. $x^2 + 6x - 7 = 0$

   e. $2x^2 + 8x = -6$

Section 5: Deriving the quadratic formula by completing the square

If you can complete the square for a general quadratic equation, you will derive a formula you can use to solve any quadratic equation. Start with $ax^2 + bx + c = 0$, and follow the steps you used in Section 4.

STANDARD TO VERTEX FORM

GEORGIA STANDARDS OF EXCELLENCE

MGSE9-12.A.SSE.2 Use the structure of an expression to rewrite it in different equivalent forms. For example, see $x^4 – y^4$ as $(x^2)^2 – (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 – y^2)(x^2 + y^2)$.

MGSE9-12.A.SSE.3 Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.

MGSE9-12.A.SSE.3b Complete the square in a quadratic expression to reveal the maximum and minimum value of the function defined by the expression.

MGSE9-12.F.IF.7 Graph functions expressed algebraically and show key features of the graph both by hand and by using technology.

MGSE9-12.F.IF.7a Graph linear and quadratic functions and show intercepts, maxima, and minima (as determined by the function or by context).

MGSE9-12.F.IF.8 Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.
(Focus on quadratic functions; compare with linear and exponential functions studied in Coordinate Algebra.)

MGSE9-12.F.IF.8a Use the process of factoring and completing the square in a quadratic function to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context. *For example, compare and contrast quadratic functions in standard, vertex, and intercept forms.*

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them by requiring students to interpret and make meaning of a problem and find a logical starting point, and to monitor their progress and change their approach to solving the problem, if necessary.

2. Reason abstractly and quantitatively by requiring students to make sense of quantities and their relationships to one another in problem situations.

3. Construct viable arguments and critique the reasoning of others by engaging students on discussion of why they agree or disagree with responses, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

7. Look for and make use of structure by expecting students to apply rules, look for patterns and analyze structure.

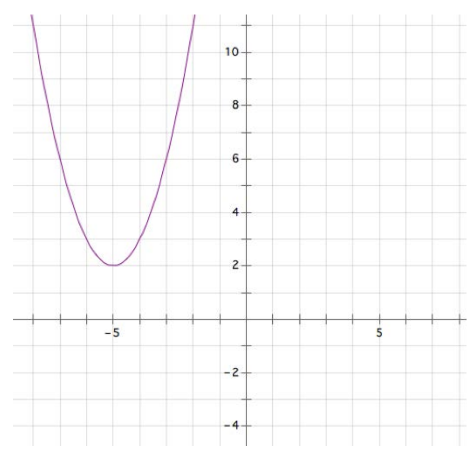
Standard to Vertex Form

In this task you will learn to identify key features of quadratic functions by completing the square

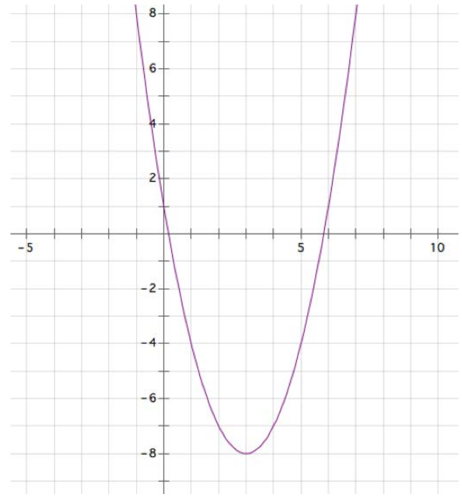
In addition to solving equations, completing the square can be helpful in identifying horizontal and vertical shifts in the graph of a function. For instance, suppose you want to graph $f(x) = x^2 + 6x + 5$. We can complete the square to write it in vertex form, so we want it to look like $f(x) = a(x - h)^2 + k$. We complete the square to find the $(x - h)^2$ part, and in doing so, we also find k. Look at $x^2 + 6x$. We know from our work earlier that we can add 9 to make a perfect square trinomial. But we can't just add 9 to an equation. We can add 9 and subtract 9 (because then we're just adding zero). So, we have $f(x) = (x^2 + 6x + 9) + 5 - 9$. When we simplify, we get $f(x) = (x + 3)^2 - 4$. So the graph of this function will be shifted three to the left and four down.

Find the horizontal and vertical shifts by completing the square and graph each of these:

1. $f(x) = x^2 + 10x + 27$
   $f(x) = (x^2 + 10x + 25) + 27 - 25$
   $f(x) = (x + 5)^2 + 2$
   Graph is shifted 5 to the left and up 2.



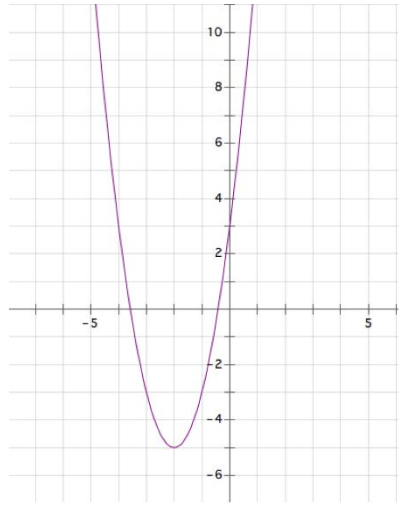
2. $f(x) = x^2 - 6x + 1$
   $f(x) = (x^2 - 6x + 9) + 1 - 9$
   $f(x) = (x - 3)^2 - 8$
   Graph is shifted 3 to the right and down 8.



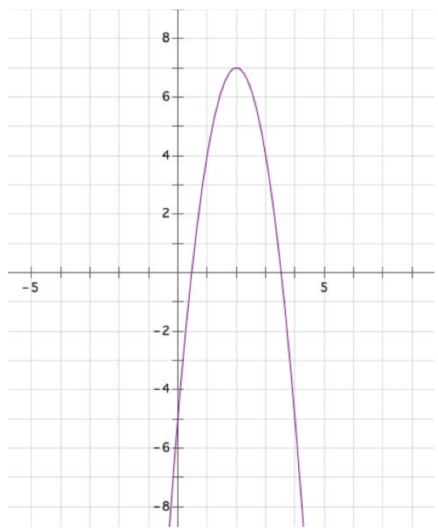
When the leading coefficient is not 1, we have to be even more careful when changing from standard to vertex form. However, the ideas are the same. We want to create a perfect square trinomial and write the equation in vertex form. For example, say we have $f(x) = 3x^2 + 6x + 5$. This time, we need to factor the leading coefficient out of the first two terms and then complete the square. So, we have $f(x) = 3(x^2 + 2x \quad) + 5$. Completing the square on $x^2 + 2x$ means we need to add 1. But if we add 1 inside the parentheses, we are actually adding three $(3 \cdot 1)$, so we have to add 3 and subtract 3: $f(x) = 3(x^2 + 2x + 1) + 5 - 3$. Simplifying, we have $f(x) = 3(x + 1)^2 + 2$.

Find the horizontal and vertical shifts as well as any stretches or shrinks by completing the square and graph each of these:

3. $f(x) = 2x^2 + 8x + 3$
   $\mathbf{f(x) = 2(x^2 + 4x) + 3}$
   $\mathbf{f(x) = 2(x^2 + 4x + 4) + 3 - 8}$
   $\mathbf{f(x) = 2(x + 2)^2 - 5}$
   Graph is shifted 2 to the left and down 5 and has a vertical stretch of 2.



4. $f(x) = -3x^2 + 12x - 5$
   $\mathbf{f(x) = -3(x^2 - 4x) - 5}$
   $\mathbf{f(x) = -3(x^2 - 4x + 4) - 5 + 12}$
   $\mathbf{f(x) = -3(x - 2)^2 + 7}$
   Graph is shifted 2 to the right and down 5 and has a vertical stretch of 2.



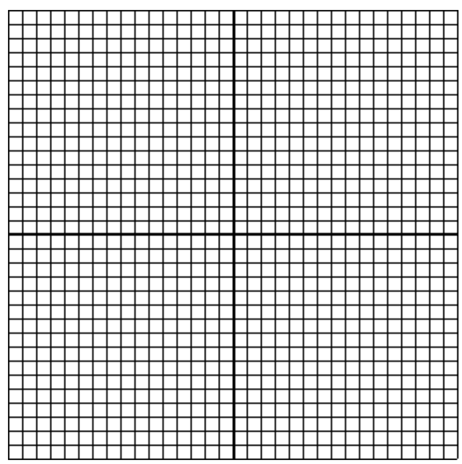
STANDARD TO VERTEX FORM

In this task you will learn to identify key features of quadratic functions by completing the square

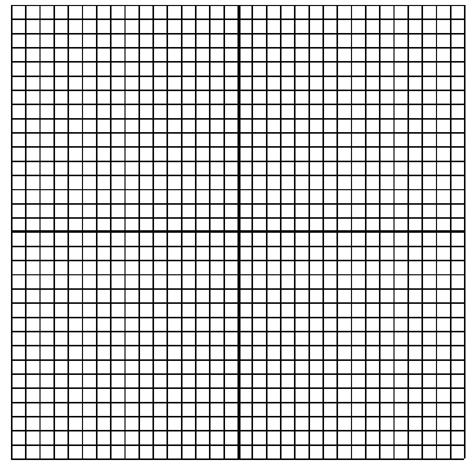
In addition to solving equations, completing the square can be helpful in identifying horizontal and vertical shifts in the graph of a function. For instance, suppose you want to graph $f(x) = x^2 + 6x + 5$. We can complete the square to write it in vertex form, so we want it to look like $f(x) = a(x - h)^2 + k$. We complete the square to find the $(x - h)^2$ part, and in doing so, we also find k. Look at $x^2 + 6x$. We know from our work earlier that we can add 9 to make a perfect square trinomial. But we can't just add 9 to an equation. We can add 9 and subtract 9 (because then we're just adding zero). So, we have $f(x) = (x^2 + 6x + 9) + 5 - 9$. When we simplify, we get $f(x) = (x + 3)^2 - 4$. So the graph of this function will be shifted three to the left and four down.

Find the horizontal and vertical shifts by completing the square and graph each of these:

5. $f(x) = x^2 + 10x + 27$



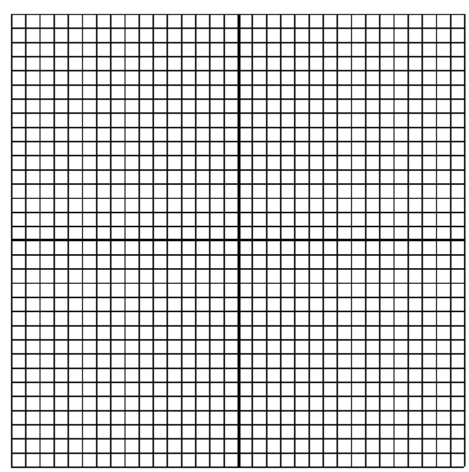
6. $f(x) = x^2 - 6x + 1$



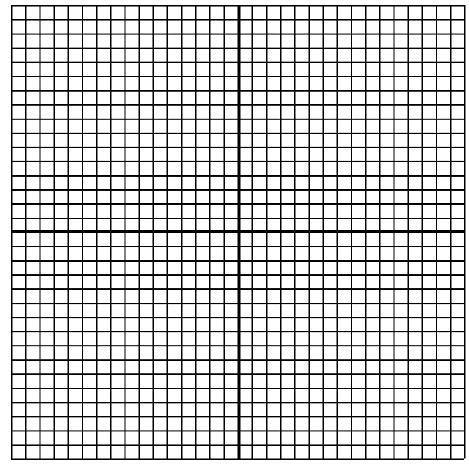
When the leading coefficient is not 1, we have to be even more careful when changing from standard to vertex form. However, the ideas are the same. We want to create a perfect square trinomial and write the equation in vertex form. For example, say we have $f(x) = 3x^2 + 6x + 5$. This time, we need to factor the leading coefficient out of the first two terms and then complete the square. So, we have $f(x) = 3(x^2 + 2x\ \ ) + 5$. Completing the square on $x^2 + 2x$ means we need to add 1. But if we add 1 inside the parentheses, we are actually adding three $(3 \cdot 1)$, so we have to add 3 and subtract 3: $f(x) = 3(x^2 + 2x + 1) + 5 - 3$. Simplifying, we have $f(x) = 3(x + 1)^2 + 2$.

Find the horizontal and vertical shifts by completing the square and graph each of these:

7. $f(x) = 2x^2 - 8x + 3$



8. $f(x) = -3x^2 + 12x - 5$



JUST THE RIGHT BORDER

GEORGIA STANDARDS OF EXCELLENCE

MGSE9-12.A.SSE.3 Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.

MGSE9-12.A.SSE.3b Complete the square in a quadratic expression to reveal the maximum or minimum value of the function defined by the expression.

MGSE9-12.A.REI.4a Use the method of completing the square to transform any quadratic equation in x into an equation of the form $(x - p)^2 = q$ that has the same solutions. Derive the quadratic formula from $ax^2 + bx + c = 0$.

MGSE9-12.A.REI.4b Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, factoring, completing the square, and the quadratic formula, as appropriate to the initial form of the equation (limit to real number solutions).

MGSE9-12.F.IF.4 Using tables, graphs, and verbal descriptions, interpret the key characteristics of a function which models the relationship between two quantities. Sketch a graph showing key features including: intercepts; interval where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; ~~and periodicity~~.

MGSE9-12.F.IF.8 Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.

STANDARDS FOR MATHEMATICAL PRACTICE

1. Make sense of problems and persevere in solving them by requiring students to interpret and make meaning of a problem and find a logical starting point, and to monitor their progress and change their approach to solving the problem, if necessary.

2. Reason abstractly and quantitatively by requiring students to make sense of quantities and their relationships to one another in problem situations.

4. Model with mathematics by expecting students to apply the mathematics concepts they know in order to solve problems arising in everyday situations, and reflect on whether the results are sensible for the given scenario.

5. Use appropriate tools strategically by expecting students to consider available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a compass, a calculator, software, etc.

7. Look for and make use of structure by expecting students to apply rules, look for patterns and analyze structure.

8. Look for and express regularity in repeated reasoning by expecting students to understand broader applications and look for structure and general methods in similar situations.

Common Student Misconceptions

1. Students may believe that factoring and completing the square are isolated techniques within a unit of quadratic equations. Teachers should help students to see the value of these skills in the context of solving higher degree equations and examining different function families.

2. Students may think that the minimum (the vertex) of the graph of $y = (x + 5)^2$ is shifted to the right of the minimum (the vertex) of the graph $y = x^2$ due to the addition sign. Students should explore examples both analytically and graphically to overcome this misconception.

3. Some students may believe that the minimum of the graph of a quadratic function always occur at the y–intercept.

4. Some students may think that rewriting equations into various forms (taking square roots, completing the square, using quadratic formula and factoring) are isolated techniques within a unit of quadratic equations. Teachers should help students see the value of these skills in the context of solving higher degree equations and examining different families of functions.

5. Students may believe that it is reasonable to input any x–value into a function, so they will need to examine multiple situations in which there are various limitations to the domains.

6. Students may also believe that the slope of a linear function is merely a number used to sketch the graph of the line. In reality, slopes have real–world meaning, and the idea of a rate of change is fundamental to understanding major concepts from geometry to calculus.

7. Students may believe that each family of functions (e.g., quadratic, square root, etc.) is independent of the others, so they may not recognize commonalities among all functions and their graphs.

8. Students may also believe that skills such as factoring a trinomial or completing the square are isolated within a unit on polynomials, and that they will come to understand the usefulness of these skills in the context of examining characteristics of functions.

9. Additionally, student may believe that the process of rewriting equations into various forms is simply an algebra symbol manipulation exercise, rather than serving a purpose of allowing different features of the function to be exhibited.

Teacher Notes:

Introduction:

This task guides students in learning to apply the quadratic formula to solve quadratic equations. The teacher should teach the skill of Complete the Square before the task is introduced (may use the preceding task to do so). The Quadratic Formula should be developed by using Complete the Square. It opens with an applied problem that leads to a quadratic equation with irrational solutions. Students are guided through the application of the quadratic formula to this particular equation and then practice using the quadratic formula with a variety of equations including some that could be solved by factoring or extraction of square roots. Students are introduced to the discriminant, analyzing the information it provides when coefficients are rational numbers and when coefficients are real numbers. Students are asked to use the quadratic formula to solve a few quadratic equations with some irrational coefficients. The task ends with two applied problems that students should solve on their own.

In this task, students are expected to classify quadratic equations with a negative discriminant as equations having no real solution. Students will also find complex solutions to quadratic equations. Students also review the connection between real solutions of a quadratic equation in standard form and x–intercepts of the corresponding quadratic function.

This task provides a guided discovery for the following:

$$\begin{aligned} 2t^2 - 3t - 9 &= 2t^2 - 6t + 3t - 9 \\ &= 2t(t-3) + 3(t-3) \\ &= (t-3)(2t+3) \end{aligned}$$

Quadratic formula: The solution(s) of the quadratic equation of the form $ax^2 + bx + c = 0$, where a, b, and c are real numbers with $a \neq 0$, is

Given an quadratic equation of the form, $ax^2 + bx + c = 0$ *where a, b, and c are real numbers with,* $a \neq 0$.

$b^2 - 4ac > 0$ *if and only if the equation has two real solutions,*

$b^2 - 4ac = 0$ *if and only if the equation has one real solution, and*

$b^2 - 4ac < 0$ if and only if the equation has no real number solution - Extension

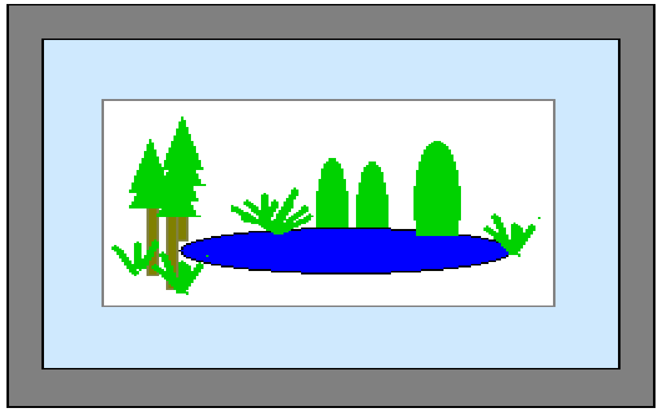
When a, b, and c are rational numbers, if $b^2 - 4ac$ *is the square of a rational number, then the solution(s) are rational, and if* $b^2 - 4ac$ *is not the square of a rational number, then both solutions are irrational.*

Supplies Needed:

Calculator

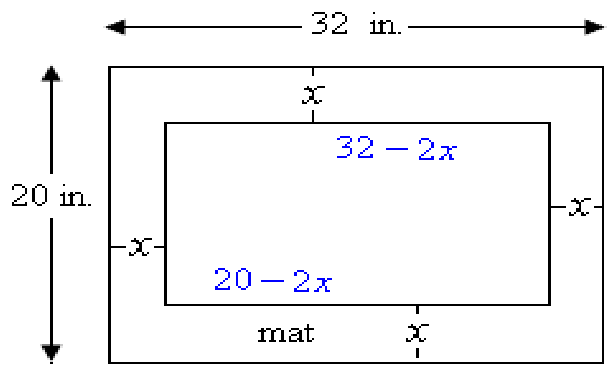
Graphing utility

1. Hannah is an aspiring artist who enjoys taking nature photographs with her digital camera. Her mother, Cheryl, frequently attends estate sales in search of unique decorative items. Last month Cheryl purchased an antique picture frame that she thinks would be perfect for framing one of Hannah's recent photographs. The frame is rather large, so the photo needs to be enlarged. Cheryl wants to mat the picture. One of Hannah's art books suggest that mats should be designed so that the picture takes up 50% of the area inside the frame and the mat covers the other 50%. The inside of the picture frame is 20 inches by 32 inches. Cheryl wants Hannah to enlarge, and crop if necessary, her photograph so that it can be matted with a mat of uniform width and, as recommended, take up 50% of the area inside the mat. See the image at the right.



   a. Let x denote the width of the mat for the picture. Model this situation with a diagram. Write an equation in x that models this situation.

   Comment(s):



   The equation is based on recognition that the information provides two different ways to express the area covered by the picture: (i) 50% of the area inside the frame, and (ii) length of the picture area times the width of the picture area. Teachers may need to remind students that mathematical models often depend on expressing the same quantity in two different ways.

   Solution(s):

   x = width of mat in inches

   $20 – 2x$ = width of the picture area

   $32 – 2x$ = length of the picture area

   ***Equation:* $(20 – 2x)(32 – 2x) = 0.50(20)(32)$, or $(20 – 2x)(32 – 2x) = 320$**

b. Put the equation from part a in the standard form $ax^2 + bx + c = 0$. Can this equation be solved by factoring (using integers)?

Comment(s):

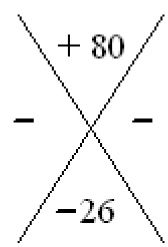
Based on the instructions, it is expected that students will probably stop working as soon as they get a quadratic equation in standard form. It will be simpler to apply the quadratic formula if they simplify the equation by dividing through by the GCF, but there is no expectation that they will do so at this step.

Solution(s):

$$\begin{aligned}(20-2x)(32-2x)&=320\\ 640-64x-40x+4x^2&=320\\ 4x^2-104x+320&=0\end{aligned}$$

Comment(s):

Students need to show that there are no integer pairs that meet the required conditions for factoring the expression over the integers.



Solution(s):

$$\begin{aligned}4x^2-104x+320&=0\\ 4\left(x^2-26x+80\right)&=0\\ x^2-26x+80&=0\end{aligned}$$

In order to solve the equation by factoring, we would need to factor the expression $x^2-26x+80$. In order to factor this expression we need two factors of 80 that add to 26. The ways of factoring 80 as a product of two integers are: 1·80, 2·40, 4·20, 5·16, 8·10. None of these pairs of integers adds to 26, so the expression does not factor over the integers.

c. The quadratic formula can be used to solve quadratic equations that cannot be solved by factoring over the integers. Using the simplest equivalent equation in standard form, identify a, b, and c from the equation in part b and find $b^2 - 4ac$; then substitute these values in the quadratic formula to find the solutions for x. Give exact answers for x and approximate the solutions to two decimal places.

Comment(s):

The exercise asks students to use the simplest equivalent equation so that students can focus on the basics of using the quadratic formula. Students are not required to simplify the answers, but some students will simplify so the simplification is shown to include the range of possible correct answers. At this point, teachers should focus students on using the formula correctly and understanding how to find the two correct decimal approximations. Taking the steps to find the decimal approximations helps students to learn the quadratic formula with understanding that expressions like $\frac{26 \pm \sqrt{356}}{2}$ ***correspond to two specific and different real numbers.***

Solution(s):

$x^2 - 26x + 80 = 0$ ***is the simplest equivalent equation.***

$a = 1$, $b = -26$. $c = 80$; $b^2 - 4ac = (-26)^2 - 4(80) = 676 - 320 = 356$

$$x = \frac{26 \pm \sqrt{356}}{2} \approx 22.43,\ 3.57$$

Simplifying the exact answer,

$$x = \frac{26 \pm \sqrt{4}\sqrt{89}}{2} = \frac{26 \pm 2\sqrt{89}}{2} = \frac{26}{2} \pm \frac{2\sqrt{89}}{2} = 13 \pm \sqrt{89} \approx 13 \pm 9.43$$

d. To the nearest tenth of an inch, what should be the width of the mat and the dimensions for the photo enlargement?

Comment(s):

This part requires students to realize that only one of the solutions to the equation is meaningful in this physical situation. Such a realization requires a higher level of comprehension than is needed to reject a negative solution when the answer for a physical situation must be a positive number. Note that the rounded answer leads to a photo area that is smaller than the required 320 inches because rounding to the nearest tenth requires rounding down. If any students are concerned about the results of a check similar to the one below, suggest that they give answers to the nearest thousandth of an inch and check these to get an area of 320.001424 square inches.. The item does not ask for answers to the nearest thousandth of an inch because it is unrealistic for the situation to assume that Cheryl and Hannah would have the tools to measure more accurately than the nearest tenth of an inch.

Note to teachers: The solution below includes a check of the answer by using the stated answer in the words of the problem to see if it is consistent with the original information. This is not listed as a separate step because students need to internalize that, whenever time permits, they need to do such checking when giving the final answer to an applied problem.

Solution(s):

Decimal approximations of the values of x are 22.43 and 3.57.

The measure of the width inside the frame is only 20 inches, so the mat cannot be 22.43 inches wide.

Thus, to the nearest tenth of inch, the mat is 3.6 inches wide and the photo enlargement is 20 – 7.2 = 12.8 inches by 32 – 7.2 = 24.8 inches.

Check of the solution: (12.8 in.)(24.8 in.) = 317.44 sq. in. ≈ 320 sq. in., which is the required 50% of the area inside the frame.

2. The quadratic formula can be very useful in solving problems. Thus, it should be practiced enough to develop accuracy in using it and to allow you to commit the formula to memory. Use the quadratic formula to solve each of the following quadratic equations, even if you could solve the equation by other means. Begin by identifying a, b, and c and finding $b^2 - 4ac$; then substitute these values into the formula.

a. $4z^2 + z - 6 = 0$

b. $t^2 + 2t + 8 = 0$

c. $3x^2 + 15x = 12$

d. $25w^2 + 9 = 30w$

e. $7x^2 = 10x$

f. $\frac{t}{2} + \frac{7}{t} = 2$

g. $3\left(2p^2 + 5\right) = 23p$

h. $12z^2 = 90$

Comment(s):

Students are told to identify a, b, and c, and find b^2 – 4ac before substituting values into the quadratic formula. Textbooks often demonstrate the use of the quadratic formula with substitution of a, b, and c into the formula. This set up usually requires lots of parentheses and several steps before the expression under the square root is simplified. Some teachers have found that having students calculate b^2 – 4ac before starting to substitute into the formula promotes better understanding and accuracy, perhaps because it leads to less rote behavior. The solutions below show calculation of b^2 – 4ac using calculation of ac first. This approach seems to help student accuracy because, at each stage of the calculation, students are making a decision about the sign of the product of two signed numbers. When teachers model using the formula, they are encouraged to use the steps as shown in the solutions below and to verbalize the substitution in the quadratic formula in a manner similar to the following.

"The opposite of b plus or minus the square root of the quantity b^2 – 4ac all divided by twice a."

In preparation for complex number solutions (Algebra II) in the form a + bi, and because students seem to understand better, when simplifying answers requires reducing to lowest terms, the solutions below show separating the solutions into two separate rational expressions and reducing each individually.

Solution(s):

a. $a = 4,\ b = 1,\ c = -6;\ b^2 - 4ac = (1)^2 - 4(-24) = 1 + 96 = 97$

$$x = \frac{-1 \pm \sqrt{97}}{8} \quad or \quad -\frac{1}{8} \pm \frac{\sqrt{97}}{8}$$

b. $a = 1,\ b = 2,\ c = 8;\ b^2 - 4ac = (2)^2 - 4(8) = 4 - 32 = -28$

$$x = \frac{-2 \pm \sqrt{-28}}{2}$$

Since $\sqrt{-28}$ ***is not a real number, there is <u>no real number solution</u>.***

c. ***We put the equation in standard form::*** $3x^2 + 15x - 12 = 0$

$a = 3,\ b = 15,\ c = -12;\ b^2 - 4ac = (15)^2 - 4(-36) = 225 + 144 = 369$

$$x = \frac{-15 \pm \sqrt{369}}{6}$$

Simplifying the solution,

$$x = \frac{-15 \pm \sqrt{9}\sqrt{41}}{6} = \frac{-15 \pm 3\sqrt{41}}{6} = \frac{-15}{6} \pm \frac{3\sqrt{41}}{6} = \frac{-5}{2} \pm \frac{\sqrt{41}}{2}$$

Alternately, if we divide the standard form of the equation by the GCF of 3 to obtain $x^2 + 5x - 4 = 0$ ***:***

$a = 1,\ b = 5,\ c = -4;\ b^2 - 4ac = (5)^2 - 4(-4) = 25 + 16 = 41$

$$x = \frac{-5 \pm \sqrt{41}}{2} \quad or \quad -\frac{5}{2} \pm \frac{\sqrt{41}}{2}$$

d. ***We put the equation in standard form:*** $25w^2 - 30w + 9 = 0$

$a = 25,\ b = -30,\ c = 9;\ b^2 - 4ac = (-30)^2 - 4(225) = 900 - 900 = 0$

$$w = \frac{30 \pm \sqrt{0}}{50} = \frac{30}{50} = \frac{3}{5}$$

e. ***We put the equation in standard form:*** $7x^2 - 10x = 0$

$a = 7,\ b = -10,\ c = 0;\ b^2 - 4ac = (-10)^2 - 4(0) = 100$

$$x = \frac{10 \pm \sqrt{100}}{14} = \frac{10 \pm 10}{14} = \frac{10+10}{14},\ \frac{10-10}{14} = \frac{20}{14},\ \frac{0}{14} = \frac{10}{7},\ 0$$

f. ***Starting with the equation*** $\frac{t}{2} + \frac{7}{t} = 2$, ***we multiply both sides of the equation by the least common denominator, 2t, to obtain:***

$$2t\left(\frac{t}{2}\right) + 2t\left(\frac{7}{t}\right) = 2t(2)$$
$$t^2 + 14 = 4t$$
$$t^2 - 4t + 14 = 0$$

$a = 1,\ b = -4,\ c = 14;\ b^2 - 4ac = (-4)^2 - 4(14) = 16 - 56 = -40$

$$t = \frac{4 \pm \sqrt{-40}}{2}$$

Since $\sqrt{-40}$ ***is not a real number, there is <u>no real number solution</u>.***

g. ***We put the equation in standard form:*** $6p^2 + 15 = 23p$

$$6p^2 - 23p + 15 = 0$$

$a = 6,\ b = -23,\ c = 15;\ b^2 - 4ac = (-23)^2 - 4(90) = 529 - 360 = 169$

$$p = \frac{23 \pm \sqrt{169}}{12} = \frac{23 \pm 13}{12} = \frac{36}{12}, \frac{10}{12} = 3, \frac{5}{6}$$

h. ***We put the equation in standard form:*** $12z^2 - 90 = 0$

$a = 12,\ b = 0,\ c = -90;\ b^2 - 4ac = (0)^2 - 4(12)(-90) = 4320$

$$z = \frac{0 \pm \sqrt{4320}}{24} = \pm\frac{\sqrt{144}\sqrt{30}}{24} = \pm\frac{12\sqrt{30}}{24} = \pm\frac{\sqrt{30}}{2}$$

Alternately, if we divide the standard form of the equation by the GCF of 3 to obtain $4z^2 - 30 = 0$***:***

$a = 4,\ b = 0,\ c = -30;\ b^2 - 4ac = (0)^2 - 4(4)(-30) = 480$

$$z = \frac{0 \pm \sqrt{480}}{8} = \pm\frac{\sqrt{16}\sqrt{30}}{8} = \pm\frac{4\sqrt{30}}{8} = \pm\frac{\sqrt{30}}{2}$$

3. The expression $b^2 - 4ac$ in the quadratic formula is called the discriminant of the quadratic equation in standard form. All of the equations in item 2 had values of a, b, and c that are rational numbers. Answer the following questions for quadratic equations in standard form when a, b, and c are rational numbers. Make sure that your answers are consistent with the solutions from item 2.

   a. What kind of number is the discriminant when there are two real number solutions to a quadratic equation?

   b. What kind of number is the discriminant when the two real number solutions to a quadratic equation are rational numbers?

   c. What kind of number is the discriminant when the two real number solutions to a quadratic equation are irrational numbers?

d. Could a quadratic equation with rational coefficients have one rational solution and one irrational solution? Explain your reasoning.

e. What kind of number is the discriminant when there is only one real number solution?

f. What kind of number is the discriminant when there is no real number solution to the equation?

Comment(s):

This item emphasizes that students are stating results in the case that a, b, and c are rational numbers. When coefficients are rational, students usually find an equivalent equation with integer coefficients before applying the quadratic formula, but, in stating general results, it is simpler to assume rational coefficients. In fact, it would be good to lead the class through at least one example of direct application of the quadratic formula to rational coefficients; one such example is provided immediately below.

Solve $\frac{2}{3}q^2 + \frac{1}{4}q = \frac{1}{6}$ using the quadratic formula.

Solution(s):

Method 1 – Using LCD to find an equivalent equation with integer coefficients

$$12\left(\frac{2}{3}q^2\right) + 12\left(\frac{1}{4}q\right) = 12\left(\frac{1}{6}\right)$$

$$8q^2 + 3q = 2$$

$$8q^2 + 3q - 2 = 0$$

$$a = 8,\ b = 3,\ c = -2,\ b^2 - 4ac = 9 - 4(-16) = 9 + 64 = 73$$

$$q = \frac{-3 \pm \sqrt{73}}{16}$$

Method 2 – Using the rational coefficients in the quadratic formula

$$\frac{2}{3}q^2+\frac{1}{4}q-\frac{1}{6}=0$$

$$a=\frac{2}{3},\ b=\frac{1}{4},\ c=-\frac{1}{6},$$

$$b^2-4ac=\frac{1}{16}-4\left(\frac{2}{3}\right)\left(-\frac{1}{6}\right)=\frac{1}{16}+\frac{4}{9}=\frac{9}{144}+\frac{64}{144}=\frac{73}{144}$$

$$q=\frac{-\frac{1}{4}\pm\sqrt{\frac{73}{144}}}{\frac{4}{3}}=\frac{-\frac{1}{4}\pm\frac{\sqrt{73}}{12}}{\frac{4}{3}}=\left(-\frac{1}{4}\pm\frac{\sqrt{73}}{12}\right)\cdot\frac{3}{4}$$

$$q=-\frac{3}{16}\pm\frac{\sqrt{73}}{16}$$

In working with quadratic equations with rational coefficients, whether or not they use an equivalent equation with integer coefficients, students should realize that, when all coefficients are rational, irrational solutions arise with the discriminant is not the square of any rational number. This idea reduces to a perfect square discriminant when integer coefficients are used. Students should also realize that, since any irrational solutions arise when the square root of the determinant is not rational, irrational solutions come in pairs.

Students should see that, when all the coefficients are rational numbers, a perfect square determinant (or a rational number determinant whose numerator and denominator are both perfect squares) implies that the solutions will be rational. An important consequence of this result is that a perfect square determinant indicates that the equation can be solved by factoring. Students will not be able to deduce this result until they learn the Factor Theorem but may make such a conjecture based on their experience with factoring.

Most of the quadratic equations students solve involve rational coefficients. Students often see the properties they will list in answering the parts of this item but do not comprehend that they apply only in the case of rational coefficients. This item is designed to avoid such misconceptions.

Students are allowed to build their understanding of the determinant in stages. In item 7, after they have solved a few quadratic equations that have some irrational number coefficients, they will revisit statements about the discriminant when the coefficients are known to be real numbers but not necessarily rational.

Note that consideration of quadratic equations with complex number coefficients is beyond the scope of Algebra I since finding square roots of complex numbers requires trigonometric functions and DeMoirve's Theorem.

Note that the questions are asked in such a way that students have to see that the important concept is whether the discriminant is positive, zero, or negative (extension).

Solution(s):

a. ***When there are two real number solutions, the discriminant is positive.***

b. ***When there are two rational number solutions, the discriminant is the square of a nonzero rational number, that is, numerator and denominator are perfect squares.***

c. ***When there are two irrational number solutions, the discriminant is not a perfect square of a rational number.***

d. ***No. If a quadratic equation has one rational and one irrational solution, it has two real number solutions so the discriminant is positive. In this case, it is either the square of a rational number or it is not. If it is the square of a rational number, there are two rational number solutions. If it is not the square of a rational number, there are two irrational solutions. Neither possible case leads to one rational and one irrational solution.***

e. ***When there is only one real number solution, the discriminant is 0. In this case, the solution to the quadratic equation is $\frac{-b}{2a}$, which is a rational number when a and b are rational numbers.***

f. ***When there is no real number solution, the discriminant is a negative number.***

4. There are many ways to show why the quadratic formula always gives the solution(s) to any quadratic equation with real number coefficients. You can work through one of these by responding to the parts below. Start by assuming that each of *a*, *b*, and *c* is a real number, that $a \neq 0$, and then consider the quadratic equation $ax^2 + bx + c = 0$.
   a. Why do we assume that $a \neq 0$?

   b. Form the corresponding quadratic function, $f(x) = ax^2 + bx + c$, and put the formula for $f(x)$ in vertex form, expressing *k* in the vertex form as a single rational expression.

   c. Use the vertex form to solve for *x*–intercepts of the graph and simplify the solution. Hint: Consider two cases, $a > 0$ and $a < 0$, in simplifying $\sqrt{a^2}$.

Comment(s):

All classes need to see a derivation of the quadratic formula. This alternate item 4 provides a brief guide for students to verify the quadratic formula for themselves. Based on the characteristics of each class, teachers should make the decision whether to use alternative item 4 at this point, at a later point after students have worked with the quadratic formula further, or to conduct a teacher–led discussion of the derivation.

Solution(s):

a. ***If a = 0, the equation is of the form $bx + c = 0$, which is a linear equation and not quadratic. A quadratic equation must have a term with the square of the variable.***

b. ***In the Protein Bar Toss Part 2, we found that the x–coordinate of the vertex of the function $f(x) = ax^2 + bx + c$ is given by $x = \frac{-b}{2a}$. Then,***

$$\begin{aligned} f\left(\frac{-b}{2a}\right) &= a\left(\frac{-b}{2a}\right)^2 + b\left(\frac{-b}{2a}\right) + c \\ &= a\left(\frac{b^2}{4a^2}\right) + b\left(\frac{-b}{2a}\right) + c \\ &= \frac{b^2}{4a} + \frac{-b^2}{2a} + c \\ &= \frac{b^2}{4a} + \frac{-2b^2}{4a} + \frac{4ac}{4a} \\ &= \frac{-b^2 + 4ac}{4a} \end{aligned}$$

Thus, the vertex is $\left(\frac{-b}{2a}, \frac{-b^2+4ac}{4a}\right)$ ***and the vertex form is*** $f(x) = a\left(x + \frac{b}{2a}\right)^2 + \frac{-b^2+4ac}{4a}$.

c. ***To find the x–intercepts, we solve*** $f(x) = 0$. ***Substituting the vertex form of the formula for*** $f(x)$ ***yields***

$$a\left(x+\frac{b}{2a}\right)^2+\frac{-b^2+4ac}{4a}=0$$

$$a\left(x+\frac{b}{2a}\right)^2=\frac{-\left(-b^2+4ac\right)}{4a}$$

$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$$

$$x+\frac{b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a^2}}$$

$$=\begin{cases}\pm\dfrac{\sqrt{b^2-4ac}}{2a}, & \text{for } a>0\\ \pm\dfrac{\sqrt{b^2-4ac}}{2(-a)}, & \text{for } a<0\end{cases}$$

$$=\begin{cases}\pm\dfrac{\sqrt{b^2-4ac}}{2a}, & \text{for } a>0\\ \mp\dfrac{\sqrt{b^2-4ac}}{2a}, & \text{for } a<0\end{cases}$$

$$=\pm\frac{\sqrt{b^2-4ac}}{2a}$$

Thus, $x=\dfrac{-b}{2a}\pm\dfrac{\sqrt{b^2-4ac}}{2a}=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

5. Use the quadratic formula to solve the following equations with real number coefficients. Approximate each real, but irrational, solution correct to hundredths.

a. $x^2 + \sqrt{5}x + 1 = 0$

b. $3q^2 - 5q + 2\pi = 0$

c. $3t^2 + 11 = 2\sqrt{33}t$

d. $9w^2 = \sqrt{13}w$

Comment(s):

These examples remind students that irrational numbers are also real numbers and give them some specific examples on which to test whether all the statements that are true of the discriminant when the coefficients of the quadratic equation are rational numbers continue to be true for arbitrary real number coefficients. The exercises have been limited to avoid multiple levels of square root. The goal is to extend student understanding beyond rational coefficients. However, once students have done these examples, they should see that they can use any real number coefficients with limited simplification of the exact answer, as happens part b involving π.

Solution(s):

a. $a = 1$, $b = \sqrt{5}$, $c = 1$; ***thus,*** $b^2 - 4ac = \left(\sqrt{5}\right)^2 - 4(1) = 5 - 4 = 1$

$$x = \frac{-\sqrt{5} \pm \sqrt{1}}{2} = \frac{-\sqrt{5} + 1}{2} \approx -0.62 \text{ or } \frac{-\sqrt{5} - 1}{2} \approx -1.62$$

b. $a = 3$, $b = 5$, $c = 2\pi$; ***thus,*** $b^2 - 4ac = (5)^2 - 4(6\pi) = 25 - 24\pi \approx -50.40$

$$q = \frac{-5 \pm \sqrt{25 - 24\pi}}{6} \approx \frac{-5 \pm \sqrt{-50.40}}{6}$$, ***not real numbers***

There is no real number solution.

c. ***Putting the equation in standard form yields*** $3t^2 - 2\sqrt{33}t + 11 = 0$. ***Then,***

$a = 3, b = -2\sqrt{33}, c = 11;\ \ b^2 - 4ac = \left(-2\sqrt{33}\right)^2 - 4(33) = 4(33) - 4(33) = 0$

$$t = \frac{2\sqrt{33} \pm \sqrt{0}}{6} = \frac{2\sqrt{33}}{6} = \frac{\sqrt{33}}{3} \approx 1.91$$

d. ***Putting the equation in standard form yields*** $9w^2 - \sqrt{13}w = 0$. ***Then,***

$a = 9, b = -\sqrt{13}, c = 0;\ \ b^2 - 4ac = \left(-\sqrt{13}\right)^2 - 4(0) = 13 - 0 = 13$

$$w = \frac{\sqrt{13} \pm \sqrt{13}}{18} = \frac{\sqrt{13} + \sqrt{13}}{18} \text{ or } \frac{\sqrt{13} - \sqrt{3}}{18} = \frac{2\sqrt{13}}{18} \text{ or } \frac{0}{18}$$

$$w = 0 \text{ or } \frac{\sqrt{13}}{9} \approx 0.40$$

6. Verify each answer for item 5 by using a graphing utility to find the *x*–intercept(s) of an appropriate quadratic function.
Comment(s):

Students need to put equations in standard form before creating the corresponding function. Students are to verify the solutions by checking that the x–intercepts are consistent with the solutions. Students should be encouraged to trace their graphs to verify that the x–intercept values agree with their solutions.

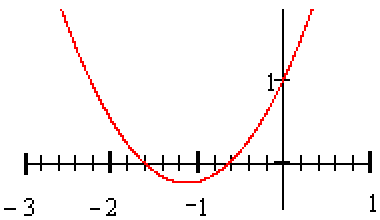
This item gives an excellent opportunity to point out the connection between the formula for the x–coordinate of the vertex of a quadratic function and the solution for x–intercepts using the quadratic formula. When we solve for x–intercepts, if there are any, they are of the form $-\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$. ***If there are two intercepts, then*** $-\frac{b}{2a}$ ***is the midpoint between these two symmetrically placed points of the graph. But then the vertex must be at this x–value because it is on the line of symmetry, so the x–coordinate of the vertex is*** $-\frac{b}{2a}$.

If there is only one x–intercept, then the parabolic shape of the graph forces that one the x–intercept to be the vertex and, again, the x–coordinate of the vertex is $-\dfrac{b}{2a}$.

Solution(s):

Equation: $x^2+\sqrt{5}\,x+1=0$

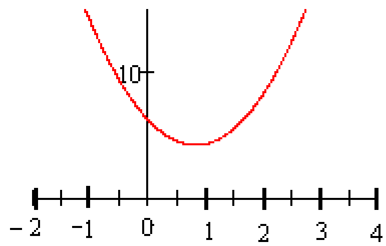
Function: $f(x)=x^2+\sqrt{5}\,x+1$



The x-intercepts correspond to approximate solutions of -0.62 *and* -1.62.

Equation: $3q^2-5q+2\pi=0$

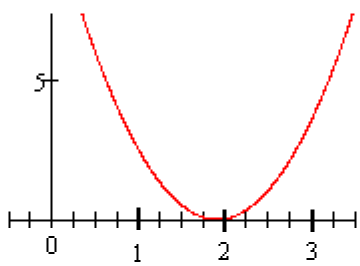
Function: $f(q)=3q^2-5q+2\pi$



There is no x-intercept and no real number solution.

Equation: $3t^2+11=2\sqrt{33}\,t$

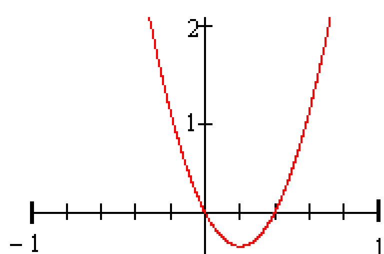
Function: $f(t)=3t^2-2\sqrt{33}\,t+11$



The x-intercept occurs at the vertex of the parabola and corresponds to the approximate solution of 1.91.

Equation: $9w^2=\sqrt{13}\,w$

Function: $f(w)=9w^2-\sqrt{13}\,w$



The x-intercepts correspond to the approximate solutions of 0 *and* 0.4.

a. Put the function for item 5, part c, in vertex form. Use the vertex form to find the *t*–intercept.

Comment(s):

This part guides students to see that the quadratic polynomial in the equation is a multiple of a perfect square trinomial with real coefficients. From this form, students can solve the equation by extraction of roots, as demonstrated in the solution for the t–intercept below.

Solution(s):

$$f(t) = 3t^2 - 2\sqrt{33}t + 11$$

vertex: $t = \frac{2\sqrt{33}}{6} = \frac{\sqrt{33}}{3}$;

$$f\left(\frac{\sqrt{33}}{3}\right) = 3\left(\frac{\sqrt{33}}{3}\right)^2 - 2\sqrt{33}\left(\frac{\sqrt{33}}{3}\right) + 11$$
$$= 3\left(\frac{33}{9}\right) - \frac{2}{3}(33) + 11 = \frac{33 - 66 + 33}{3} = \frac{0}{3} = 0$$

So the vertex form of the function is $f(t) = 3\left(t - \frac{\sqrt{33}}{3}\right)^2$*, and, thus,*

the t–intercept is found by solving $f(t) = 0$.

$$3\left(t - \frac{\sqrt{33}}{3}\right)^2 = 0 \qquad \textit{Thus,} \quad \left(t - \frac{\sqrt{33}}{3}\right)^2 = \frac{0}{3} = 0.$$

$$t - \frac{\sqrt{33}}{3} = 0$$

$$t = \frac{\sqrt{33}}{3}$$

b. Solve the equation from item 5, part d, by factoring.

Comment(s):

This part makes sure that students notice that solving this equation did not require use of the quadratic formula.

Solution(s):

$$9w^2 - \sqrt{13}w = 0$$
$$w\left(9w - \sqrt{13}\right) = 0$$
$$9w - \sqrt{13} = 0 \qquad \text{or} \quad w = 0$$
$$9w = \sqrt{13}$$
$$w = \frac{\sqrt{13}}{9}$$

7. Answer the following questions for quadratic equations in standard from where *a*, *b*, and *c* are real numbers.

   a. What kind of number is the discriminant when there are two real number solutions to a quadratic equation?

   b. Could a quadratic equation with real coefficients have one rational solution and one irrational solution? Explain your reasoning.

   c. What kind of number is the discriminant when there is only one real number solution?

   d. What kind of number is the discriminant when there is no real number solution to the equation?

   e. Summarize what you know about the relationship between the determinant and the solutions of a quadratic of the form $ax^2 + bx + c = 0$ where *a*, *b*, and *c* are real numbers with $a \neq 0$ into a formal statement using biconditionals.

Comment(s):

As indicated for item 3, this item asks students to explore the information provided by the discriminant when the coefficients are real numbers but not necessarily rational numbers.

Solution(s):

a. When there are two real number solutions, the discriminant is positive.

b. Yes, item 5, part d, provided just such an example. In this case the "b" is equal to $\sqrt{b^2 - 4ac}$. Thus, one solution is 0, a rational number, and the other solution is irrational.

c. When there is only one real number solution, the discriminant is zero.

d. When there is no real number solution, the discriminant is a negative number.

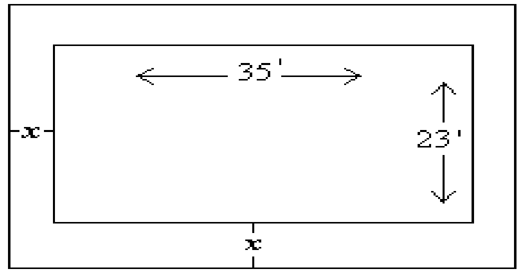
e. Given an equation of the form $ax^2 + bx + c = 0$, where a, b, and c are real numbers with $a \neq 0$,

$b^2 - 4ac > 0$ if and only if the equation has two real solutions,

$b^2 - 4ac = 0$ if and only if the equation has one real solution, and

Extension: $b^2 - 4ac < 0$ if and only if the equation has no real number solution. When a, b, and c are rational numbers, if $b^2 - 4ac$ is the square of a rational number, then the solution(s) are rational, and if $b^2 - 4ac$ is not the square of a rational number, then both solutions are irrational.

8. A landscape designer included a cloister (a rectangular garden surrounded by a covered walkway on all four sides) in his plans for a new public park. The garden is to be 35 feet by 23 feet and the total area enclosed by the garden and walkway together is 1200 square feet. To the nearest inch, how wide does the walkway need to be?



Comment(s):

This problem is similar to the opening problem, but this time the inner dimensions are given and twice the width of the walk must be added (instead of subtracted) to form the dimensions for the whole area. There is an issue of units here. Numbers are smaller if the unknown width of the walk is assumed to be measured in feet and then converted to inches at the end.

Solution(s):

Let x ***= the width of the walkway, in feet***

then $35 + 2x$ ***= length of the whole area,***

and $23 + 2x$ ***= width of the whole area.***

$$(23+2x)(35+2x)=1200$$
$$805+46x+70x+4x^2=1200$$
$$4x^2+116x-395=0$$
$$a=4,\ b=116,\ c=-395$$
$$b^2-4ac=116^2-4(4)(-395)=13456+6320=19776$$
$$x=\frac{-116\pm\sqrt{19776}}{8}$$
$$x\approx\frac{-116\pm 140.6271666}{8}$$
$$x\approx\frac{24.627166}{8}\quad or\quad \frac{-256.6271666}{8}$$

The negative solution to the equation does not apply to the physical situation.

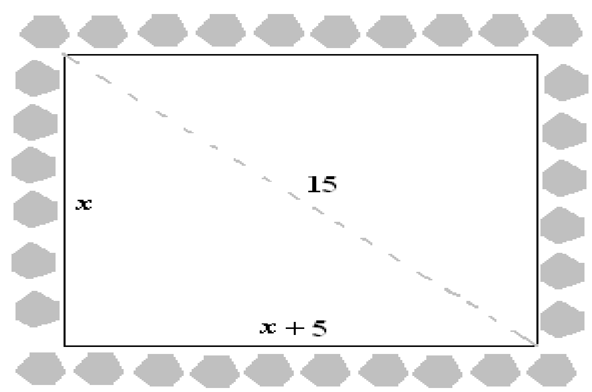
Thus, $x\approx 3.078395831$ ***feet.***

$3.078395831(12)=36.94074997$ ***Hence, to the nearest inch the walk should be 37 inches.***

Checking the answer:

$$[23+2(3.078395831)][35+2(3.078395831)]\approx(29.156791662)(41.156791662)$$
$$\approx 1200$$

9. In another area of the park, there will be an open rectangular area of grass surrounded by a flagstone path. If a person cuts across the grass to get from the southeast corner of the area to the northwest corner, the person will walk a distance of 15 yards. If the length of the area is 5 yards more than the width, to the nearest foot, how much distance is saved by cutting across rather than walking along the flagstone path?



Comment(s):

In working this problem, students should neglect the extra length caused by walking on the path slightly outside the edge of the rectangular area.

Solution(s):

Let x = width of the rectangular area,

then

x + 5 = length of the rectangular area

We use the Pythagorean Theorem to find x,

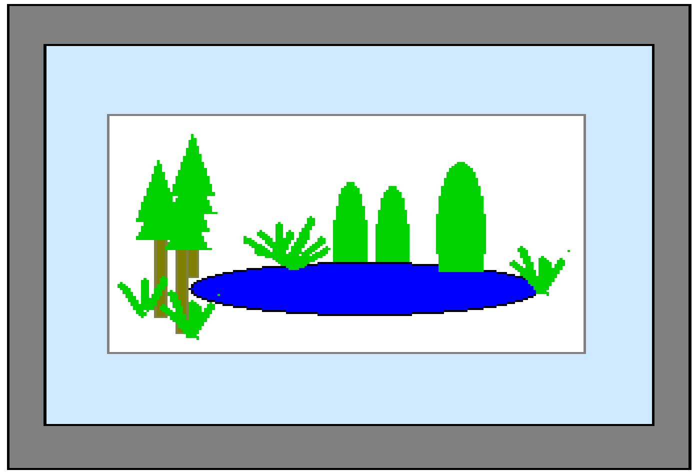
$$\begin{aligned} x^2 + (x+5)^2 &= 15^2 \\ x^2 + x^2 + 2x + 25 &= 225 \\ 2x^2 + 2x - 200 &= 0 \\ x^2 + x - 100 &= 0 \end{aligned}$$

$$\begin{aligned} a = 1,\ b = 1,\ c &= -100, \\ b^2 - 4ac = 1^2 - 4(-100) &= 401 \\ x &= \frac{-1 \pm \sqrt{401}}{2} \\ x &\approx 9.5125,\ -10.5125 \end{aligned}$$

The negative solution is not meaningful as a length.

Thus, the rectangular area is approximately* 9.5125 *yards wide and* 14.5125 *yards long. Walking around the path is a distance of* 25.025 *yards. Thus, cutting across the diagonal saves* 25.025 – 15 = 10.025 *yards. Since there are* 3 *feet in a yard and* 3(10.025*)* = 30.075*, to the nearest foot, a person saves* 30 *feet by cutting across the area.

JUST THE RIGHT BORDER



1. Hannah is an aspiring artist who enjoys taking nature photographs with her digital camera. Her mother, Cheryl, frequently attends estate sales in search of unique decorative items. Last month Cheryl purchased an antique picture frame that she thinks would be perfect for framing one of Hannah's recent photographs. The frame is rather large, so the photo needs to be enlarged. Cheryl wants to mat the picture. One of Hannah's art books suggest that mats should be designed so that the picture takes up 50% of the area inside the frame and the mat covers the other 50%. The inside of the picture frame is 20 inches by 32 inches. Cheryl wants Hannah to enlarge, and crop if necessary, her photograph so that it can be matted with a mat of uniform width and, as recommended, take up 50% of the area inside the mat. See the image at the right.

a. Let x denote the width of the mat for the picture. Model this situation with a diagram. Write an equation in x that models this situation.

b. Put the equation from part a in the standard form $ax^2 + bx + c = 0$. Can this equation be solved by factoring (using integers)?

c. The quadratic formula can be used to solve quadratic equations that cannot be solved by factoring over the integers. Using the simplest equivalent equation in standard form, identify a, b, and c from the equation in part b and find $b^2 - 4ac$; then substitute these values in the quadratic formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, to find the solutions for x. Give exact answers for x and approximate the solutions to two decimal places.

d. To the nearest tenth of an inch, what should be the width of the mat and the dimensions for the photo enlargement?

2. The quadratic formula can be very useful in solving problems. Thus, it should be practiced enough to develop accuracy in using it and to allow you to commit the formula to memory. Use the quadratic formula to solve each of the following quadratic equations, even if you could solve the equation by other means. Begin by identifying *a*, *b*, and *c* and finding $b^2 - 4ac$; then substitute these values into the formula.

a. $4z^2 + z - 6 = 0$	b. $t^2 + 2t + 8 = 0$
c. $3x^2 + 15x = 12$	d. $25w^2 + 9 = 30w$
e. $7x^2 = 10x$	f. $\frac{t}{2} + \frac{7}{t} = 2$
g. $3\left(2p^2 + 5\right) = 23p$	h. $12z^2 = 90$

3. The expression $b^2 - 4ac$ in the quadratic formula is called the discriminant of the quadratic equation in standard form. All of the equations in item 2 had values of *a*, *b*, and *c* that are rational numbers. Answer the following questions for quadratic equations in standard form when a, *b*, and *c* are rational numbers. Make sure that your answers are consistent with the solutions from item 2.
   a. What kind of number is the discriminant when there are two real number solutions to a quadratic equation?

   b. What kind of number is the discriminant when the two real number solutions to a quadratic equation are rational numbers?

   c. What kind of number is the discriminant when the two real number solutions to a quadratic equation are irrational numbers?

   d. Could a quadratic equation with rational coefficients have one rational solution and one irrational solution? Explain your reasoning.

   e. What kind of number is the discriminant when there is only one real number solution? What kind of number do you get for the solution?

   f. What kind of number is the discriminant when there is no real number solution to the equation?

4. There are many ways to show why the quadratic formula always gives the solution(s) to any quadratic equation with real number coefficients. You can work through one of these by responding to the parts below. Start by assuming that each of *a*, *b*, and *c* is a real number, that $a \neq 0$, and then consider the quadratic equation $ax^2 + bx + c = 0$.

   a. Why do we assume that $a \neq 0$?

   b. Form the corresponding quadratic function, $f(x) = ax^2 + bx + c$, and put the formula for f(x) in vertex form, expressing *k* in the vertex form as a single rational expression.

   c. Use the vertex form to solve for *x*–intercepts of the graph and simplify the solution. Hint: Consider two cases, $a > 0$ and $a < 0$, in simplifying $\sqrt{a^2}$.

5. Use the quadratic formula to solve the following equations with real number coefficients. Approximate each real, but irrational, solution correct to hundredths.

   a. $x^2 + \sqrt{5}x + 1 = 0$

   b. $3q^2 - 5q + 2\pi = 0$

   c. $3t^2 + 11 = 2\sqrt{33}t$

   d. $9w^2 = \sqrt{13}w$

6. Verify each answer for item 5 by using a graphing utility to find the *x*–intercept(s) of an appropriate quadratic function.

   a. Put the function for item 5, part c, in vertex form. Use the vertex form to find the *x*–intercept.

   b. Solve the equation from item 5, part d, by factoring.

7. Answer the following questions for quadratic equations in standard from where *a*, *b*, and *c* are real numbers.

   a. What kind of number is the discriminant when there are two real number solutions to a quadratic equation?

   b. Could a quadratic equation with real coefficients have one rational solution and one irrational solution? Explain your reasoning.

   c. What kind of number is the discriminant when there is only one real number solution?

   d. What kind of number is the discriminant when there is no real number solution to the equation?

   e. Summarize what you know about the relationship between the determinant and the solutions of a quadratic of the form $ax^2 + bx + c = 0$ where *a*, *b*, and *c* are real numbers with $a \neq 0$ into a formal statement using biconditionals.

8. A landscape designer included a cloister (a rectangular garden surrounded by a covered walkway on all four sides) in his plans for a new public park. The garden is to be 35 feet by 23 feet and the total area enclosed by the garden and walkway together is 1200 square feet. To the nearest inch, how wide does the walkway need to be?

9. In another area of the park, there will be an open rectangular area of grass surrounded by a flagstone path. If a person cuts across the grass to get from the southeast corner of the area to the northwest corner, the person will walk a distance of 15 yards. If the length of the area is 5 yards more than the width, to the nearest foot, how much distance is saved by cutting across rather than walking along the flagstone path?

Characteristics of Quadratic Functions

GEORGIA STANDARDS OF EXCELLENCE

MGSE9-12.F.IF.4 Using tables, graphs, and verbal descriptions, interpret the key characteristics of a function which models the relationship between two quantities. Sketch a graph showing key features including: intercepts; interval where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; ~~and periodicity.~~

MGSE9-12.F.IF.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*

MGSE9-12.F.IF.7a Graph linear and quadratic functions and show intercepts, maxima, and minima (as determined by the function or by context).

MGSE9-12.F.BF.3 ~~Identify the effect on the graph of replacing f(x) by f(x) + k, k f(x), f(kx), and f(x + k) for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology.~~ Include recognizing even and odd functions from their graphs and algebraic expressions for them.

STANDARDS FOR MATHEMATICAL PRACTICE

2. Reason abstractly and quantitatively by requiring students to make sense of quantities and their relationships to one another in problem situations.

4. Model with mathematics by expecting students to apply the mathematics concepts they know in order to solve problems arising in everyday situations, and reflect on whether the results are sensible for the given scenario.

5. Use appropriate tools strategically by expecting students to consider available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a compass, a calculator, software, etc.

6. Attend to precision by requiring students to calculate efficiently and accurately; and to communicate precisely with others by using clear mathematical language to discuss their reasoning.

7. Look for and make use of structure by expecting students to apply rules, look for patterns and analyze structure.

8. Look for and express regularity in repeated reasoning by expecting students to understand broader applications and look for structure and general methods in similar situations.

Common Student Misconceptions

1. Students may believe that it is reasonable to input any x-value into a function, so they will need to examine multiple situations in which there are various limitations to the domains.

2. Students may also believe that the slope of a linear function is merely a number used to sketch the graph of the line. In reality, slopes have real-world meaning, and the idea of a rate of change is fundamental to understanding major concepts from geometry to calculus.

3. Students may believe that each family of functions (e.g., quadratic, square root, etc.) is independent of the others, so they may not recognize commonalities among all functions and their graphs.

4. Students may also believe that skills such as factoring a trinomial or completing the square are isolated within a unit on polynomials, and that they will come to understand the usefulness of these skills in the context of examining characteristics of functions.

5. Additionally, student may believe that the process of rewriting equations into various forms is simply an algebra symbol manipulation exercise, rather than serving a purpose of allowing different features of the function to be exhibited.

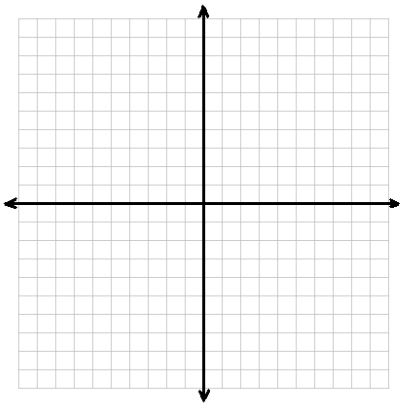
Teacher Notes:

This problem set is provided to focus students into automating the information that can be gleaned from different representations of quadratic functions. These problems could be used as an assessment of students' understanding of the important concepts of this unit. Certainly, teachers can skip this problem set if students have demonstrated competence in summarizing the information available from quadratic functions. Problems can be worked with or without technology.

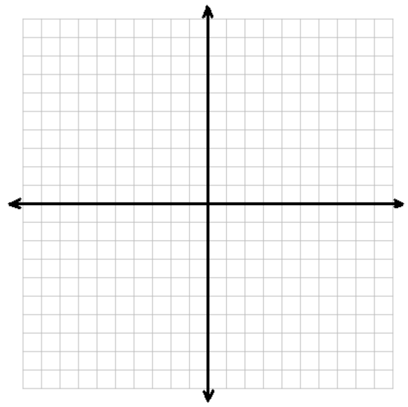
Characteristics of Quadratic Functions

Complete a table, graph, and investigate the following functions.

a) $y = x^2 + 16x + 28$



b) $y = x^2 - 11x + 10$



State the following…

Domain: ***All real numbers***

Range: ***$y \geq -36$***

Zeros: ***(-2,0) and (-14,0)***

Y-Intercept: ***(0, 28)***

Interval of Increase: ***$x \geq -8$***

Interval of Decrease: ***$x \leq -8$***

Maximum: ***None***

Minimum: ***(-8,-36)***

End Behavior: ***Up on both ends***

Even/Odd/Neither: ***Neither***

Domain: ***All real numbers***

Range: ***$y \geq -20.25$***

Zeros: ***(1,0) and (10,0)***

Y-Intercept: ***(0,10)***

Interval of Increase: ***$x \geq 5.5$***

Interval of Decrease: ***$x \leq 5.5$***

Maximum: ***None***

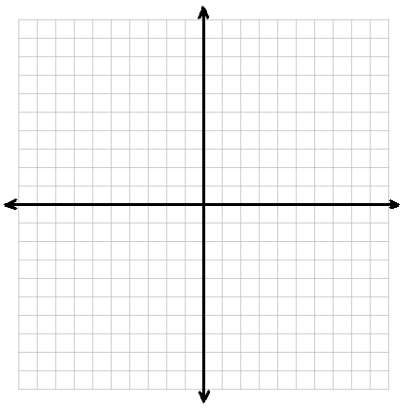
Minimum: ***(5.5,-20.25)***

End Behavior: ***Up on both ends***

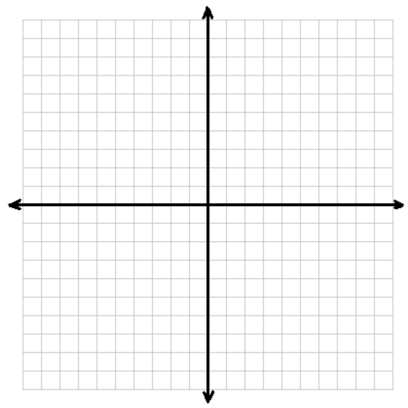
Even/Odd/Neither: ***Neither***

Complete a table, graph, and investigate the following functions.

c) $y = x^2 - 5x + 6$



d) $y = 5x^2 - 10x + 20$



State the following…

c)

Domain: ***All real numbers***

Range: ***$y \geq -.25$***

Zeros: ***(2,0) and (3,0)***

Y-Intercept: ***(0,6)***

Interval of Increase: ***$x \geq 2.5$***

Interval of Decrease: ***$x \leq 2.5$***

Maximum: ***None***

Minimum: ***(2.5,-.25)***

End Behavior: ***Up on both ends***

Even/Odd/Neither: ***Neither***

d)

Domain: ***All real numbers***

Range: ***$y \geq 15$***

Zeros: ***None***

Y-Intercept: ***(0,20)***

Interval of Increase: ***$x \geq 1$***

Interval of Decrease: ***$x \leq 1$***

Maximum: ***None***

Minimum: ***(1,15)***

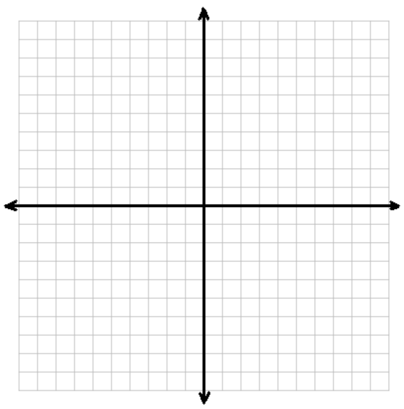
End Behavior: ***Up on both ends***

Even/Odd/Neither: ***Neither***

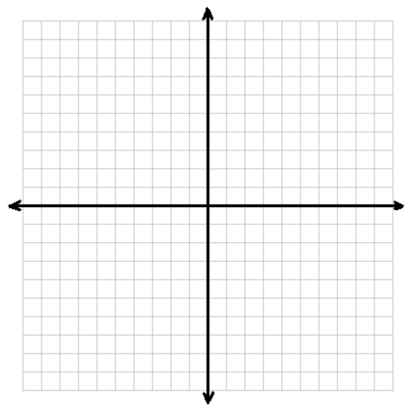
Characteristics of Quadratic Functions

Complete a table, graph, and investigate the following functions.

a) $y = x^2 + 16x + 28$



b) $y = x^2 - 11x + 10$

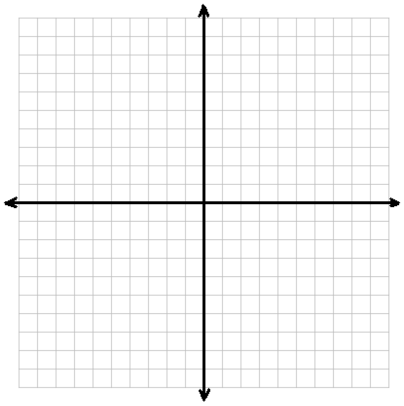


State the following…

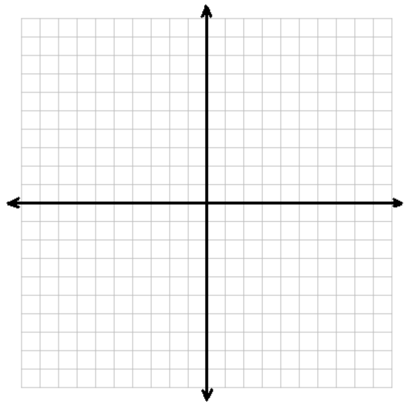
Domain:	Domain:
Range:	Range:
Zeros:	Zeros:
Y-Intercept:	Y-Intercept:
Interval of Increase:	Interval of Increase:
Interval of Decrease:	Interval of Decrease:
Maximum:	Maximum:
Minimum:	Minimum:
End Behavior:	End Behavior:
Even/Odd/Neither:	Even/Odd/Neither:

Complete a table, graph, and investigate the following functions.

c) $y = x^2 - 5x + 6$



d) $y = 5x^2 - 10x + 20$



State the following…

Domain:

Range:

Zeros:

Y-Intercept:

Interval of Increase:

Interval of Decrease:

Maximum:

Minimum:

End Behavior:

Even/Odd/Neither:

Domain:

Range:

Zeros:

Y-Intercept:

Interval of Increase:

Interval of Decrease:

Maximum:

Minimum:

End Behavior:

Even/Odd/Neither:

Seeing Structure in Expressions (Short Cycle Task)

Source: Balanced Assessment Materials from Mathematics Assessment Project
http://www.map.mathshell.org/materials/download.php?fileid=832

ESSENTIAL QUESTIONS:

- How do you see structure in expressions?

TASK COMMENTS:

Tasks and lessons from the Mathematics Assessment Project are specifically designed to help teachers effectively formatively assess their students. The way the tasks and lessons are designed gives the teacher a clear understanding of what the students are able to do and not do. Within the lesson, teachers will find suggestions and question prompts that will help guide students towards understanding. For more information access the MAP website:
http://www.map.mathshell.org/materials/background.php?subpage=summative

The task, *Seeing Structure in Expressions*, is a Mathematics Assessment Project Assessment Task that can be found at the website:
http://www.map.mathshell.org/materials/tasks.php?taskid=290&subpage=novice

The PDF version of the task can be found at the link below:
http://www.map.mathshell.org/materials/download.php?fileid=832

The scoring rubric can be found at the following link:
http://www.map.mathshell.org/materials/download.php?fileid=833

STANDARDS ADDRESSED IN THIS TASK:

MGSE9–12.A.SSE.1 Interpret expressions that represent a quantity in terms of its context. *(Focus on quadratic expressions; compare with linear and exponential functions studied in Coordinate Algebra.)*

MGSE9–12.A.SSE.1a Interpret parts of an expression, such as terms, factors, and coefficients, in context. *(Focus on quadratic expressions; compare with linear and exponential functions studied in Coordinate Algebra.)*

MGSE9–12.A.SSE.1b Given situations which utilize formulas or expressions with multiple terms and/or factors, interpret the meaning (in context) of individual terms or factors. *(Focus on quadratic expressions; compare with linear and exponential functions studied in Coordinate Algebra.)*

MGSE9-12.A.SSE.2 Use the structure of an expression to rewrite it in different equivalent forms. For example, see $x^4 – y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 – y^2) (x^2 + y^2)$.

MGSE9-12.A.SSE.3 Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.

MGSE9-12.A.SSE.3a Factor any quadratic expression to reveal the zeros of the function defined by the expression.

MGSE9-12.A.SSE.3b Complete the square in a quadratic expression to reveal the maximum and minimum value of the function defined by the expression.

Standards for Mathematical Practice

This task uses all of the practices with emphasis on:

2. Reason abstractly and quantitatively by requiring students to make sense of quantities and their relationships to one another in problem situations.

6. Attend to precision by requiring students to calculate efficiently and accurately; and to communicate precisely with others by using clear mathematical language to discuss their reasoning.