Mathematics Assessment Project

Formative Assessment Lesson Materials

Steps to Solving Equations

MARS Shell Center
University of Nottingham & UC Berkeley
Alpha Version

Please Note:

These materials are still at the "alpha" stage and are not expected to be perfect. The revision process concentrated on addressing any major issues that came to light during the first round of school trials of these early attempts to introduce this style of lesson to US classrooms. In many cases, there have been very substantial changes from the first drafts and new, untried, material has been added.

We suggest that you check with the Nottingham team before releasing any of this material outside of the core project team.

If you encounter errors or other issues in this version, please send details to the MAP team c/o map.feedback@mathshell.org.

Steps to Solving Equations

Mathematical goals

This lesson unit is intended to help you assess how well students are able to:

- Form and solve linear equations involving factorizing and using the distributive law.

In particular, this unit aims to help you identify and assist students who have difficulties in:

- Using variables to represent quantities in a real-world or mathematical problem.
- Solving word problems leading to equations of the form $px + q = r$ and $p(x + q) = r$.

Common Core State Standards

This lesson involves *mathematical content* in the standards from across the grades, with emphasis on:

7.EE: Use properties of operations to generate equivalent expressions.
Solve real-life and mathematical problems using numerical and algebraic expressions and equations

This lesson involves a range of *mathematical practices,* with emphasis on:

1. Make sense of problems and persevere in solving them.
4. Model with mathematics.

Introduction

The lesson unit is structured in the following way:

- Before the lesson, students attempt the assessment task individually. You then review students' responses and formulate questions that will help them improve their work.
- During the lesson, students work collaboratively in pairs or threes, matching equations to stories and then ordering the steps used to solve these equations. Throughout their work, students explain their reasoning to their peers.
- Finally, students work individually again to review their work and attempt a second, similar task to the initial assessment task.

Materials required

- Each individual student will need a copy of the pre-assessment task *Express Yourself!*, a copy of the post-assessment task, *Express Yourself Again!,* a mini-whiteboard, an eraser, and a pen.
- For each small group of students provide cut up copies of *Card Set A: Stories*, *Card Set B: Equations*, *Card Set C: Steps to Solving*, *Card Set D: Methods,* a large sheet of paper for making a poster, a marker, and a glue stick.
- There are also some projector resources to help with whole-class discussion.

Time needed

Twenty minutes before the lesson for the assessment task, a one-hour lesson, and 20 minutes in a follow-up lesson (or for homework). All timings are approximate, depending on the needs of your students.

Before the lesson

Pre-Assessment task: *Express Yourself!* (20 minutes)

Have the students do this task, in class or for homework, a day or more before the formative assessment lesson. This will give you an opportunity to assess the work, and identify students who have misconceptions or need other forms of help. You should then be able to target your help more effectively in the follow-up lesson.

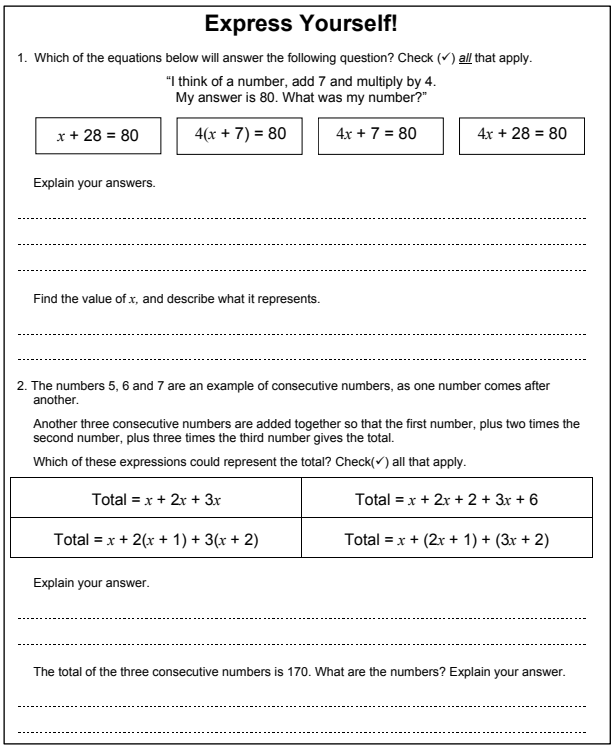
Give each student a copy of *Express Yourself!*

Introduce the task briefly, and help the class to understand what they are being asked to do.

> *Spend fifteen minutes working individually, answering these questions.*
> *Show all your work on the sheet.*
> *Make sure you explain your answers really clearly.*

It is important that students answer the questions without assistance, as far as possible.

Students should not worry too much if they cannot understand or do everything, because you will teach a lesson using a similar task, which should help them. Explain to students that by the end of the next lesson, they should expect to answer questions such as these. This is their goal.



Express Yourself!

1. Which of the equations below will answer the following question? Check (✓) *all* that apply.

"I think of a number, add 7 and multiply by 4.
My answer is 80. What was my number?"

$x + 28 = 80$	$4(x + 7) = 80$	$4x + 7 = 80$	$4x + 28 = 80$

Explain your answers.

Find the value of x, and describe what it represents.

2. The numbers 5, 6 and 7 are an example of consecutive numbers, as one number comes after another.

Another three consecutive numbers are added together so that the first number, plus two times the second number, plus three times the third number gives the total.

Which of these expressions could represent the total? Check(✓) all that apply.

Total = $x + 2x + 3x$	Total = $x + 2x + 2 + 3x + 6$
Total = $x + 2(x + 1) + 3(x + 2)$	Total = $x + (2x + 1) + (3x + 2)$

Explain your answer.

The total of the three consecutive numbers is 170. What are the numbers? Explain your answer.

Assessing students' responses

Collect students' responses to the task. Make some notes on what their work reveals about their current levels of understanding and difficulties. The purpose of this is to forewarn you of the issues that will arise during the lesson, so that you may prepare carefully.

We suggest that you do not score students' work. The research shows that this is counterproductive, as it encourages students to compare scores, and distracts their attention from how they may improve their mathematics.

Instead, help students to make further progress by asking questions that focus attention on aspects of their work. Some suggestions for these are given in the table on the next page. These have been drawn from common difficulties observed in trials of this unit.

We strongly recommend that you write your own lists of questions, based on your own students' work, using the ideas in the *Common issues* table. You may choose to write questions on each student's work. If you do not have time for this, then prepare a few questions that apply to most students and write these on the board when the assessment task is revisited.

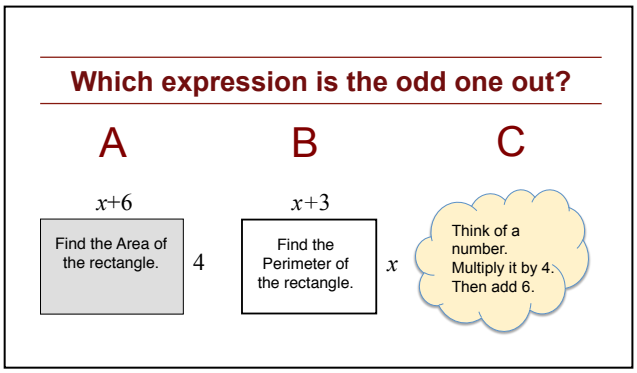
Common issues:	Suggested questions and prompts:
Student applies operations in the wrong order (Q1) For example: The student chooses $4x + 7 = 80$ as an appropriate equation. Or: The student chooses $x + 28 = 80$ as an appropriate equation.	• *In this expression, what is the first thing that happens to the number I am thinking of? Then what happens?* • *What does x represent? What are you adding 7 to?* • *Is adding 7 and then multiplying by 4 the same as adding 28? How could you check this?*
Student does not recognize all relevant expressions (Q1) For example: The student chooses ***4(x + 7) = 80*** as the only appropriate equation.	• *How else could you write the expression 4(x +7)?*
Student assumes the three numbers are equal (Q2) For example: The student selects *Total = x + 2x + 3x* as an appropriate expression.	• *What does "consecutive" mean?* • *What does x represent?* • *Can you try some numbers to check that this works?*
Student does not multiply all terms in the bracket (Q2) For example: The student selects *Total = x + (2x + 1) + (3x + 2)* as an appropriate expression.	• *What does x represent?* • *How do you write "one more than x" using algebra? Now read the question again: what happens next? What happens if you add two of these numbers together?*
Student calculates an incorrect value for *x* (Q1, Q2)	• *If you substitute your value of x into the left hand side of the equation, does it equal the number on the right hand side?* • *How will you check whether your value for x is correct?*
Student does not distinguish between area and perimeter (Q3) For example: The student writes an expression for the area instead of the perimeter of the rectangles in *Diagrams C* and *D*. Or: The student writes an expression for the perimeter instead of the area of the rectangles in *Diagrams A* and *B*.	• *How do you calculate the area of a rectangle?* • *What does perimeter mean?* • *Does your expression represent the area or the perimeter of this rectangle?*
Student does not interpret the solution For example: The student does not realize that *x* represents the number first thought of (Q1). Or: The student does not recognize that *x =27* is the first of the three consecutive numbers found (Q2).	• *You have found that x = 27. Read the question again. What are the **three** consecutive numbers?*
Student completes the task	• *Can you make up a situation that would lead to the equation 4(x+3)=16?* • *Could you solve these equations using a different method? What would the method be?*

Suggested lesson outline

Whole class introduction: Odd One Out? (10 minutes)

Give each student a mini-whiteboard, pen, and wipe.

Display the first slide, *Odd One Out?* from the projector resources.

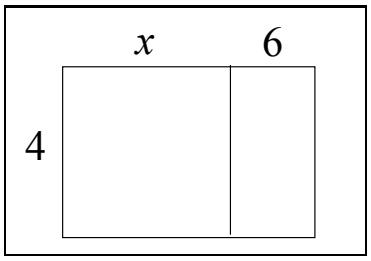


Which of the three expressions is the odd one out? Write the letter A, B, or C on your whiteboard. Be prepared to explain your thinking!

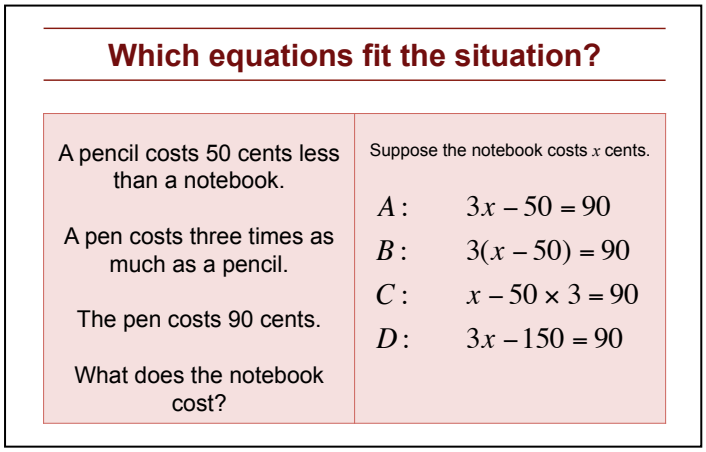
Give students a few minutes to discuss in pairs, then share suggestions.

Students should identify A as the odd one out, since this one results in $4(x+6)$ or $4x+24$, whereas the other two result in $4x+6$. Make sure students understand and can explain the distributive law using the diagram on the second slide.

Can you see why it is 4x+24? [See the diagram as two pieces]
Can you think of another way of writing 4x+24?
Can you see why it is 4(x+6)?
[Width multiplied by the whole length]



Display the third slide from the projector resources:



Ask students to identify the equations that best fit the situation.

Try to write expressions for the cost of the pencil and then the pen in terms of x.
Which equations fit the situation?

Students may struggle with this initially. Help them to see how the equation may be built up. The cost of the pencil is $x - 50$. The pen costs three times this: $3(x - 50)$ (and not $3x - 50$ or $x - 50 \times 3$). Link to the previous slide on the distributive law to show that $3x - 150 = 90$ is also true.

Explain to students that in the next activity they will be trying to match situations to equations in a similar way.

Collaborative activity: Matching *Card Sets A* and *B* (15 minutes)

Organize students into groups of two or three. For each group provide a cut up copy of the *Card Set A: Stories*, *Card Set B: Equations*, and a large sheet of poster paper.

> *The six stories have also been written as equations using algebra.*
> *Work together in your group.*
> *Your job is to match each story with an equation.*
>
> *You may need to try writing some expressions before you can find the matching equation. Use your whiteboard to do this.*
>
> *You don't need to stick anything in place at this stage.*

It does not matter if students are unable to match all six stories. It is more important that they can explain and justify their choices confidently.

While students work in small groups you have two tasks: to make a note of students approaches to the task, and to support student reasoning.

Make a note of student approaches to the task

Listen and watch students carefully. Note any common errors in algebra and computation. Do students use the distributive law correctly? Do they only multiply part of an expression? Notice the ways students check to see if their card-match is correct. Do they substitute back into the equations? Do they know which value x represents?

Notice the quality and depth of students' explanations. Are students satisfied just to match the cards, or do they explain choices? Do they challenge each other if they disagree on a matched pair?

Support student reasoning

Prompt students to explain what expressions mean clearly.

> *What does x represent in this story?*
> *What information do you have? What do you need to find out?*
>
> *You've decided how you're going to write [how old James is/the score for Paper 1/the cost of a strawberry chew]. What's the connection between [his age and his dad's age/the scores/the costs of a chew and a lollipop]? How does that help you to write [his dad's age/the score for paper 2/a lollipop]?*

Encourage students to explain their reasoning carefully, and check that all group members are able to justify each choice.

> *Jean, you matched those cards. Terry, do you agree that those cards match? Explain please.*

If students finish quickly, ask them to write their own, different stories to match the equation cards.

Sharing work (5 minutes)

As students finish matching the cards, ask students to jot down their matched pairs on their whiteboards (for example, Story 1 with E1, Story 5 with E2, etc.). Then ask one student from each group to visit another group. This way they can compare their own matches with another group's.

The students remaining at their desks are to explain their reasoning for the matched cards.

Students may now want to make changes or additions to their matches, especially if they have visited a group that has matched up different stories to their own. If this is the case, it is important that students are able to

explain the new match. They should not just assume that another group's matches are correct, without exploring the reasoning used.

Collaborative activity: Matching *Card Sets C* and *D* (20 minutes)

Give each group of students a marker, and a glue stick.

> *Put the cards E5 and E6 and the story card you've matched with them on one side.*
> *You already have a large sheet of paper. I would like you to divide your paper into four quarters.*
>
> *You are going to work with equation cards E1 – E4.*
> *Stick one at the top of each section, along with the matched-story.*
> *If you haven't managed to match all four of the equation cards with a story, just stick down the four equation cards.*

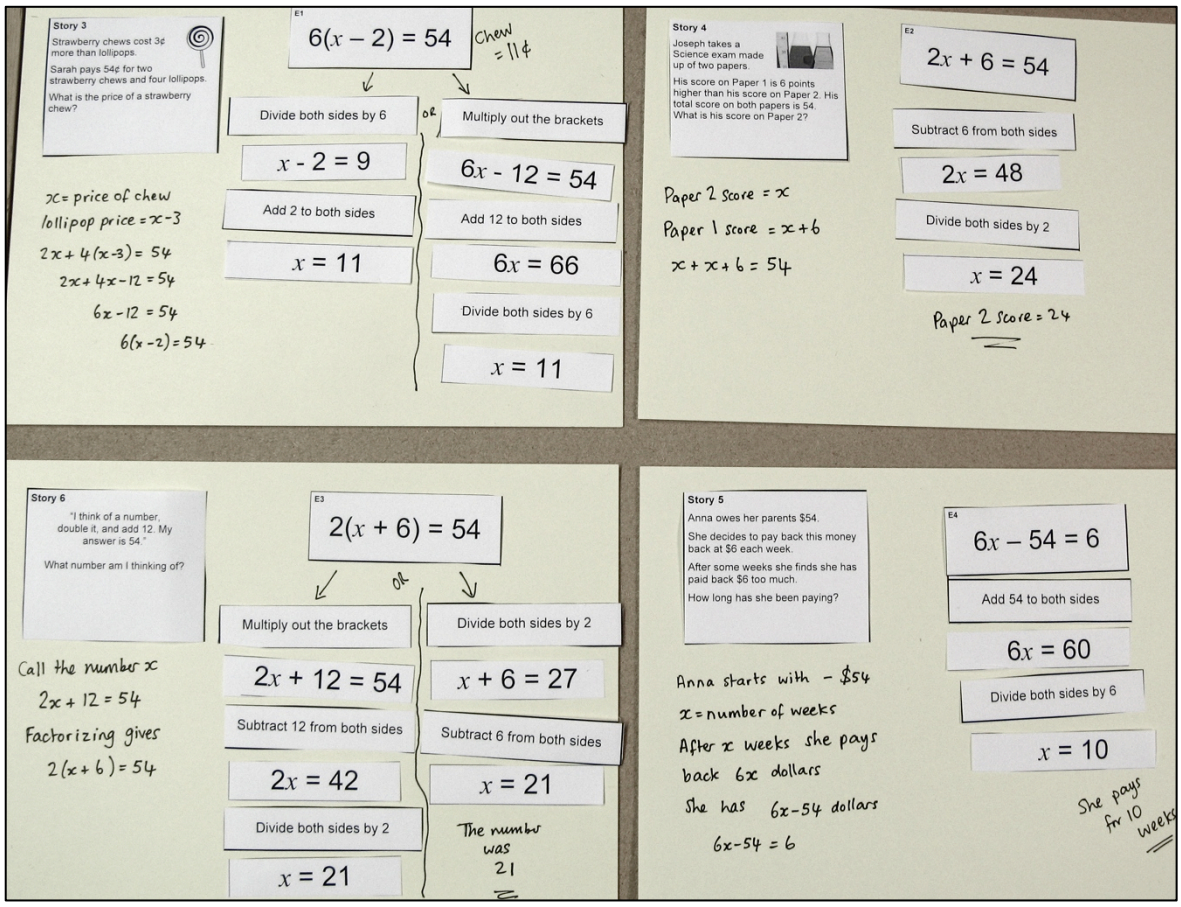
Students don't need to stick the last two sets of cards in place, as they are not used in the second matching activity. If the sheets of paper you have provided are very large, they may wish to do this, nevertheless.

For each group provide a cut-up copy of *Card Set C: Steps to Solving* and *Card Set D: Methods*.

> *You are going to explore the steps to solving these four equations.*
>
> *Choose a description for the method involved, followed by the equation this produces. Repeat this until you finally reach a solution.*
>
> *If you find there is more than one method for solving an equation, stick the two solutions side-by-side.*

Once students have completed this work, they can finish any matching of pairs. Then encourage them to add explanations to their posters to show how they arrived at an equation for each of their chosen stories.

As students work, support them as before. Walk around, watch, and listen, and ask students to explain each step of the solution process. The finished poster may look something like this:



Whole-class discussion (10 minutes)

Select two or three students from different groups that have completed a solution for *Equation Cards E1* and/or *E3*. Ask them to explain why there are two methods for solving these equations.

Which of the two methods is the most efficient?

Which method do you prefer? Why?

Is there a different method that could be used to solve these equations?

Students may prefer to clear the brackets, even though this creates an extra step in the solution process.

*What do you need to remember when using the distributive property to clear brackets? [To multiply **every** term in the bracket by the term outside.]*

How else could we clear the brackets? [E.g. $2(x + 3) = (x + 3) + (x + 3) = x + 3 + x + 3$.]

The focus of this discussion is not to promote a particular method, but instead, to explore the processes involved in a range of different approaches.

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Follow-up lesson: Review individual solutions to the assessment task (20 minutes)

Give students their response to the original assessment task *Express Yourself!* and a copy of the task, *Express Yourself Again!*.

If you have not added questions to individual pieces of work then write your list of questions on the board. Students should select from this list only the questions they think appropriate to their own work.

Look at your written script and think about what you have learned since you did this task.

Make some notes on what you have learned during the lesson.

Now have a go at the second sheet, Express Yourself Again! Can you use what you have learned to answer these questions?

Express Yourself Again!

1. Which of the equations below will answer the following question? Check (√) *all* that apply.

"I think of a number, add 8 and multiply by 3.
My answer is 66. What was my number?"

$x + 24 = 66$	$3x + 8 = 66$	$3x + 24 = 66$	$3(x + 8) = 66$

Explain your answers.

Find the value of x, and describe what it represents.

2. The numbers 5, 6 and 7 are an example of consecutive numbers, as one number comes after another.

Another three consecutive numbers are added together, then the sum is multiplied by three.

Some of the equations below represent the total using algebra. Check (√) *all* that apply.

Total = $3x + 3x + 1 + 3x + 2$	Total = $3x + 3x + 3 + 3x + 6$
Total = $3x + 3(x + 1) + 3(x+2)$	Total = $x + x + 3 + x + 6$

Explain your answers.

The total of the equation is 162. What are the consecutive numbers? Explain your answer.

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Solutions

Pre-Assessment task: *Express Yourself!*

1. The task is to write this sentence using algebra: "I think of a number, add 7 and multiply by 4. My answer is 80." $4(x + 7) = 80$ and $4x + 28 = 80$ are two ways of representing this.
   Students choosing the other two equations may have not applied the distributive property to one of the terms in the left-hand expression.
   $x = 13$ represents the number I was thinking of.
2. Total = $x + 2x + 2 + 3x + 6$ and Total = $x + 2(x + 1) + 3(x + 2)$ are the expressions that match the sentence.
   Students may simplify the expression, before solving the equation:
   $6x + 8 = 170$
   $6x = 162$
   $x = 27$.
   The consecutive numbers are 27, 28 and 29.
3. Diagram A does not describe the algebraic expression $4x + 4$. The area of this rectangle is $4x + 8$.

Lesson task

In the first card matching activity, these are the correct pairs:

Story 1 → E5.
Story 2 → E6.
Story 3 → E1.
Story 4 → E2.
Story 5 → E4.
Story 6 → E3.

These are the matches that provide the "steps to solving" the Equations on Cards E1 to E4:

E1 $6(x – 2) = 54$

Method 1:
$6(x – 2) = 54$
Divide both sides by 6
$x – 2 = 9$
Add 2 to both sides
$x = 11$.

Method 2:
$6(x – 2) = 54$
Multiply out the brackets
$6x – 12 = 54$
Add 12 to both sides
$6x = 66$
Divide both sides by *6*
$x = 11$.

A strawberry chew costs 11¢ (and a lollipop costs 8¢).

E2 $2x + 6 = 54$

$2x + 6 = 54$
Subtract 6 from both sides
$2x = 48$
Divide both sides by *2*
$x = 24$.

The score for Paper 2 was 24 marks.

E3 **$2(x + 6) = 54$**

Method 1:
$2(x + 6) = 54$
Multiply out the brackets
$2x + 12 = 54$
Subtract 12 from both sides
$2x = 42$
Divide both sides by 2
$x = 21$.

Method 2:
$2(x + 6) = 54$
Divide both sides by 2
$x + 6 = 27$
Subtract 6 from both sides
$x = 21$.

The number I was thinking of was 21.

E4 **$6x – 54 = 6$**

$6x – 54 = 6$
Add 54 to both sides
$6x = 60$
Divide both sides by 6
$x = 10$.

She has been paying for 10 weeks

Post-Assessment task: *Express Yourself Again!*

1. Alicia's statement can be represented by these equations:
$3x + 24=66$, and $3(x + 8)=66$.
Students choosing the other two equations may have not applied the distributive property to one of the terms in the left-hand expression.
x represents the number Alicia first thought of.
The method used to solve the equation will depend on which representation the student chooses to work with. The correct solution is $x=14$.

2. The expressions for three consecutive numbers are x, $x + 1$, $x + 2$.
Total = $3x + 3x + 3 + 3x + 6$ and Total = $3x + 3(x + 1) + 3(x + 2)$ are the expressions that match the sentence.
Summing the terms gives $x + x + 1 + x + 2$. Students may simplify this before or after multiplying by three:

$3(x + x + 1 + x + 2) = 162$	or	$3(3x + 3) = 162$
$3x + 3x + 3 + 3x + 6 = 162$	or	$9x + 9 = 162$
$9x + 9 = 162$		

From this point, the solution methods are the same.
$9x = 153$
$x = 17$.

3. Diagram A: the perimeter is $2((2x + 1) + 3) = 2(2x + 4) = 4x + 8$.
Diagram B: the perimeter is $2((2x + 2) + 3)=2(2x + 5)=4x + 10$.
Diagram C: the area is $2(x + 2) \times 2 = 4(x + 2) = 4x + 8$.
Diagram D: the perimeter of the rectangle is $2((x + 3) + (x + 1)) = 2(2x + 4) = 4x + 8$.
Thus the diagrams that match the expression are A, C, and D.

Express Yourself!

1. Which of the equations below will answer the following question? Check (✓) *all* that apply.

"I think of a number, add 7 and multiply by 4.
My answer is 80. What was my number?"

$x + 28 = 80$	$4(x + 7) = 80$	$4x + 7 = 80$	$4x + 28 = 80$

Explain your answers.

Find the value of x, and describe what it represents.

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2. The numbers 5, 6 and 7 are an example of consecutive numbers, as one number comes after another.

Another three consecutive numbers are added together so that the first number, plus two times the second number, plus three times the third number gives the total.

Which of these expressions could represent the total? Check(✓) all that apply.

Total = $x + 2x + 3x$	Total = $x + 2x + 2 + 3x + 6$
Total = $x + 2(x + 1) + 3(x + 2)$	Total = $x + (2x + 1) + (3x + 2)$

Explain your answer.

The total of the three consecutive numbers is 170. What are the numbers? Explain your answer.

Express Yourself! (Continued)

3. Here are four diagrams:

Diagram A

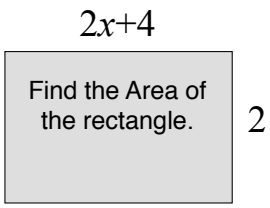


Diagram B

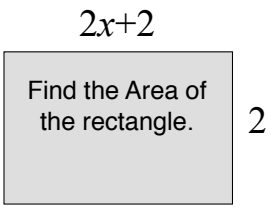


Diagram C

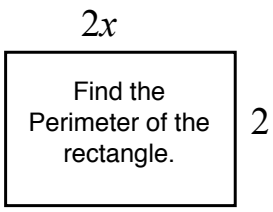
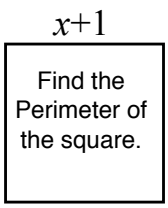


Diagram D

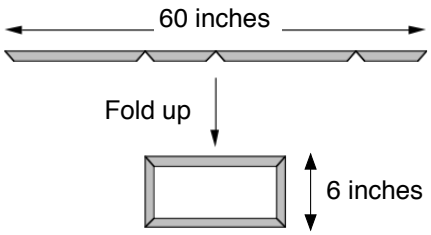


Which diagram **does not** represent the expression $4x + 4$?

Explain your answer.

Card Set A: Stories

Story 1



60 inches of plastic are folded to make a picture frame.

The height of the finished frame is 6 inches. How long is the frame?

Story 2

Tom is 57 years old.

Tom has a son called James.

In three years time Tom will be twice as old as James.

How old is James?

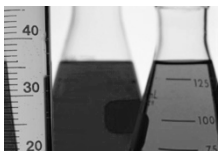
Story 3

Strawberry chews cost 3¢ more than lollipops.

Sarah pays 54¢ for two strawberry chews and four lollipops.

What is the price of a strawberry chew?

Story 4



Joseph takes a Science exam made up of two papers.

His score on Paper 1 is 6 points higher than his score on Paper 2. His total score on both papers is 54. What is his score on Paper 2?

Story 5

Anna owes her parents $54.

She decides to pay back this money back at $6 each week.

After some weeks she finds she has paid back $6 too much.

How long has she been paying?

Story 6

"I think of a number, double it, and add 12. My answer is 54."

What number am I thinking of?

Card Set B: Equations

E1 $6(x - 2) = 54$	**E2** $2x + 6 = 54$
E3 $2(x + 6) = 54$	**E4** $6x - 54 = 6$
E5 $2x + 12 = 60$	**E6** $2(x + 3) = 60$

Card Set C: Steps to Solving

$6x = 60$	$2x = 48$
$6x = 66$	$x - 2 = 9$
$x = 24$	$x + 6 = 27$
$x = 11$	$x = 21$
$6x - 12 = 54$	$2x = 42$
$x = 10$	$x = 11$
$2x + 12 = 54$	$x = 21$

Card Set D: Methods

Add 12 to both sides	Multiply out the brackets
Divide both sides by 2	Add 54 to both sides
Subtract 6 from both sides	Divide both sides by 6
Divide both sides by 2	Multiply out the brackets
Divide both sides by 6	Divide both sides by 2
Subtract 6 from both sides	Subtract 12 from both sides
Add 2 to both sides	Divide both sides by 6

Express Yourself Again!

1. Which of the equations below will answer the following question? Check (✓) *all* that apply.

"I think of a number, add 8 and multiply by 3.
My answer is 66. What was my number?"

$x + 24 = 66$	$3x + 8 = 66$	$3x + 24 = 66$	$3(x + 8) = 66$

Explain your answers.

Find the value of x, and describe what it represents.

---

2. The numbers 5, 6 and 7 are an example of consecutive numbers, as one number comes after another.

   Another three consecutive numbers are added together, then the sum is multiplied by three.

   Some of the equations below represent the total using algebra. Check (✓) *all* that apply.

Total = $3x + 3x + 1 + 3x + 2$	Total = $3x + 3x + 3 + 3x + 6$
Total = $3x + 3(x + 1) + 3(x+2)$	Total = $x + x + 3 + x + 6$

   Explain your answers.

   The total of the equation is 162. What are the consecutive numbers? Explain your answer.

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3. Look at the four diagrams below:

Diagram A

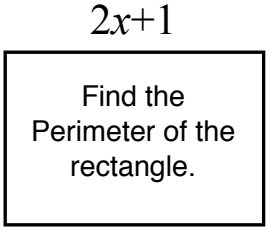


Diagram B

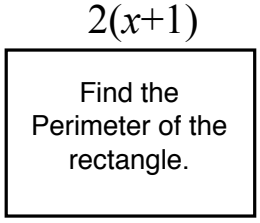


Diagram C

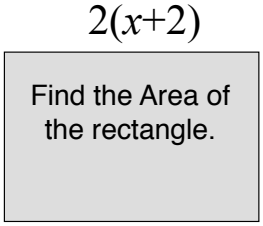
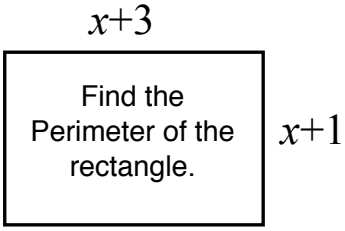


Diagram D



Check (✓) **every** diagram that represents the expression:

$$4x + 8$$

Explain your answers.

Which expression is the odd one out?

A

$x+6$

Find the Area of the rectangle.

4

B

$x+3$

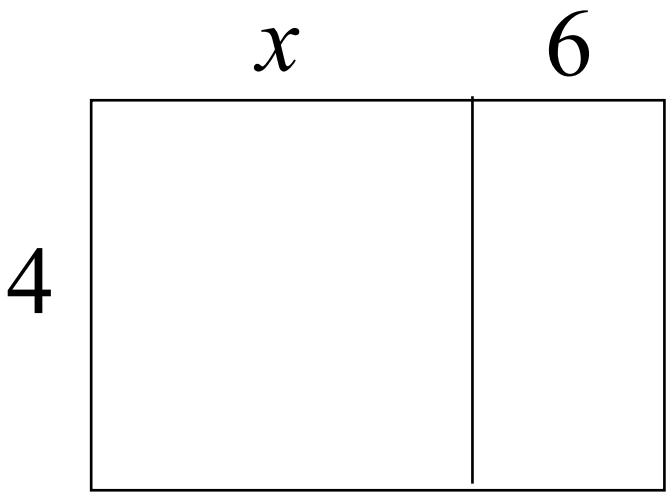
Find the Perimeter of the rectangle.

x

C

Think of a number.
Multiply it by 4.
Then add 6.

What is the area of this shape?



Which equations fit the situation?

A pencil costs 50 cents less than a notebook.

A pen costs three times as much as a pencil.

The pen costs 90 cents.

What does the notebook cost?

Suppose the notebook costs x cents.

A: $3x - 50 = 90$

B: $3(x - 50) = 90$

C: $x - 50 \times 3 = 90$

D: $3x - 150 = 90$