Georgia
Standards of Excellence
Curriculum Frameworks

Mathematics

Accelerated GSE Analytic Geometry B/
Advanced Algebra

Unit 2: Geometric and Algebraic Connections

Georgia Department of Education

Richard Woods, Georgia's School Superintendent
"Educating Georgia's Future"

Unit 2
Geometric and Algebraic Connections

Table of Contents

OVERVIEW

In this unit students will:

- use Algebra to model Geometric ideas
- spend time developing equations from geometric definition of circles.
- address equations in standard and general forms
- graph by hand and by using graphing technology
- develop the idea of algebraic proof in conjunction with writing formal geometric proofs

Although the units in this instructional framework emphasize key standards and big ideas at specific times of the year, routine topics such as estimation, mental computation, and basic computation facts should be addressed on an ongoing basis. Ideas related to the eight practice standards should be addressed constantly as well. This unit provides much needed content information and excellent learning activities. However, the intent of the framework is not to provide a comprehensive resource for the implementation of all standards in the unit. A variety of resources should be utilized to supplement this unit. The tasks in this unit framework illustrate the types of learning activities that should be utilized from a variety of sources. To assure that this unit is taught with the appropriate emphasis, depth, and rigor, it is important that the **"Strategies for Teaching and Learning"** and the tasks listed under **"Evidence of Learning"** be reviewed early in the planning process.

STANDARDS ADDRESSED IN THIS UNIT

Mathematical standards are interwoven and should be addressed throughout the year in as many different units and activities as possible in order to emphasize the natural connections that exist among mathematical topics. Students will not only have the opportunity to use different forms of circles and parabolas, but also to derive the formulas for themselves. This should lead to a deeper understanding of the conceptual ideas of circles and parabolas. Throughout the process, students will be developing formulas and algebraic proofs.

KEY STANDARDS

Translate between the geometric description and the equation for a conic section

MGSE9-12.G.GPE.1 Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.

Use coordinates to prove simple geometric theorems algebraically

MGSE9-12.G.GPE.4 Use coordinates to prove simple geometric theorems algebraically. *For example, ~~prove or disprove that a figure defined by four given points in the coordinate plane is a rectangle;~~ prove or disprove that the point (1, √3) lies on the circle centered at the origin and containing the point (0,2).*
(Focus on ~~quadrilaterals, right triangles, and~~ circles.)

Apply geometric concepts in modeling situations

MGSE9-12.G.MG.1 Use geometric shapes, their measures, and their properties to describe objects
(e.g., modeling a tree trunk or a human torso as a cylinder)

MGSE9-12.G.MG.2 Apply concepts of density based on area and volume in modeling situations
(e.g., persons per square mile, BTUs per cubic foot)

MGSE9-12.G.MG.3 Apply geometric methods to solve design problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios).

STANDARDS FOR MATHEMATICAL PRACTICE

Refer to the Comprehensive Course Overview for more detailed information about the Standards for Mathematical Practice.

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

SMP = Standards for Mathematical Practice

ENDURING UNDERSTANDINGS

- Derive the formula for a circle using the Pythagorean Theorem
- Apply algebraic formulas and ideas to geometric figures and definitions
- Extend knowledge of quadratic function and their characteristics to maximize profit or minimize cost in the real world.

- Model everyday objects using three dimensional shapes and describe the object using characteristics of the shape.
- Solve real world problems that can be modeled using density, area, and volume concepts.

ESSENTIAL QUESTIONS

- How can I use the Pythagorean Theorem to derive the equation of a circle?
- How are the graph of a circle and its equation related?
- How are the equation of a circle and its graph related?
- How can I prove properties of geometric figures algebraically?
- How can I minimize cost and maximize the volume of a topless box?

CONCEPTS/SKILLS TO MAINTAIN

It is expected that students will have prior knowledge/experience related to the concepts and skills identified below. It may be necessary to pre-assess in order to determine if time needs to be spent on conceptual activities that help students develop a deeper understanding of these ideas.

- number sense
- computation with whole numbers and decimals, including application of order of operations
- addition and subtraction of common fractions with like denominators
- applications of the Pythagorean Theorem
- usage of the distance formula, including distance between a point and a line
- finding a midpoint
- graphing on a coordinate plane
- completing the square
- operations with radicals
- methods of proof

SELECTED TERMS AND SYMBOLS

The following terms and symbols are often misunderstood. These concepts are not an inclusive list and should not be taught in isolation. However, due to evidence of frequent difficulty and misunderstanding associated with these concepts, instructors should pay particular attention to them and how their students are able to explain and apply them.

The definitions below are for teacher reference only and are not to be memorized by the students. Students should explore these concepts using models and real life examples. Students should understand the concepts involved and be able to recognize and/or demonstrate them with words, models, pictures, or numbers.

The website below is interactive and includes a math glossary.

http://intermath.coe.uga.edu/dictnary/homepg.asp
Definitions and activities for these and other terms can be found on the Intermath website. Links to external sites are particularly useful.

- **Center of a Circle**: The point inside the circle that is the same distance from all of the points on the circle.
- **Circle:** The set of all points in a plane that are the same distance, called the radius, from a given point, called the center. Standard form: $(x-h)^2+(y-k)^2=r^2$
- **Diameter**: The distance across a circle through its center. The line segment that includes the center and whose endpoints lie on the circle.
- **Pythagorean Theorem**: A theorem that states that in a right triangle, the square of the length of the hypotenuse equals the sum of the squares of the lengths of the legs.
- **Radius:** The distance from the center of a circle to any point on the circle. Also, the line segment that has the center of the circle as one endpoint and a point on the circle as the other endpoint.
- **Standard Form of a Circle:** $(x-h)^2+(y-k)^2=r^2$, where (h,k) is the center and r is the radius.

EVIDENCE OF LEARNING

By the conclusion of this unit, students should be able to demonstrate the following competencies:

- Write the equation for a circle given information such as a center, radius, point on the circle, etc.
- Prove simple geometric properties using coordinates.

FORMATIVE ASSESSMENT LESSONS (FAL)

Formative Assessment Lessons are intended to support teachers in formative assessment. They reveal and develop students' understanding of key mathematical ideas and applications. These lessons enable teachers and students to monitor in more detail their progress towards the targets of the standards. They assess students' understanding of important concepts and problem solving performance, and help teachers and their students to work effectively together to move each student's mathematical reasoning forward.

More information on Formative Assessment Lessons may be found in the Comprehensive Course Overview.

SPOTLIGHT TASKS

A Spotlight Task has been added to each GSE mathematics unit in the Georgia resources for middle and high school. The Spotlight Tasks serve as exemplars for the use of the Standards for Mathematical Practice, appropriate unit-level Georgia Standards of Excellence, and research-based pedagogical strategies for instruction and engagement. Each task includes teacher commentary and support for classroom implementation. Some of the Spotlight Tasks are revisions of existing Georgia tasks and some are newly created. Additionally, some of the Spotlight Tasks are 3-Act Tasks based on 3-Act Problems from Dan Meyer and Problem-Based Learning from Robert Kaplinsky.

3-ACT TASKS

A Three-Act Task is a whole group mathematics task consisting of 3 distinct parts: an engaging and perplexing Act One, an information and solution seeking Act Two, and a solution discussion and solution revealing Act Three.

More information along with guidelines for 3-Act Tasks may be found in the Comprehensive Course Overview.

TASKS

The following tasks represent the level of depth, rigor, and complexity expected of all Analytic Geometry students. These tasks, or tasks of similar depth and rigor, should be used to demonstrate evidence of learning. It is important that all elements of a task be addressed throughout the learning process so that students understand what is expected of them. While some tasks are identified as a performance task, they may also be used for teaching and learning (learning/scaffolding task).

Task Name	**Task Type** ***Grouping Strategy***	**Content Addressed**	**SMPs Addressed**
Rolling Cups (Spotlight Task)	Performance Task *Large Group/Partner Task*	Introducing the unit along with choosing appropriate mathematics to solve a non-routine problem	1, 2, 3, 4
Deriving the General Equation of Circles	Performance Task *Large Group/Partner Task*	Generalizing the formula for the equation of a circle.	2, 3, 7, 8

Equations of Circles – 1 (FAL)	Formative Assessment Lesson	Translate between geometric features of a circle and its equation.	1, 5, 7
Equations of Circles – 2 (FAL)	Formative Assessment Lesson	Translate between a circle's equation and its geometric features.	1, 7
Converting Standard to General Form	Learning Task *Individual/Partner Task*	Algebraic manipulations necessary to change an equation from standard form to general form.	2, 3, 7, 8
Completing the Square in a Circle?	Learning Task *Individual/Partner Task*	Completing the square to find the center and radius of a given circle.	2, 3, 7, 8
Graphing Circles on a Graphing Calculator	Extension Task *Partner/Small Group Task*	Using technology to graph a circle.	2, 3, 7, 8
Radio Station Listening Areas	Performance Task *Individual/Partner Task*	Real-world applications of writing the equation of a circle.	2, 3, 7, 8
Algebraic Proof	Learning Task *Individual/Partner Task*	Using coordinates to prove simple geometric theorems algebraically.	2, 3, 7
A Day at the Beach	Performance Task *Individual/Partner Task*	Determine the volume relationships of cylinders, pyramids, cones, and spheres	1, 4, 6
How Many Cells are in the Human Body?	Performance Task *Partner Task*	Apply the concepts of mass, volume, and density in a real-world context.	2, 3, 5, 6
Maximize Volume	Learning Task *Individual/Partner Task*	Maximize Volume	2-3, 7-8
Culminating Task: Dr. Cone's New House	Performance Task *Individual/Partner Task*	Write the equations of circles and parabolas and use coordinates to prove simple geometric theorems algebraically.	2, 3, 7, 8

Rolling Cups

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Modification of Source: Lesson Materials from Mathematics Assessment Project
http://map.mathshell.org/materials/download.php?fileid=1254

ESSENTIAL QUESTIONS:

- How do you choose appropriate mathematics to solve a non-routine problem?
- How do you generate useful data by systematically controlling variables?
- How do you develop experimental and analytical models of a physical situation?

TASK COMMENTS:

Tasks and lessons from the Mathematics Assessment Project are specifically designed to help teachers effectively formatively assess their students. The way the tasks and lessons are designed gives the teacher a clear understanding of what the students are able to do and not do. Within the lesson, teachers will find suggestions and question prompts that will help guide students towards understanding. For more information access the MAP website:
http://www.map.mathshell.org/materials/background.php?subpage=formative

The task that this task is based on, *Modeling: Rolling Cups*, is a Formative Assessment Lesson (FAL) that can be found at the website:
http://map.mathshell.org/materials/lessons.php?taskid=428&subpage=problem

The FAL document provides a clear lesson design, from the opening of the lesson to the closing of the lesson.

The PDF version of the task can be found at the link below:
http://map.mathshell.org/materials/download.php?fileid=1254

STANDARDS ADDRESSED IN THIS TASK:

MGSE9-12.G.MG.1 Use geometric shapes, their measures, and their properties to describe objects (e.g., modeling a tree trunk or a human torso as a cylinder).

MGSE9-12.G.MG.3 Apply geometric methods to solve design problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios).

Standards for Mathematical Practice
This lesson uses all of the practices with emphasis on:

1. Make sense of problems and persevere in solving them by requiring students to interpret and make meaning of a problem and find a logical starting point, and to monitor their progress and change their approach to solving the problem, if necessary.

2. Reason abstractly and quantitatively by requiring students to make sense of quantities and their relationships to one another in problem situations.

3. Construct viable arguments and critique the reasoning of others by engaging students on discussion of why they agree or disagree with responses, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

4. Model with mathematics by expecting students to apply the mathematics concepts they know in order to solve problems arising in everyday situations, and reflect on whether the results are sensible for the given scenario.

More information along with guidelines for 3-Act Tasks may be found in the Comprehensive Course Guide.

Act I

Have the students watch a video of each cup being rolled. If you have cups, you can also bring them in and have students roll their own cups along with watching the video. If the MAP website must be used, as opposed to the youtube link or source video, only show the parts of each cup video that shows the cup rolling. A youtube link is provided below for the video or Act I – Rolling Cups.mov can be used.

After students have watched the video, leave the video on loop and ask the students to discuss questions that they might ask about the situation (e.g., the dimensions of the glass, how long it takes to complete a roll, how big the roll circle is etc.). Keep a running tab on the whiteboard of the questions that students generate. If not posed, introduce, "How big is the roll circle?" as the focal question. As the students work on the problem, leave the video playing on loop.

Watch the presented video, which can be found at:
http://youtu.be/psMYzAqRJVQ
or at:
http://real.doe.k12.ga.us/vod/gso/math/Act-I-Rolling-Cups.mp4

Act II

With the focal question introduced, have students work in groups pursuing this question, as well as the other questions that they have raised. Write, "What information do you need?" and remind them they should be asking this as they go along. The most relevant information is:
Green Cup:
Diameter: 3 inches
Slant Length: 3.5 inches
Bottom Diameter: 2 inches
Clear Short Cup:
Diameter 3.5 inches
Slant Length: 3.75 inches
Botton Diamter: 3 inches

Soup Can:
Diameter 3 inches
Slant Length: 4.25 inches

Tall Cup:
Diameter: 2.5 inches
Slant Length: 5.75 inches
Bottom Diameter: 2 inches

This information is on the video at: http://real.doe.k12.ga.us/vod/gso/math/videos/Act-II-Rolling-Cups-with-Cup-Measurements.mp4

As you work on your problems, think about and determine what information you need.

Act III

Video reveal is done, using Act III – Rolling Cups.mp4, the MAP website, or the following Youtube link.

Watch the presented video, which can be found at:
http://youtu.be/d9yCNzA4fDg
or at
http://real.doe.k12.ga.us/vod/gso/math/Act-III-Rolling-Cups.mp4

The following questions can be used as extension questions. Later in the unit, you can also return to this task and have them define equations for the roll circles using the center of the circle as the origin.

Design your own cup and then determine what its roll radius will be.

Choose a roll radius and design at least 2 cups with this roll radius.

Rolling Cups

Act I

Watch the presented video, which can be found at:
http://youtu.be/psMYzAqRJVQ
or at:
http://real.doe.k12.ga.us/vod/gso/math/Act-I-Rolling-Cups.mp4

Act II

As you work on your problems, think about and determine what information you need.

Act III

Watch the presented video, which can be found at:
http://youtu.be/d9yCNzA4fDg
or at:
http://real.doe.k12.ga.us/vod/gso/math/Act-III-Rolling-Cups.mp4

Design your own cup and then determine what its roll radius will be.

Choose a roll radius and design at least 2 cups with this roll radius.

Deriving the General Equation of a Circle

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Standard Addressed in this Task
MGSE9-12.G.GPE.1 Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.

Standards for Mathematical Practice
2. Reason abstractly and quantitatively by requiring students to make sense of quantities and their relationships to one another in problem situations.

3. Construct viable arguments and critique the reasoning of others by engaging students on discussion of why they agree or disagree with responses, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

7. Look for and make use of structure by expecting students to apply rules, look for patterns and analyze structure.

8. Look for and express regularity in repeated reasoning by expecting students to understand broader applications and look for structure and general methods in similar situations.

Common Student Misconceptions
1. Because new vocabulary is being introduced in this cluster, remembering the names of the conic sections can be problematic for some students.

2. The Euclidean distance formula involves squared, subscripted variables whose differences are added. The notation and multiplicity of steps can be a serious stumbling block for some students.

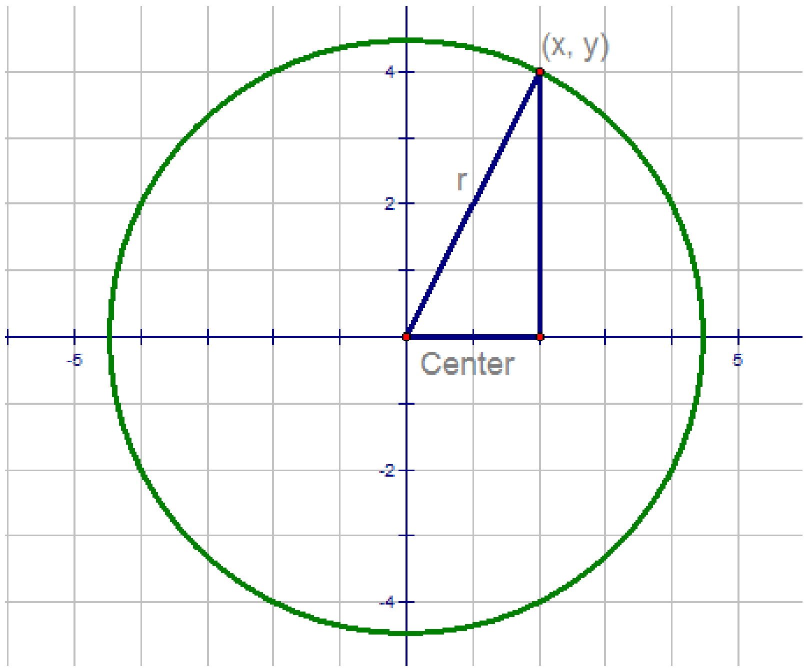
3. The method of completing the square is a multi-step process that takes time to assimilate.

4. A geometric demonstration of completing the square can be helpful in promoting conceptual understanding

Teacher Notes
This task is designed to walk a student through the process of generalizing the formula for the equation of a circle. Hopefully the teacher will be needed less and less as the students become more familiar with the process. As a teacher, it is not your job to provide the students with the answer, but to encourage them to persevere through the problem solving process. Students will benefit greatly from practice outside of class through thoughtful homework assignments that develop fluency with the equations.

Part 1: Finding the Radius

Consider the circle below. Notice the center is at the origin and a point is on the circle (x, y).



Answer the following questions or perform the requested constructions.

1. Construct a line segment from the center to the point (x, y) on the circle and label it “r”. What is this line segment called?

   Solution
   The line segment is the radius of the circle

2. Construct a right triangle with r as the hypotenuse. What are the coordinates of the point (x, y)?

   Solution
   The point is (2, 4)

3. What is the measure of r? Explain your method for calculating it.

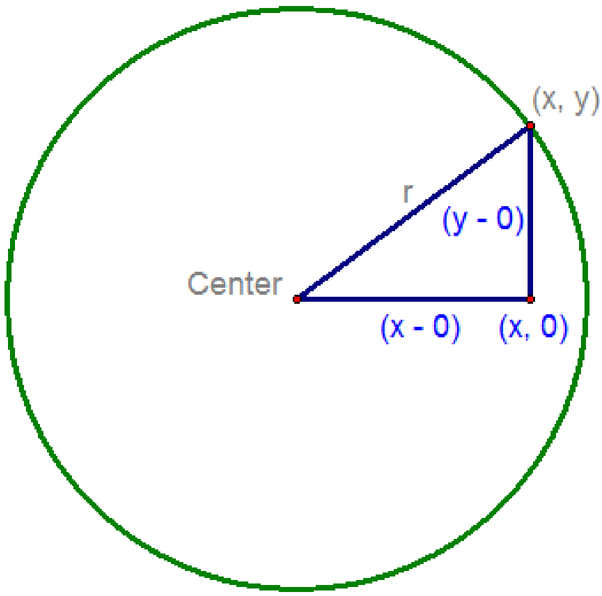
   Solution
   The measure can be found several ways. One way is the Pythagorean Theorem:

$$2^2 + 4^2 = c^2$$
$$20 = c^2$$
$$c = \sqrt{20} = 2\sqrt{5}$$

Another possibility is by using the distance formula:

$$d = \sqrt{(2-0)^2 + (4-0)^2}$$
$$d = \sqrt{20} = 2\sqrt{5}$$

Part 2: Circles Centered at the Origin.
Consider the circle below. The center is located at the origin.



Answer the following questions or perform the requested constructions.

1. Construct a radius from the center to the point (x, y). Label it "r".

2. Construct a right triangle with r as the hypotenuse. What are the coordinates of the point where the legs meet?

   Comments
   It is important here that students begin the process of generalizing the point. This is at the heart of deriving a formula. Don't be afraid to spend a little extra time on developing the idea that this is not a specific point, but could be any point on the circle.

   Solution
   The point is (x, 0)
3. Write an expression for the distance from the center to the point from #2. Label the triangle accordingly.

   Solution
   (x – 0)

4. Write an expression for the distance from (x, y) to the point from #2. Label the triangle accordingly.

Solution
(y – 0)

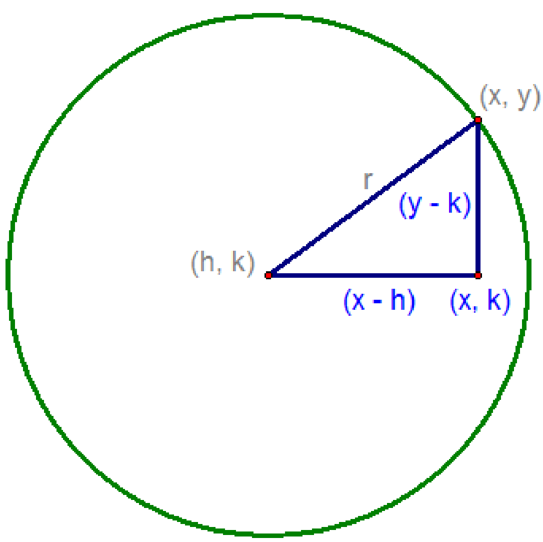
5. Now use your method from part one to write an expression for r^2

Solution
From the Pythagorean Theorem: $(x-0)^2+(y-0)^2=r^2$

Part 3: Circles centered anywhere!

In the previous section, you found that $x^2+y^2=r^2$. This is the general equation for a circle centered at the origin. However, circles are not always centered at the origin. Use the following circle and directions to find the general equation for a circle centered anywhere.

Solution
This is the main point of the activity. By now, the students should have examples for reference and may be able to complete part 3 on their own.



Answer the following questions and perform the requested constructions.

1. Construct a radius between (h, k) and (x, y). Then create a right triangle with the radius as the hypotenuse. Find the coordinates for the point where the legs meet.

Solution
(x, k)

2. Write an expression for the distance between (x, y) and the point from #1. Label the triangle.

   Solution
   (y – k)

3. Write an expression for the distance between (h, k) and the point from #1. Label the triangle.

   Solution
   (x – h)

4. Now write an expression for r^2.

   Solution
   From the Pythagorean Theorem: $(x-h)^2+(y-k)^2=r^2$

Deriving the General Equation of a Circle

Standard Addressed in this Task
MGSE9-12.G.GPE.1 Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.

Standards for Mathematical Practice
2. Reason abstractly and quantitatively by requiring students to make sense of quantities and their relationships to one another in problem situations.

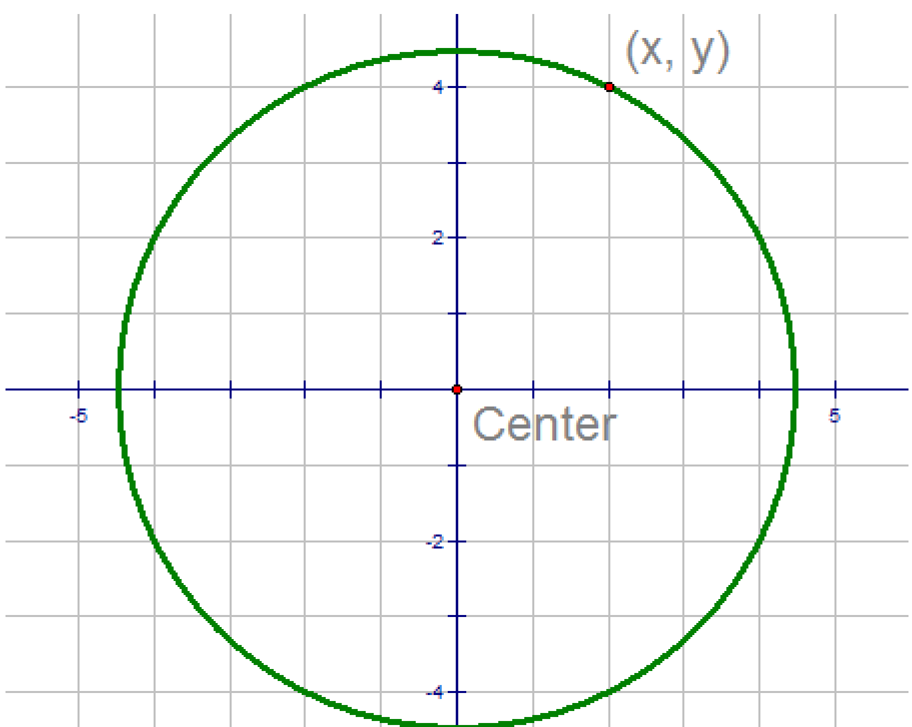
3. Construct viable arguments and critique the reasoning of others by engaging students on discussion of why they agree or disagree with responses, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

7. Look for and make use of structure by expecting students to apply rules, look for patterns and analyze structure.

8. Look for and express regularity in repeated reasoning by expecting students to understand broader applications and look for structure and general methods in similar situations.

Part 1: Finding the Radius

Consider the circle below. Notice the center is at the origin and a point is on the circle (x, y).



Answer the

following questions or perform the requested constructions.

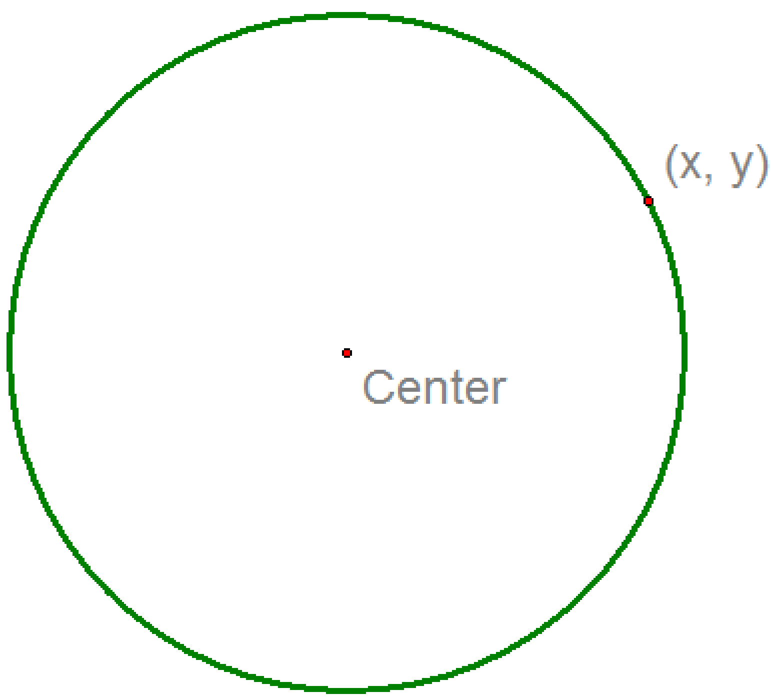
1. Construct a line segment from the center to the point (x, y) on the circle and label it “r”. What is this line segment called?

2. Construct a right triangle with r as the hypotenuse. What are the coordinates of the point (x, y)?

3. What is the measure of r? Explain your method for calculating it.

Part 2: Circles Centered at the Origin.

Consider the circle below. The center is located at the origin.



Answer the following questions or perform the requested constructions.

1. Construct a radius from the center to the point (x, y). Label it “r”.

2. Construct a right triangle with r as the hypotenuse. What are the coordinates of the point where the legs meet?

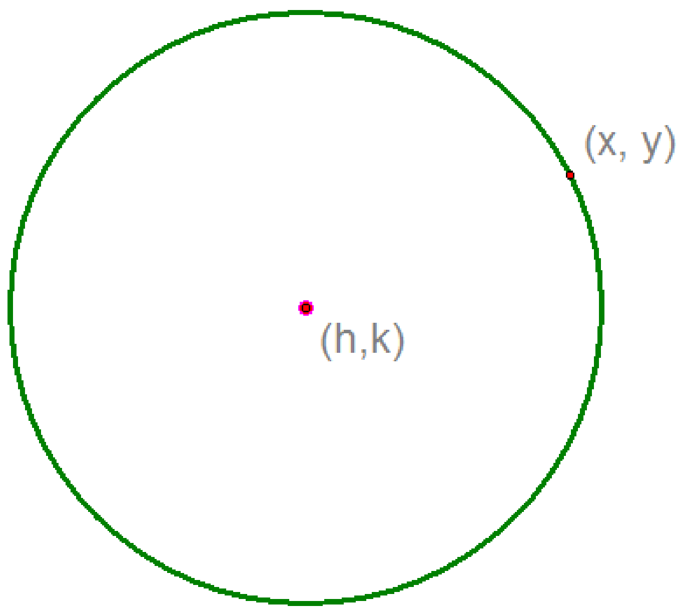
3. Write an expression for the distance from the center to the point from #2. Label the triangle accordingly.

4. Write an expression for the distance from (x, y) to the point from #2. Label the triangle accordingly.

5. Now use your method from part one to write an expression for r^2

Part 3: Circles centered anywhere!

In the previous section, you found that $x^2+y^2=r^2$. This is the general equation for a circle centered at the origin. However, circles are not always centered at the origin. Use the following circle and directions to find the general equation for a circle centered anywhere.



Answer the following questions and perform the requested constructions.

1. Construct a radius between (h, k) and (x, y). Then create a right triangle with the radius as the hypotenuse. Find the coordinates for the point where the legs meet.

2. Write an expression for the distance between (x, y) and the point from #1. Label the triangle.

3. Write an expression for the distance between (h, k) and the point from #1. Label the triangle.

4. Now write an expression for r^2.

Formative Assessment Lesson: Equations of Circles – 1

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Source: Formative Assessment Lesson Materials from Mathematics Assessment Project
http://map.mathshell.org/materials/download.php?fileid=1202

ESSENTIAL QUESTIONS:

- How do you use the Pythagorean theorem to derive the equation of a circle?
- How do you translate between the geometric features of circles and their equations?

TASK COMMENTS:

Tasks and lessons from the Mathematics Assessment Project are specifically designed to help teachers effectively formatively assess their students. The way the tasks and lessons are designed gives the teacher a clear understanding of what the students are able to do and not do. Within the lesson, teachers will find suggestions and question prompts that will help guide students towards understanding. For more information access the MAP website:
http://www.map.mathshell.org/materials/background.php?subpage=formative

The task, *Equations of Circles - 1*, is a Formative Assessment Lesson (FAL) that can be found at the website: http://map.mathshell.org/materials/lessons.php?taskid=406&subpage=concept

The FAL document provides a clear lesson design, from the opening of the lesson to the closing of the lesson.

The PDF version of the task can be found at the link below:
http://map.mathshell.org/materials/download.php?fileid=1202

STANDARDS ADDRESSED IN THIS TASK:

Translate between the geometric description and the equation for a conic section

MGSE9-12.G.GPE.1 Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.

Standards for Mathematical Practice
This lesson uses all of the practices with emphasis on:

1. Make sense of problems and persevere in solving them by requiring students to interpret and make meaning of a problem and find a logical starting point, and to monitor their progress and change their approach to solving the problem, if necessary.

5. Use appropriate tools strategically by expecting students to consider available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a compass, a calculator, software, etc.

7. Look for and make use of structure by expecting students to apply rules, look for patterns and analyze structure.

Formative Assessment Lesson: Equations of Circles – 2 [Back to Task Table]

Source: Formative Assessment Lesson Materials from Mathematics Assessment Project
http://map.mathshell.org/materials/download.php?fileid=1247

ESSENTIAL QUESTIONS:

- How do you translate between the equations of circles and their geometric features?
- How do you sketch a circle from its equation?

TASK COMMENTS:

Tasks and lessons from the Mathematics Assessment Project are specifically designed to help teachers effectively formatively assess their students. The way the tasks and lessons are designed gives the teacher a clear understanding of what the students are able to do and not do. Within the lesson, teachers will find suggestions and question prompts that will help guide students towards understanding. For more information access the MAP website:
http://www.map.mathshell.org/materials/background.php?subpage=formative

The task, *Equations of Circles - 2*, is a Formative Assessment Lesson (FAL) that can be found at the website http://map.mathshell.org/materials/lessons.php?taskid=425&subpage=concept

The FAL document provides a clear lesson design, from the opening of the lesson to the closing of the lesson.

The PDF version of the task can be found at the link below:
http://map.mathshell.org/materials/download.php?fileid=1247

STANDARDS ADDRESSED IN THIS TASK:
Translate between the geometric description and the equation for a conic section

MGSE9-12.G.GPE.1 Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.

Standards for Mathematical Practice
This lesson uses all of the practices with emphasis on:

1. Make sense of problems and persevere in solving them by requiring students to interpret and make meaning of a problem and find a logical starting point, and to monitor their progress and change their approach to solving the problem, if necessary.

7. Look for and make use of structure by expecting students to apply rules, look for patterns and analyze structure

Converting Standard Form to General Form

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Standard Addressed in this Task

MGSE9-12.G.GPE.1 Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.

Standards for Mathematical Practice
2. Reason abstractly and quantitatively by requiring students to make sense of quantities and their relationships to one another in problem situations.

3. Construct viable arguments and critique the reasoning of others by engaging students on discussion of why they agree or disagree with responses, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

7. Look for and make use of structure by expecting students to apply rules, look for patterns and analyze structure.

8. Look for and express regularity in repeated reasoning by expecting students to understand broader applications and look for structure and general methods in similar situations.

Common Student Misconceptions
1. Because new vocabulary is being introduced in this cluster, remembering the names of the conic sections can be problematic for some students.
2. The Euclidean distance formula involves squared, subscripted variables whose differences are added. The notation and multiplicity of steps can be a serious stumbling block for some students.
3. The method of completing the square is a multi-step process that takes time to assimilate.
4. A geometric demonstration of completing the square can be helpful in promoting conceptual understanding.

Teacher Notes
This task is a brief look at the algebraic manipulations necessary to change an equation from standard form to general form. This task should in no way take the place of regular classroom instruction when it comes to writing the equation of a circle. Students should feel comfortable writing the equation of a circle given a center and a radius.

In Task 1, you used the Pythagorean Theorem to derive: $(x-h)^2+(y-k)^2=r^2$, which is known as the **Standard Form of a Circle**.

By expanding the binomial terms this equation can be written as

$x^2 - 2hx + h^2 + y^2 - 2ky + k^2 = r^2$ or

$x^2 + y^2 - 2hx - 2ky + h^2 + k^2 - r^2 = 0$

Then, by using variables for coefficients, and realizing that h^2, k^2 and r^2 are all real numbers and can be added, we derive the **General Form equation of a Circle**:

$$Ax^2 + By^2 + Cx + Dy + E = 0$$

******Note: In order to be a circle, A and B must be equal.*

Occasionally, it becomes necessary to convert the equation of a circle from Standard to General Form. Take the circle with a center at (3, 4) and a radius of 6, for example.

The Standard Equation would be: $(x-3)^2 + (y-4)^2 = 6^2$

By expanding the binomial terms, we would then have: $x^2 - 6x + 9 + y^2 - 8y + 16 = 36$.

Grouping the monomials according to degree would yield:
$x^2 + y^2 - 6x - 8y + 9 + 16 - 36 = 0$

Through arithmetic, the General Form equation would be: $x^2 + y^2 - 6x - 8y - 11 = 0$

Write the General form equations for the following circles:

1. A circle with center *(1, -6)* and radius *4* ***Solution:*** $x^2 + y^2 - 2x + 12y + 21 = 0$
2. A circle with center *(6, 8)* and radius *10* ***Solution:*** $x^2 + y^2 - 12x - 16y = 0$
3. A circle with center *(0, 3)* and radius $2\sqrt{3}$ ***Solution:*** $x^2 + y^2 - 6y - 3 = 0$
4. A circle with center *(-0.5, 5.5)* and radius *8.4* ***Solution:*** $x^2 + y^2 + x - 11y - 40.06 = 0$
5. A circle with center *(a, b)* and radius *c* ***Solution:***
   $x^2 + y^2 - 2ax - 2by + a^2 + b^2 - c^2 = 0$

Converting General Form to Standard Form

Standard Addressed in this Task
MGSE9-12.G.GPE.1 Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.

Standards for Mathematical Practice
2. Reason abstractly and quantitatively by requiring students to make sense of quantities and their relationships to one another in problem situations.

3. Construct viable arguments and critique the reasoning of others by engaging students on discussion of why they agree or disagree with responses, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

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$x^2+y^2-2hx-2ky+h^2+k^2-r^2=0$

Then, by using variables for coefficients, and realizing that h^2, k^2 and r^2 are all real numbers and can be added, we derive the **General Form equation of a Circle**:

$$Ax^2+By^2+Cx+Dy+E=0$$

***Note:** *In order to be a circle, A and B must be equal.*

Occasionally, it becomes necessary to convert the equation of a circle from Standard to General Form. Take the circle with a center at (3, 4) and a radius of 6, for example.

The Standard Equation would be: $(x-3)^2+(y-4)^2=6^2$

By expanding the binomial terms, we would then have: $x^2 - 6x + 9 + y^2 - 8y + 16 = 36$.
Grouping the monomials according to degree would yield:
$x^2 + y^2 - 6x - 8y + 9 + 16 - 36 = 0$

Through arithmetic, the General Form equation would be: $x^2 + y^2 - 6x - 8y - 11 = 0$

Write the General form equations for the following circles:

1. A circle with center *(1, -6)* and radius *4*
2. A circle with center *(6, 8)* and radius *10*
3. A circle with center *(0, 3)* and radius $2\sqrt{3}$
4. A circle with center *(-0.5, 5.5)* and radius *8.4*
5. A circle with center *(a, b)* and radius *c*

Completing the Square in a Circle?

[Back to Task Table]

Standard Addressed in this Task
MGSE9-12.G.GPE.1 Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.

Standards for Mathematical Practice
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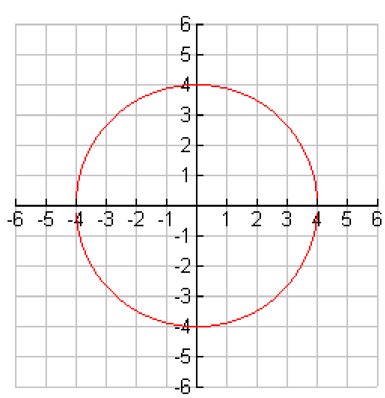
Common Student Misconceptions
1. Because new vocabulary is being introduced in this cluster, remembering the names of the conic sections can be problematic for some students.
2. The Euclidean distance formula involves squared, subscripted variables whose differences are added. The notation and multiplicity of steps can be a serious stumbling block for some students.
3. The method of completing the square is a multi-step process that takes time to assimilate.
4. A geometric demonstration of completing the square can be helpful in promoting conceptual understanding.

1. Write equations for the following circle graphs in both standard form and general form.

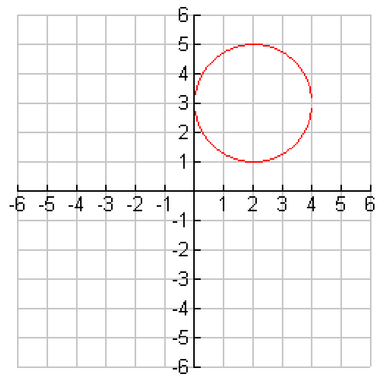
a.



Solution:
center at (0, 0) and radius r = 4

$x^2 + y^2 = 4$ **(standard form)**
$x^2 + y^2 - 4 = 0$ **(general form)**

b.



Solution:
center at (2, 3) and radius r = 2

$(\mathbf{x - 2})^2 + (\mathbf{y - 3})^2 = 2^2$ **(standard form)**

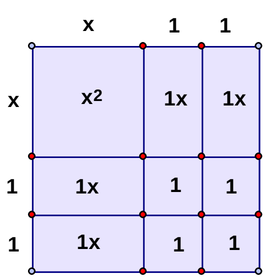
$x^2 - 4x + 4 + y^2 - 6y + 9 = 4$

$x^2 + y^2 - 4x - 6y + 9 = 0$ (general form)

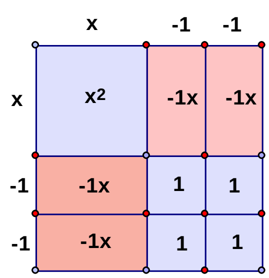
Teacher Notes:
Using a geometric area model approach will lend a visual to the idea of "completing the square". Students often get wrapped up in an algorithm for completing the square and never understand the idea behind it. Some students may find this confusing if they have not been exposed to algebra tiles earlier in their curriculum. This is only a suggested presentation, not the only way to teach it.

To change from general form to standard form, it is necessary to complete the square for x and y. **Completing the square** is an algebraic tool used to change equations of conic sections given in general form, $Ax^2 + Cy^2 + Dx + Ey + F = 0$, to standard form, $(\mathbf{x - h})^2 + (\mathbf{y - k})^2 = \mathbf{r}^2$. Standard form is the form used to graph conic sections.

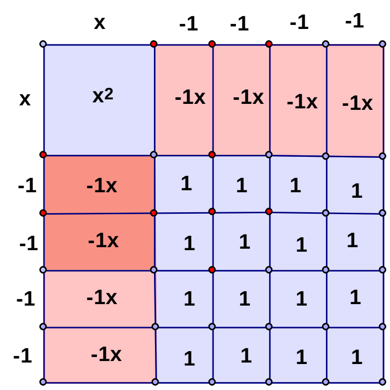
Perfect squares are numbers or expressions which have exactly two identical factors.
(2)(2) = 4 (-5)(-5) = 25 $(3x)(3x) = 9x^2$ $(-6y)(-6y) = 36y^2$ $(x + 2)(x + 2) = x^2 + 4x + 4$
Consider the following geometric area models of three perfect squares. The area is given as both factors and as a quadratic expression.



area = (x + 2)(x + 2)
$= x^2 + 4x + 4$



area = (x - 2)(x - 2)
$= x^2 - 4x + 4$

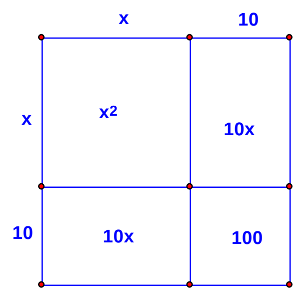


area = (x - 4)(x - 4)
$= x^2 - 8x + 16$

2. Find the products of the following expressions.

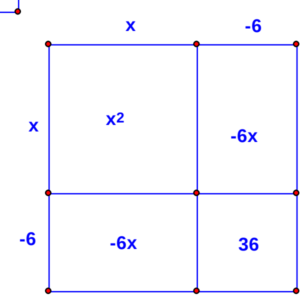
   a. $(x + 1)^2 = (x + 1)(x + 1) =$ ***Solution:*** $x^2 + 1x + 1x + 1 = x^2 + 2x + 1$
   b. $(x – 3)^2 = (x – 3)(x – 3) =$ ***Solution:*** $x^2 - 3x - 3x + 9 = x^2 - 6x + 9$
   c. $(x – 5)^2 = (x – 5)(x – 5) =$ ***Solution:*** $x^2 - 5x - 5x + 25 = x^2 - 10x + 25$
   d. $(x + 7)^2 = (x + 7)(x + 7) =$ ***Solution:*** $x^2 + 7x + 7x + 49 = x^2 + 14x + 49$
   e. $(x + n)^2 = (x + n)(x + n) =$ ***Solution:*** $x^2 + nx + nx + n^2 = x^2 + 2nx + n^2$

3. Each of the products in #2 is a perfect square. Use the results of #2 to complete each of the squares and show their factored forms. Include Geometric diagrams to illustrate the perfect squares.



   a. $x^2 + 20x +$ ____ $= (x +$ ____$)^2$

   Solution: $x^2 + 20x + 100 = (x + 10)^2$



   b. $x^2 – 12x +$ ____ $= (x –$ ____$)^2$

   Solution: $x^2 - 12x + 36 = (x - 6)^2$

   c. $x^2 + 18x +$ ____ $= (x +$ ____$)^2$ ***Solution:***
   $x^2 + 18x + 81 = (x + 9)^2$

   d. $x^2 – 7x +$ ____ $= (x –$ ____$)^2$ ***Solution:*** $x^2 - 7x + \frac{49}{4} = \left(x - \frac{7}{2}\right)^2$

   e. $x^2 + 2nx +$ ____ $= (x +$ ____$)^2$ ***Solution:*** $x^2 + 2nx + n^2 = (x + n)^2$

In order to graph a circle given in general form, it is necessary to change to standard form. In order to rewrite $x^2 + y^2 + 2x - 4y - 11 = 0$ in standard form to facilitate graphing, it is necessary to complete the square for both x and y.

Teacher Notes:
The teacher should emphasize here the benefits of having the equation expressed in standard form. It is easier to see the center and the radius, therefore it is easier to graph.

$\mathbf{x^2 + y^2 + 2x - 4y - 11 = 0}$
$(x^2 + 2x \quad) + (y^2 - 4y \quad) = 11$ group the x and y terms

$(x^2 + 2x + 1) + (y^2 - 4y + 4) = 11 + 1 + 4$ complete the square on x and y; balance the equation by adding 1 and 4 to both sides of the equation

$\mathbf{(x + 1)^2 + (y - 2)^2 = 16}$ factor

Circle with center at (-1, 2) and radius 4

To change $x^2 + y^2 + 2x - 4y - 11 = 0$ to standard form, it is necessary to remove a factor of 2 before completing the square for both x and y.

$\mathbf{2x^2 + 2y^2 - 4x + 6y - 4 = 0}$ remove a factor of 2

$(x^2 - 2x \quad) + (y^2 + 3y \quad) = 2$ group the x and y terms

$(x^2 - 2x + 1) + (y^2 + 3y + {}^9/_4) = 2 + 1 + {}^9/_4$ complete the square on x and y; balance the equation by adding 1 and ${}^9/_4$ to both sides of the equation

$(x - 1)^2 + (y + {}^3/_2)^2 = {}^{21}/_4$ factor

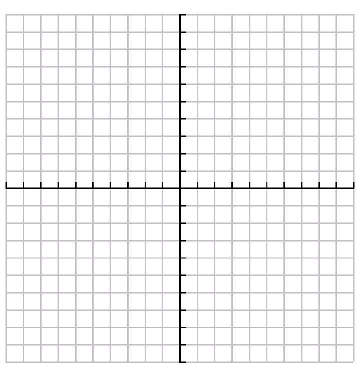
$\mathbf{(x - 1)^2 + (y + 1.5)^2 = 5.25}$

circle with center at (1, -1.5) and radius 5.25

4. Change the following equations to standard form. Graph the circles; identify the centers and the radii.

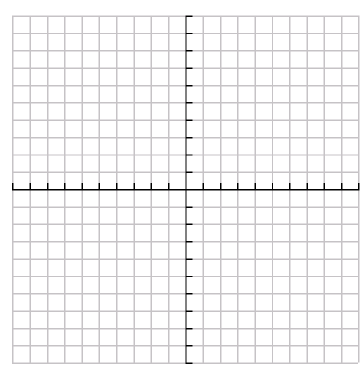
a. $x^2 + y^2 + 2x + 4y - 20 = 0$

$(x + 1)^2 + (y + 2)^2 = 25$
center (-1, -2) radius 5



b. $x^2 + y^2 - 4y = 0$

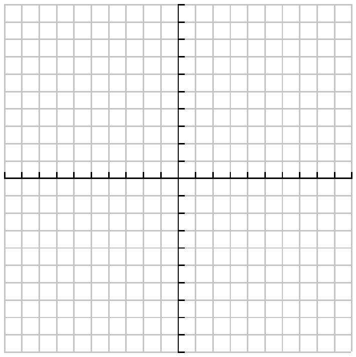
$(x + 0)^2 + (y - 2)^2 = 4$
center (0, 2) radius 2



c. $x^2 + y^2 - 6x - 10y = 2$

$(x - 3)^2 + (y - 5)^2 = 36$

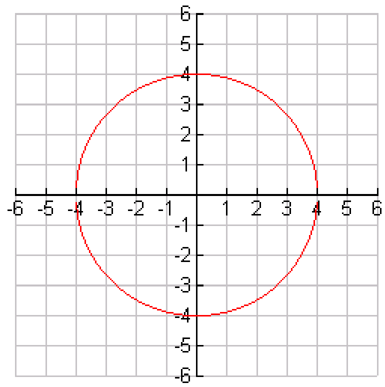
center (3, 5) radius 6



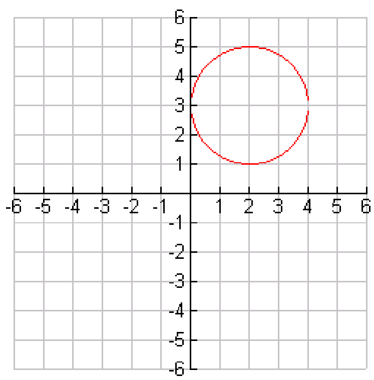
Completing the Square in a Circle?

1. Write equations for the following circle graphs in both standard form and general form.

a.



b.



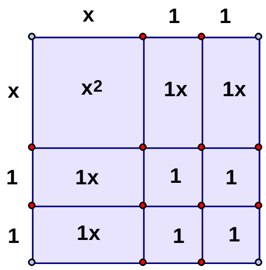
2. Take a moment to compare your General form and Standard form equations. Which form would be easier to graph? Why do you think so?

In Task 2, you converted the Standard form equation to a General form equation. Today you will convert from General form to Standard form.

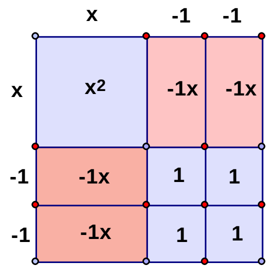
To change from general form to standard form, it is necessary to "complete the square" for x and y. **Completing the square** is an algebraic tool used to change equations of circles given in general form, $Ax^2 + By^2 + Cx + Dy + E = 0$, to standard form, $(x-h)^2 + (y-k)^2 = r^2$. Standard form is the form used to graph circles.
Perfect squares are numbers or expressions which have exactly two identical factors.

$(2)(2) = 4$ $(-5)(-5) = 25$ $(3x)(3x) = 9x^2$ $(x + 2)(x + 2) = x^2 + 4x + 4$

Consider the following geometric area models of three perfect squares. The area is given as both factors and as a quadratic expression.



area = (x + 2)(x + 2)
= x2 + 4x + 4



area = (x - 2)(x - 2)
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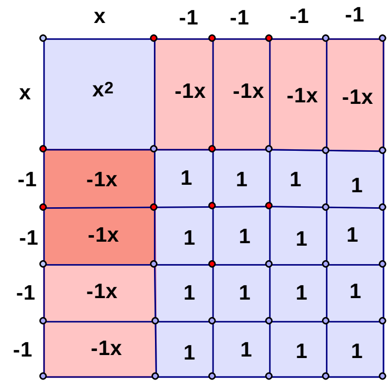


area = (x - 4)(x - 4)
= x2 - 8x + 16

3. Find the products of the following expressions.

   a. $(x + 1)^2 = (x + 1)(x + 1) =$
   b. $(x - 3)^2 = (x - 3)(x - 3) =$
   c. $(x - 5)^2 = (x - 5)(x - 5) =$
   d. $(x + 7)^2 = (x + 7)(x + 7) =$
   e. $(x + n)^2 = (x + n)(x + n) =$

4. Each of the products in #3 is a perfect square. Use the results of #3 to complete each of the squares and show their factored forms. Include Geometric diagrams to illustrate the perfect squares.

   a. $x^2 + 20x + \_\_\_\_ = (x + \_\_\_\_)^2$

   b. $x^2 - 12x + \_\_\_\_ = (x - \_\_\_\_)^2$

   c. $x^2 + 18x + \_\_\_\_ = (x + \_\_\_\_)^2$

   d. $x^2 - 7x + \_\_\_\_ = (x - \_\_\_\_)^2$

   e. $x^2 + 2nx + \_\_\_\_ = (x + \_\_\_\_)^2$

In order to graph a circle given in general form, it is necessary to change to standard form. In order to rewrite $x^2 + y^2 + 2x - 4y - 11 = 0$ in standard form to facilitate graphing, it is necessary to complete the square for both x and y.

$\mathbf{x^2 + y^2 + 2x - 4y - 11 = 0}$	
$(x^2 + 2x \quad) + (y^2 - 4y \quad) = 11$	group the x and y terms
$(x^2 + 2x + 1) + (y^2 - 4y + 4) = 11 + 1 + 4$	complete the square on x and y balance the equation by adding 1 and 4 to both sides of the equation
$\mathbf{(x + 1)^2 + (y - 2)^2 = 16}$	factor

circle with center at (-1, 2) and radius 4

To change $x^2 + y^2 + 2x - 4y - 11 = 0$ to standard form, it is necessary to remove a factor of 2 before completing the square for both x and y.

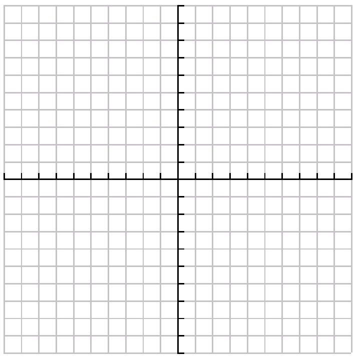
$\mathbf{2x^2 + 2y^2 - 4x + 6y - 4 = 0}$	remove a factor of 2
$(x^2 - 2x \quad) + (y^2 + 3y \quad) = 2$	group the x and y terms
$(x^2 - 2x + 1) + (y^2 + 3y + {}^9/_4) = 2 + 1 + {}^9/_4$	complete the square on x and y balance the equation by adding 1 and ${}^9/_4$ to both sides of the equation
$(x - 1)^2 + (y + {}^3/_2)^2 = {}^{21}/_4$	factor

$\mathbf{(x - 1)^2 + (y + 1.5)^2 = 5.25}$

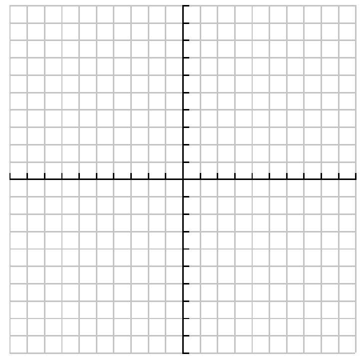
circle with center at (1, -1.5) and radius 5.25

5. Change the following equations to standard form. Graph the circles; identify the centers and the radii.

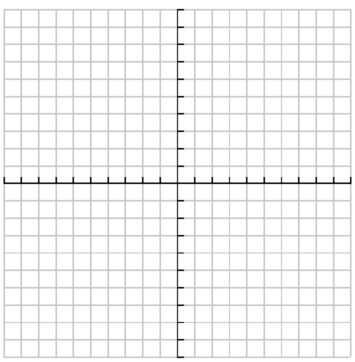
a. $x^2 + y^2 + 2x + 4y - 20 = 0$



b. $x^2 + y^2 - 4y = 0$



c. $x^2 + y^2 - 6x - 10y = 2$



Graphing Circles on a Graphing Calculator

Back to Task Table

Standard Addressed in this Task

MGSE9-12.G.GPE.1 Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.

Standards for Mathematical Practice
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Common Student Misconceptions
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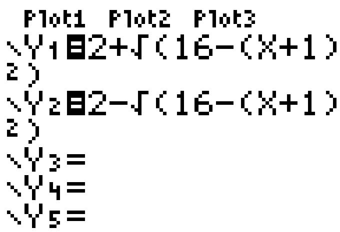
2. The Euclidean distance formula involves squared, subscripted variables whose differences are added. The notation and multiplicity of steps can be a serious stumbling block for some students.
3. The method of completing the square is a multi-step process that takes time to assimilate.
4. A geometric demonstration of completing the square can be helpful in promoting conceptual understanding

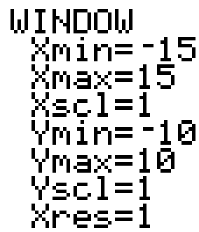
Teacher Notes:
This is an extension activity. Ultimately it is an exercise in symbolic manipulation, but you may find it useful as a way to get students to use technology.

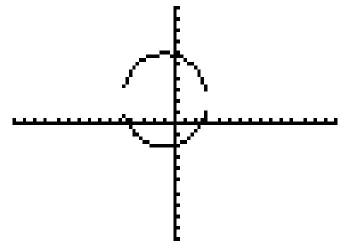
To graph the circle $x^2 + y^2 + 2x - 4y - 11 = 0$ using a TI83/TI84 it is necessary to solve for y after changing the equation to standard form.

$$(x + 1)^2 + (y - 2)^2 = 16$$
$$(y - 2)^2 = 16 - (x + 1)^2$$
$$\sqrt{(y-2)^2} = \pm\sqrt{16-(x+1)^2}$$
$$y - 2 = \pm\sqrt{16-(x+1)^2}$$
$$y = 2 \pm \sqrt{16-(x+1)^2}$$

Enter this result as two functions y1 = 2 + $\sqrt{(16-(x+1)^2)}$ and y2 = 2 − $\sqrt{(16-(x+1)^2)}$. In order to minimize the distortion caused by the rectangular screen of the graphing calculator, use a window with an x to y ratio of 3 to 2. Otherwise circles appear as ellipses.







1. Write the equations as you would enter them in a graphing calculator and list an appropriate graphing window to show the entire circle graph.

a. $x^2 + y^2 + 2x + 4y - 20 = 0$ ***Solution:*** $y_1 = -2 + \sqrt{\left(25-(x+1)^2\right)}$, $y_2 = -2 - \sqrt{\left(25-(x+1)^2\right)}$ [-15, 15] by [-10, 10]

b. $x^2 + y^2 - 4y = 0$ ***Solution:*** $y_1 = 2 + \sqrt{\left(4-x^2\right)}$, $y_2 = 2 - \sqrt{\left(4-x^2\right)}$ [-6, 6] by [-4, 4]

c. $x^2 + y^2 - 6x - 10y = 2$ ***Solution:*** $y_1 = 5 + \sqrt{\left(36-(x-3)^2\right)}$, $y_2 = 5 - \sqrt{\left(36-(x-3)^2\right)}$ [-18, 18] by [-12, 12]

Graphing Circles on a Graphing Calculator

Standard Addressed in this Task
MGSE9-12.G.GPE.1 Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.

Standards for Mathematical Practice
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7. Look for and make use of structure by expecting students to apply rules, look for patterns and analyze structure.

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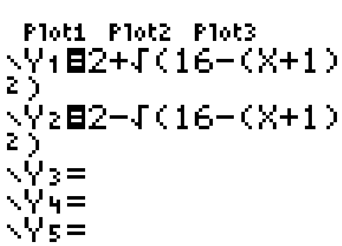
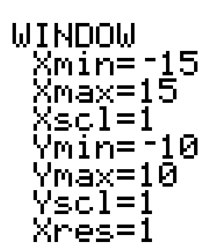
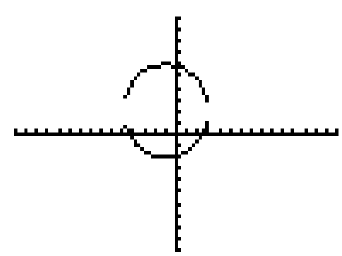
To graph the circle $x^2 + y^2 + 2x - 4y - 11 = 0$ using a TI83/TI84 it is necessary to solve for y after changing the equation to standard form.

$$(x + 1)^2 + (y - 2)^2 = 16$$
$$(y - 2)^2 = 16 - (x + 1)^2$$
$$\sqrt{(y-2)^2} = \pm\sqrt{16-(x+1)^2}$$
$$y - 2 = \pm\sqrt{16-(x+1)^2}$$
$$y = 2 \pm \sqrt{16-(x+1)^2}$$

Enter this result as two functions $y1 = 2 + \sqrt{(16-(x+1)^2)}$ and $y2 = 2 - \sqrt{(16-(x+1)^2)}$. In order to minimize the distortion caused by the rectangular screen of the graphing calculator, use a window with an x to y ratio of 3 to 2. Otherwise circles appear as ellipses.







1. Write the equations as you would enter them in a graphing calculator and list an appropriate graphing window to show the entire circle graph.

a. $x^2 + y^2 + 2x + 4y - 20 = 0$

b. $x^2 + y^2 - 4y = 0$

c. $x^2 + y^2 - 6x - 10y = 2$

Radio Station Listening Areas

Back to Task Table

Standard Addressed in this Task
MGSE9-12.G.GPE.1 Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.

Standards for Mathematical Practice
2. Reason abstractly and quantitatively by requiring students to make sense of quantities and their relationships to one another in problem situations.

3. Construct viable arguments and critique the reasoning of others by engaging students on discussion of why they agree or disagree with responses, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

7. Look for and make use of structure by expecting students to apply rules, look for patterns and analyze structure.

8. Look for and express regularity in repeated reasoning by expecting students to understand broader applications and look for structure and general methods in similar situations.

Common Student Misconceptions
1. Because new vocabulary is being introduced in this cluster, remembering the names of the conic sections can be problematic for some students.
2. The Euclidean distance formula involves squared, subscripted variables whose differences are added. The notation and multiplicity of steps can be a serious stumbling block for some students.
3. The method of completing the square is a multi-step process that takes time to assimilate.
4. A geometric demonstration of completing the square can be helpful in promoting conceptual understanding

Teacher Notes:
This activity is an application of writing equations of circles. It gives students a taste of a real life application of the formula that they wrote in the first task.

1. Radio signals emitted from a transmitter form a pattern of concentric circles. Write equations for three concentric circles.

Solution:
Answers vary. The student should realize that concentric circles have the same center, but different radii.

2. Randy listens to radio station WYAY from Atlanta. Randy's home is located 24 miles east and 32 miles south of the radio station's transmitter. His house is located on the edge of WYAY's maximum broadcast range.

a. When a radio signal reaches Randy's house, how far has it traveled? Sketch WYAY's listening area of the partial map of Georgia given. On the map let Atlanta's WYAY have coordinates (0, 0) and use the scale as 100 miles = 60 mm.

Solution:
See map. Randy's house at (24mi, -32mi) or (14.4mm, -19.2mm)

a. Find an equation which represents the station's maximum listening area.

Solution:
miles: $x^2 + y^2 = 40^2$ ***or mm:*** $x^2 + y^2 = 24^2$

c. Determine four additional locations on the edge of WYAY's listening area, give coordinates correct to tenths.

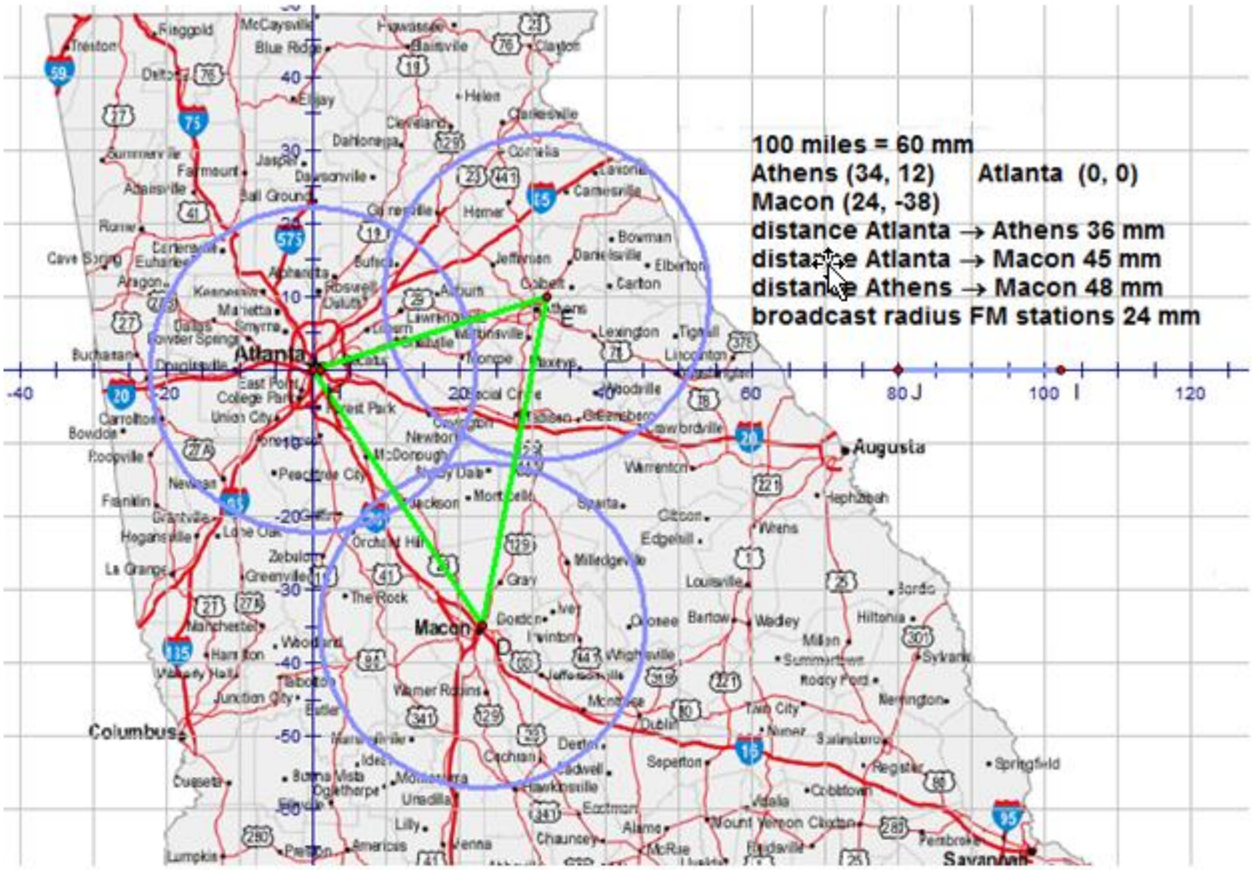
Solution:
some possible solutions in miles: (±26, ±30.4) (±22, ±33.4) (±19, ±35.2)

Teacher Notes:
Before starting this exercise, check the scale on the map. Some distortion occurred when the map was pasted into Geometer's Sketchpad to set the scale. However values should be close enough to allow students to draw graphs on the map and locate points.

3. Randy likes to listen to country music. Several of his friends have suggested that in addition to WYAY, he try station WXAG in Athens and WDEN in Macon. WYAY, WXAG, and WDEN are FM stations which normally have an average broadcast range of 40 miles. Use the map included with the indicated measures to answer the following questions.

a. Given the location of Randy's home, can he expect to pick up radio signals from WXAG and WDEN? Explain how you know.

Solution:
Randy's home should be close to Jackson. He should get signals from Macon, but not from Athens.



Broadcast area of WXAG in Athens is given by the equation $(x - 56.7)^2 + (y - 20)^2 = 40^2$ or $(x - 34)^2 + (y - 12)^2 = 24^2$

Broadcast area of WDEN in Macon is given by the equation $(x - 40)^2 + (y + 63.3)^2 = 40^2$ or $(x - 24)^2 + (y + 38)^2 = 24^2$

Broadcast areas drawn to scale in mms.

Radio Station Listening Areas

Standard Addressed in this Task
MGSE9-12.G.GPE.1 Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.

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   a. When a radio signal reaches Randy's house, how far has it traveled? Sketch WYAY's listening area of the partial map of Georgia given. On the map let Atlanta's WYAY have coordinates (0, 0) and use the scale as 100 miles = 60 mm.

   b. Find an equation which represents the station's maximum listening range.

   c. Determine four additional locations on the edge of WYAY's listening area, give coordinates correct to tenths.

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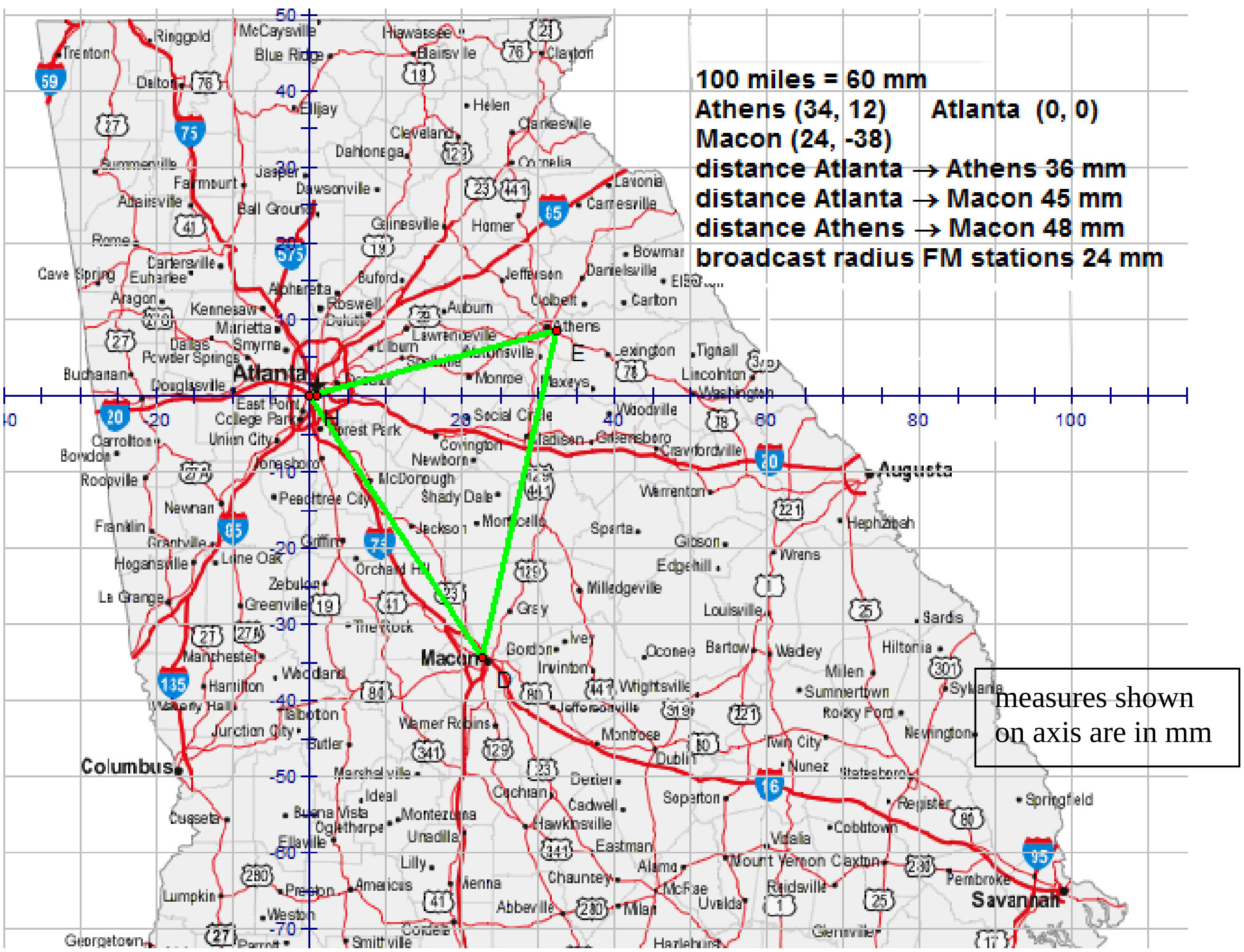

Algebraic Proof

Back to Task Table

Standard Addressed in this Task
MGSE9-12.G.GPE.4 Use coordinates to prove simple geometric theorems algebraically. *For example, ~~prove or disprove that a figure defined by four given points in the coordinate plane is a rectangle;~~ prove or disprove that the point (1, √3) lies on the circle centered at the origin and containing the point (0,2).*
(Focus on ~~quadrilaterals, right triangles, and~~ circles.)

Standards for Mathematical Practice
2. Reason abstractly and quantitatively by requiring students to make sense of quantities and their relationships to one another in problem situations.

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7. Look for and make use of structure by expecting students to apply rules, look for patterns and analyze structure.

Common Student Misconceptions
Students may claim that a vertical line has infinite slopes. This suggests that infinity is a number. Since applying the slope formula to a vertical line leads to division by zero, we say that the slope of a vertical line is undefined. Also, the slope of a horizontal line is 0.
Students often say that the slope of vertical and/or horizontal lines is “no slope,” which is incorrect

Teacher Notes:
The importance of this standard cannot be overstated. Using Algebra to prove ideas and confirm hypotheses is a critical mathematical thinking skill. Do not focus on the specific proofs presented in this task, but rather the overarching idea of Algebraic proof. The examples presented here are meant as a starting point. More examples should be presented in class.

Already in Analytic Geometry you have been exposed to and written proofs about geometric theorems and properties. It is now time to mix in some algebraic proofs. For this unit, we will restrict our algebraic proofs to problems involving the coordinate plane.

First, we should examine our “toolbox” to see what math concepts we have at our disposal for these types of proofs:

Distance Formula: Useful for determining distances between two points.

Slope Formula: Useful for determining if lines are parallel or perpendicular.

Substitution: Useful for determining if points satisfy given equations.

1. Proof #1: Prove or disprove that the point $\left(1, \sqrt{3}\right)$ lies on the circle centered at the origin and passing through the point (0, 2).

Teacher Notes:
The teacher must avoid the temptation to just substitute and move on. The idea behind these types of proofs is to understand that the points given are equidistant from the center. Always take the algebra back to the geometry and the definition.

a. What do we need to show in order to prove or disprove this statement?

Solution:
Students need to show that the two points are equidistant from the center. This could be shown in multiple ways, including using the equation of a circle.

b. Write an equation for the circle described in the problem.

Comments: This is only one approach to solving the problem. It also refers back to skills introduced at the beginning of the unit. While this proof is step-by-step you should challenge students to complete similar proofs on their own.

Solution:

$$\left(0-0\right)^2+(2-0)^2=r^2$$
$$4=r^2$$
$$2=r$$
$$x^2+y^2=2^2$$

c. Substitute the point in for the equation and comment on the results. Did you prove the statement or disprove it?

Solution:

$$(1)^2+\left(\sqrt{3}\right)^2=4$$

Because the equation is true, the given point is the same distance from the center as the given point on the circle.

Now you are ready to try some on your own. Use the questions above as a guide and write algebraic proofs for the following.

2. Prove or disprove that the point A(10, 3) lies on a circle centered at C(5, -2) and passing through the point B(6, 5).

   Solution:
   The point does lie on the circle. The equation of the circle is: $(x-5)^2+(y+2)^2=50$
   Substituting in the point gives: $(10-5)^2+(3+2)^2=50$

Algebraic Proof

Standard Addressed in this Task
MGSE9-12.G.GPE.4 Use coordinates to prove simple geometric theorems algebraically. *For example, ~~prove or disprove that a figure defined by four given points in the coordinate plane is a rectangle;~~ prove or disprove that the point (1, √3) lies on the circle centered at the origin and containing the point (0,2).*
(Focus on ~~quadrilaterals, right triangles, and~~ circles.)

Standards for Mathematical Practice
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a. What do we need to show in order to prove or disprove this statement?

b. Write an equation for the circle described in the problem.

c. Substitute the point in for the equation and comment on the results. Did you prove the statement or disprove it?

d. Now use the facts in a-c to write a paragraph proof.

Now you are ready to try some on your own. Use the questions above as a guide and write algebraic proofs for the following.

2. Prove or disprove that the point A(10, 3) lies on a circle centered at C(5, -2) and passing through the point B(6, 5).

A Day at the Beach

Back to Task Table

Source: NYC Department of Education
https://www.weteachnyc.org/media2016/filer_public/f3/12/f312f45e-a203-4400-8648-c40a2b090cb9/g10_math_a_day_at_the_beach.pdf

Mathematical Goals

- To visualize and identify the dimensions of geometric shapes
- To determine the volume relationships of cylinders, pyramids, cones, and spheres
- To justify geometric arguments

Essential Questions

- Which geometric shape will fill a pail full of sand faster?

TASK COMMENTS

The students will identify and label geometric shapes and apply the appropriate formulas and measurements to calculate the volume of the figures. After finding this information, students will determine which geometric figure will fill a pail full of sand faster. In this task, the student routinely interprets the mathematical results, applies geometric concepts in the context of the situation, reflects on whether the results make sense and uses all appropriate tools strategically.
http://schools.nyc.gov/NR/rdonlyres/C03D80B2-9213-43A9-AAA3-BB0032C62F4F/139657/NYCDOE_G10_ADayattheBeach_FINAL1.pdf

The task, *A Day at the Beach*, is a Performance task that can be found at the website:
https://www.weteachnyc.org/media2016/filer_public/f3/12/f312f45e-a203-4400-8648-c40a2b090cb9/g10_math_a_day_at_the_beach.pdf

The scoring rubric can be found at the following link:
https://www.weteachnyc.org/media2016/filer_public/f3/12/f312f45e-a203-4400-8648-c40a2b090cb9/g10_math_a_day_at_the_beach.pdf

GEORGIA STANDARDS OF EXCELLENCE

MGSE9-12.G.MG.1 Use geometric shapes, their measures, and their properties to describe objects (e.g., modeling a tree trunk or a human torso as a cylinder).

MGSE9-12.G.MG.3 Apply geometric methods to solve design problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios).

STANDARDS FOR MATHEMATICAL PRACTICE

This task uses all of the practices with emphasis on:

1. **Make sense of problems and persevere in solving them** by requiring students to interpret and make meaning of a problem and find a logical starting point, and to monitor their progress and change their approach to solving the problem, if necessary.
4. **Model with mathematics** by expecting students to apply the mathematics concepts they know in order to solve problems arising in everyday situations, and reflect on whether the results are sensible for the given scenario.
6. **Attend to precision** by requiring students to calculate efficiently and accurately; and to communicate precisely with others by using clear mathematical language to discuss their reasoning.

Grouping

- Individual/Partner

Time Needed

- 90 minutes

How Many Cells are in the Human Body?

Back to Task Table

Source: Illustrative Mathematics
https://www.illustrativemathematics.org/content-standards/HSG/MG/A/1/tasks/1146

Mathematical Goals

- To apply the concepts of mass, volume, and density in a real-world context.

Essential Questions

- About how many cells are in the human body?

TASK COMMENTS

The purpose of this task is to help students apply the concepts of mass, volume, and density in a real-world context. The task allows someone to estimate the volume of a person to make calculations on the number of cells. However, in order to better adapt this task to the Georgia Standards of Excellence, students can make the comparison of the shape of a human to that of a geometric figure, for example a human torso could compare to a cylinder, make the appropriate calculations to determine the number of cells are in the human body.
https://www.illustrativemathematics.org/content-standards/HSG/MG/A/1/tasks/1146

The task, *How many cells are in the human body?*, is a Performance task that can be found at the website: https://www.illustrativemathematics.org/content-standards/HSG/MG/A/1/tasks/1146

GEORGIA STANDARDS OF EXCELLENCE

MGSE9-12.G.MG.1 Use geometric shapes, their measures, and their properties to describe objects (e.g., modeling a tree trunk or a human torso as a cylinder).

MGSE9-12.G.MG.2 Apply concepts of density based on area and volume in modeling situations (e.g, persons per square mile, BTUs per cubic foot).

STANDARDS FOR MATHEMATICAL PRACTICE

This task uses all of the practices with emphasis on:

2. **Make sense of problems and persevere in solving them** by requiring students to interpret and make meaning of a problem and find a logical starting point, and to monitor their progress and change their approach to solving the problem, if necessary.
3. **Reason abstractly and quantitatively** by requiring students to make sense of quantities and their relationships to one another in problem situations.

5. **Model with mathematics** by expecting students to apply the mathematics concepts they know in order to solve problems arising in everyday situations, and reflect on whether the results are sensible for the given scenario.

6. **Attend to precision** by requiring students to calculate efficiently and accurately; and to communicate precisely with others by using clear mathematical language to discuss their reasoning.

Grouping

- Partner

Time Needed

- 60-90 minutes

Maximize Volume

[Back to Task Table]

Standard Addressed in this Task
MGSE9-12.G.MG.3 Apply geometric methods to solve design problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios).

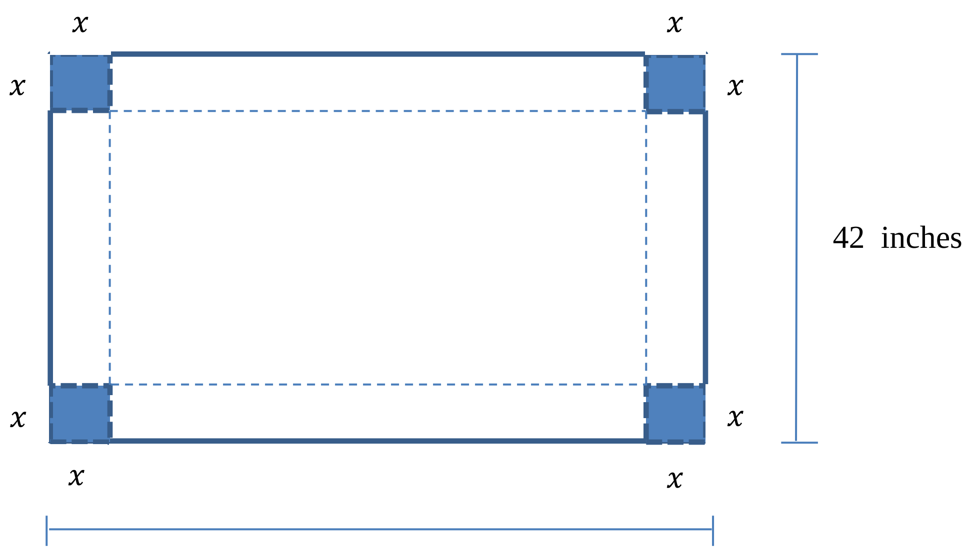
Standards for Mathematical Practice
2. Reason abstractly and quantitatively by requiring students to make sense of quantities and their relationships to one another in problem situations.

3. Construct viable arguments and critique the reasoning of others by engaging students on discussion of why they agree or disagree with responses, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

7. Look for and make use of structure by expecting students to apply rules, look for patterns and analyze structure.

8. Look for and express regularity in repeated reasoning by expecting students to understand broader applications and look for structure and general methods in similar situations.

Mary needs open-topped boxes to store her excess inventory at year's end. Mary purchases large rectangles of thick cardboard with a length of 78 inches and width of 42 inches to make the boxes. Mary is interested in maximizing the volume of the boxes and wants to know what size squares to cut out at each corner of the cardboard (which will allow the corners to be folded up to form the box) in order to do this.



(a) Volume is a three-dimension What is the third dimension that the value x represents?

x represents the height of the box

(b) Using the table below, choose five values of x and find the corresponding volumes.

Answers vary.

x	Length	Width	Volume

You tested several different values of x above, and calculated five different volumes. There is a way to guarantee that you use dimensions that will maximize volume, and now we're going to work through that process.

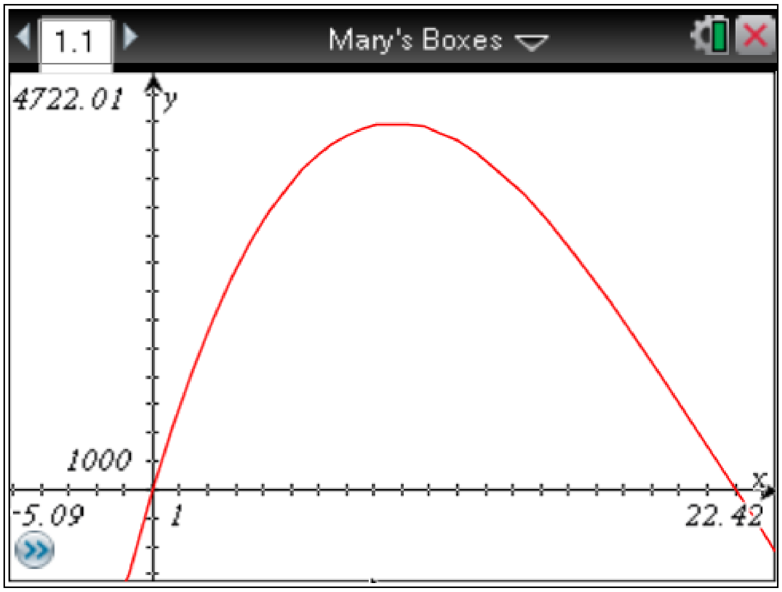
(c) Write an equation for volume in terms of the three dimensions of the box.

$$V = x(78 - 2x)(42 - 2x)$$

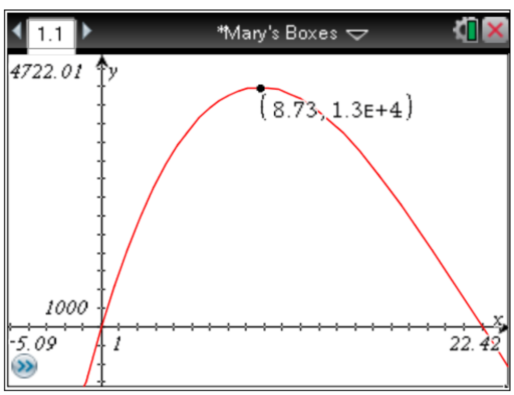
$$V = x(3276 - 240x + 4x^2)$$

$$V = 3276x - 240x^2 + 4x^3$$

(d) Graph the equation from part (c).



(e) From your graph, what are the values of the three dimensions that maximize the volume of the box? What is the maximum volume of the box?



From the graph, it appears that the maximum occurs at approximately $(\mathbf{8.73, 13000})$***, so the maximum volume would be 13000 cubic inches with a height of 8.73 inches, a length of 60.54 inches, and a width of 24.54 inches.***

Maximize Volume

Standard Addressed in this Task
MGSE9-12.G.MG.3 Apply geometric methods to solve design problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios).

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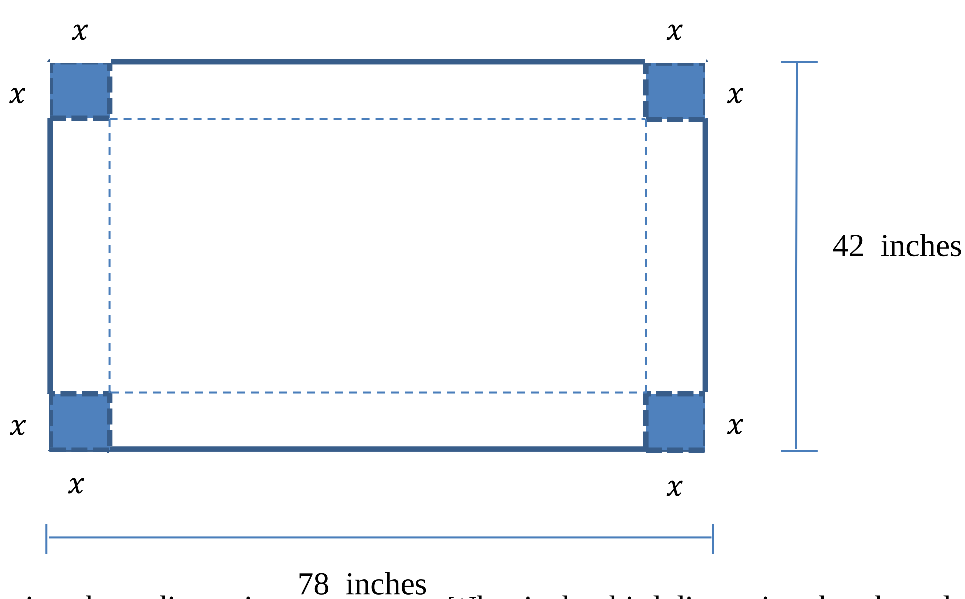


(a) Volume is a three-dimensior [illegible] What is the third dimension that the value x represents?

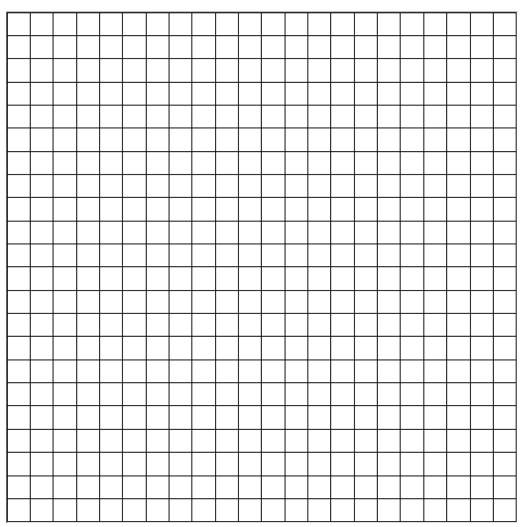
(b) Using the table below, choose five values of x and find the corresponding volumes.

x	**Length**	**Width**	**Volume**

You tested several different values of x above, and calculated five different volumes. There is a way to guarantee that you use dimensions that will maximize volume, and now we're going to work through that process.

(c) Write an equation for volume in terms of the three dimensions of the box.

(d) Graph the equation from part (c).



(e) From your graph, what are the values of the three dimensions that maximize the volume of the box? What is the maximum volume of the box?

Culminating Task: Dr. Cone's New House

[Back to Task Table]

Standard Addressed in this Task
MGSE9-12.G.GPE.1 Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.

MGSE9-12.G.GPE.4 Use coordinates to prove simple geometric theorems algebraically. *For example, ~~prove or disprove that a figure defined by four given points in the coordinate plane is a rectangle;~~ prove or disprove that the point (1, √3) lies on the circle centered at the origin and containing the point (0,2).*
(Focus on ~~quadrilaterals, right triangles, and~~ circles.)

Standards for Mathematical Practice
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Teacher Notes:

This performance task is designed to apply students' understanding of equations of parabolas and circles. While completing this task, students should take care to write the equations and fit the graphs in the required specifications. This is a great time to revisit parts of a parabola, as the x-intercepts are likely to be where the doorframes are. Scale will be an important aspect of this task. There are many possible designs for the entry way.

A local mathematician, Dr. Cone, has hired your architecture firm to design his new house. Because your boss knows you are in Analytic Geometry, he has put you in charge of the design for the entrance of the house. The mathematician has given some very unconventional requests for the design of the entrance:

- He wants two doors, both shaped like parabolas.
- He wants at least two windows, both shaped like circles.

You also know the following information:

- The dimensions of the front entrance way are 18 feet long and 10 feet tall.
- A local window and door manufacturer can produce any shape window or door, given an equation for the shape.

State and Local guidelines also state:

- All entryways to residential property must be greater than or equal to 7 feet in height.

Using a piece of graph paper, draw a design for the entry way of the house. Be sure to label all important points for the builder. Include a "Specifications Sheet" that includes equations of the figures for the window and door manufacturer.

Teacher Notes:

Once students have completed their plans, hand them the information below. This will give them a chance to prove algebraically that the box will or will not fit in the door.

Dr. Cone has come to you after seeing your design and expresses a concern. He has just ordered a new transmogrifer and he is worried that it will not fit in the front door. The transmogrifer ships in a box that is 3'x3'x9'. According to your design, will the box fit in one of your doors?

Culminating Task: Dr Cone's New House

Standard Addressed in this Task

MGSE9-12.G.GPE.1 Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.

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State and Local guidelines also state:

- All entryways to residential property must be greater than or equal to 7 feet in height.

Using a piece of graph paper, draw a design for the entry way of the house. Be sure to label all important points for the builder. Include a "Specifications Sheet" that includes equations of the figures for the window and door manufacturer.

Dr. Cone has come to you after seeing your design and expresses a concern. He has just ordered a new transmogrifer and he is worried that it will not fit in the front door. The transmogrifer ships in a box that is 3'x3'x9'. According to your design, will the box fit in one of your doors?