Georgia
Standards of Excellence
Curriculum Frameworks

Mathematics

Accelerated GSE Analytic Geometry B/
Advanced Algebra

Unit 6: Polynomial Functions

Georgia Department of Education
Richard Woods, Georgia's School Superintendent
"Educating Georgia's Future"

Unit 6
Polynomial Functions

Table of Contents

OVERVIEW

In this unit students will:

- use polynomial identities to solve problems
- use complex numbers in polynomial identities and equations
- understand and apply the rational Root Theorem
- understand and apply the Remainder Theorem
- understand and apply The Fundamental Theorem of Algebra
- understand the relationship between zeros and factors of polynomials
- represent, analyze, and solve polynomial functions algebraically and graphically

In this unit, students continue their study of polynomials by identifying zeros and making connections between zeros of a polynomial and solutions of a polynomial equation. Students will see how the Fundamental Theorem of Algebra can be used to determine the number of solutions of a polynomial equation and will find all the roots of those equations. Students will graph polynomial functions and interpret the key characteristics of the function.

STANDARDS ADDRESSED IN THIS UNIT

Mathematical standards are interwoven and should be addressed throughout the year in as many different units and activities as possible in order to emphasize the natural connections that exist among mathematical topics.

Use complex numbers in polynomial identities and equations.

MGSE9-12.N.CN.9 Use the Fundamental Theorem of Algebra to find all roots of a polynomial equation.

Interpret the structure of expressions

MGSE9-12.A.SSE.1 Interpret expressions that represent a quantity in terms of its context.

MGSE9-12.A.SSE.1a Interpret parts of an expression, such as terms, factors, and coefficients, in context.

MGSE9-12.A.SSE.1b Given situations which utilize formulas or expressions with multiple terms and/or factors, interpret the meaning (in context) of individual terms or factors.

MGSE9-12.A.SSE.2 Use the structure of an expression to rewrite it in different equivalent forms. For example, see $x^4 – y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 – y^2) (x^2 + y^2)$.

Understand the relationship between zeros and factors of polynomials

MGSE9-12.A.APR.2 Know and apply the Remainder Theorem: For a polynomial p(x) and a number a, the remainder on division by x – a is p(a), so p(a) = 0 if and only if (x – a) is a factor of p(x).

MGSE9-12.A.APR.3 Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial.

Use polynomial identities to solve problems

MGSE9-12.A.APR.4 Prove polynomial identities and use them to describe numerical relationships. *For example, the polynomial identity* $(x^2 + y^2)^2 = (x^2 - y^2)^2 + (2xy)^2$ *can be used to generate Pythagorean triples.*

Interpret functions that arise in applications in terms of the context

MGSE9-12.F.IF.4 Using tables, graphs, and verbal descriptions, interpret the key characteristics of a function which models the relationship between two quantities. Sketch a graph showing key features including: intercepts; interval where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; ~~and periodicity~~. *(Limit to polynomial functions.)*

Analyze functions using different representations

MGSE9-12.F.IF.7 Graph functions expressed algebraically and show key features of the graph both by hand and by using technology. *(Limit to polynomial functions.)*

MGSE9-12.F.IF.7c Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior.

RELATED STANDARDS

Perform arithmetic operations on polynomials.

MGSE9-12.A.APR.1 Add, subtract, and multiply polynomials; understand that polynomials form a system analogous to the integers in that they are closed under these operations.

Solve systems of equations.

MGSE9-12.A.REI.7 Solve a simple system consisting of a linear equation and a quadratic polynomial equation in two variables algebraically and graphically.

Represent and solve equations and inequalities graphically.

MGSE9-12.A.REI.11 Using graphs, tables, or successive approximations, show that the solution to the equation f(x) = g(x) is the x-value where the y-values of f(x) and g(x) are the same.

Use complex numbers in polynomial identities and equations.

MGSE9-12.N.CN.8 Extend polynomial identities to include factoring with complex numbers. *For example, rewrite* $x^2 + 4$ *as* $(x + 2i)(x - 2i)$.

Build new functions from existing functions

MGSE9-12.F.BF.3 Identify the effect on the graph of replacing f(x) by f(x) + k, k f(x), f(kx), and f(x + k) for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them.

STANDARDS FOR MATHEMATICAL PRACTICE

Refer to the Comprehensive Course Overview for more detailed information about the Standards for Mathematical Practice.

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

ENDURING UNDERSTANDINGS

- Viewing an expression as a result of operations on simpler expressions can sometimes clarify its underlying structure.
- Factoring and other forms of writing polynomials should be explored.
- The Fundamental Theorem of Algebra is not limited to what can be seen graphically; it applies to real and complex roots.
- Real and complex roots of higher degree polynomials can be found using the Factor Theorem, Remainder Theorem, Rational Root Theorem, and Fundamental Theorem of Algebra, incorporating complex and radical conjugates.
- A system of equations is not limited to linear equations; we can find the intersection between a line and a polynomial
- Asking when two functions have the same value for the same input leads to an equation; graphing the two functions allows for finding approximate solutions to the equation.

ESSENTIAL QUESTIONS

- What is the Remainder Theorem and what does it tell us?
- What is the Rational Root Theorem and what does it tell us?
- What is the Fundamental Theorem Algebra and what does it tell us?
- How can we solve polynomial equations?
- Which sets of numbers can be solutions to polynomial equations?
- What is the relationship between zeros and factors?
- What characteristics of polynomial functions can be seen on their graphs?
- How can we solve a system of a linear equation with a polynomial equation?

CONCEPTS/SKILLS TO MAINTAIN

It is expected that students will have prior knowledge/experience related to the concepts and skills identified below. It may be necessary to pre-assess in order to determine if time needs to be spent on conceptual activities that help students develop a deeper understanding of these ideas.

- Combining like terms and simplifying expressions
- Long division
- The distributive property
- The zero property
- Properties of exponents
- Simplifying radicals with positive and negative radicands
- Factoring quadratic expressions
- Solving quadratic equations by factoring, taking square roots, using the quadratic formula and utilizing graphing calculator technology to finding zeros/ x-intercepts
- Observing symmetry, end-behaviors, and turning points (relative maxima and relative minima) on graphs

SELECT TERMS AND SYMBOLS

The following terms and symbols are often misunderstood. These concepts are not an inclusive list and should not be taught in isolation. However, due to evidence of frequent difficulty and misunderstanding associated with these concepts, instructors should pay particular attention to them and how their students are able to explain and apply them.

The definitions below are for teacher reference only and are not to be memorized by the students. Students should explore these concepts using models and real life examples. Students should understand the concepts involved and be able to recognize and/or demonstrate them with words, models, pictures, or numbers.

The websites below are interactive and include a math glossary suitable for high school children. **Note – At the high school level, different sources use different definitions. Please preview any website for alignment to the definitions given in the frameworks**
http://www.amathsdictionaryforkids.com/
This web site has activities to help students more fully understand and retain new vocabulary.
http://intermath.coe.uga.edu/dictnary/homepg.asp
Definitions and activities for these and other terms can be found on the Intermath website.

- **Coefficient:** a number multiplied by a variable.

- **Degree:** the greatest exponent of its variable

- **End Behavior**: the value of f(x) as x approaches positive and negative infinity

- **Fundamental Theorem of Algebra:** every non-zero single-variable polynomial with complex coefficients has exactly as many complex roots as its degree, if each root is counted up to its multiplicity.

- **Multiplicity**: the number of times a root occurs at a given point of a polynomial equation.

- **Pascal's Triangle:** an arrangement of the values of ${}_nC_r$ in a triangular pattern where each row corresponds to a value of n

- **Polynomial:** a mathematical expression involving a sum of nonnegative integer powers in one or more variables multiplied by coefficients. A polynomial in one variable with constant coefficients can be written in $a_nx^n + a_{n-1}x^{n-1} + ...a_2x^2 + a_1x + a_0$ form.

- **Rational Root Theorem:** a theorem that provides a complete list of all possible rational roots of a polynomial equation. It states that every rational zero of the polynomial equation f(x) = $a_nx^n + a_{n-1}x^{n-1} + ...a_2x^2 + a_1x + a_0$,where all coefficients are integers, has the following form: $\dfrac{p}{q} = \dfrac{\text{factors of constant term } a_0}{\text{factors of leading coefficient } a_n}$

- **Relative Minimum**: a point on the graph where the function is increasing as you move away from the point in the positive and negative direction along the horizontal axis.

- **Relative Maximum**: a point on the graph where the function is decreasing as you move away from the point in the positive and negative direction along the horizontal axis.

- **Remainder Theorem:** states that the remainder of a polynomial f(x) divided by a linear divisor (x – c) is equal to f(c).

- **Roots**: solutions to polynomial equations.

- **Synthetic Division:** Synthetic division is a shortcut method for dividing a polynomial by a linear factor of the form (x – a). It can be used in place of the standard long division algorithm.

- **Zero**: If f(x) is a polynomial function, then the values of x for which f(x) = 0 are called the **zeros** of the function. Graphically, these are the x intercepts.

EVIDENCE OF LEARNING

By the conclusion of this unit, students should be able to demonstrate the following competencies:

- apply the Remainder Theorem to determine zeros of polynomial functions
- utilize the Rational Root Theorem to determine possible zeros to polynomial functions
- solve polynomial equations using algebraic and graphing calculator methods
- apply the Fundamental Theorem of Algebra to determine the number of zeros of polynomial functions
- construct rough graphs of polynomial functions, displaying zeros, relative maxima's, and end-behaviors
- identify key features of graphs of polynomial functions
- find the intersection of a linear and a polynomial equation

FORMATIVE ASSESSMENT LESSONS (FAL)

Formative Assessment Lessons are intended to support teachers in formative assessment. They reveal and develop students' understanding of key mathematical ideas and applications. These lessons enable teachers and students to monitor in more detail their progress towards the targets of the standards. They assess students' understanding of important concepts and problem solving performance, and help teachers and their students to work effectively together to move each student's mathematical reasoning forward.

More information on Formative Assessment Lessons may be found in the Comprehensive Course Guide.

SPOTLIGHT TASKS

A Spotlight Task has been added to each GSE mathematics unit in the Georgia resources for middle and high school. The Spotlight Tasks serve as exemplars for the use of the Standards for Mathematical Practice, appropriate unit-level Georgia Standards of Excellence, and research-based pedagogical strategies for instruction and engagement. Each task includes teacher commentary and support for classroom implementation. Some of the Spotlight Tasks are revisions of existing Georgia tasks and some are newly created. Additionally, some of the Spotlight Tasks are 3-Act Tasks based on 3-Act Problems from Dan Meyer and Problem-Based Learning from Robert Kaplinsky

3-ACT TASKS

A Three-Act Task is a whole group mathematics task consisting of 3 distinct parts: an engaging and perplexing Act One, an information and solution seeking Act Two, and a solution discussion and solution revealing Act Three.

More information along with guidelines for 3-Act Tasks may be found in the Comprehensive Course Guide.

TASKS

The following tasks represent the level of depth, rigor, and complexity expected of all Algebra II/Advanced Algebra students. These tasks, or tasks of similar depth and rigor, should be used to demonstrate evidence of learning. It is important that all elements of a task be addressed throughout the learning process so that students understand what is expected of them. While some tasks are identified as a performance task, they may also be used for teaching and learning (learning/scaffolding task).

Task Name	Task Type *Grouping Strategy*	Content Addressed	SMPs Addressed
Divide and Conquer	Practice Task *Individual/Partner Task*	Polynomial division (long and synthetic) and synthetic substitution	1, 6
Factor's, Zeros, and Roots: Oh My!	Practice Task *Individual/Partner Task*	Explore the relationship between factors, zeros, and roots of polynomial functions	1, 6
Representing Polynomials	Formative Assessment Lesson *Partner Task*	Graph polynomial functions and shows key features of the graph, by hand	1-8
The Canoe Trip	Learning Task *Flexible Grouping*	Interpret key features of an application	1, 3, 7
Trina's Triangles	Learning Task *Individual/Partner*	Use polynomials to prove numeric relationships	1, 3, 7
Polynomial Patterns	Practice Task *Individual/Partner Task*	Graph polynomial functions and show key features of the graph, using technology	1, 2, 5
Culminating Task: Polynomial Project Task Part 1	Performance Task *Individual/Partner Task*	Find and analyze zeros and factors of polynomial functions *(It is appropriate to administer this after the Factor's, Zeros, and Roots: Oh My! Task)*	1-8
Culminating Task: Polynomial Project Task Part 2	Performance Task *Individual/Partner Task*	Graph polynomial functions using technology and use graphs to solve real-life problems *(It is appropriate to administer this after the Polynomial Patterns Task)*	1-8

Divide and Conquer

Back to Task Table

Mathematical Goals

- Divide polynomials using long division
- Divide polynomials using synthetic division
- Determine if polynomial division is closed
- Evaluate polynomials using synthetic substitution by applying the Remainder Theorem

Georgia Standards of Excellence

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Standards for Mathematical Practice

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Introduction

In this task, we are going to perform division with polynomials and determine whether this operation is closed. We will make the connection of division with integers to division with polynomials by looking at the process of long division. We will then explore an alternate method for finding quotients when the divisor can be written in $(x-k)$ form. Finally, we will explore an application of the remainder theorem by performing synthetic substitution in order to evaluate polynomials for given variables of interest.

Materials

- Pencil
- Handout
- Graphing Calculator

You may have noticed that in the first task, We've Got to Operate, we only performed addition, subtraction, and multiplication of polynomials. Now we are ready to explore polynomial division.

1. First, let's think about something we learned in elementary school, long division. Find the quotient using long division and describe what you do in each step:

a. $19\overline{)2052}$ ***108***

b. $46\overline{)3768}$ ***81*** $\frac{42}{46}$ **** may need to review how to express a remainder in fraction form***

2. Now, we are going to use the same idea to explore polynomial division. Specifically, read the example below and determine if this process is similar to the methods you described in your process of long division above.

Stapel, Elizabeth. "Polynomial Long Division." Purplemath. Available from http://www.purplemath.com/modules/polydiv2.htm. Accessed 25 May 2012

Polynomial Long Division

If you're dividing a polynomial by something more complicated than just a simple monomial, then you'll need to use a different method for the simplification. That method is called "long (polynomial) division", and it works just like the long (numerical) division you did back in elementary school, except that now you're dividing with variables.

- **Divide $x^2 - 9x - 10$ by $x + 1$**

  Think back to when you were doing long division with plain old numbers. You would be given one number that you had to divide into another number. You set up the division symbol, inserted the two numbers where they belonged, and then started making guesses. And you didn't guess the whole answer right away; instead, you started working on the "front" part (the larger place values) of the number you were dividing.

Long division for polynomials works in much the same way:

First, I set up the division: For the moment, I'll ignore the other terms and look just at the leading x of the divisor and the leading x^2 of the dividend.	$x+1\overline{)x^2 \quad -9x-10}$
If I divide the leading x^2 inside by the leading x in front, what would I get? I'd get an x. So I'll put an x on top:	$\begin{array}{r} x \qquad\qquad \\ x+1\overline{)x^2 \quad -9x-10} \end{array}$
Now I'll take that x, and multiply it through the divisor, $x + 1$. First, I multiply the x (on top) by the x (on the "side"), and carry the x^2 underneath:	$\begin{array}{r} x \qquad\qquad \\ x+1\overline{)x^2 \quad -9x-10} \\ x^2 \qquad\qquad\quad \end{array}$
Then I'll multiply the x (on top) by the 1 (on the "side"), and carry the $1x$ underneath:	$\begin{array}{r} x \qquad\qquad \\ x+1\overline{)x^2 \quad -9x-10} \\ x^2 \quad +1x \qquad \end{array}$
Then I'll draw the "equals" bar, so I can do the subtraction. To subtract the polynomials, I *change all the signs* in the second line...	$\begin{array}{r} x \qquad\qquad \\ x+1\overline{)x^2 \quad -9x-10} \\ \underline{-x^2 \quad \overset{-}{+}1x \qquad} \end{array}$
...and then I add down. The first term (the x^2) will cancel out:	$\begin{array}{r} x \qquad\qquad \\ x+1\overline{)x^2 \quad -9x-10} \\ \underline{-x^2 \quad \overset{-}{+}1x \qquad} \\ -10x \qquad \end{array}$
I need to remember to carry down that last term, the "subtract ten", from the dividend:	$\begin{array}{r} x \qquad\qquad \\ x+1\overline{)x^2 \quad -9x-10} \\ \underline{-x^2 \quad \overset{-}{+}1x \qquad} \\ -10x-10 \end{array}$
Now I look at the x from the divisor and the new leading term, the $-10x$, in the bottom line of the division. If I divide the $-10x$ by the x, I would end up with a -10, so I'll put that on top:	$\begin{array}{r} x \;-10 \qquad \\ x+1\overline{)x^2 \quad -9x-10} \\ \underline{-x^2 \quad \overset{-}{+}1x \qquad} \\ -10x-10 \end{array}$

Now I'll multiply the -10 (on top) by the leading x (on the "side"), and carry the $-10x$ to the bottom:	$\begin{array}{r} x-10 \\ x+1\overline{)x^2-9x-10} \\ -x^2 \mp 1x \\ \hline -10x-10 \\ -10x \end{array}$
...and I'll multiply the -10 (on top) by the 1 (on the "side"), and carry the -10 to the bottom:	$\begin{array}{r} x-10 \\ x+1\overline{)x^2-9x-10} \\ -x^2 \mp 1x \\ \hline -10x-10 \\ -10x-10 \end{array}$
I draw the equals bar, and *change the signs* on all the terms in the bottom row:	$\begin{array}{r} x-10 \\ x+1\overline{)x^2-9x-10} \\ -x^2 \mp 1x \\ \hline -10x-10 \\ +10x+10 \\ \hline \end{array}$
Then I add down:	$\begin{array}{r} x-10 \\ x+1\overline{)x^2-9x-10} \\ -x^2 \mp 1x \\ \hline -10x-10 \\ +10x+10 \\ \hline 0 \end{array}$

Then the solution to this division is: $\boldsymbol{x - 10}$

Since the remainder on this division was zero (that is, since there wasn't anything left over), the division came out "even". When you do regular division with numbers and the division comes out even, it means that the number you divided by is a factor of the number you're dividing. For instance, if you divide 50 by 10, the answer will be a nice neat "5" with a zero remainder, because 10 is a factor of 50. In the case of the above polynomial division, the zero remainder tells us that $x + 1$ is a factor of $x^2 - 9x - 10$, which you can confirm by factoring the original quadratic dividend, $x^2 - 9x - 10$.

3. Your teacher will now guide you through several of these to practice long division.

a. $\begin{array}{r} x+2 \\ x^2+3x-1\overline{)x^3+5x^2+5x-2} \end{array}$

b. $\begin{array}{r} 3x-4+\dfrac{5}{2x+3} \\ 2x+3\overline{)6x^2+x-7} \end{array}$

c. $(x^3 + 6x^2 - 5x + 20) \div (x^2 + 5)$

$x + 6 + \dfrac{-10x - 10}{x^2 + 5}$

d. $(4x^4 - 3x^3 - 7x^2 - 4x - 9) \div (x - 2)$

$4x^3 + 5x^2 + 3x + 2 + \dfrac{-5}{x - 2}$

4. As you can see, long division can be quite tedious. Now, let us consider another way to find quotients called synthetic division. Unfortunately, synthetic division is defined only when the divisor is linear.

a. In which problem(s) above is synthetic division defined? ***Problems 3b and 3d***

b. The next part of this task will explore how it works and why it only works when there is a linear divisor.

The following excerpt is taken from:
J.M. Kingston, *Mathematics for Teachers of the Middle Grades*, John Wiley & Sons, Inc., NY, 1966, p. 203-205.

9.04 SYNTHETIC DIVISION AND SYNTHETIC SUBSTITUTION

The labor of dividing a polynomial by $x - t$ can be reduced considerably by eliminating the symbols that occur repetitiously in the procedure. Let us consider the following division:

$$
\begin{array}{r|l}
 & 4x^3 + 5x^2 + 3x + 2 \\
\hline
x - 2 & 4x^4 - 3x^3 - 7x^2 - 4x - 9 \\
 & \underline{4x^4 - 8x^3} \\
 & \qquad 5x^3 - 7x^2 \\
 & \qquad \underline{5x^3 - 10x^2} \\
 & \qquad\qquad 3x^2 - 4x \\
 & \qquad\qquad \underline{3x^2 - 6x} \\
 & \qquad\qquad\qquad 2x - 9 \\
 & \qquad\qquad\qquad \underline{2x - 4} \\
 & \qquad\qquad\qquad\quad - 5
\end{array}
$$

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We may streamline this division, as follows, leaving out the various powers of x but maintaining the coefficients in their proper places.

$$
\begin{array}{r|rrrrr}
\multicolumn{1}{r}{} & 4 & 5 & 3 & 2 & \\
\cline{2-6}
-2 & 4 & -3 & -7 & -4 & -9 \\
\multicolumn{1}{r}{} & & \underline{-8} & & & \\
\multicolumn{1}{r}{} & & 5 & & & \\
\multicolumn{1}{r}{} & & & \underline{-10} & & \\
\multicolumn{1}{r}{} & & & 3 & & \\
\multicolumn{1}{r}{} & & & & \underline{-6} & \\
\multicolumn{1}{r}{} & & & & 2 & \\
\multicolumn{1}{r}{} & & & & & \underline{-4} \\
\multicolumn{1}{r}{} & & & & & -5
\end{array}
$$

The above arrangement may be "collapsed" to give the following:

$$
\begin{array}{r|rrrrr}
-2 & 4 & -3 & -7 & -4 & -9 \\
 & & -8 & -10 & -6 & -4 \\
\hline
 & 4 & 5 & 3 & 2 & -5
\end{array}
$$

Note that:

$$
\begin{aligned}
-8 &= 4(-2) \\
-10 &= 5(-2) \\
-6 &= 3(-2) \\
-4 &= 2(-2)
\end{aligned}
$$

Since it is generally easier to add than to subtract, we shall replace -2 by 2 and add, rather than subtract, in each column beginning with the second from the left. Hence we have the final streamlined division known as *synthetic division*:

$$
\begin{array}{r|rrrrr}
2 & 4 & -3 & -7 & -4 & -9 \\
 & & 8 & 10 & 6 & 4 \\
\hline
 & 4 & 5 & 3 & 2 & -5
\end{array}
$$

There are several points to be noted in connection with this procedure:

(1) The number in the upper left-hand corner is "t", if we are dividing by $x - t$.

(2) The top row consists of coefficients of terms of the dividend polynomial in order of descending degree. Any missing term in the sequence must be indicated by a zero coefficient. For example, we shall treat $5x^4 + 3x$ as $5x^4 + 0x^3 + 0x^2 + 3x + 0$.

(3) The left-hand coefficient in the top row is merely "brought down" to the third row.

(4) The procedure is then one of "multiply by t and add."

(5) The third row, except for the right-hand number, consists of the coefficients of powers of x in the quotient polynomial, in order of descending degree.

(6) The right-hand number in the third row is the remainder, when the divisor is $x - t$, which, by the Remainder Theorem, also represents the value of the dividend polynomial at $x = t$.

(7) In view of the Remainder Theorem the process is known equally well as *synthetic substitution*.

5. Your teacher will now guide you through several of these to practice synthetic division.

a. $x-2\overline{)x^3+2x^2-6x-9}$

$x^2 + 4x + 2 + \frac{-5}{x-2}$

b. $x+3\overline{)x^3+2x^2-6x-9}$

$x^2 - x - 3$

c. $(x^4-16x^2+x+4)\div(x+4)$

$x^3 - 4x^2 + 1$

d. $(4x^4-3x^3-7x^2-4x-9)\div(x-2)$

$4x^3 + 5x^2 + 3x + 2 + \frac{-5}{x-2}$

6. Compare you answers in problem 3d to your solution to 5d. Is it the same? Why? **Yes**

7. Look back at all of your quotients in problems 3 and 5. Is the operation of division closed for polynomials? In other words, are the results of the division operation always an element in the set of polynomials? Why or why not? ***No, not all quotients are polynomials (the quotients that had remainders are not polynomials)***

8. One way to evaluate polynomial functions is to use direct substitution. For instance, $f(x) = 2x^4 - 8x^2 + 5x - 7$ can be evaluated when $x = 3$ as follows:
$f(3) = 2(3)^4 - 8(3)^2 + 5(3) - 7 = 162 - 72 + 15 - 7 = 98$.

However, there is another way to evaluate a polynomial function called synthetic substitution. Since the Remainder Theorem states that the remainder of a polynomial f(x) divided by a linear divisor (x – c) are equal to f(c), the value of the last number on the right corner should give an equivalent result. Let's see.

3	2	0	-8	5	-7
		6	18	30	105
	2	6	10	35	98

Use synthetic substitution to evaluate the following. You can confirm your results with direct substitution using a calculator.

a. $f(x) = 2x^4 + x^3 - 3x^2 + 5x - 8,\ x = -1$ ***-15***

b. $f(x) = -3x^3 + 7x^2 - 4x + 8,\ x = 3$ ***-22***

c. $f(x) = 3x^5 - 2x^2 + x,\ x = 2$ ***90***

d. $f(x) = -x^4 + 8x^3 + 13x - 4,\ x = -2$ ***-110***

Divide and Conquer

Introduction

In this task, we are going to perform division with polynomials and determine whether this operation is closed. We will make the connection of division with integers to division with polynomials by looking at the process of long division. We will then explore an alternate method for finding quotients when the divisor can be written in $(x-k)$ form. Finally, we will explore an application of the remainder theorem by performing synthetic substitution in order to evaluate polynomials for given variables of interest.

Materials

- Pencil
- Handout
- Graphing Calculator

You may have noticed that in the first task, We've Got to Operate, we only performed addition, subtraction, and multiplication of polynomials. Now we are ready to explore polynomial division.

1. First, let's think about something we learned in elementary school, long division. Find the quotient using long division and describe what you do in each step:

a. $19\overline{)2052}$

b. $46\overline{)3768}$

2. Now, we are going to use the same idea to explore polynomial division. Specifically, read the example below and determine if this process is similar to the methods you described in your process of long division above.

Polynomial Long Division

Stapel, Elizabeth. "Polynomial Long Division." Purplemath. Available from http://www.purplemath.com/modules/polydiv2.htm. Accessed 25 May 2012

If you're dividing a polynomial by something more complicated than just a simple monomial, then you'll need to use a different method for the simplification. That method is called "long (polynomial) division", and it works just like the long (numerical) division you did back in elementary school, except that now you're dividing with variables.

- **Divide $x^2 - 9x - 10$ by $x + 1$**

  Think back to when you were doing long division with plain old numbers. You would be given one number that you had to divide into another number. You set up the division symbol, inserted the two numbers where they belonged, and then started making guesses. And you didn't guess the whole answer right away; instead, you started working on the "front" part (the larger place values) of the number you were dividing.

  Long division for polynomials works in much the same way:

  \

First, I set up the division: For the moment, I'll ignore the other terms and look just at the leading x of the divisor and the leading x^2 of the dividend.	$x+1\overline{)x^2 \; -9x-10}$
If I divide the leading x^2 inside by the leading x in front, what would I get? I'd get an x. So I'll put an x on top:	$\begin{array}{r} x \\ x+1\overline{)x^2 \; -9x-10} \end{array}$
Now I'll take that x, and multiply it through the divisor, $x + 1$. First, I multiply the x (on top) by the x (on the "side"), and carry the x^2 underneath:	$\begin{array}{r} x \\ x+1\overline{)x^2 \; -9x-10} \\ x^2 \end{array}$
Then I'll multiply the x (on top) by the 1 (on the "side"), and carry the $1x$ underneath:	$\begin{array}{r} x \\ x+1\overline{)x^2 \; -9x-10} \\ x^2 \; +1x \end{array}$
Then I'll draw the "equals" bar, so I can do the subtraction. To <u>subtract</u> the polynomials, I *change all the signs* in the second line...	$\begin{array}{r} x \\ x+1\overline{)x^2 \; -9x-10} \\ \underline{-x^2 \; \overset{-}{+}1x } \end{array}$

...and then I add down. The first term (the x^2) will cancel out:	$\begin{array}{rr} & x \qquad\qquad \\ x+1\big) & \overline{x^2 \;\; -9x-10} \\ & \underline{-x^2 \;\; \overset{-}{+}1x \qquad} \\ & -10x \qquad \end{array}$
I need to remember to carry down that last term, the "subtract ten", from the dividend:	$\begin{array}{rr} & x \qquad\qquad \\ x+1\big) & \overline{x^2 \;\; -9x-10} \\ & \underline{-x^2 \;\; \overset{-}{+}1x \qquad} \\ & -10x-10 \end{array}$
Now I look at the x from the divisor and the new leading term, the $-10x$, in the bottom line of the division. If I divide the $-10x$ by the x, I would end up with a -10, so I'll put that on top:	$\begin{array}{rr} & x \;-10 \quad \\ x+1\big) & \overline{x^2 \;\; -9x-10} \\ & \underline{-x^2 \;\; \overset{-}{+}1x \qquad} \\ & -10x-10 \end{array}$
Now I'll multiply the -10 (on top) by the leading x (on the "side"), and carry the $-10x$ to the bottom:	$\begin{array}{rr} & x \;-10 \quad \\ x+1\big) & \overline{x^2 \;\; -9x-10} \\ & \underline{-x^2 \;\; \overset{-}{+}1x \qquad} \\ & -10x-10 \\ & -10x \qquad \end{array}$
...and I'll multiply the -10 (on top) by the 1 (on the "side"), and carry the -10 to the bottom:	$\begin{array}{rr} & x \;-10 \quad \\ x+1\big) & \overline{x^2 \;\; -9x-10} \\ & \underline{-x^2 \;\; \overset{-}{+}1x \qquad} \\ & -10x-10 \\ & -10x-10 \end{array}$
I draw the equals bar, and *change the signs* on all the terms in the bottom row:	$\begin{array}{rr} & x \;-10 \quad \\ x+1\big) & \overline{x^2 \;\; -9x-10} \\ & \underline{-x^2 \;\; \overset{-}{+}1x \qquad} \\ & -10x-10 \\ & \underline{+10x+10} \end{array}$
Then I add down:	$\begin{array}{rr} & x \;-10 \quad \\ x+1\big) & \overline{x^2 \;\; -9x-10} \\ & \underline{-x^2 \;\; \overset{-}{+}1x \qquad} \\ & -10x-10 \\ & \underline{+10x+10} \\ & 0 \end{array}$

Then the solution to this division is: $\boldsymbol{x - 10}$

Since the remainder on this division was zero (that is, since there wasn't anything left over), the division came out "even". When you do regular division with numbers and the division comes out even, it means that the number you divided by is a factor of the number you're dividing. For instance, if you divide 50 by 10, the answer will be a nice neat "5" with a zero remainder, because 10 is a factor of 50. In the case of the above polynomial division, the zero remainder tells us that $x + 1$ is a factor of $x^2 - 9x - 10$, which you can confirm by factoring the original quadratic dividend, $x^2 - 9x - 10$.

3. Your teacher will now guide you through several of these to practice long division.

a. $x^2 + 3x - 1 \overline{\smash{\big)} x^3 + 5x^2 + 5x - 2}$

b. $2x + 3 \overline{\smash{\big)} 6x^2 + x - 7}$

c. $(x^3 + 6x^2 - 5x + 20) \div (x^2 + 5)$

d. $(4x^4 - 3x^3 - 7x^2 - 4x - 9) \div (x - 2)$

4. As you can see, long division can be quite tedious. Now, let us consider another way to find quotients called synthetic division. Unfortunately, synthetic division is defined only when the divisor is linear.

a. In which problem(s) above is synthetic division defined?

b. The next part of this task will explore how it works and why it only works when there is a linear divisor.

The following excerpt is taken from:
J.M. Kingston, *Mathematics for Teachers of the Middle Grades*, John Wiley & Sons, Inc., NY, 1966, p. 203-205.

9.04 SYNTHETIC DIVISION AND SYNTHETIC SUBSTITUTION

The labor of dividing a polynomial by $x - t$ can be reduced considerably by eliminating the symbols that occur repetitiously in the procedure. Let us consider the following division:

$$
\begin{array}{r|l}
 & 4x^3 + 5x^2 + 3x + 2 \\ \hline
x - 2 & 4x^4 - 3x^3 - 7x^2 - 4x - 9 \\
 & \underline{4x^4 - 8x^3} \\
 & \quad\quad 5x^3 - 7x^2 \\
 & \quad\quad \underline{5x^3 - 10x^2} \\
 & \quad\quad\quad\quad 3x^2 - 4x \\
 & \quad\quad\quad\quad \underline{3x^2 - 6x} \\
 & \quad\quad\quad\quad\quad\quad 2x - 9 \\
 & \quad\quad\quad\quad\quad\quad \underline{2x - 4} \\
 & \quad\quad\quad\quad\quad\quad\quad -5
\end{array}
$$

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We may streamline this division, as follows, leaving out the various powers of x but maintaining the coefficients in their proper places.

$$
\begin{array}{r|rrrrr}
 & 4 & 5 & 3 & 2 & \\ \hline
-2 & 4 & -3 & -7 & -4 & -9 \\
 & & \underline{-8} & & & \\
 & & 5 & & & \\
 & & & \underline{-10} & & \\
 & & & 3 & & \\
 & & & & \underline{-6} & \\
 & & & & 2 & \\
 & & & & & \underline{-4} \\
 & & & & & -5
\end{array}
$$

The above arrangement may be "collapsed" to give the following:

$$
\begin{array}{r|rrrrr}
-2 & 4 & -3 & -7 & -4 & -9 \\
 & & -8 & -10 & -6 & -4 \\ \hline
 & 4 & 5 & 3 & 2 & -5
\end{array}
$$

Note that:

$$\begin{aligned} -8 &= 4(-2) \\ -10 &= 5(-2) \\ -6 &= 3(-2) \\ -4 &= 2(-2) \end{aligned}$$

Since it is generally easier to add than to subtract, we shall replace -2 by 2 and add, rather than subtract, in each column beginning with the second from the left. Hence we have the final streamlined division known as *synthetic division*:

$$\begin{array}{r|rrrrr} \underline{2} & 4 & -3 & -7 & -4 & -9 \\ & & 8 & 10 & 6 & 4 \\ \hline & 4 & 5 & 3 & 2 & -5 \end{array}$$

There are several points to be noted in connection with this procedure:

(1) The number in the upper left-hand corner is "t", if we are dividing by $x - t$.

(2) The top row consists of coefficients of terms of the dividend polynomial in order of descending degree. Any missing term in the sequence must be indicated by a zero coefficient. For example, we shall treat $5x^4 + 3x$ as $5x^4 + 0x^3 + 0x^2 + 3x + 0$.

(3) The left-hand coefficient in the top row is merely "brought down" to the third row.

(4) The procedure is then one of "multiply by t and add."

(5) The third row, except for the right-hand number, consists of the coefficients of powers of x in the quotient polynomial, in order of descending degree.

(6) The right-hand number in the third row is the remainder, when the divisor is $x - t$, which, by the Remainder Theorem, also represents the value of the dividend polynomial at $x = t$.

(7) In view of the Remainder Theorem the process is known equally well as *synthetic substitution.*

5. Your teacher will now guide you through several of these to practice synthetic division.

a. $x-2\overline{\smash{)}x^3+2x^2-6x-9}$ b. $x+3\overline{\smash{)}x^3+2x^2-6x-9}$

c. $(x^4-16x^2+x+4)\div(x+4)$ d. $(4x^4-3x^3-7x^2-4x-9)\div(x-2)$

6. Compare you answers in problem 3d to your solution to 5d. Is it the same? Why?

7. Look back at all of your quotients in problems 3 and 5. Is the operation of division closed for polynomials? In other words, are the results of the division operation always an element in the set of polynomials? Why or why not?

8. One way to evaluate polynomial functions is to use direct substitution. For instance, $f(x)=2x^4-8x^2+5x-7$ can be evaluated when $x=3$ as follows:
$f(3)=2(3)^4-8(3)^2+5(3)-7=162-72+15-7=98$.

However, there is another way to evaluate a polynomial function called synthetic substitution. Since the Remainder Theorem states that the remainder of a polynomial f(x) divided by a linear divisor (x – c) is equal to f(c), the value of the last number on the right corner should give an equivalent result. Let's see.

$$\begin{array}{r|rrrrr} 3 & 2 & 0 & -8 & 5 & -7 \\ & & 6 & 18 & 30 & 105 \\ \hline & 2 & 6 & 10 & 35 & 98 \end{array}$$

We can see that the remainder is equivalent to the solution to this problem! Use synthetic substitution to evaluate the following. You can confirm your results with direct substitution using a calculator.

a. $f(x) = 2x^4 + x^3 - 3x^2 + 5x - 8,\ x = -1$

b. $f(x) = -3x^3 + 7x^2 - 4x + 8,\ x = 3$

c. $f(x) = 3x^5 - 2x^2 + x,\ x = 2$

d. $f(x) = -x^4 + 8x^3 + 13x - 4,\ x = -2$

Factors, Zeros, and Roots: Oh My!

Back to Task Table

Mathematical Goals

- Know and apply the Remainder Theorem
- Know and apply the Rational Root Theorem
- Know and apply the Factor Theorem
- Know and apply the Fundamental Theorem of Algebra
- Identify zeros and factors of polynomial functions
- Classify roots as rational, irrational, real, and/or imaginary

Georgia Standards of Excellence

Interpret the structure of expressions.

MGSE9-12.A.SSE.1 Interpret expressions that represent a quantity in terms of its context.

MGSE9-12.A.SSE.1a Interpret parts of an expression, such as terms, factors, and coefficients, in context.

MGSE9-12.A.SSE.1b Given situations which utilize formulas or expressions with multiple terms and/or factors, interpret the meaning (in context) of individual terms or factors.

MGSE9-12.A.SSE.2 Use the structure of an expression to rewrite it in different equivalent forms. For example, see $x^4 – y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 – y^2)(x^2 + y^2)$.

Perform arithmetic operations on polynomials.

MGSE9-12.A.APR.1 Add, subtract, and multiply polynomials; understand that polynomials form a system analogous to the integers in that they are closed under these operations.

Use complex numbers in polynomial identities and equations.

MGSE9-12.N.CN.9 Use the Fundamental Theorem of Algebra to find all roots of a polynomial equation.

Understand the relationship between zeros and factors of polynomials.

MGSE9-12.A.APR.2 Know and apply the Remainder Theorem: For a polynomial p(x) and a number a, the remainder on division by x – a is p(a), so p(a) = 0 if and only if (x – a) is a factor of p(x).

MGSE9-12.A.APR.3 Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial.

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Model with mathematics.
4. Look for and make use of structure.
5. Look for and express regularity in repeated reasoning.

<u>Notes on Factors, Zeros, and Roots Task</u>: This task is designed to draw on student's prior knowledge. Students have solved quadratics with a variety of methods, now the natural progression of the degree of the polynomials is larger than before. Students will look to use technology to analyze the graphs of the polynomial and incorporate their knowledge of solving with the additional methods that are presented. BUT before we launch students down the road of using technology to analyze higher order polynomials make sure to dive into the other important concepts involved like the use of the discriminant (which is revisited), the process of long division and synthetic division is walked through step by step, the remainder theorem, and the rational root theorem. If used appropriately, this task will allow teachers to introduce the rational root theorem. This task is designed to be completed primarily without technology. As a teacher guiding students through these concepts, be prepared to leap onto opportunities where students work to integrate the technology. It can be difficult to incorporate the rational root theorem into application when technology is the focus. This task is intended to give the student the opportunity to consider the options, understand the options, and make appropriate choices throughout the process.

Teachers should use this task as guided instruction for their students. The design is for the teacher to introduce and model the use of these techniques and then to use the variety of additional functions to provide students with ample opportunity to attempt this work on their own and demonstrate their work for the teacher and other students.

Solving polynomials that have a degree greater than those solved in previous courses is going to require the use of skills that were developed when we previously solved quadratics. Let's begin by taking a look at some second degree polynomials and the strategies used to solve them. These equations have the form $ax^2 + bx + c = 0$, and when they are graphed the result is a parabola.

This section is intended to be a brief opportunity to revisit factoring. Additional practice for warm-ups could be helpful.

1. Factoring is used to solve quadratics of the form $ax^2 + bx + c = 0$ when the roots are rational. Find the roots of the following quadratic functions:

a. $f(x) = x^2 - 5x - 14$

b. $f(x) = x^2 - 64$

c. $f(x) = 6x^2 + 7x - 3$

d. $f(x) = 3x^2 + x - 2$

Solutions:
a. x = 7 and -2
b. x = 8 and -8
c. $x = \frac{-3}{2}$ ***and*** $\frac{1}{3}$
d. $x = \frac{2}{3}$ ***and -1***

Again, this next section should be a brief opportunity to engage prior knowledge of the quadratic formula. Be careful not to bog down in the review material. It will important to weave this skill review into the overall development of this topic.

2. Another option for solving a quadratic whether it is factorable but particularly when it is not is to use the quadratic formula. Remember, a quadratic equation written in $ax^2 + bx + c = 0$ has solution(s) $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Also remember that $b^2 - 4ac$ is the discriminant and gives us the ability to determine the nature of the roots.

$$b^2 - 4ac \begin{cases} > 0 & 2 \quad real\, roots \\ = 0 & 1 \quad real\ root \\ < 0 & 0 \quad real\, roots\,(imaginary) \end{cases}$$

Find the roots for each of the following. Also, describe the number and nature of these roots.

a. $f(x) = 4x^2 - 2x + 9$

b. $f(x) = 3x^2 + 4x - 8$

c. $f(x) = x^2 - 5x + 9$

Solutions

a. $x = \dfrac{2 \pm i\sqrt{35}}{4}$***, two imaginary solutions***

b. $x = \dfrac{-2 \pm 2\sqrt{7}}{3}$***, two irrational solutions***

c. $x = \dfrac{5 \pm i\sqrt{11}}{2}$***, two imaginary solutions***

3. Let's take a look at the situation of a polynomial that is one degree greater. When the polynomial is a third degree, will there be any similarities when we solve?

Look to direct students toward the fact that a second degree polynomial (quadratic) yields two solutions (of some type) , therefore a cubic would yield three solutions.
Draw on or look to develop students understanding of factored form of a polynomial. This concept has been developed in Analytic Geometry. If students do not have a good grasp of factored form, now is the time to make this connection. Look back at the first section of this task and use some of the quadratics provided to develop this concept.

Suppose we want to find the roots of $f(x) = x^3 + 2x^2 - 5x - 6$. By inspecting the graph of the function, we can see that one of the roots is distinctively 2. Since we know that x = 2 is a solution to $f(x)$, we also know that $x - 2$ is a factor of the expression $x^3 + 2x^2 - 5x - 6$. This means that if we divide $x^3 + 2x^2 - 5x - 6$ by $x - 2$ there will be a remainder of zero. Let's confirm this with synthetic substitution:

$$\begin{array}{r|rrrr} 2 & 1 & 2 & -5 & -6 \\ & & 2 & 8 & 6 \\ \hline & 1 & 4 & 3 & 0 \end{array}$$

Let's practice synthetic division before we tackle how to solve cubic polynomials in general. Do the following division problems synthetically.

a. $\dfrac{10x^3 - 17x^2 - 7x + 2}{x - 2}$

b. $\dfrac{x^3 + 3x^2 - 10x - 24}{x + 4}$

c. $\dfrac{x^3 - 7x - 6}{x + 1}$

Solutions

a. $10x^2 + 3x - 1$
b. $x^2 - x - 6$
c. $x^2 - x - 6$

The main thing to notice about solving cubic polynomials (or higher degree polynomials) is that a polynomial that is divisible by $x - k$ has a root at *k*. Synthetic division applied to a polynomial and a factor result in a zero for the remainder. This leads us to the Factor Theorem, which states A polynomial $f(x)$ has a factor $x - k$ if and only if $f(k) = 0$.

Solving cubic polynomials can be tricky business sometimes. A graphing utility can be a helpful tool to identify some roots, but in general there is no simple formula for solving cubic polynomials like the quadratic formula aids us in solving quadratics.

Direct student efforts toward developing approaches to solving polynomials. This idea can be more abstract than many have done.

There is however a tool that we can use for helping us to identify Rational Roots of the polynomial in question.

Many teachers have not used the rational root theorem very much. The opportunity here is to direct students to apply abstract concepts to a function. This task jumps right into a cubic, but it might be good to start with a quadratic that will demonstrate the use. Though the calculator is great tool, look to start this process without it. See what your students can do.

4. The Rational Root Theorem states that any rational solutions to a polynomial will be in the form of $\frac{p}{q}$ where p is a factor of the constant term of the polynomial (the term that does not show a variable) and q is a factor of the leading coefficient.
This is actually much simpler than it appears at first glance.

a. Let us consider the polynomial $f(x) = x^3 - 5x^2 - 4x + 20$

Identify p (all the factors of 20): ***1, -1, 20, -20, 2, -2, 10, -10, 4, -4, 5, -5,***

Identify q (all the factors of the lead coefficient, 1): ***1, -1***

Identify all possible combinations of $\frac{p}{q}$: ***1, -1, 20, -20, 2, -2, 10, -10, 4, -4, 5, -5***

If $f(x) = x^3 - 5x^2 - 4x + 20$ is going to factor, then one of these combinations is going to "work", that is, the polynomial will divide evenly. So the best thing to do is employ a little trial and error. Let's start with the smaller numbers, they will be easier to evaluate.
Substitute for x: 1, -1, 2, -2, 4, -4 …20, -20.

Don't ignore the fact that some polynomials can still be factored by grouping.

Why would substituting these values in for x be a useful strategy?

Simple evaluating of values in the function is helpful. Connecting that fact that f(x) = 0 is a definition of a solution of a function. f(x) not equal to zero is a counterexample.

Why do we not have to use synthetic division on every one?

This question is intended to be a lead in for the introduction of the Remainder Theorem.

Let us define what the Remainder Theorem states and how it helps us.

***Remainder Theorem:* An application of polynomial long division. It states that the remainder of a polynomial f(x) divided by a linear divisor x – a is equal to f(a).**

Hopefully, you did not get all the way to -20 before you found one that works. Actually, 2 should have worked. Once there is one value that works, we can go from there.
Use the factor $x-2$ to divide $f(x)$. This should yield:

$$f(x)=x^3-5x^2-4x+20=(x-2)(x^2-3x-10)$$

By factoring the result we can find all the factors: $f(x)=x^3-5x^2-4x+20=$ $(x-2)(x+2)(x-5)$

Therefore the roots are 2, -2, and 5.

What could be done if this portion was not factorable?

Direct students back to the quadratic formula.

5. Use the Quadratic Formula

For each of the following find each of the roots, classify them and show the factors.

a. $f(x) = x^3 - 5x^2 - 4x + 20$

Possible rational roots:

Solutions: ±1, ±2, ±4, ±5, ±10, ± 20

Show work for Synthetic Division and Quadratic Formula(or Factoring):

Solutions: Roots of x = -2, 2, 5

Complete Factorization: ________________________________

Solution: (x + 2)(x – 2)(x – 5)

Roots and Classification

__________	Rational	Irrational	Real	Imaginary
__________	Rational	Irrational	Real	Imaginary
__________	Rational	Irrational	Real	Imaginary

Solutions: x = -2, 2, 5 (All three roots are Rational and Real)

b. $f(x) = x^3 + 2x^2 - 5x - 6$

Possible rational roots:

Solutions: ±1, ±2, ±3, ±6

Show work for Synthetic Division and Quadratic Formula(or Factoring):

Solutions: Roots of x = -3, -1, 2

Complete Factorization: ________________________________

Solution: (x + 3)(x + 1)(x – 2)

Roots and Classification

__________	Rational	Irrational	Real	Imaginary
__________	Rational	Irrational	Real	Imaginary
__________	Rational	Irrational	Real	Imaginary

Solutions: x = -3, -1, 2 (All three roots are Rational and Real)

c. $f(x) = 4x^3 - 7x + 3$

Possible rational roots:

Solutions: ±1, ±3, ±1/4, ±3/4, ±1/2, ±3/2

Show work for Synthetic Division and Quadratic Formula(or Factoring):

Solutions: Roots of x = -3/2, ½, 1

Complete Factorization: ______________________________________

Solution: (2x + 3)(2x – 1)(x – 1)
Note: Students may have a different form of the polynomial above if they used a different root in their synthetic division. Ask students how they can check to make sure they are equivalent polynomials even though they look different (by multiplying the factorization out).

Roots and Classification

__________	Rational	Irrational	Real	Imaginary
__________	Rational	Irrational	Real	Imaginary
__________	Rational	Irrational	Real	Imaginary

Solutions: x = -3/2, 1/2, 1 (All three roots are Rational and Real)

6. What happens when we come to a function that is a 4th degree?

Solution: Answers will vary but most should agree there will be an additional root and therefore an additional factor.

Well, just like the cubic there is no formula to do the job for us, but by extending our strategies that we used on the third degree polynomials, we can tackle any quartic function.

1st Develop your possible roots using the $\frac{p}{q}$ method.

2nd Use synthetic division with your possible roots to find an actual root. If you started with a 4th degree, that makes the dividend a cubic polynomial.

3rd Continue the synthetic division trial process with the resulting cubic. Don't forget that roots can be used more than once.

4th Once you get to a quadratic, use factoring techniques or the quadratic formula to get to the other two roots.

For each of the following find each of the roots, classify them and show the factors.

a. $f(x) = x^4 + 2x^3 - 9x^2 - 2x + 8$

Possible rational roots:

Solutions: ±1, ±2, ±4, ±8

Show work for Synthetic Division and Quadratic Formula(or Factoring):

Solutions: Roots of x = -4, -1, 1, 2

Complete Factorization: ______________________________________

Solutions: (x + 4)(x + 1)(x – 1)(x – 2)

Roots and Classification

_________	Rational	Irrational	Real	Imaginary
_________	Rational	Irrational	Real	Imaginary
_________	Rational	Irrational	Real	Imaginary
_________	Rational	Irrational	Real	Imaginary

Solutions: x = -4, -1, 1, 2 (All four roots are Rational and Real)

b. $f(x) = x^4 - 11x^3 - 13x^2 + 11x + 12$

Possible rational roots:

Solutions: ±1, ±2, ±3, ±4, ±6, ±12

Show work for Synthetic Division and Quadratic Formula(or Factoring):

Solutions: Roots of x = -1, 1, 12
Note: The root of x = -1 is a double root and thus could be used twice in the synthetic division process. Additionally, this provides students the opportunity to review the characteristics of a double root graphically.

Complete Factorization: ______________________________________

Solution: (x + 1)(x + 1)(x – 1)(x – 12) = $(x + 1)^2$(x – 1)(x – 12)

Roots and Classification

__________	Rational	Irrational	Real	Imaginary
__________	Rational	Irrational	Real	Imaginary
__________	Rational	Irrational	Real	Imaginary
_ __________	Rational	Irrational	Real	Imaginary

Solutions: x = -1, 1, 12 (All roots are Rational and Real. In addition, x = -1 is a double root.)

c. $f(x) = x^5 - 12x^4 + 49x^3 - 90x^2 + 76x - 24$

Possible rational roots:

Solutions: ±1, ±2, ±3, ±4, ±6, ±8, ±12, ±24

Show work for Synthetic Division and Quadratic Formula(or Factoring):

Solutions: x = 1, 6, 2,
Again, this function yields double roots. Reinforcing this concept is essential.

Complete Factorization: ______________________________________

Factors: (x – 1)(x – 1)(x – 6)(x – 2)(x – 2)

Roots and Classification

__________	Rational	Irrational	Real	Imaginary
__________	Rational	Irrational	Real	Imaginary
__________	Rational	Irrational	Real	Imaginary
__________	Rational	Irrational	Real	Imaginary
__________	Rational	Irrational	Real	Imaginary

Solution: All five solutions are classified as real and rational.

d. $f(x) = x^5 - 5x^4 + 8x^3 - 8x^2 + 16x - 16$

Possible rational roots:

Solutions: ±1, ±2, ±4, ±8, ±16

Show work for Synthetic Division and Quadratic Formula(or Factoring):

Solutions: 2 and $\frac{-1 \pm i\sqrt{7}}{2}$

Complete Factorization: ______________________________________

Factored form: (x – 2)(x – 2)(x – 2)(x^2 + x + 2)
Discussions about a triple root should not be ignored here.

Roots and Classification

_________	Rational	Irrational	Real	Imaginary
_________	Rational	Irrational	Real	Imaginary
_________	Rational	Irrational	Real	Imaginary
_________	Rational	Irrational	Real	Imaginary
_________	Rational	Irrational	Real	Imaginary

Solutions: All three of the (x – 2)'s should be considered Real and Rational

$x = \dfrac{-1 \pm i\sqrt{7}}{2}$ ***represents two solutions that are to be considered Imaginary.***

Notes: These problems are an introduction to the process of using synthetic division to factor higher order polynomials and thus have simpler solutions. As such, additional problems with irrational and/or imaginary solutions should also be included in the problem sets when practicing with this method.

7. Let's consider a scenario where the roots are imaginary.

Suppose that you were asked to find the roots of $f(x) = x^4 - x^3 + 3x^2 - 4x - 4$.
There are only 6 possible roots: $\pm 1, \pm 2, \pm 4$. In the light of this fact, let's take a look at the graph of this function.

Direct students back to characteristics of imaginary solutions. This is now a good opportunity to alert students of how technology is not going to always be able to produce every solution. A function of this type would be a good tool to use in assessment of this standard. Encourage students to be aware of a variety of resources that they can go to. "Tools in the tool box"

It should be apparent that none of these possible solutions are roots of the function. And without a little help at this point we are absolutely stuck. None of the strategies we have discussed so far help us at this point.

a. But consider that we are given that one of the roots of the function is 2*i*. Because roots come in pairs (think for a minute about the quadratic formula); an additional root should be -2*i*. So, let's take these values and use them for synthetic division.

b. Though the values may not be very clean, this process should work just as it did earlier. Take a moment and apply what you have been doing to this function.

Complete Factorization: ___________________________________

Roots and Classification

_________	Rational	Irrational	Real	Imaginary
_________	Rational	Irrational	Real	Imaginary
_________	Rational	Irrational	Real	Imaginary
_________	Rational	Irrational	Real	Imaginary
_________	Rational	Irrational	Real	Imaginary

Factored form: (x – 2i)(x + 2i)($x^2 - x - 1$***)***

Solutions: x = 2i, -2i, and $\frac{-1 \pm \sqrt{5}}{2}$

Factors, Zeros, and Roots: Oh My!

Solving polynomials that have a degree greater than those solved in previous courses is going to require the use of skills that were developed when we previously solved quadratics. Let's begin by taking a look at some second degree polynomials and the strategies used to solve them. These equations have the form $ax^2 + bx + c = 0$, and when they are graphed the result is a parabola.

1. Factoring is used to solve quadratics of the form $ax^2 + bx + c = 0$ when the roots are rational. Find the roots of the following quadratic functions:

a. $f(x) = x^2 - 5x - 14$

b. $f(x) = x^2 - 64$

c. $f(x) = 6x^2 + 7x - 3$

d. $f(x) = 3x^2 + x - 2$

2. Another option for solving a quadratic whether it is factorable but particularly when it is not is to use the quadratic formula. Remember, a quadratic equation written in $ax^2 + bx + c = 0$ has solution(s) $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Also remember that $b^2 - 4ac$ is the discriminant and gives us the ability to determine the nature of the roots.

$$b^2 - 4ac \begin{cases} > 0 & 2 \quad real\, roots \\ = 0 & 1 \quad real\ root \\ < 0 & 0 \quad real\, roots\, (imaginary) \end{cases}$$

Find the roots for each of the following. Also, describe the number and nature of these roots.

a. $f(x) = 4x^2 - 2x + 9$

b. $f(x) = 3x^2 + 4x - 8$

c. $f(x) = x^2 - 5x + 9$

3. Let's take a look at the situation of a polynomial that is one degree greater. When the polynomial is a third degree, will there be any similarities when we solve?

Suppose we want to find the roots of $f(x) = x^3 + 2x^2 - 5x - 6$. By inspecting the graph of the function, we can see that one of the roots is distinctively 2. Since we know that x = 2 is a solution to $f(x)$, we also know that $x - 2$ is a factor of the expression $x^3 + 2x^2 - 5x - 6$. This means that if we divide $x^3 + 2x^2 - 5x - 6$ by $x - 2$ there will be a remainder of zero. Let's confirm this with synthetic substitution:

$$\begin{array}{r|rrrr} 2 & 1 & 2 & -5 & -6 \\ & & 2 & 8 & 6 \\ \hline & 1 & 4 & 3 & 0 \end{array}$$

Let's practice synthetic division before we tackle how to solve cubic polynomials in general. Do the following division problems synthetically.

a. $\dfrac{10x^3 - 17x^2 - 7x + 2}{x - 2}$

b. $\dfrac{x^3 + 3x^2 - 10x - 24}{x + 4}$

c. $\dfrac{x^3 - 7x - 6}{x + 1}$

The main thing to notice about solving cubic polynomials (or higher degree polynomials) is that a polynomial that is divisible by $x-k$ has a root at k. Synthetic division applied to a polynomial and a factor result in a zero for the remainder. This leads us to the Factor Theorem, which states: A polynomial $f(x)$ has a factor $x-k$ if and only if $f(k)=0$.

Solving cubic polynomials can be tricky business sometimes. A graphing utility can be a helpful tool to identify some roots, but in general there is no simple formula for solving cubic polynomials like the quadratic formula aids us in solving quadratics.

There is however a tool that we can use for helping us to identify Rational Roots of the polynomial in question.

4. The Rational Root Theorem states that any rational solutions to a polynomial will be in the form of $\frac{p}{q}$ where p is a factor of the constant term of the polynomial (the term that does not show a variable) and q is a

factor of the leading coefficient. This is actually much simpler than it appears at first glance.

a. Let us consider the polynomial $f(x)=x^3-5x^2-4x+20$

Identify p (all the factors of the constant term 20) = ______________________________

Identify q (all the factors of the leading coefficient 1) = ___________________________

Identify all possible combinations of $\frac{p}{q}$: ______________________________________

If $f(x)=x^3-5x^2-4x+20$ is going to factor, then one of these combinations is going to "work", that is, the polynomial will divide evenly. So the best thing to do is employ a little trial and error. Let's start with the smaller numbers, they will be easier to evaluate.
Substitute for x: 1, -1, 2, -2, 4, -4 …20, -20.

Why would substituting these values in for x be a useful strategy?

Why do we not have to use synthetic division on every one?

Define what the Remainder Theorem states and how it helps us.

Hopefully, you did not get all the way to -20 before you found one that works. Actually, 2 should have worked. Once there is one value that works, we can go from there.
Use the factor $(x-2)$ to divide $f(x)$. This should yield:

$$f(x) = x^3 - 5x^2 - 4x + 20 = (x-2)(x^2 - 3x - 10)$$

By factoring the result we can find all the factors: $f(x) = x^3 - 5x^2 - 4x + 20 = (x-2)(x+2)(x-5)$

Therefore the roots are 2, -2, and 5.

What could be done if this portion was not factorable?

5. Use the Quadratic Formula

For each of the following find each of the roots, classify them and show the factors.

a. $f(x) = x^3 - 5x^2 - 4x + 20$

Possible rational roots:

Show work for Synthetic Division and Quadratic Formula (or Factoring):

Complete Factorization: ____________________________________

Roots and Classification

_________	Rational	Irrational	Real	Imaginary
_________	Rational	Irrational	Real	Imaginary
_________	Rational	Irrational	Real	Imaginary

b. $f(x) = x^3 + 2x^2 - 5x - 6$

Possible rational roots:

Show work for Synthetic Division and Quadratic Formula (or Factoring):

Complete Factorization: ____________________________________

Roots and Classification

__________	Rational	Irrational	Real	Imaginary
__________	Rational	Irrational	Real	Imaginary
__________	Rational	Irrational	Real	Imaginary

c. $f(x) = 4x^3 - 7x + 3$

Possible rational roots:

Show work for Synthetic Division and Quadratic Formula (or Factoring):

Complete Factorization: ____________________________________

Roots and Classification

__________	Rational	Irrational	Real	Imaginary
__________	Rational	Irrational	Real	Imaginary
__________	Rational	Irrational	Real	Imaginary

6. What happens when we come to a function that is a 4th degree?

Well, just like the cubic there is no formula to do the job for us, but by extending our strategies that we used on the third degree polynomials, we can tackle any quartic function.

1st Develop your possible roots using the $\frac{p}{q}$ method.

2nd Use synthetic division with your possible roots to find an actual root. If you started with a 4th degree, that makes the dividend a cubic polynomial.

3rd Continue the synthetic division trial process with the resulting cubic. Don't forget that roots can be used more than once.

4th Once you get to a quadratic, use factoring techniques or the quadratic formula to get to the other two roots.

For each of the following find each of the roots, classify them and show the factors.

a. $f(x) = x^4 + 2x^3 - 9x^2 - 2x + 8$

Possible rational roots:

Show work for Synthetic Division and Quadratic Formula (or Factoring):

Complete Factorization: ____________________________________

Roots and Classification

_________	Rational	Irrational	Real	Imaginary
_________	Rational	Irrational	Real	Imaginary
_________	Rational	Irrational	Real	Imaginary
_________	Rational	Irrational	Real	Imaginary

b. $f(x) = x^4 - 11x^3 - 13x^2 + 11x + 12$

Possible rational roots:

Show work for Synthetic Division and Quadratic Formula(or Factoring):

Complete Factorization: ____________________________________

Roots and Classification

__________	Rational	Irrational	Real	Imaginary
__________	Rational	Irrational	Real	Imaginary
__________	Rational	Irrational	Real	Imaginary
__________	Rational	Irrational	Real	Imaginary

c. $f(x) = x^5 - 12x^4 + 49x^3 - 90x^2 + 76x - 24$

Possible rational roots:

Show work for Synthetic Division and Quadratic Formula (or Factoring):

Complete Factorization: ____________________________________

Roots and Classification

__________	Rational	Irrational	Real	Imaginary
__________	Rational	Irrational	Real	Imaginary
__________	Rational	Irrational	Real	Imaginary
__________	Rational	Irrational	Real	Imaginary
__________	Rational	Irrational	Real	Imaginary

d. $f(x) = x^5 - 5x^4 + 8x^3 - 8x^2 + 16x - 16$

Possible rational roots:

Show work for Synthetic Division and Quadratic Formula (or Factoring):

Complete Factorization: ______________________________________

Roots and Classification

__________	Rational	Irrational	Real	Imaginary
__________	Rational	Irrational	Real	Imaginary
__________	Rational	Irrational	Real	Imaginary
__________	Rational	Irrational	Real	Imaginary
__________	Rational	Irrational	Real	Imaginary

7. Let's consider a scenario where the roots are imaginary.

Suppose that you were asked to find the roots of $f(x) = x^4 - x^3 + 3x^2 - 4x - 4$.

There are only 6 possible roots: $\pm 1, \pm 2, \pm 4$. In the light of this fact, let's take a look at the graph of this function.

It should be apparent that none of these possible solutions are roots of the function. And without a little help at this point we are absolutely stuck. None of the strategies we have discussed so far help us at this point.

a. But consider that we are given that one of the roots of the function is 2*i*. Because roots come in pairs (think for a minute about the quadratic formula); an additional root should be -2*i*. So, let's take these values and use them for synthetic division.

b. Though the values may not be very clean, this process should work just as it did earlier. Take a moment and apply what you have been doing to this function.

Complete Factorization: ______________________________________

Roots and Classification

__________	Rational	Irrational	Real	Imaginary
__________	Rational	Irrational	Real	Imaginary
__________	Rational	Irrational	Real	Imaginary
__________	Rational	Irrational	Real	Imaginary
__________	Rational	Irrational	Real	Imaginary

Representing Polynomials

Back to Task Table

This lesson connects what students learned previously using transformations and graphing quadratic equations in previous courses with graphing polynomials in this course. The emphasis is on pencil and paper graphs, not using graphing technology. (The next task will utilize technology to delve further into graphs of polynomial functions.)
Time needed: 20 minutes before the lesson for the assessment task, an 80-minute lesson (or two 40-minute lessons), and 20 minutes in a follow up lesson (or for homework). All timings are approximate, depending on the needs of your students.

Formative Assessment Lesson: Representing Polynomials

Source: Formative Assessment Lesson Materials from Mathematics Assessment Project
http://www.map.mathshell.org/materials/lessons.php?taskid=436&subpage=concept

ESSENTIAL QUESTIONS:

- What is the relationship between graphs and algebraic representations of polynomials?
- What is the connection between the zeros of polynomials, when suitable factorizations are available, and graphs of the functions defined by polynomials?
- What is the connection between transformations of the graphs and transformations of the functions obtained by replacing $f(x)$ by $f(x + \text{k})$, $f(x) + \text{k}$, $-f(x)$, $f(-x)$?

TASK COMMENTS:

Tasks and lessons from the Mathematics Assessment Project are specifically designed to help teachers effectively formatively assess their students. The way the tasks and lessons are designed gives the teacher a clear understanding of what the students are able to do and not do. Within the lesson, teachers will find suggestions and question prompts that will help guide students towards understanding. For more information access the MAP website:
http://www.map.mathshell.org/materials/background.php?subpage=formative

The task, *Representing Polynomials*, is a Formative Assessment Lesson that can be found at the website: http://www.map.mathshell.org/materials/lessons.php?taskid=436&subpage=concept
This document provides a clear lesson design, from the opening of the lesson to the closing of the lesson.

The PDF version of the task can be found at the link below:
http://www.map.mathshell.org/materials/download.php?fileid=1271

STANDARDS ADDRESSED IN THIS TASK:

Interpret the structure of expressions.

MGSE9-12.A.SSE.1 Interpret expressions that represent a quantity in terms of its context.

MGSE9-12.A.SSE.1a Interpret parts of an expression, such as terms, factors, and coefficients, in context.

MGSE9-12.A.SSE.1b Given situations which utilize formulas or expressions with multiple terms and/or factors, interpret the meaning (in context) of individual terms or factors.

MGSE9-12.A.SSE.2 Use the structure of an expression to rewrite it in different equivalent forms. For example, see $x^4 – y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 – y^2) (x^2 + y^2)$.

Understand the relationship between zeros and factors of polynomials.

MGSE9-12.A.APR.3 Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial.

Analyze functions using different representations.

MGSE9-12.F.IF.7 Graph functions expressed algebraically and show key features of the graph both by hand and by using technology. *(Limit to polynomial functions)*

MGSE9-12.F.IF.7c Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior.

Build new functions from existing functions

MGSE9-12.F.BF.3 Identify the effect on the graph of replacing f(x) by f(x) + k, k f(x), f(kx), and f(x + k) for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them.

Standards for Mathematical Practice

This lesson uses all of the practices with emphasis on:

1. Reason abstractly and quantitatively.
2. Look for and make use of structure.

The Canoe Trip

Back to Task Table

Source: Illustrative Mathematics
https://www.illustrativemathematics.org/content-standards/HSF/IF/B/4/tasks/386
Note: The task represented here is one of many that Illustrative Mathematics offers addressing the F.IF strand (Interpreting and Analyzing functions). See all the tasks for this strand at https://www.illustrativemathematics.org/content-standards/HSF/IF/B/4

Mathematical Goals

- Students will interpret rates and distances in terms of an application and express them algebraically to answer questions.

Essential Questions

- What type of function is the best model?

TASK COMMENTS

This task adds to the classic "boat in a river" problem and has students consider the utility of the model generated by the conditions given. For some values of the funtion's domain, the application does not have a reasonable interpretation. Illustrative Mathematics follows up this task with a second variation of the same problem.

The task, *The Canoe Trip, Variation 1* is a learning task that can be found at the website: https://www.illustrativemathematics.org/content-standards/HSF/IF/B/4/tasks/386

GEORGIA STANDARDS OF EXCELLENCE

Interpret functions that arise in applications in terms of the context

MGSE9-12.F.IF.4 Using tables, graphs, and verbal descriptions, interpret the key characteristics of a function which models the relationship between two quantities. Sketch a graph showing key features including: intercepts; interval where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; ~~and periodicity~~.

STANDARDS FOR MATHEMATICAL PRACTICE

This task uses all of the practices with emphasis on:

1. **Make sense of problems and persevere in solving them**
3. **Construct viable arguments and critique the reasoning of others.**
7. **Look for and make use of structure.**

Grouping

- Individual or Partner

Time Needed

- 20-30 minutes

Trina's Triangles

Back to Task Table

Source: Illustrative Mathematics
https://www.illustrativemathematics.org/content-standards/HSA/APR/C/4/tasks/594

Mathematical Goals

- Students will use a polynomial equation (identity) to explore the relationship between sets of numbers

Essential Questions

- How can the lengths of the legs of a right triangle be generated?

TASK COMMENTS

Students will investigate the validity of the following statement in this task: "Pick any two integers. Look at the sum of their squares, the difference of their squares, and twice the product of the two integers you chose. Those three numbers are the sides of a right triangle." Students are to investigate the statement and express it algebraically.

The task, *Trina's Triangles* is a Performance task that can be found at the website:
https://www.illustrativemathematics.org/content-standards/HSA/APR/C/4/tasks/594

GEORGIA STANDARDS OF EXCELLENCE

Build a function that models a relationship between two quantities

MGSE9-12.F.BF.1 Write a function that describes a relationship between two quantities.

> **MGSE9-12.F.BF.1b** Combine standard function types using arithmetic operations in contextual situations (Adding, subtracting, and multiplying functions of different types).

MGSE9-12.A.APR.4 Prove polynomial identities and use them to describe numerical relationships. *For example, the polynomial identity* $(x^2 + y^2)^2 = (x^2 - y^2)^2 + (2xy)^2$ *can be used to generate Pythagorean triples.*

STANDARDS FOR MATHEMATICAL PRACTICE

This task uses all of the practices with emphasis on:

2. **Make sense of problems and persevere in solving them**
3. **Construct viable arguments and critique the reasoning of others.**
7. **Look for and make use of structure.**

Grouping

- Individual or Partner

Time Needed

- 20-30 minutes

Polynomials Patterns Task

Back to Task Table

Mathematical Goals

- Roughly sketch the graphs of simple polynomial functions by hand
- Graph polynomial functions using technology
- Identify key features of the graphs of polynomial functions, including but not limited to, zeros, turning points, and end behaviors
- Explain why the x-coordinate of the intersection of the line $f(x)=0$ and the polynomial $g(x)=a_n x^n + a_{n-1}x^{n-1} + ...a_2 x^2 + a_1 x + a_0$ are solutions to the equation $f(x)=g(x)$

Georgia Standards of Excellence

Interpret the structure of expressions.

MGSE9-12.A.SSE.1 Interpret expressions that represent a quantity in terms of its context.

MGSE9-12.A.SSE.1a Interpret parts of an expression, such as terms, factors, and coefficients, in context.

MGSE9-12.A.SSE.1b Given situations which utilize formulas or expressions with multiple terms and/or factors, interpret the meaning (in context) of individual terms or factors.

MGSE9-12.A.SSE.2 Use the structure of an expression to rewrite it in different equivalent forms. For example, see $x^4 – y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 – y^2)(x^2 + y^2)$.

Use complex numbers in polynomial identities and equations.

MGSE9-12.N.CN.9 Use the Fundamental Theorem of Algebra to find all roots of a polynomial equation.

Represent and solve equations and inequalities graphically.

MGSE9-12.A.REI.11 Using graphs, tables, or successive approximations, show that the solution to the equation f(x) = g(x) is the x-value where the y-values of f(x) and g(x) are the same.

Solve systems of equations.

MGSE9-12.A.REI.7 Solve a simple system consisting of a linear equation and a quadratic polynomial equation in two variables algebraically and graphically.

Understand the relationship between zeros and factors of polynomials.

MGSE9-12.A.APR.2 Know and apply the Remainder Theorem: For a polynomial p(x) and a number a, the remainder on division by x – a is p(a), so p(a) = 0 if and only if (x – a) is a factor of p(x).

MGSE9-12.A.APR.3 Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial.

Analyze functions using different representations.

MGSE9-12.F.IF.7 Graph functions expressed algebraically and show key features of the graph both by hand and by using technology. *(Limit to polynomial functions)*

MGSE9-12.F.IF.7c Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior.

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Materials

- Pencil
- Handout
- Graphing Calculator
- Graph Paper

Notes on Polynomial Patterns Learning Task

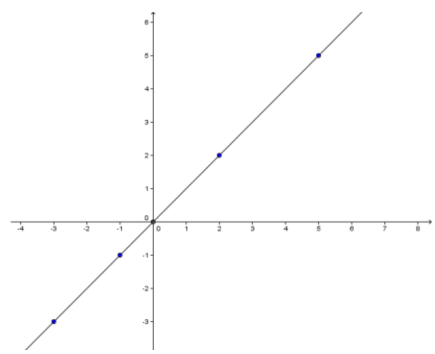
A systematic exploration of patterns found in the behavior of polynomial functions is offered in this task. The task is not set in a context, but it is set up in the spirit of mathematics as a science of patterns. Instead of telling students all of the characteristics of polynomial functions, this task is crafted to allow students to make conjectures using provided examples and then proceed to test them using other examples and their calculators. An understanding of characteristics of polynomial functions that students will gain in this task will be needed and further developed throughout the remainder of the unit in various contexts. This set of explorations can be broken apart and used at various other times throughout the unit if you prefer to do mini-investigations as the characteristics come up throughout the unit.

Polynomials Patterns Task

1. To get an idea of what polynomial functions look like, we can graph the first through fifth degree polynomials with leading coefficients of 1. For each polynomial function, make a table of 6 points and then plot them so that you can determine the shape of the graph. Choose points that are both positive and negative so that you can get a good idea of the shape of the graph. Also, include the x intercept as one of your points.

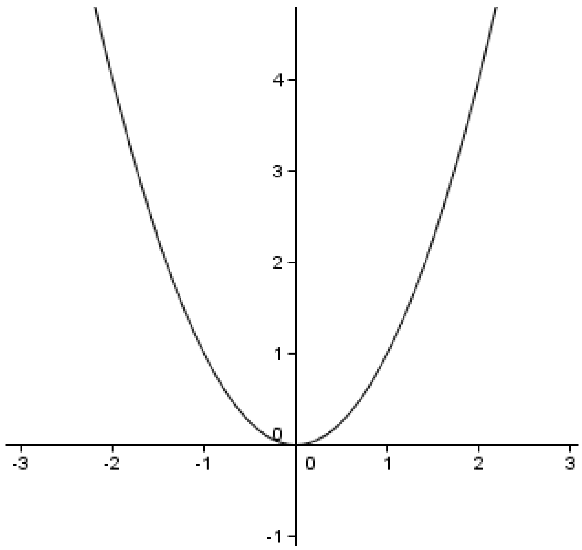
a. For example, for the first order polynomial function: $y = x^1$. You may have the following table and graph:

x	**y**
-3	-3
-1	-1
0	0
2	2
5	5
10	10

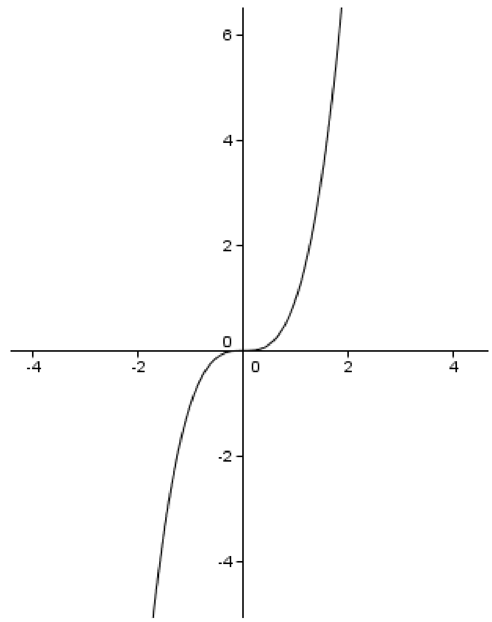


Solution:

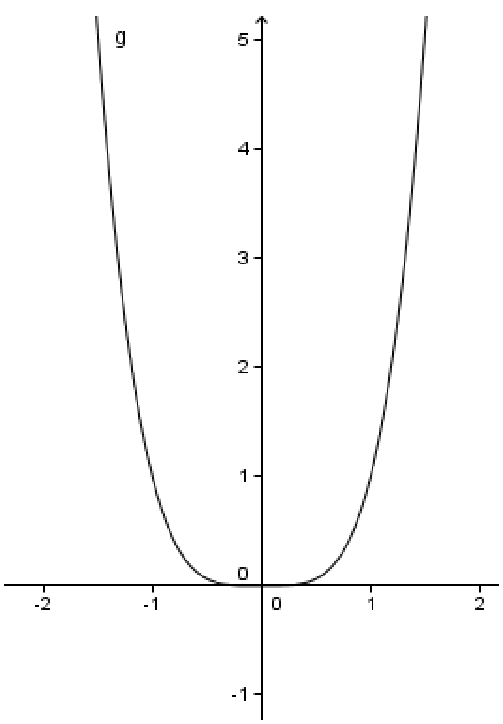
$b.\ \ y = x^2$



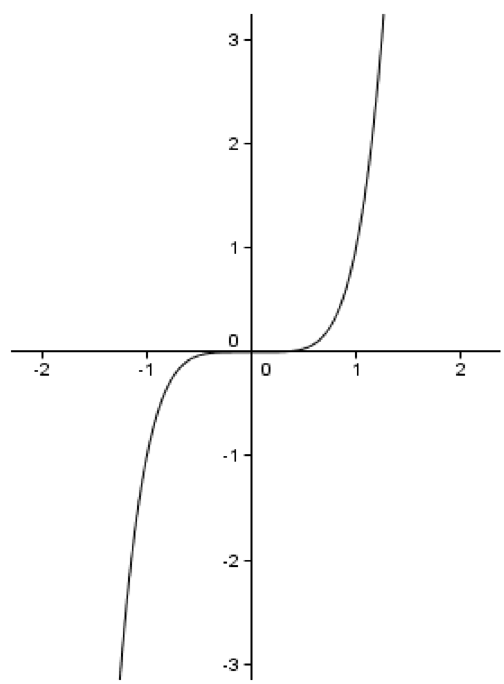
$c.\ \ y = x^3$



$d.\ \ y = x^4$



e. $y = x^5$



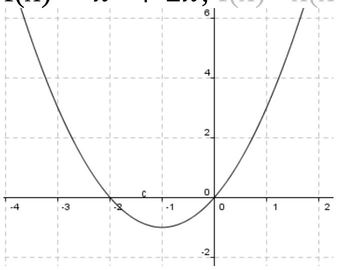
f. Compare these five graphs. By looking at the graphs, describe in your own words how $y = x^2$ is different from $y = x^4$. Also, how is $y = x^3$ different from $y = x^5$?

g. Note any other observations you make when you compare these graphs.

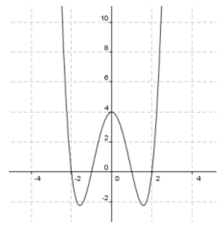
2. In order to examine their characteristics in detail so that we can find the patterns that arise in the behavior of polynomial functions, we can study some examples of polynomial functions and their graphs. Here are 8 polynomial functions and their accompanying graphs that we will use to refer back to throughout the task.

Handout of Graphs of Polynomial Functions :

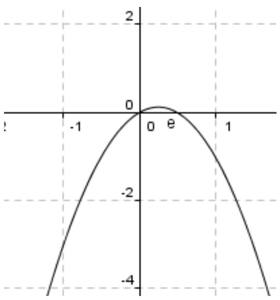
f(x) = $x^2 + 2x$; f(x)= x(x+2)



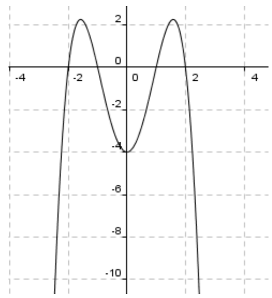
k(x) = $x^4 - 5x^2$+4 ; k(x) = (x-1)(x+1)(x-2)(x+2)



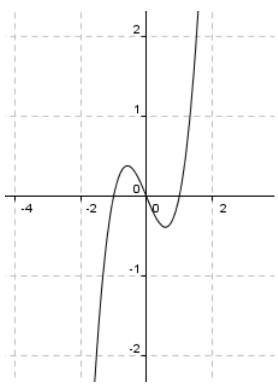
g(x) = $-2x^2 + x$; g(x)=x(-2x+1)



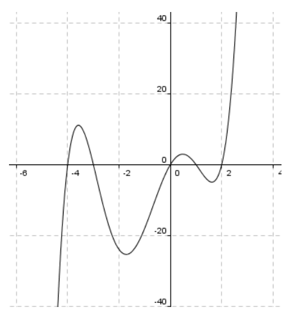
l(x) = $-(x^4 - 5x^2$+4) ;
l(x) = -(x-1)(x+1)(x-2)(x+2)



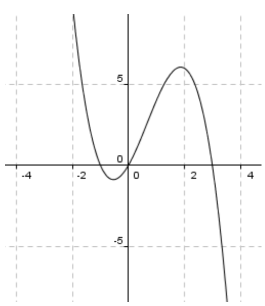
h(x) = $x^3 - x$; h(x) = $x(x-1)(x+1)$



m(x) = $\frac{1}{2}(x^5 + 4x^4 - 7x^3 - 22x^2 + 24x)$;
m(x)= $\frac{1}{2}$x(x-1)(x-2)(x+3)(x+4)

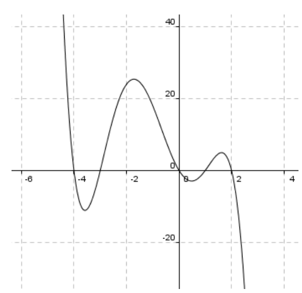


j(x)=$-x^3 + 2x^2 + 3x$; j(x) = -x(x-3)(x+1)



n(x) = $-\frac{1}{2}(x^5 + 4x^4 - 7x^3 - 22x^2 + 24x)$;
n(x)=- $\frac{1}{2}$x(x-1)(x-2)(x+3)(x+4)

Each of these equations can be re-expressed as a product of linear factors by factoring the equations, as shown below in the gray equations.



Note: Discuss the idea with students that every polynomial P(x) of degree n (n>0) can be written as the product of a constant k (k≠0) and n linear factors:

$P(x) = k(x - r_1)(x - r_2)(x - r_3)\ldots(x - r_n)$. ***Thus a polynomial equation of degree n has exactly n complex roots, namely, $r_1, r_2, r_3, \ldots, r_n$. Students will use this understanding in more depth in a later unit.***

a. List the x-intercepts of j(x) using the graph above. How are these intercepts related to the linear factors in gray?

The x-intercepts are (0,0), (3,0) and (-1,0). We can see this graphically, or we know that at x-intercepts, the value of the function is zero. We can find the x-values when j(x) equals zero using the equations: –x = 0, (x-3) = 0, and (x+1) = 0.

b. Why might it be useful to know the linear factors of a polynomial function?

You can find the zeros of the function, which are the x-intercepts. Knowing these will help you graph the function, and zeros are real solutions to the polynomial equations.

c. Although we will not factor higher order polynomial functions in this unit, you have factored quadratic functions in a previous course. For review, factor the following second degree polynomials, or quadratics.

$y = x^2 - x - 12$

$y = x^2 + 5x - 6$

$y = 2x^2 - 6x - 10$

d. Using these factors, find the roots of these three equations.

e. Sketch a graph of the three quadratic equations above without using your calculator and then use your calculator to check your graphs.

f. You can factor some polynomial equations and find their roots in a similar way.

Try this one: . $y = x^5 + x^4 - 2x^3$.

This polynomial factors into $y = x^3(x-1)(x+2)$ or if you want to show the 5 linear factors, you can write it as $y = x\,x\,x\,(x-1)(x+2)$.

What are the roots of this fifth order polynomial function?

The roots are 0, 1, and -2.

g. How many roots are there? ***3***

Why are there not five roots since this is a fifth degree polynomial? ***There are three x = 0 equations to solve and each one yields the same root, 0. This root is repeated three times.***

Note: Later the idea of multiplicity will be examined, but this example introduces the idea that even though an nth order polynomial function can be expressed as n linear factors there are not always n real roots; n, rather is the maximum number of real roots an nth order polynomial function can have. This should be discussed with students.

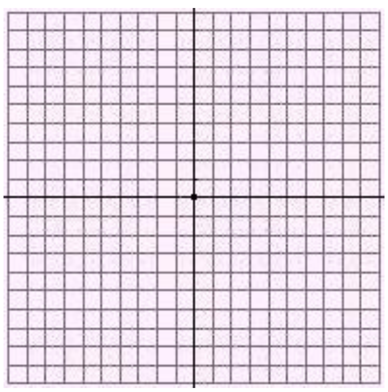
h. Check the roots by generating a graph of this equation using your calculator.

i. For other polynomial functions, we will not be able to draw upon our knowledge of factoring quadratic functions to find zeroes. For example, you may not be able to factor $= x^3 + 8x^2 + 5x - 14$, but can you still find its zeros by graphing it in your calculator? How?

Write are the zeros of this polynomial function.
You can find the x-intercepts if you graph the function. The zeros for this function are -1, 2, and -7.

3. Symmetry
The first characteristic of these 8 polynomials functions we will consider is symmetry.

a. Sketch a function you have seen before that has symmetry about the y-axis.

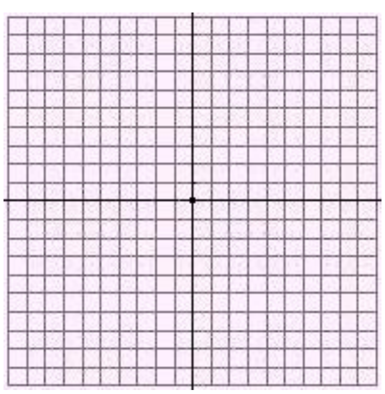


Describe in your own words what it means to have symmetry about the y-axis.

Answers may vary. Possible solution: the graph does not change if you reflect it about the y-axis.

What is do we call a function that has symmetry about the y-axis?

Even function



b. Sketch a function you have seen before that has symmetry about the origin.

Describe in your own words what it means to have symmetry about the origin.

If you rotate the graph 180 degrees around the origin, the graph does not change.

What do we call a function that has symmetry about the origin?

Odd Function

c. Using the table below and your handout of the following eight polynomial functions, classify the functions by their symmetry.

Function	Symmetry about the y axis?	Symmetry about the origin?	Even, Odd, or Neither?
f(x) = $x^2 + 2x$			***Neither***
g(x) = $-2x^2 + x$			***Neither***
h(x) = $x^3 - x$		*yes*	***Odd***
j(x)=$-x^3 + 2x^2 + 3x$			***Neither***
k(x) = $x^4 - 5x^2$+4	*yes*		***Even***
l(x) = $-(x^4 - 5x^2$+4)	*yes*		***Even***
m(x) = $\frac{1}{2}(x^5 + 4x^4 - 7x^3 - 22x^2 + 24x)$			***Neither***
n(x) = $-\frac{1}{2}(x^5 + 4x^4 - 7x^3 - 22x^2 + 24x)$			***Neither***

d. Now, sketch your own higher order polynomial function (an equation is not needed) with symmetry about the y-axis.

Answers will vary.

e. Now, sketch your own higher order polynomial function with symmetry about the origin.

Answers will vary.

f. Using these examples from the handout and the graphs of the first through fifth degree polynomials you made, why do you think an odd function may be called an odd function? Why are even functions called even functions? ***Only polynomials with an even degree can possibly be even functions. Only polynomials with an odd degree can possibly be odd functions.***

f. Why don't we talk about functions that have symmetry about the x-axis? Sketch a graph that has symmetry about the x-axis. What do you notice? ***It is not a function because it does not pass the vertical line test. That is, there exists more than one y-value for unique x-values.***

4. Domain and Range
Another characteristic of functions that you have studied is domain and range. For each polynomial function, determine the domain and range.

Function	Domain	Range
f(x) = $x^2 + 2x$		
g(x) = $-2x^2 + x$		
h(x) = $x^3 - x$		
j(x)=$-x^3 + 2x^2 + 3x$		
k(x) = $x^4 - 5x^2 + 4$		
l(x) = $-(x^4 - 5x^2 + 4)$		
m(x) = $\frac{1}{2}(x^5 + 4x^4 - 7x^3 - 22x^2 + 24x)$		
n(x) = $-\frac{1}{2}(x^5 + 4x^4 - 7x^3 - 22x^2 + 24x)$		

5. Zeros

a. We can also describe the functions by determining some points on the functions. We can find the x-intercepts for each function as we discussed before. Under the column labeled "x-intercepts" write the ordered pairs (x,y) of each intercept and record the number of intercepts in the next column. Also record the degree of the polynomial.

Function	Degree	X-intercepts	Zeros	Number of Zeros
$f(x) = x^2 + 2x$				
$g(x) = -2x^2 + x$				
$h(x) = x^3 - x$				
$j(x) = -x^3 + 2x^2 + 3x$				
$k(x) = x^4 - 5x^2 + 4$				
$l(x) = -(x^4 - 5x^2 + 4)$				
$m(x) = \frac{1}{2}(x^5 + 4x^4 - 7x^3 - 22x^2 + 24x)$				
$n(x) = -\frac{1}{2}(x^5 + 4x^4 - 7x^3 - 22x^2 + 24x)$				

b. These x-intercepts are called the *zeros* of the polynomial functions. Why do you think they have this name?

The x-intercept is a point where the function equals zero.

<u>Note:</u>
Discuss with students that x-intercepts are zeros. If f(x) is a polynomial function, then the values of x for which f(x)=0 are called the zeros of the function. Graphically these are the x intercepts. Numbers that are zeros of a polynomial function are also solutions to polynomial equations. Solutions to polynomial equations are called roots. Roots or zeros are complex numbers (either imaginary or real).

There are two reasons why the number of x-intercepts does not always equal the degree of the polynomial function. First, all zeros that are real numbers are x-intercepts, but zeros can also be complex numbers that are not represented on the x-axis (containing only real numbers). Also, there may be "two" zeros at the same x-intercept. For example, if you graph the function f(x) = (x-3)(x-3). This is a second degree polynomial function a real zero of 3 with a multiplicity of two.

Note: Students found complex zeros of polynomial functions previously; this task concentrates on the graphs of polynomial functions, but in order to sketch graphs to have a general sense of how they look, it is helpful to understand the real zeros of a polynomial function, so that idea in addressed in this unit.

c. Fill in the column labeled "Zeroes" by writing the zeroes that correspond to the x-intercepts of each polynomial function, and also record the number of zeroes each function has.

Function	Degree	X-intercepts	Zeroes	Number of Zeroes
f(x) = $x^2 + 2x$	*2*	*(0,0);(-2,0)*	*0;-2*	*2*
g(x) = $-2x^2 + x$	*2*	*(0,0);(1/2,0)*	*0; ½*	*2*
h(x) = $x^3 - x$	*3*	*(0,0);(1,0);(-1,0)*	*0; 1; -1*	*3*
j(x)=$-x^3 + 2x^2 + 3x$	*3*	*(0,0);(3,0);(-1,0)*	*0; 3; -1*	*3*
k(x) = $x^4 - 5x^2 + 4$	*4*	*(1,0);(-1,0);(2,0);(-2,0)*	*1;-1;2;-2*	*4*
l(x) = $-(x^4 - 5x^2 + 4)$	*4*	*(1,0);(-1,0);(2,0);(-2,0)*	*1;-1;2;-2*	*4*
m(x) = $\frac{1}{2}(x^5 + 4x^4 - 7x^3 - 22x^2 + 24x)$	*5*	*(0,0);(1,0);(2,0);(-3,0);(-4,0)*	*0;1;2;-3;-4*	*5*
n(x) = $-\frac{1}{2}(x^5 + 4x^4 - 7x^3 - 22x^2 + 24x)$	*5*	*(0,0);(1,0);(2,0);(-3,0);(-4,0)*	*0;1;2;-3;-4*	*5*

d. Make a conjecture about the relationship of degree of the polynomial and number of zeroes.

Students will probably conjecture that the degree of the polynomial is equal to the number of zeroes.

e. Test your conjecture by graphing the following polynomial functions using your calculator: $y = x^2$, $y = x^2(x-1)(x+4)$, $y = x(x-1)^2$.

Function	Degree	X-Intercepts	Zeroes	Number of Zeroes
$y = x^2$	*2*	(0,0)	*0*	*1*
$y = x^2(x-1)(x+4)$	*4*	(0,0); (0,-1);(0-4)	*0; 1; -4*	*3*
$y = x(x-1)^2$	*3*	*(0,0); (0, 1)*	*0; 1*	*2*

How are these functions different from the functions in the table?

Students should recognize that in these functions a zero appears more than once, so the number of x-intercepts is fewer. Here the idea of multiplicity of roots can be discussed with these examples. It may be useful to return to this discussion and look at some graphs where the multiplicity of some roots are 2, 3, 4, and 5 so students can see that zeros with even multiplicity behave differently (do not cross x axis) from roots with odd multiplicity (flatten out graph at that point).

Now amend your conjecture about the relationship of the degree of the polynomial and the number of x-intercepts.

The degree is equal to the maximum possible number of x-intercepts a given polynomial function can have. They could have fewer, as in the above examples, but never more. These 8

graphs are examples of polynomial functions whose graphs have the maximum number of x-intercepts a polynomial of that degree can have.

Make a conjecture for the maximum number of x-intercepts the following polynomial function will have: $p(x) = 2x^{11} + 4x^6 - 3x^2$

Eleven.

6. End Behavior

In determining the range of the polynomial functions, you had to consider the *end behavior* of the functions: the value of $f(x)$ as x approaches infinity and negative infinity.

Polynomials exhibit patterns of end behavior that are helpful in sketching polynomial functions.

a. Graph the following on your calculator. Make a rough sketch next to each one and answer the following:

- Is the **degree** even or odd?
- Is the leading coefficient, the coefficient on the term of highest degree, positive or negative?
- Does the graph rise or fall on the left? On the right?

1. $y = x$
2. $y = x^2$
3. $y = -3x$
4. $y = 5x^4$
5. $y = x^3$
6. $y = 2x^5$
7. $y = -x^2$
8. $y = -3x^4$
9. $y = -x^3$
10. $y = -2x^5$
11. $y = -3x^6$
12. $y = 7x^3$

b. Write a conjecture about the **end behavior**, whether it rises or falls at the ends, of a function of the form $f(x) = ax^n$ for each pair of conditions below. Then test your conjectures on some of the 8 polynomial functions graphed on your handout.

Condition a: When *n* is even and *a* > 0,

The end behavior is the same for both ends. It increases towards infinity.

Condition b: When *n* is even and *a* < 0,

The end behavior is the same for both ends. It decreases towards negative infinity.

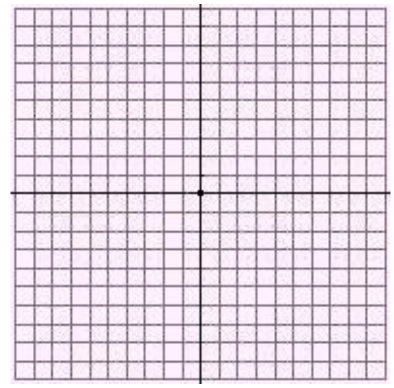
Condition c: When *n* is odd and *a* > 0,

The end behavior is different for each end. It starts from negative infinity and goes up towards positive infinity.

Condition d: When *n* is odd and *a* < 0,

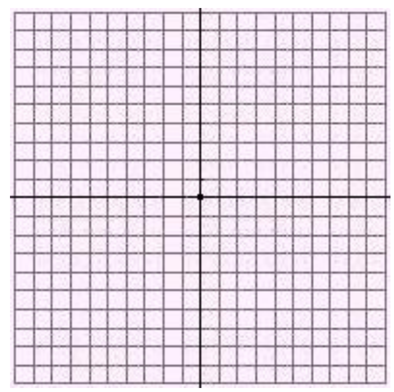
The end behavior is different for each end. It starts from positive infinity and goes down towards negative infinity.

c. Based on your conjectures in part (b), sketch a fourth degree polynomial function with a negative leading coefficient.



Answers will vary.

d. Now sketch a fifth degree polynomial with a positive leading coefficient.



Answers will vary.

Note we can sketch the graph with the end behavior even though we cannot determine where and how the graph behaves otherwise without an equation or without the zeros.

e. If we are given the real zeros of a polynomial function, we can combine what we know about end behavior to make a rough sketch of the function.

Sketch the graph of the following functions using what you know about end behavior and zeros:

f. $f(x) = (x - 2)(x - 3)$

g. $g.\ f(x) = -x(x - 1)(x + 5)(x - 7)$

7. Critical Points Other points of interest in sketching the graph of a polynomial function are the points where the graph begins or ends increasing or decreasing. Recall what it means for a point of a function to be an *absolute minimum* or an *absolute maximum*.

a. Which of the twelve graphs from part 6a have an absolute maximum? ***7, 8, 11***

b. Which have an absolute minimum? ***2, 4***

c. What do you notice about the degree of these functions?

They are even degree polynomial functions.

d. Can you ever have an absolute maximum AND an absolute minimum in the same function? If so, sketch a graph with both. If not, why not?

For polynomial functions, absolute maximum/minimum values do not exist for odd degree polynomial functions because the end behaviors are opposite, one extends to infinity and the other to negative infinity so the highest and lowest points over the domain are not defined but rather reach to infinity. For even degree polynomial functions the end behavior is the same, both extending to infinity OR negative infinity, so an even degree polynomial will have an absolute maximum or an absolute minimum, but not both.

e. For each of the following graphs from the handout, locate the turning points and the related intervals of increase and decrease, as you have determined previously for linear and quadratic polynomial functions.

Then record which turning points are *relative minimum* (the lowest point on a given portion of the graph) and *relative maximum* (the highest point on a given portion of the graph) values.

<u>Note:</u>
Students have previously not determined relative minimum and maximum values, so relative extrema will need to be distinguished from absolute extrema through discussion and examples.

Function	Degree	Turning Points	Intervals of Increase	Intervals of Decrease	Relative Minimum	Relative Maximum
f(x)	***2***	***(-1,-1)***	***(-1,inf)***	***(-inf,-1)***	***(-1,-1)***	
h(x)	***3***	***(.58,-.38)*** ***(-.58,.38)***	***(-inf, -.58)*** ***(.58, inf)***	***(-.58, .58)***	***(.58, -.38)***	***(-.58, .38)***
k(x)	***4***	***(0,4)*** ***(-1.6, -2.3)*** ***(1.6, -2.3)***	***(-1.6, 0)*** ***(1.6, inf)***	***(-inf, -1.6)*** ***(0, 1.6)***	***(-1.6, -2.3)*** ***(1.6, -2.3)***	***(0,4)***
n(x)	***5***	***(-3.6, -11.1)*** ***(-1.7, 25.4)*** ***(0.5, -3)*** ***(1.6, 5)***	***(-3.6, -1.7)*** ***(0.5, 1.6)***	***(-inf, -3.6)*** ***(-1.7 ,0.5)*** ***(1.6, +inf)***	***(0.5, -3)*** ***(-3.6, -11.1)***	***(-1.7, 25.4)*** ***(1.6, 5)***

f. Make a conjecture about the relationship of the degree of the polynomial and the number of turning points that the polynomial has. Recall that this is the maximum number of turning points a polynomial of this degree can have because these graphs are examples in which all zeros have a multiplicity of one.

The most turning points you can have is (n-1), where n is the degree of the polynomial.

g. Sometimes points that are relative minimums or maximums are also absolute minimums or absolute maximum. Are any of the relative extrema in your table also absolute extrema?

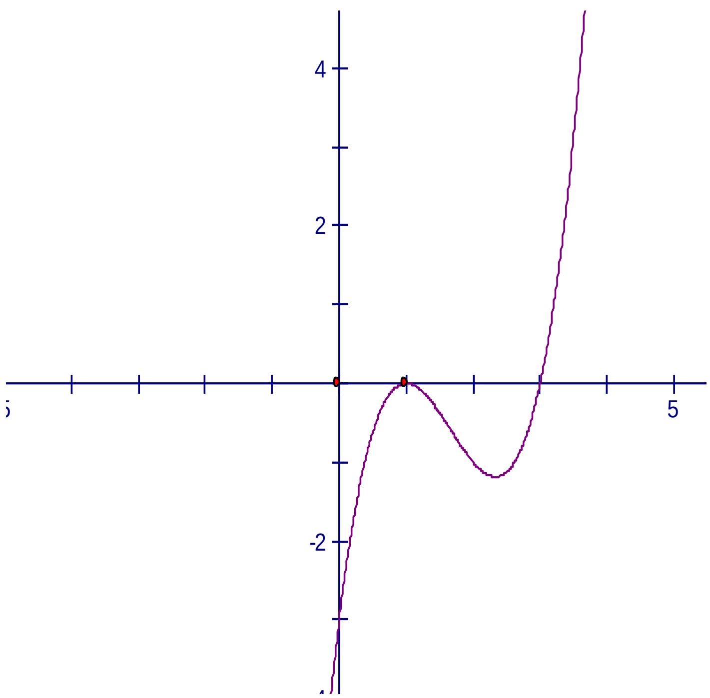
Yes, for f(x), (-1,.-1) is also an absolute minimum.

8. Putting it all Together:

Now that you have explored various characteristics of polynomial functions, you will be able to describe and sketch graphs of polynomial functions when you are given their equations.

a. If I give you the function: $f(x) = (x-3)(x-1)^2$ then what can you tell me about the graph of this function? Make a sketch of the graph of this function, describe its end behavior, and locate its critical point and zeroes.

Solution:
We know the function has zeroes at (1,0) and (3,0). Because (1,0) has a multiplicity of 2 (even multiplicity), it does not cross the x-axis at that point. It does cross the x-axis at (3,0). Because the degree is odd with a positive leading coefficient, we know the graph begins from negative infinity and ends going towards positive infinity. From this information, we can sketch the graph without a calculator or computer program.



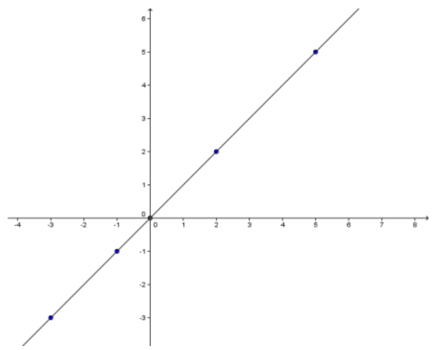
<u>Note</u>:Students should practice constructing other graphs in this manner to get an idea for how knowing the various characteristics of a graph can help you produce a rough sketch of the function without a calculator.

Polynomials Patterns Task

1. To get an idea of what polynomial functions look like, we can graph the first through fifth degree polynomials with leading coefficients of 1. For each polynomial function, make a table of 6 points and then plot them so that you can determine the shape of the graph. Choose points that are both positive and negative so that you can get a good idea of the shape of the graph. Also, include the x intercept as one of your points.

a. For example, for the first order polynomial function: $y = x^1$. You may have the following table and graph:

x	y
-3	-3
-1	-1
0	0
2	2
5	5
10	10

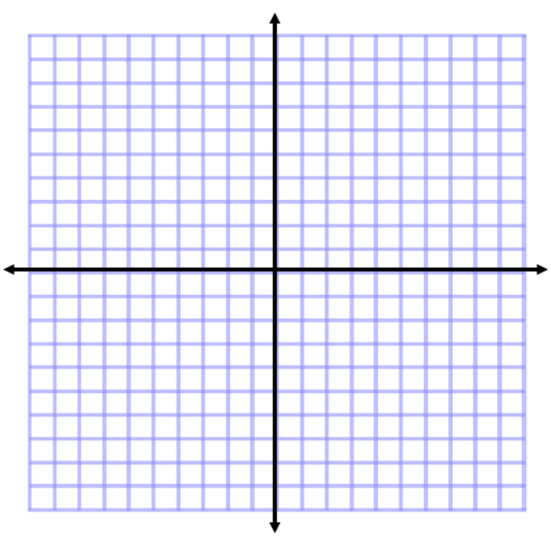


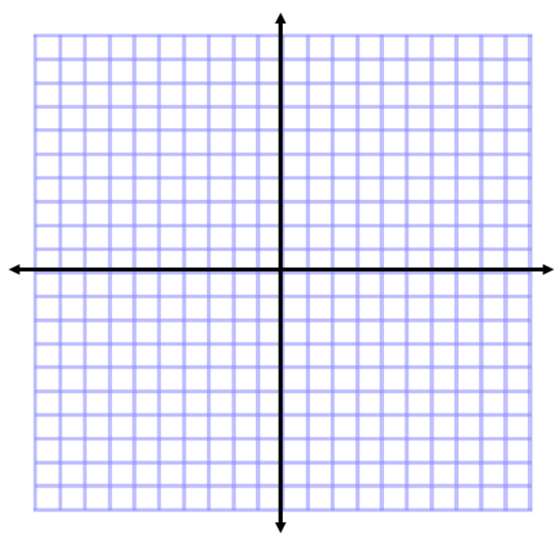
b. $y = x^2$

x	y

c. $y = x^3$

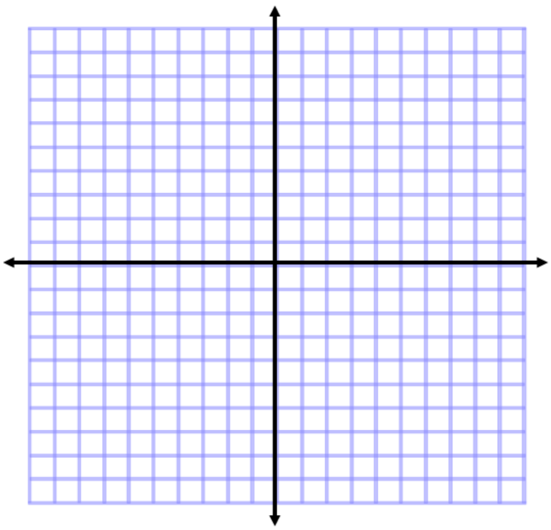
x	y





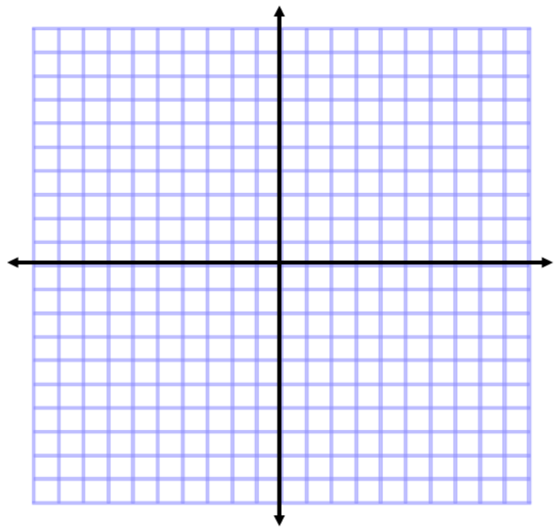
d. $y = x^4$

x	y



e. $y = x^5$

x	y



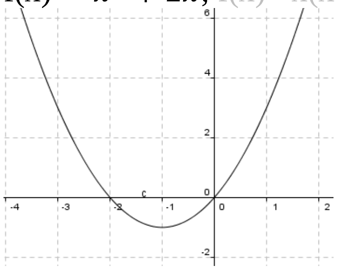
f. Compare these five graphs. By looking at the graphs, describe in your own words how $y = x^2$ is different from $y = x^4$. Also, how is $y = x^3$ different from $y = x^5$?

g. Note any other observations you make when you compare these graphs.

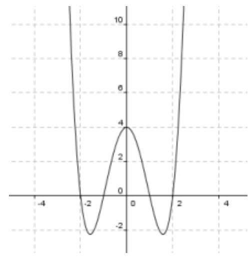
2. In order to examine their characteristics in detail so that we can find the patterns that arise in the behavior of polynomial functions, we can study some examples of polynomial functions and their graphs. Here are 8 polynomial functions and their accompanying graphs that we will use to refer back to throughout the task.

Handout of Graphs of Polynomial Functions :

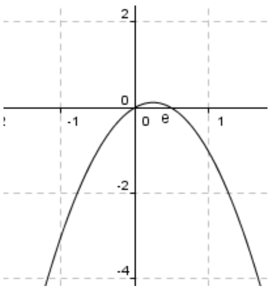
f(x) = $x^2 + 2x$; f(x)= x(x+2)



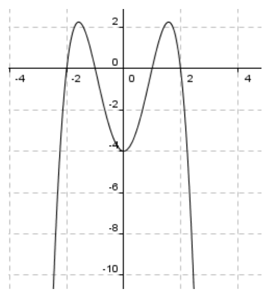
k(x) = $x^4 - 5x^2$+4 ; k(x) = (x-1)(x+1)(x-2)(x+2)



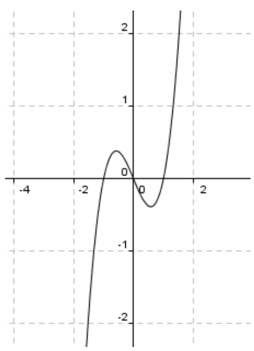
g(x) = $-2x^2 + x$; g(x)=x(-2x+1)



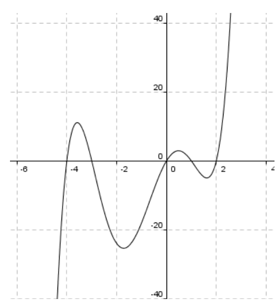
l(x) = $-(x^4 - 5x^2$+4) ;
l(x) = -(x-1)(x+1)(x-2)(x+2)



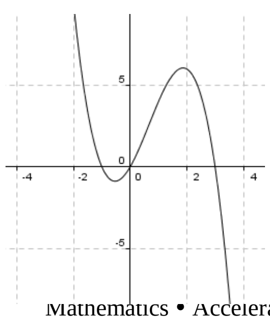
h(x) = $x^3 - x$; h(x) = $x(x-1)(x+1)$



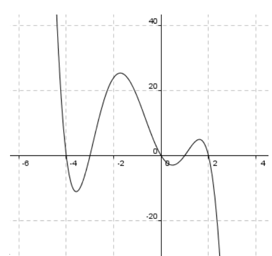
m(x) = $\frac{1}{2}(x^5 + 4x^4 - 7x^3 - 22x^2 + 24x)$;
m(x)= $\frac{1}{2}$x(x-1)(x-2)(x+3)(x+4)



j(x)=$-x^3 + 2x^2 + 3x$; j(x) = -x(x-3)(x+1)



n(x) = $-\frac{1}{2}(x^5 + 4x^4 - 7x^3 - 22x^2 + 24x)$;
n(x)=- $\frac{1}{2}$x(x-1)(x-2)(x+3)(x+4)



Each of these equations can be re-expressed as a product of linear factors by factoring the equations, as shown below in the gray equations.

a. List the x-intercepts of j(x) using the graph above. How are these intercepts related to the linear factors in gray?

b. Why might it be useful to know the linear factors of a polynomial function?

c. Although we will not factor higher order polynomial functions in this unit, you have factored quadratic functions in a previous course. For review, factor the following second degree polynomials, or quadratics.

$y = x^2 - x - 12$

$y = x^2 + 5x - 6$

$y = 2x^2 - 6x - 10$

d. Using these factors, find the roots of these three equations.

e. Sketch a graph of the three quadratic equations above without using your calculator and then use your calculator to check your graphs.

f. You can factor some polynomial equations and find their roots in a similar way.

Try this one: . $y = x^5 + x^4 - 2x^3$.

What are the roots of this fifth order polynomial function?

g. How many roots are there?

Why are there not five roots since this is a fifth degree polynomial?

h. Check the roots by generating a graph of this equation using your calculator.

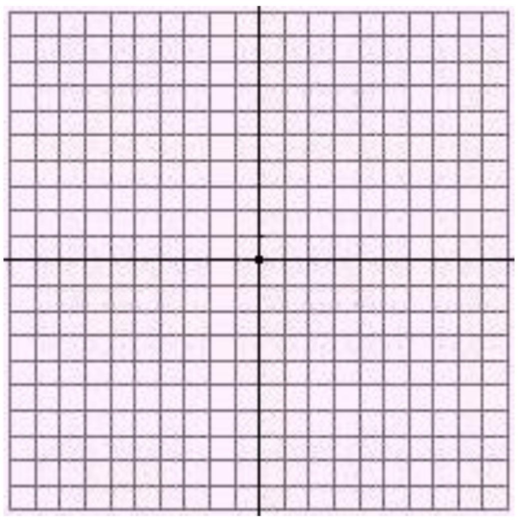
i. For other polynomial functions, we will not be able to draw upon our knowledge of factoring quadratic functions to find zeroes. For example, you may not be able to factor $= x^3 + 8x^2 + 5x - 14$, but can you still find its zeros by graphing it in your calculator? How?

Write are the zeros of this polynomial function.

3. Symmetry

The first characteristic of these 8 polynomials functions we will consider is symmetry.

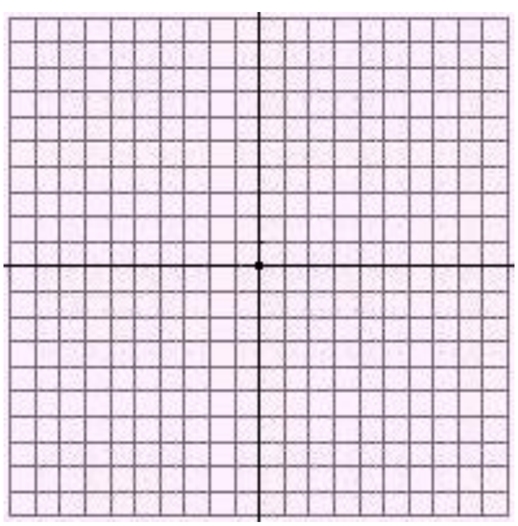
a. Sketch a function you have seen before that has symmetry about the y-axis.



Describe in your own words what it means to have symmetry about the y-axis.

What is do we call a function that has symmetry about the y-axis?

b. Sketch a function you have seen before that has symmetry about the origin.



Describe in your own words what it means to have symmetry about the origin.

What do we call a function that has symmetry about the origin?

c. Using the table below and your handout of the following eight polynomial functions, classify the functions by their symmetry.

Function	Symmetry	Symmetry	Even, Odd,

	about the y axis?	about the origin?	or Neither?
f(x) = $x^2 + 2x$			
g(x) = $-2x^2 + x$			
h(x) = $x^3 - x$			
j(x)=$-x^3 + 2x^2 + 3x$			
k(x) = $x^4 - 5x^2+4$			
l(x) = $-(x^4 - 5x^2+4)$			
m(x) = $\frac{1}{2}(x^5 + 4x^4 - 7x^3 - 22x^2 + 24x)$			
n(x) = $-\frac{1}{2}(x^5 + 4x^4 - 7x^3 - 22x^2 + 24x)$			

d. Now, sketch your own higher order polynomial function (an equation is not needed) with symmetry about the y-axis.

e. Now, sketch your own higher order polynomial function with symmetry about the origin.

f. Using these examples from the handout and the graphs of the first through fifth degree polynomials you made, why do you think an odd function may be called an odd function? Why are even functions called even functions?

g. Why don't we talk about functions that have symmetry about the x-axis? Sketch a graph that has symmetry about the x-axis. What do you notice?

4. Domain and Range
Another characteristic of functions that you have studied is domain and range. For each polynomial function, determine the domain and range.

Function	Domain	Range
f(x) = $x^2 + 2x$		
g(x) = $-2x^2 + x$		
h(x) = $x^3 - x$		
j(x)=$-x^3 + 2x^2 + 3x$		
k(x) = $x^4 - 5x^2$+4		
l(x) = $-(x^4 - 5x^2$+4)		
m(x) = $\frac{1}{2}(x^5 + 4x^4 - 7x^3 - 22x^2 + 24x)$		
n(x) = $-\frac{1}{2}(x^5 + 4x^4 - 7x^3 - 22x^2 + 24x)$		

5. Zeros
a. We can also describe the functions by determining some points on the functions. We can find the x-intercepts for each function as we discussed before. Under the column labeled "x-intercepts" write the ordered pairs (x,y) of each intercept and record the number of intercepts in the next column. Also record the degree of the polynomial.

Function	Degree	X-intercepts	Zeros	Number of Zeros
f(x) = x^2+ 2x				
g(x) = $-2x^2 + x$				
h(x) = $x^3 - x$				
j(x)=$-x^3 + 2x^2 + 3x$				
k(x) = $x^4 - 5x^2$+4				
l(x) = $-(x^4 - 5x^2$+4)				
m(x) = $\frac{1}{2}(x^5 + 4x^4 - 7x^3 - 22x^2 + 24x)$				
n(x) = $-\frac{1}{2}(x^5 + 4x^4 - 7x^3 - 22x^2 + 24x)$				

b. These x-intercepts are called the *zeros* of the polynomial functions. Why do you think they have this name?

c. Fill in the column labeled "Zeroes" by writing the zeroes that correspond to the x-intercepts of each polynomial function, and also record the number of zeroes each function has.

d. Make a conjecture about the relationship of degree of the polynomial and number of zeroes.

e. Test your conjecture by graphing the following polynomial functions using your calculator: $y = x^2$, $y = x^2(x-1)(x+4)$, $y = x(x-1)^2$.

Function	Degree	X-Intercepts	Zeroes	Number of Zeroes
$y = x^2$		(0,0)		
$y = x^2(x-1)(x+4)$		(0,0); (0,-1);(0-4)		
$y = x(x-1)^2$				

How are these functions different from the functions in the table?

Now amend your conjecture about the relationship of the degree of the polynomial and the number of x-intercepts.

Make a conjecture for the maximum number of x-intercepts the following polynomial function will have: $p(x) = 2x^{11} + 4x^6 - 3x^2$

6. End Behavior

In determining the range of the polynomial functions, you had to consider the *end behavior* of the functions, that is the value of $f(x)$ as x approaches infinity and negative infinity.

Polynomials exhibit patterns of end behavior that are helpful in sketching polynomial functions.

a. Graph the following on your calculator. Make a rough sketch next to each one and answer the following:

- Is the **degree** even or odd?
- Is the leading coefficient, the coefficient on the term of highest degree, positive or negative?
- Does the graph rise or fall on the left? On the right?

1. $y = x$

7. $y = -x^2$

2. $y = x^2$

8. $y = -3x^4$

3. $y = -3x$

4. $y = 5x^4$

5. $y = x^3$

6. $y = 2x^5$

9. $y = -x^3$

10. $y = -2x^5$

11. $y = -3x^6$

12. $y = 7x^3$

b. Write a conjecture about the **end behavior**, whether it rises or falls at the ends, of a function of the form $f(x) = ax^n$ for each pair of conditions below. Then test your conjectures on some of the 8 polynomial functions graphed on your handout.

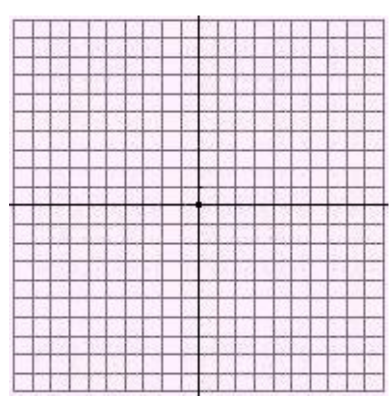
Condition a: When *n* is even and *a* > 0,

Condition b: When *n* is even and *a* < 0,

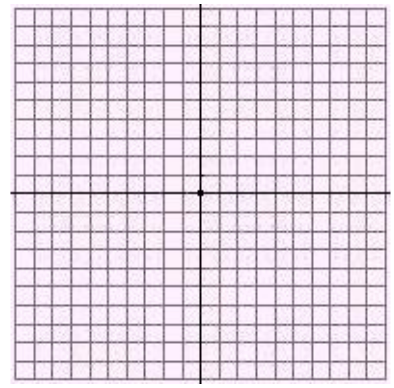
Condition c: When *n* is odd and *a* > 0,

Condition d: When *n* is odd and *a* < 0,

c. Based on your conjectures in part (b), sketch a fourth degree polynomial function with a negative leading coefficient.



d. Now sketch a fifth degree polynomial with a positive leading coefficient.



Note we can sketch the graph with the end behavior even though we cannot determine where and how the graph behaves otherwise without an equation or without the zeros.

e. If we are given the real zeros of a polynomial function, we can combine what we know about end behavior to make a rough sketch of the function.

Sketch the graph of the following functions using what you know about end behavior and zeros:

$$f.\ \ f(x) = (x-2)(x-3)$$

h. $g.\ \ f(x) = -x(x-1)(x+5)(x-7)$

7. Critical Points Other points of interest in sketching the graph of a polynomial function are the points where the graph begins or ends increasing or decreasing. Recall what it means for a point of a function to be an *absolute minimum* or an *absolute maximum*.

a. Which of the twelve graphs from part 6a have an absolute maximum?

b. Which have an absolute minimum?

c. What do you notice about the degree of these functions?

d. Can you ever have an absolute maximum AND an absolute minimum in the same function? If so, sketch a graph with both. If not, why not?

e. For each of the following graphs from the handout, locate the turning points and the related intervals of increase and decrease, as you have determined previously for linear and quadratic polynomial functions.

Then record which turning points are *relative minimum* (the lowest point on a given portion of the graph) and *relative maximum* (the highest point on a given portion of the graph) values.

Function	Degree	Turning Points	Intervals of Increase	Intervals of Decrease	Relative Minimum	Relative Maximum
f(x)						
h(x)						
k(x)						
n(x)						

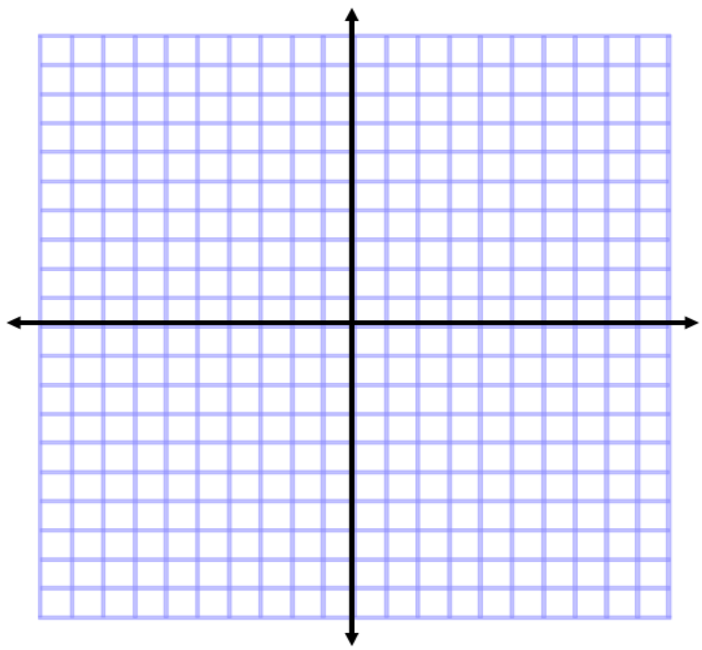
f. Make a conjecture about the relationship of the degree of the polynomial and the number of turning points that the polynomial has. Recall that this is the maximum number of turning points a polynomial of this degree can have because these graphs are examples in which all zeros have a multiplicity of one.

g. Sometimes points that are relative minimums or maximums are also absolute minimums or absolute maximum. Are any of the relative extrema in your table also absolute extrema?

8. Putting it all Together:

Now that you have explored various characteristics of polynomial functions, you will be able to describe and sketch graphs of polynomial functions when you are given their equations.

a. If I give you the function: $f(x) = (x-3)(x-1)^2$ then what can you tell me about the graph of this function? Make a sketch of the graph of this function, describe its end behavior, and locate its critical point and zeroes.



Polynomial Project Culminating Task: Part 1

Back to Task Table

Mathematical Goals

- Find zeros of polynomial functions using both graphic and algebraic methods
- Define and identify complex conjugates
- Apply the Remainder Theorem, Factor Theorem and the Fundamental Theorem in order to find all solutions
- Perform long and synthetic division
- Make a conjecture about using synthetic division when the leading coefficient of the linear divisor does not equal one

Georgia Standards of Excellence

Interpret the structure of expressions.

MGSE9-12.A.SSE.1 Interpret expressions that represent a quantity in terms of its context.

MGSE9-12.A.SSE.1a Interpret parts of an expression, such as terms, factors, and coefficients, in context.

MGSE9-12.A.SSE.1b Given situations which utilize formulas or expressions with multiple terms and/or factors, interpret the meaning (in context) of individual terms or factors.

MGSE9-12.A.SSE.2 Use the structure of an expression to rewrite it in different equivalent forms. For example, see $x^4 – y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 – y^2) (x^2 + y^2)$.

Perform arithmetic operations on polynomials.

MGSE9-12.A.APR.1 Add, subtract, and multiply polynomials; understand that polynomials form a system analogous to the integers in that they are closed under these operations.

Use complex numbers in polynomial identities and equations.

MGSE9-12.N.CN.8 Extend polynomial identities to include factoring with complex numbers. *For example, rewrite $x^2 + 4$ as $(x + 2i)(x – 2i)$.*

MGSE9-12.N.CN.9 Use the Fundamental Theorem of Algebra to find all roots of a polynomial equation.

Represent and solve equations and inequalities graphically.

MGSE9-12.A.REI.11 Using graphs, tables, or successive approximations, show that the solution to the equation f(x) = g(x) is the x-value where the y-values of f(x) and g(x) are the same.

Understand the relationship between zeros and factors of polynomials.

MGSE9-12.A.APR.2 Know and apply the Remainder Theorem: For a polynomial p(x) and a number a, the remainder on division by x – a is p(a), so p(a) = 0 if and only if (x – a) is a factor of p(x).

MGSE9-12.A.APR.3 Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial.

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Look for and make use of structure.
6. Look for and express regularity in repeated reasoning.

Materials

- Pencil
- Handout
- Graphing Calculator

Polynomial Project Culminating Task: Part 1

I. Finding and Exploring Zeros of Polynomial Functions

A. Find all of the zeros of the polynomial functions

1. $f(x) = x^4 + 5x^2 - 36$
2. $f(x) = x^4 - 12x^2 + 27$
3. $f(x) = x^3 - 3x^2 + 2x$
4. $f(x) = x^3 - 3x^2 - 5x + 15$
5. $f(x) = x^3 - 4x^2 - 3x + 12$
6. $f(x) = x^4 + 2x^3 + x^2 + 8x - 12$
7. $f(x) = x^3 - 2x^2 - 2x + 1$
8. $f(x) = 12x^3 - 32x^2 - 145x + 25$

B. Which functions above have zeros that are **complex conjugates**?

C. How do you determine whether zeros of a polynomial are complex conjugates?

D. If you are asked to write a polynomial function with zeros of 4 and $3 - 2i$, what is the **least degree** it could have?

II. Polynomial Division $\left(4x^4 - 20x^3 + 23x^2 + 5x - 6\right) \div (x - 3)$

A. Find the quotient using long division. **Show work!**

B. Find the quotient using synthetic division. **Show work!**

C. Why do you think you need to subtract in the process of long division, but add when doing synthetic division?

D. Under what conditions can you use synthetic division to find a quotient?

E. What is the remainder to this problem and what information does it provide about $(x - 3)$?

F. List **all** of the zeros of the equation $f(x) = 4x^4 - 20x^3 + 23x^2 + 5x - 6$.

III. Polynomial Division $\left(4x^4 - x^2 - 2x + 1\right) \div (2x - 3)$

A. Find the quotient using long division. **Show work!**

B. Find the quotient using synthetic division. **Show work!**

C. Explain how it is still possible to use synthetic division when the leading coefficient of the divisor is not equal to 1. Be sure to clearly indicate what you must do to the quotient when you use synthetic division with a linear divisor that does not have its leading coefficient equal 1. (Hint: Compare your results in IIIB to your result in IIIA.)

IV. Grading

A. Write all answers and show work on the answer sheet provided to you.

B. There are 20 questions & each answer is worth a maximum of 5 points

Answer Sheet

I. Finding and Exploring Zeros of Polynomial Functions

A.

1. ***2, -2, 3i, -3i***

2. ***3, -3,*** $\pm\sqrt{3}$

3. ***0, 2, 1***

4. ***3,*** $\pm\sqrt{5}$

5. ***4,*** $\pm\sqrt{3}$

6. ***1, -3, 2i, -2i***

7. ***-1,*** $\frac{3\pm\sqrt{5}}{2}$

8. ***5,*** $-\frac{5}{2},\frac{1}{6}$

B. ***#'s 1 and 6***

C. ***They are solutions that come in a+bi and a – bi form (answers may vary)***

D. ***3rd degree or cubic***

II. Polynomial Division

$\left(4x^4-20x^3+23x^2+5x-6\right)\div\left(x-3\right)$

A. ***$4x^3 - 8x^2 - x + 2$***

B. ***$4x^3 - 8x^2 - x + 2$***

C. ***because it is easier to add integers than to subtract (answers may vary)***

D. ***when the divisor is linear or can be written in (x – c) form (answers may vary)***

E. ***The remainder is zero therefore (x – 3) is a factor of the polynomial***

F. ***2, 3,*** $-\frac{1}{2},\frac{1}{2}$

III. Polynomial Division

$\left(4x^4-x^2-2x+1\right)\div\left(2x-3\right)$

A. ***$2x^3 + 3x^2 + 4x + 5 +$*** $\frac{16}{2x-3}$

B. ***$4x^3 + 6x^2 + 8x + 10 +$*** $\frac{16}{2x-3}$

C. ***You must divide the quotient by the leading coefficient. You do not alter the remainder.***

Answer Sheet

I. Finding and Exploring Zeros of Polynomial Functions

A.

1. ______________________________
2. ______________________________
3. ______________________________
4. ______________________________
5. ______________________________
6. ______________________________
7. ______________________________
8. ______________________________

B. ______________________________

C. ______________________________

____

D. ______________________________

II. Polynomial Division

$\left(4x^4 - 20x^3 + 23x^2 + 5x - 6\right) \div \left(x - 3\right)$

A.

B.

C. ______________________________

____

____

D. ______________________________

____

E. ______________________________

____

F. ______________________________

III. Polynomial Division

$\left(4x^4 - x^2 - 2x + 1\right) \div \left(2x - 3\right)$

A.

B.

C. ______________________________

____

____

Polynomial Project Culminating Task: Part 2

Back to Task Table

Mathematical Goals

- Classify a polynomial by degree and state the number of terms
- Identify key features of the graph of a polynomial functions
- Graph a polynomial function using technology
- Roughly sketch the graphs of a simple polynomial functions by hand
- Solve real-world application problems by analyzing the graph of the polynomial

Georgia Standards of Excellence

Interpret the structure of expressions.

MGSE9-12.A.SSE.1 Interpret expressions that represent a quantity in terms of its context.

MGSE9-12.A.SSE.1a Interpret parts of an expression, such as terms, factors, and coefficients, in context.

Represent and solve equations and inequalities graphically.

MGSE9-12.A.REI.11 Using graphs, tables, or successive approximations, show that the solution to the equation f(x) = g(x) is the x-value where the y-values of f(x) and g(x) are the same.

Understand the relationship between zeros and factors of polynomials.

MGSE9-12.A.APR.3 Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial.

Analyze functions using different representations.

MGSE9-12.F.IF.7 Graph functions expressed algebraically and show key features of the graph both by hand and by using technology. *(Limit to polynomial functions.)*

MGSE9-12.F.IF.7c Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior.

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Model with mathematics.
3. Use appropriate tools strategically.
4. Attend to precision.
5. Look for and make use of structure.

Materials

- Pencil

- Handout
- Graphing Calculator

Polynomial Project Culminating Task: Part 2

It is recommended that students write solutions in interval notation, but have included alternative response formats as well

I. Explore the graph of the polynomial function $f(x) = x^4 + x^3 - 11x^2 - 9x + 18$

A. Classification

1. Classify this polynomial by degree: ***quartic***

2. State the number of terms it has: ***five***

B. Symmetry

1. Is the function symmetric about the y-axis? ***no***

2. Is the function symmetric about the origin? ***no***

3. Is the function even, odd, or neither? ***neither***

C. Domain and Range

1. State the domain of the function: ***all real numbers or*** $(-\infty, \infty)$

2. State the range of the function: $x \geq -20.97$ ***or*** $[-20.97, \infty)$

D. X-Intercepts, Factors, and Zeros

1. Identify all the x-intercepts of the graph of the polynomial function: ***(-3,0), (-2,0), (1,0), (3,0)***
2. Identify all the factors of the polynomial function: ***(x+3)(x+2)(x – 1)(x – 3)***

3. Identify all the zeros or roots of the polynomial function: ***-3, -2, 1, 3***

4. How many zeros does the function have? ***four***

E. End-Behaviors

1. Describe the left end-behavior of the graph of this function:
The function is increasing or $f(x) \to \infty$ as $x \to -\infty$

2. Describe the right end-behavior of the graph of this function:
The function is increasing or $f(x) \to \infty$ as $x \to \infty$

F. Relative Maximums and Minimums

1. Identify all of the relative minimum(s) in the graph of the polynomial function:

    (-2.6,-4.9) (2.2, -21)

2. Identify all of the relative maximum(s) in the graph of the polynomial function:
    (-0.4, 19.8)

G. Intervals of Increase or Decrease

1. Identify all of the intervals of increase in the graph of the polynomial function:

    -2.56<x<-0.4 or x>2.21 ***or*** $(-2.56,-0.4)\cup(2.21,\infty)$

2. Identify all of the intervals of decrease in the graph of the polynomial function:

    x<-2.56 or -0.4<x<2.21 ***or*** $(-\infty,-2.56)\cup(-0.4,2.21)$

II. Problem Solving and Applications

A. The average monthly cable TV rate from 1980 to 2000 can be modeled by the function $R(t)=-0.0036t^3+0.13t^2-0.073t+7.7$ where R(t) is the monthly rate in dollars and t is the number of years since 1980. Explore the behavior of the graph for the years from 1980 to 2010.

1. Describe the right end-behavior of the graph of this function:

    The function is decreasing or $f(x)\to-\infty$ as $x\to\infty$

2. What would you estimate the average monthly cable rate was in 2000? ***$29.44***

3. Use the graph to estimate the year that the average monthly cable rate peaked or was at its highest. ***x = 23.7 so in 2003***

B. The average amount of oranges, in pounds, eaten per person each year in the U.S. from 1991 to 1996 can be modeled by $f(x)=0.298x^3-2.73x^2+7.05x+8.45$ where x is the number of years since 1991.

1. Identify any extreme points on the graph in the interval $0\le x\le5$.

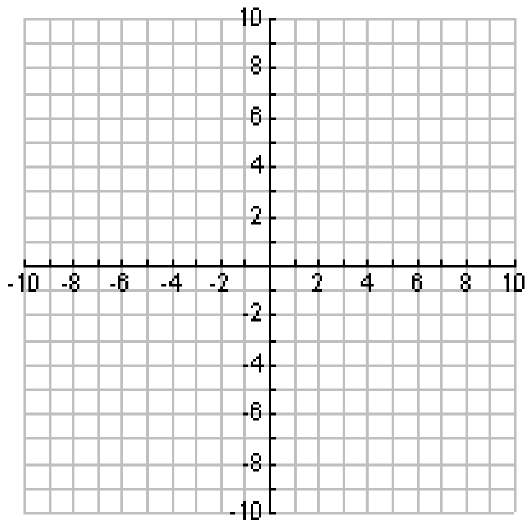
    (1.85, 14.04) and (4.25, 11.98)

2. What real-life meaning do these points have?

   In 1992, each person consumed about 14 oranges. This was the highest level of consumption per person over the time period observed. In 1995, each person consumed about 12 oranges per year. This was the lowest value over the time period.

C. Construct a polynomial having the following characteristics: fourth degree, positive leading coefficient, and four real zeros. Sketch the graph of your polynomial below **and** label the x-intercepts and all points of extrema on the graph.

Answers will vary



Polynomial Project Culminating Task: Part 2

I. Explore the graph of the polynomial function $f(x) = x^4 + x^3 - 11x^2 - 9x + 18$

A. Classification

1. Classify this polynomial by degree: _________________

2. State the number of terms it has: _________________

B. Symmetry

1. Is the function symmetric about the y-axis? ________________

2. Is the function symmetric about the origin? _________________

3. Is the function even, odd, or neither? _________________

C. Domain and Range

1. State the domain of the function: _________________

2. State the range of the function: _________________

D. X-Intercepts, Factors, and Zeros

1. Identify all the x-intercepts of the graph of the polynomial function: _______________

2. Identify all the factors of the polynomial function: _________________

3. Identify all the zeros or roots of the polynomial function: _________________

4. How many zeros does the function have? _________________

E. End-Behaviors

1. Describe the left end-behavior of the graph of this function: _________________

2. Describe the right end-behavior of the graph of this function: _________________

F. Relative Maximums and Minimums

1. Identify all of the relative minimum(s) in the graph of the polynomial function:

____

2. Identify all of the relative maximum(s) in the graph of the polynomial function:

____

G. Intervals of Increase or Decrease

1. Identify all of the intervals of increase in the graph of the polynomial function:

____

2. Identify all of the intervals of decrease in the graph of the polynomial function:

____

II. Problem Solving and Applications

A. The average monthly cable TV rate from 1980 to 2000 can be modeled by the function $R(t) = -0.0036t^3 + 0.13t^2 - 0.073t + 7.7$ where R(t) is the monthly rate in dollars and t is the number of years since 1980. Explore the behavior of the graph for the years from 1980 to 2010.

1. Describe the right end-behavior of the graph of this function:______________________

2. What would you estimate the average monthly cable rate was in 2000? ______________

3. Use the graph to estimate the year that the average monthly cable rate peaked or was at its highest. _____________________

B. The average amount of oranges, in pounds, eaten per person each year in the U.S. from 1991 to 1996 can be modeled by $f(x) = 0.298x^3 - 2.73x^2 + 7.05x + 8.45$ where x is the number of years since 1991.

3. Identify any extreme points on the graph in the interval $0 \le x \le 5$.

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4. What real-life meaning do these points have? ______________________________

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C. Construct a polynomial having the following characteristics: fourth degree, positive leading coefficient, and four real zeros. Sketch the graph of your polynomial below **and** label the x-intercepts and all turning points on the graph.