Georgia
Standards of Excellence
Curriculum Frameworks

Mathematics

Accelerated GSE Analytic Geometry B/
Advanced Algebra

Unit 7: Rational and Radical Relationships

Georgia Department of Education

Richard Woods, Georgia's School Superintendent
"Educating Georgia's Future"

Unit 7
Rational and Radical Relationships

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OVERVIEW

In this unit students will:

- Explore Rational and Radical Functions
- Determine rational numbers extend the arithmetic of integers by allowing division by all numbers except zero. Similarly, rational expressions extend the arithmetic of polynomials by allowing division by all polynomials except the zero polynomial
- Notice the arithmetic of rational expressions is governed by the same rules as the arithmetic of rational numbers
- Investigate the properties of simple rational and radical functions and then expand their knowledge of the graphical behavior and characteristics of more complex rational functions
- Recall and make use of their knowledge of polynomial functions as well as compositions of functions to investigate the characteristics of these more complex rational functions
- Solve equations and inequalities involving rational and radical functions
- Understand that not all solutions generated algebraically are actually solutions to the equations and extraneous solutions will be explored
- Apply these rational and radical functions with an emphasis on interpretation of real world phenomena as it relates to certain characteristics of the rational expressions

Although the units in this instructional framework emphasize key standards and big ideas at specific times of the year, routine topics such as estimation, mental computation, and basic computation facts should be addressed on an ongoing basis. Ideas related to the eight process standards should be addressed constantly as well. To assure that this unit is taught with the appropriate emphasis, depth, and rigor, it is important that the tasks listed under "Evidence of Learning" be reviewed early in the planning process. A variety of resources should be utilized to supplement this unit. This unit provides much needed content information, but excellent learning activities as well. The tasks in this unit illustrate the types of learning activities that should be utilized from a variety of sources.

STANDARDS ADDRESSED IN THIS UNIT

Mathematical standards are interwoven and should be addressed throughout the year in as many different units and activities as possible in order to emphasize the natural connections that exist among mathematical topics.

KEY STANDARDS

Rewrite rational expressions

MGSE9-12.A.APR.7 Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a nonzero rational expression; add, subtract, multiply, and divide rational expressions.

Create equations that describe numbers or relationships

MGSE9-12.A.CED.1 Create equations and inequalities in one variable and use them to solve problems. Include equations arising from ~~linear, quadratic,~~ simple rational, ~~and exponential~~ functions ~~(integer inputs only)~~.

MGSE9-12.A.CED.2 Create ~~linear, quadratic, and exponential~~ equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. *(Limit to rational and radical functions. The phrase "in two or more variables" refers to formulas like the compound interest formula, in which* $A = P(1 + r/n)^{nt}$ *has multiple variables.)*

Understand solving equations as a process of reasoning and explain the reasoning

MGSE9-12.A.REI.2 Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.

Interpret functions that arise in applications in terms of the context

MGSE9-12.F.IF.4 Using tables, graphs, and verbal descriptions, interpret the key characteristics of a function which models the relationship between two quantities. Sketch a graph showing key features including: intercepts; interval where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; ~~and periodicity~~. *(Limit to radical and rational functions.)*

MGSE9-12.F.IF.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of* *person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function. (Limit to radical and rational functions.)*

<u>Analyze functions using different representations</u>

MGSE9-12.F.IF.7 Graph functions expressed algebraically and show key features of the graph both by hand and by using technology. *(Limit to radical and rational functions.)*

> **MGSE9-12.F.IF.7b** Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions.
>
> **MGSE9-12.F.IF.7d** Graph rational functions, identifying zeros and asymptotes when suitable factorizations are available, and showing end behavior.

<u>RELATED STANDARDS</u>

MGSE9-12.A.SSE.1a Interpret parts of an expression, such as terms, factors, and coefficients, in context.

MGSE9-12.A.SSE.2 Use the structure of an expression to rewrite it in different equivalent forms. For example, see $x^4 – y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 – y^2) (x^2 + y^2)$.

MGSE9-12.F.IF.9 Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one function and an algebraic expression for another, say which has the larger maximum.*

<u>Build a function that models a relationship between two quantities</u>

MGSE9-12.F.BF.1 Write a function that describes a relationship between two quantities.

MGSE9-12.F.BF.1a Determine an explicit expression and the recursive process (steps for calculation) from context. *For example, if Jimmy starts out with $15 and earns $2 a day, the explicit expression "2x +15" can be described recursively (either in writing or verbally) as " to find out how much money Jimmy will have tomorrow, you add $2 to his total today."* $J_n = J_{n-1} + 2, J_0 = 15$

STANDARDS FOR MATHEMATICAL PRACTICE

Refer to the Comprehensive Course Overview for more detailed information about the Standards for Mathematical Practice.

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

ENDURING UNDERSTANDINGS

- Recognize rational functions as the division of two polynomial functions and rewrite a rational expression
- Find the sum, difference, product, and quotient of rational expressions
- Graph rational and radical functions
- Interpret graphs and discover characteristics of rational functions
- Solve rational and radical equations algebraically and graphically
- Solve rational inequalities

ESSENTIAL QUESTIONS

- How can we extend arithmetic properties and processes to algebraic expressions and how can we use these properties and processes to solve problems?
- How do the polynomial pieces of a rational function affect the characteristics of the function itself?
- How are horizontal asymptotes, slant asymptotes, and vertical asymptotes alike and different?

- Why are all solutions not necessarily the solution to an equation? How can you identify these extra solutions?
- Why is it important to set a rational inequality to 0 before solving?

CONCEPTS/SKILLS TO MAINTAIN

It is expected that students will have prior knowledge/experience related to the concepts and skills identified below. It may be necessary to pre-assess in order to determine if time needs to be spent on conceptual activities that help students develop a deeper understanding of these ideas.

- Computation with fractions
- Factoring polynomials
- Solving linear and quadratic equations

SELECTED TERMS AND SYMBOLS

The following terms and symbols are often misunderstood. These concepts are not an inclusive list and should not be taught in isolation. However, due to evidence of frequent difficulty and misunderstanding associated with these concepts, instructors should pay particular attention to them and how their students are able to explain and apply them.

The definitions below are for teacher reference only and are not to be memorized by the students. Students should explore these concepts using models and real life examples. Students should understand the concepts involved and be able to recognize and/or demonstrate them with words, models, pictures, or numbers.

The websites below are interactive and include a math glossary suitable for elementary children. **Note – At the elementary level, different sources use different definitions. Please preview any website for alignment to the definitions given in the frameworks.**

http://www.teachers.ash.org.au/jeather/maths/dictionary.html
This web site has activities to help students more fully understand and retain new vocabulary.

http://intermath.coe.uga.edu/dictnary/homepg.asp
Definitions and activities for these and other terms can be found on the Intermath website.

- **Algebra:** The branch of mathematics that deals with relationships between numbers, utilizing letters and other symbols to represent specific sets of numbers, or to describe a pattern of relationships between numbers.

- **Coefficient:** A number multiplied by a variable.

- **Equation:** A number sentence that contains an equality symbol.

- **Expression:** A mathematical phrase involving at least one variable and sometimes numbers and operation symbols.
- **Extraneous Solutions:** A solution of the simplified form of the equation that does not satisfy the original equation.
- **Inequality**: Any mathematical sentence that contains the symbols > (greater than), < (less than), $\leq$ (less than or equal to), or $\geq$ (greater than or equal to).
- **Polynomial:** A mathematical expression involving the sum of terms made up of variables to nonnegative integer powers and real-valued coefficients.
- **Radical Function:** A function containing a root. The most common radical functions are the square root and cube root functions, $f(x) = \sqrt{x}$ *and* $g(x) = \sqrt[3]{x}$.
- **Rational Function:** The quotient of two polynomials, P(z) and Q(z), where $R(z) = \frac{P(z)}{Q(z)}$.
- **Reciprocal:** Two numbers whose product is one. For example, $m \, x \, \frac{1}{m} = 1$
- **Variable:** A letter or symbol used to represent a number.

EVIDENCE OF LEARNING

By the conclusion of this unit, students should be able to demonstrate the following competencies:

- Rewrite rational expressions in different forms
- Add, subtract, multiply, and divide rational expressions
- Solve rational and radical equations
- Solve rational inequalities
- Graph rational and radical functions and identify key characteristics
- Interpret solutions to graphs and equations given the context of the problem

SPOTLIGHT TASKS

A Spotlight Task has been added to each GSE mathematics unit in the Georgia resources for middle and high school. The Spotlight Tasks serve as exemplars for the use of the Standards for Mathematical Practice, appropriate unit-level Georgia Standards of Excellence, and research-based pedagogical strategies for instruction and engagement. Each task includes teacher commentary and support for classroom implementation. Some of the Spotlight Tasks are revisions of existing Georgia tasks and some are newly created. Additionally, some of the Spotlight Tasks are 3-Act Tasks based on 3-Act Problems from Dan Meyer and Problem-Based Learning from Robert Kaplinsky.

3-ACT TASKS

A Three-Act Task is a whole group mathematics task consisting of 3 distinct parts: an engaging and perplexing Act One, an information and solution seeking Act Two, and a solution discussion and solution revealing Act Three.

More information along with guidelines for 3-Act Tasks may be found in the Comprehensive Course Overview.

TASKS

The following tasks represent the level of depth, rigor, and complexity expected of all Algebra II/Advanced Algebra students. These tasks, or tasks of similar depth and rigor, should be used to demonstrate evidence of learning. It is important that all elements of a task be addressed throughout the learning process so that students understand what is expected of them. While some tasks are identified as a performance task, they may also be used for teaching and learning (learning/scaffolding task).

Task Name	Task Type *Grouping Strategy*	Content Addressed
Operations with Rational Expressions	Learning Task *Partner/Small Group Task*	Add, subtract, multiply and divide rational expressions
Characteristics of Rational Functions	Learning Task *Partner/Small Group Task*	Investigating the characteristics of the graphs of rational functions
Horizontal Asymptotes: How do we find them?	Learning Task *Partner/Small Group Task*	Exploring end behavior in rational functions
Graphing Rational Functions Without a Calculator	Performance Task *Individual/Partner Task*	Graphing rational functions by hand
Jogging Into the Wind	Learning/Performance *Individual/Partner Task*	Constructing functions that represent a quantity of interest in context. Interpreting features of functions in light of a context.
Hank's Hot Dog Stand	Performance Task *Individual/Partner Task*	Interpret a simple rational function and restrict its domain based on the context of a problem
That's Radical Dude	Learning Task *Partner/Small Group Task*	Graphing square and cube roots
Let's Get to "Work"	Performance Task *Individual/Partner Task*	Solving and applying rational and radical equations
Sailing Into the Wind (Spotlight Task)	Performance Task *Individual/Partner Task*	Simplifying radical expressions
Extraneous Solutions	Learning Task *Partner/Small Group Task*	More work with rational and radical equations focusing on extraneous solutions
To Bracket or Not to Bracket	Learning Task *Partner/Small Group Task*	Solving rational inequalities

Culminating Task: NFL Passer Rating	Performance Task *Individual/Partner Task*	Real life situation involving rational equations
Culminating Task: Create-A-Can	Performance Task *Individual/Partner Task*	Real life situation involving rational equations (alternative task)

Operations with Rational Expressions Task

Back to Task Table

Math Goals

- Simplify rational expressions
- Add and subtract rational expressions
- Multiply and divide rational expressions

Georgia Standards of Excellence

MGSE9-12.A.APR.7 Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a nonzero rational expression; add, subtract, multiply, and divide rational expressions.

Standards for Mathematical Practice

1. Attend to precision
2. Look for and make use of structure
3. Look for and express regularity in repeated reasoning

Introduction

This task allows students to discuss with a partner fraction operations in order to extend those ideas to rational expressions. This task is just the introduction. Students need practice with various types of problems in order to become proficient with the skills introduced here. You can add complexity by making the numerator and denominators harder to factor. Likewise, you can also simplify the concepts for struggling learners by using polynomials that involve only simple factoring.

Operations with Rational Expressions

Thinking about operations with rational numbers, or fractions, will help us perform addition, subtraction, multiplication, and division with rational expressions. We will use examples involving fractions to help us extend our thinking to dealing with fractions with variables, or rational expressions.

- Simplifying Rational Expressions

Think about the fraction $\frac{108}{210}$. What operation do we use to rewrite this fraction in simplest form?

Answers may vary, but you want students to realize that they use division to simplify a fraction.

What is the possible obstacle in using this operation to simplify fractions?

If a student chooses a number that is not the greatest common factor of the numerator and denominator, then they will have to repeat the process multiple times.

Let's try to simplify another way. Find the prime factorization of the numerator and denominator of the fraction above. Use this form of the numerator and denominator to quickly simplify the fraction.

$$\frac{\cancel{2}\cdot 2\cdot \cancel{3}\cdot 3\cdot 3}{\cancel{2}\cdot \cancel{3}\cdot 5\cdot 7} = \frac{2\cdot 3\cdot 3}{5\cdot 7} = \frac{18}{35}$$

Now let's think about the fraction: $\frac{x^2-9}{x^2+7x+12}$. How can we use the idea of prime factorization to help us simplify this rational expression?

We want students to see that if we factor the numerator and denominator then we can easily eliminate the common factors. Solution: $\frac{x-3}{x+4}$

Try this one: $\frac{4-x}{x-4}$ How can factoring help us simplify this rational expression?

In this case, students need to see that factoring -1 from the numerator or the denominator will help them simplify the fraction since the numerator and denominator are opposites. Solution: -1.

- Multiplying and Dividing Rational Expressions

We now need to think about multiplying fractions. Take a minute to discuss with a partner how you would solve the following problem. Try to find more than one way and show your results below:

$$\frac{4}{14}\cdot\frac{24}{10}$$

Answers will vary. When the class shares out the various strategies make sure the examples include simplifying the fractions before you multiply as well as simply multiplying straight across and then simplifying the product.

Which method from above do you think would be easiest to extend to multiplication of rational expressions? Why?

Students should indicate that simplifying first is easier. In order to extend this to polynomials, students will need to factor.

Take a look at $\frac{5x^2}{x^2-4} \cdot \frac{x+2}{10x^3}$. Use your ideas from above to help you multiply these two fractions.

Make sure that students factor each denominator and numerator completely before they eliminate common factors. This is also a good time to remind students of the rules of exponents. Solution: $\frac{1}{2x(x-2)}$

Now try this one: $\frac{3x+6}{x^2-9} \cdot \frac{4x+12}{6x^2+12x}$

Solution: $\frac{2}{x(x-3)}$

What if we change the fraction multiplication problem that we started with to a division problem? Talk to your partner about how to solve the problem below:

$$\frac{4}{14} \div \frac{24}{10}$$

Answers will vary, but you want to make sure that students eventually use the correct terminology for dividing fractions, "multiply by the reciprocal".

What is the one difference in solving a fraction division problem versus a fraction multiplication problem?

They must multiply by the reciprocal of the second fraction before they can complete the problem using the strategies that we developed for multiplication above.

Apply that idea to this problem: $\frac{4x+8}{8x} \div \frac{x^2-4}{6x^2}$

Solution: $\frac{3x}{x-2}$

Let's try one more: $\frac{x^2-2x-15}{3x^2+12x} \div \frac{x^2-9}{x^2+4x}$

Solution: $\frac{x-5}{3(x-3)}$. ***It is important to point out to students that our goal is to completely simplify the fraction so it is unnecessary to distribute the 3 in the denominator.***

- Adding and Subtracting Rational Expressions

The idea of using the processes for operations with fractions to guide us as we operated with rational functions continues, but addition and subtraction may seem a little more involved. Just like with fractions it is necessary to have common denominators in both rational expressions before you can add or subtract. Think about the fraction addition problem $\frac{3}{10}+\frac{1}{6}$. What is the least common denominator (LCD)?

Solution: The LCD is 30.

You might be able to quickly realize that 30 is the LCD, but why is it 30? Turn to your partner and explain a couple of ways of finding a common denominator.

Students might try to explain that the way to find a common denominator is to multiply the denominators, but it should be pointed out that will not necessarily give you the least common denominator (like the example above).

When thinking about denominators like *x + 2* or *x – 3* it becomes important to understand what makes a LCD. In the fraction problem above, you might have been able to say that 30 is the LCD because it is the smallest number that both 10 and 6 divide into, but how do you create that number if it isn't obvious? (Hint: Think about prime factorization.)

It is important for students to realize that the least common denominator is the number that contains all the factors of the two denominators the least number of times. For example, the prime factorization of 10 is 2×5 and the prime factorization of 6 is 2×3 so 2×3×5 = 30 is the LCD.

When dealing with rational expressions, factoring is key. You must find all of the factors of each denominator to know what the LCD should be. Let's try some. Find the LCD for the following problems:

a. $\frac{3}{5a}, \frac{b}{4a^2}$

Solution: $\boldsymbol{20a^2}$

b. $\frac{4}{x+5}, \frac{3}{x-5}$

Solution: $\boldsymbol{(x+5)(x-5)}$

c. $\frac{2x}{x+2}, \frac{x+1}{x^2-3x-10}$

Solution: $\boldsymbol{(x+2)(x-5)}$

d. $\frac{7}{x}, \frac{5}{2x^2+3x}$

Solution: $\boldsymbol{x(2x+3)}$

e. $\frac{x-9}{x^2+8x+16}, \frac{x}{x^2+7x+12}$

Solution: $\boldsymbol{(x+4)^2(x+3)}$

Once you find the LCD, you complete the operation just like you would with fractions. Try these problems:

f. $\frac{3}{x+3} + \frac{2}{x-3}$

Solution: $\boldsymbol{\frac{5x-3}{(x+3)(X-3)}}$

g. $\frac{6x+7}{x^2-4} + \frac{2}{x-2}$

Solution: $\boldsymbol{\frac{8X+11}{(X+2)(X-2)}}$

h. $\frac{10}{6m^2} - \frac{2n}{5m^3}$

Solution: $\boldsymbol{\frac{50m-12n}{30m^3}}$

i. $\frac{6}{8a+4} + \frac{3a}{8}$

Solution: $\boldsymbol{\frac{3a^2+6a+12}{8(a+2)}}$ ***or*** $\boldsymbol{\frac{3(a^2+2a+4)}{8(a+2)}}$

j. $\frac{x}{x^2+5x+6} - \frac{2}{x^2+3x+2}$

Solution: $\boldsymbol{\frac{x^2-x-6}{(x+1)(x+2)(x+3)} = \frac{x-3}{(x+1)(x+3)}}$***It is important for students to see that the final solution requires more factoring in order to have simplest form.***

Operations with Rational Expressions

Thinking about operations with rational numbers, or fractions, will help us perform addition, subtraction, multiplication, and division with rational expressions. We will use examples involving fractions to help us extend our thinking to dealing with fractions with variables, or rational expressions.

- Simplifying Rational Expressions

Think about the fraction $\frac{108}{210}$. What operation do we use to rewrite this fraction in simplest form?

What is the possible obstacle in using this operation to simplify fractions?

Let's try to simplify another way. Find the prime factorization of the numerator and denominator of the fraction above. Use this form of the numerator and denominator to quickly simplify the fraction.

Now let's think about the fraction: $\frac{x^2-9}{x^2+7x+12}$. How can we use the idea of prime factorization to help us simplify this rational expression?

Try this one: $\frac{4-x}{x-4}$ How can factoring help us simplify this rational expression?

- Multiplying and Dividing Rational Expressions

We now need to think about multiplying fractions. Take a minute to discuss with a partner how you would solve the following problem. Try to find more than one way and show your results below:

$$\frac{4}{14} \cdot \frac{24}{10}$$

Which method from above do you think would be easiest to extend to multiplication of rational expressions? Why?

Take a look at $\frac{5x^2}{x^2-4} \cdot \frac{x+2}{10x^3}$. Use your ideas from above to help you multiply these two fractions.

Now try this one: $\frac{3x+6}{x^2-9} \cdot \frac{4x+12}{6x^2+12x}$

What if we change the fraction multiplication problem that we started with to a division problem? Talk to your partner about how to solve the problem below:

$$\frac{4}{14} \div \frac{24}{10}$$

What is the one difference in solving a fraction division problem versus a fraction multiplication problem?

Apply that idea to this problem: $\frac{4x+8}{8x} \div \frac{x^2-4}{6x^2}$

Let's try one more: $\frac{x^2-2x-15}{3x^2+12x} \div \frac{x^2-9}{x^2+4x}$

- Adding and Subtracting Rational Expressions

The idea of using the processes for operations with fractions to guide us as we operated with rational functions continues, but addition and subtraction may seem a little more involved. Just like with fractions it is necessary to have common denominators in both rational expressions before you can add or subtract. Think about the fraction addition problem $\frac{3}{10} + \frac{1}{6}$. What is the least common denominator (LCD)?

You might be able to quickly realize that 30 is the LCD, but why is it 30? Turn to your partner and explain a couple of ways of finding a common denominator.

When thinking about denominators like $x + 2$ or $x - 3$ it becomes important to understand what makes a LCD. In the fraction problem above, you might have been able to say that 30 is the

LCD because it is the smallest number that both 10 and 6 divide into, but how do you create that number if it isn't obvious? (Hint: Think about prime factorization.)

When dealing with rational expressions, factoring is key. You must find all of the factors of each denominator to know what the LCD should be. Let's try some. Find the LCD for the following problems:

a. $\frac{3}{5a}, \frac{b}{4a^2}$

b. $\frac{4}{x+5}, \frac{3}{x-5}$

c. $\frac{2x}{x+2}, \frac{x+1}{x^2-3x-10}$

d. $\frac{7}{x}, \frac{5}{2x^2+3x}$

e. $\frac{x-9}{x^2+8x+16}, \frac{x}{x^2+7x+12}$

Once you find the LCD, you complete the operation just like you would with fractions. Try these problems:

f. $\frac{3}{x+3} + \frac{2}{x-3}$

g. $\frac{6x+7}{x^2-4} + \frac{2}{x-2}$

h. $\frac{10}{6m^2} - \frac{2n}{5m^3}$

i. $\frac{6}{8a+4} + \frac{3a}{8}$

j. $\frac{x}{x^2+5x+6} - \frac{2}{x^2+3x+2}$

Characteristics of Rational Functions Task

Back to Task Table

Math Goals

- Find and explain characteristics of rational functions, including domain, range, zeros, points of discontinuity, asymptotes, and end behavior.

Georgia Standards of Excellence

MGSE9-12.F.IF.7 Graph functions expressed algebraically and show key features of the graph both by hand and by using technology.

MGSE9-12.F.IF.7d Graph rational functions, identifying zeros and asymptotes when suitable factorizations are available, and showing end behavior.

MGSE9-12.F.IF.9 Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one function and an algebraic expression for another, say which has the larger maximum. (Limit to radical and rational functions.)*

Standards for Mathematical Practice

1. Use appropriate tools strategically
2. Attend to precision
3. Look for and make use of structure
4. Look for and express regularity in repeated reasoning

Introduction

This task could be done as a student focused task or the instructor may choose to lead the students through the beginning of the activity to remind them of knowledge of polynomial functions they should already have. In each example, students are asked to remember previously learned material and then apply it to a new situation. Graphing utilities are used to investigate end behavior since the topic of horizontal asymptotes has not yet been discussed for complex rational functions, however, students should start making conclusions based on their investigations. As the examples progress, different characteristics of rational functions will emerge as will the importance of zeros and undefined values in determining behavior of the example. The most important piece of the task is the summary page where students have the opportunity to explain how to: find vertical asymptotes; find intercepts; determine domain and range; and, to explain the relationship between a horizontal asymptote and end behavior.

Characteristics of Rational Functions

Now that we have worked with rational expressions, it is time to look at rational functions themselves. Since a rational function is the quotient of two polynomial functions it is important to first look at the characteristics of the individual polynomials.

Let's investigate the polynomial $g(x) = x^2 + 3x - 10$. What facts can you write about g(x)?

The polynomial is a quadratic function, so it has a degree of 2. g(x) also has a positive leading coefficient.

What is the Domain? How do you determine the Domain?

The domain is all real numbers or $(-\infty, \infty)$. The domain of a function is the set of x-values that result in a real y-value.

What is the Range? How do you determine the Range?

The range of the function is $[12.25, \infty)$. Using our knowledge of quadratic functions, we can find what we know to be the least y-value (minimum) of the function through algebraic means or using a graphing calculator, and then include all values equal to or greater than the minimum value.

Where are the Roots or Zeros found? What are some different ways you know to find them?

Roots or zeros: x = -5, 2. Roots can be found by setting a function equal to zero and then factoring to solve for x, or using the quadratic formula, or by using a graphing utility to locate the x-intercepts of a curve.

What is the End Behavior? How do you know?

The y-values of the function approach infinity as x approaches both negative and positive infinity. This is true for all positive lead coefficient polynomials with an even degree. In limit notation, the end behavior would be $\lim_{x \to -\infty} g(x) = \infty$ ***and*** $\lim_{x \to \infty} g(x) = \infty$.

Let's investigate $f(x) = x + 1$. What facts can you write about f(x)?

f(x) is a linear function with a positive lead coefficient and a degree of 1.

What is the Domain?

The domain of f is all real numbers or $(-\infty, \infty)$.

What is the Range?

The range of f is all real numbers, so $(-\infty, \infty)$.

What are the Roots or Zeros?

There is only one root found when x = -1.

What is the End Behavior? How do you know?

The end behavior of f would be $\lim_{x\to-\infty} f(x) = -\infty$ and $\lim_{x\to\infty} f(x) = \infty$. We know this because all odd degree polynomials with a positive lead coefficient have this end behavior.

Now let's consider the case of the rational function $r(x) = \frac{f(x)}{g(x)}$ where *f* and *g* are the polynomial functions above. Write the expression for the function r(x).

What is the domain of *r*(*x*)? Which function, *f* or *g*, affects the domain the most? Why?

The domain of r is $(-\infty, 5) \cup (-5, 2) \cup (2, \infty)$. The domain of r is most affected by the function g because the rule for r "works" for every x-value except for the two x-values -5 and 2. This is because these two x-values are zeros for the function g and thus, the function r would be dividing by zero if we included them. Therefore, r is undefined when x = -5 or 2. The graph of r will have vertical asymptotes at x = -5 and 2.

What do you think the range of *r*(*x*) will be? Why is this so difficult to determine?

The range of a rational function is often difficult to determine without looking at a graph of the function. The purpose of this activity is to provide students with the skills needed to piece together a picture of a rational function without the use of a graphing utility, so that they can determine the range of a rational function with greater ease.

What are the roots or zeros of *r*(*x*)? Which function helps you find them?

The roots of r can be found be setting the function equal to zero. The only way a rational function can equal zero is if the numerator of the function is equal to zero (if the denominator equals zero, then the function is undefined at the x-value where this occurs). By setting f equal to zero, we know that r has a root when x = -1.

What do you think the end behavior will be? Why?

The end behavior of a rational function can also be difficult to determine. Both f and g have end behaviors that approach infinity or negative infinity. Only function f approaches negative

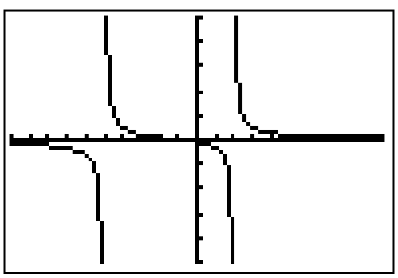
infinity; function g approaches infinity when x approaches both negative and positive infinity. Students then need to decide what this means for the function r. The real question they need to ask is which function, f or g, approaches infinity faster? Since f is a degree 1 polynomial and g is a degree 2 polynomial, we can be certain that g is approaching infinity faster than f (this is a great time to discuss how we know this for sure). This means that as x approaches infinity or negative infinity, r(x) will be approaching 0 since g will be much larger than f. In limit notation we would say $\lim_{x \to \pm\infty} r(x) = 0$***. This will be looked at further later in the activity with the aid of a graphing utility and horizontal asymptotes themselves will be focused on in a separate activity.***

Where will $r(x)$ intersect the y-axis? How do you know?

A function will intersect the y-axis when the function is evaluated at x = 0. In other words, r(0) will give the value of the y-intercept. In this case $r(0) = \frac{-1}{10}$.

Now let's look at the graph of $r(x)$ using your calculator.

Solution:



What value does $r(x)$ approach as x approaches infinity? Negative infinity? How could you describe this behavior?

r appears to be approaching the x-axis (so y = 0) as x approaches positive and negative infinity. Since the curve approaches the value of 0 but does not actually equal this value as x gets infinitely big, students should recognize this as an example of an asymptote. However, instead of a vertical asymptote like at x = -5 and 2, this is a horizontal asymptote.

What occurs at the x-values when $g(x) = 0$? Do you think this will happen every time the denominator is equal to zero?

Vertical asymptotes occur at the x-values when g(x) = 0. Anytime a rational function is undefined, there will either be a vertical asymptote or a point of discontinuity in the graph. A point of discontinuity occurs when there is a duplicate linear factor in the numerator as well as in the denominator. When the factors cancel, there is not an asymptote, but rather a single point on the curve that does not exist.

At what *x*-values does *r*(*x*) change signs (either + to – or vice-versa)? What else occurs at these *x*-values?

r(x) changes signs when x = -5, -1, and 2. Each of these x-values is the location of a vertical asymptote or a zero for the function r. By knowing the location of these "critical points" and by using a simple sign chart, we can come up with a rough sketch of r without our graphing utility.

Based on the graph from your calculator, what is the range of *r*(*x*)?

Based on the piece of the curve between the vertical asymptotes, the range will be all real numbers or $(-\infty, \infty)$.

When is the function increasing? decreasing? Are there any local extrema? How can you be certain?

Based on the graph, we can see that r(x) is never increasing as x moves from negative infinity to positive infinity. That means that r(x) is decreasing over the domain $(-\infty, \mathbf{5}) \cup (-\mathbf{5}, \mathbf{2}) \cup (\mathbf{2}, \infty)$. ***(r cannot increase nor decrease at an undefined x-value). There cannot be any local extremum for the function r(x) since there is no change in behavior from increasing to decreasing or vice-versa.***

Let's try a few more problems and see if we can discover any patterns…

1. Let $f(x) = 5$ and $g(x) = x^2 - 6x + 8$. Let $r(x) = \dfrac{f(x)}{g(x)}$.

What is the domain of *r*(*x*)? Which function, *f* or *g*, affects the domain the most? Why?

D: $(-\infty, \mathbf{2}) \cup (\mathbf{2}, \mathbf{4}) \cup (\mathbf{4}, \infty)$. ***The undefined values are zeros of the function g(x) which can be found by factoring, graphing, etc.***

What do you think the range of *r*(*x*) will be?

Answers will vary here, some students may think all real numbers like the first example, others may suggest leaving out some y-values. They will not be certain until they produce by hand or with a graphing utility a more accurate graph.

What are the roots or zeros of *r*(*x*)? Which function helps you find them?

The roots of r(x) are the zeros of the function f(x) = 5. Since f has no zeros, the function r has no zeros.

What do you think the end behavior will be? Why?

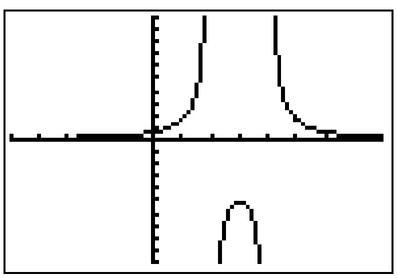
Depending on the discussion during the example problem, students may or may not have a better idea of what will happen to r(x) as x approaches positive and negative infinity. In this case, however, the numerator never changes its values, so it should be much easier to see that as x increases, the value of r will approach zero. In limit notation, $\lim_{x \to \pm\infty} r(x) = 0$.

Where will $r(x)$ intersect the y-axis? How do you know?

The y-intercept of r occurs when x = 0, so $r(0) = \frac{5}{8}$.

Now let's look at the graph of $r(x)$ using your calculator.

Solution:



What value does $r(x)$ approach as x approaches infinity? Negative infinity? How could you describe this behavior? Why do you think this happens?

Using the graph, we can see that as x approaches positive and negative infinity, r(x) is approaching 0 in both directions. Once again, this can be described as a horizontal asymptote for the function r(x) at y = 0.

What occurs at the x-values when $g(x) = 0$? Do you think this will happen every time the denominator is equal to zero?

Whenever g(x) = 0, there should be a vertical asymptote at those x-values. The only exception occurs when there is a duplicate factor in the numerator. In this case, there will be a "hole" in the graph at the undefined x-value.

At what x-values does $r(x)$ change signs (either + to – or vice-versa)? What else occurs at these x-values?

r(x) changes signs only at the vertical asymptotes which are located at x = 2 and x = 4. Since there are no zeros for r, the only sign changes occur at the vertical asymptotes.

When is the function increasing? decreasing? Are there any local extrema? How can you be certain?

The function is increasing over the interval from $(-\infty, 2) \cup (2, 3)$ and decreasing over the interval from $(3, 4) \cup (4, \infty)$. There is a local maximum at the point (3, -5), and we are certain of this because the function r changes from increasing y-values to decreasing y-values when x = 3.

Based on the graph from your calculator, what is the range of $r(x)$?

Using the graph as a guide, the range of the function will be $(-\infty, -5] \cup (0, \infty)$. We need to include y = -5 because this is the local maximum value and the function actually equals -5 when x = 3. We do not include y = 0 because r(x) never equals zero since the numerator is a constant.

2. Let $r(x) = \dfrac{2x^2 + 7x - 4}{x^3 - 1}$.

What is the domain of $r(x)$?

D: $(-\infty, 1) \cup (1, \infty)$. x = 1 is the only zero of the denominator.

What do you think the range of $r(x)$ will be?

Answers will vary although the graph will be necessary for most students to determine the range.

What are the roots or zeros of $r(x)$?

By factoring the numerator and setting it equal to zero, we can find the roots of x = ½ and -4.

What do you think the end behavior will be?

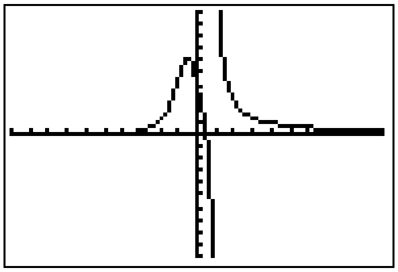
As in the previous problem, the degree of the denominator is greater than the degree of the numerator, so the y-values should approach zero as x increases towards infinity. In limit notation, $\lim_{x \to \pm\infty} r(x) = 0$.

Where will $r(x)$ intersect the y-axis? How do you know?

The y-intercept of r occurs when x = 0, so r(0) = 4.

Now let's look at the graph of $r(x)$ using your calculator.

Solution:



What value does $r(x)$ approach as x approaches infinity? Negative infinity? How could you describe this behavior? Why do you think this happens?

Using the graph, we can see that as x approaches positive and negative infinity, r(x) is approaching 0 in both directions. Once again, this can be described as a horizontal asymptote for the function r(x) at y = 0.

What occurs at the x-values when the denominator is equal to zero? Do you think this will happen every time?

Whenever the denominator is equal to zero, there should be a vertical asymptote at those x-values. The only exception occurs when there is a duplicate factor in the numerator. In this case, there will be a "hole" in the graph at the undefined x-value.

At what x-values does $r(x)$ change signs (either + to – or vice-versa)? What else occurs at these x-values?

r(x) changes signs at the vertical asymptote which is located at x = 1 and at the zeros of the function located at x = -4 and ½ .

When is the function increasing? decreasing? Are there any local extrema? How can you be certain?

Before we can answer the increasing and decreasing question, we really need to analyze the behavior of the function as x approaches negative infinity. The y-values are positive until x = -4 and then they become negative. However, the y-values of the curve are supposed to be approaching 0, so at some point, the curve must turn back towards the x-axis. This indicates a local minimum for our function. This minimum is easy to miss if one simply looks at the graph without thinking about the behavior of the function. The local minimum of the function is located at (-7.795, -.133) and a local maximum at (-.514, 6.224). Therefore, the

function is increasing from (-7.795, -.514) and decreasing from $(-\infty, -7.795) \cup (-.514, 1) \cup (1, \infty)$***. The extremum are located at the x-values where the function changes from increasing to decreasing or vice-versa.***

Based on the graph from your calculator, what is the range of $r(x)$?

Using the graph as a guide, the range of the function will be (- ∞, ∞). For x-values less than 1, all y-values from negative infinity to 6.224 are possible. For x-values greater than 1, all y-values from 0 to positive infinity are possible. This means that the range of the entire function is all real numbers or (- ∞, ∞).

3. Let $r(x) = \dfrac{4x+1}{4-x}$.

What is the domain of $r(x)$?

D: $(-\infty, 4) \cup (4, \infty)$. ***x = 4 is the only zero of the denominator.***

What do you think the range of $r(x)$ will be?

Answers will vary although the graph will be necessary for most students to determine the range.

What are the roots or zeros of $r(x)$?

The root of r is located at x = -¼.

What do you think the end behavior will be?

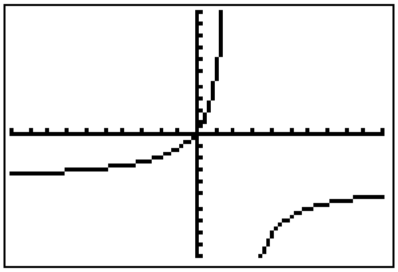
The degree of the numerator is equal to the degree of the denominator, so by comparison of large values of x, we can see that the function will be approaching the value of y = -4.

Where will $r(x)$ intersect the y-axis? How do you know?

r(x) will intersect the y-axis when x = 0, so r(0) = ¼.

Now let's look at the graph of $r(x)$ using your calculator.

Solution:



What value does $r(x)$ approach as x approaches infinity? Negative infinity? How could you describe this behavior? Why do you think this happens?

Using the graph, we can see that as x approaches positive and negative infinity, r(x) is approaching 4 in both directions. Once again, this can be described as a horizontal asymptote for the function r(x) at y = 4.

What occurs at the x-values when the denominator is equal to zero? Do you think this will happen every time?

Whenever the denominator is equal to zero, there should be a vertical asymptote at those x-values. The only exception occurs when there is a duplicate factor in the numerator. In this case, there will be a "hole" in the graph at the undefined x-value.

At what x-values does $r(x)$ change signs (either + to – or vice-versa)? What else occurs at these x-values?

r(x) changes signs at the vertical asymptote which is located at x = 4 and at the zero of the function located at x = ¼.

When is the function increasing? decreasing? Are there any local extrema? How can you be certain?

Based on the graph, we can see that r(x) is never decreasing as x moves from negative infinity to positive infinity. That means that r(x) is increasing over the defined values $(-\infty, 4) \cup (4, \infty)$ ***(r cannot increase nor decrease at an undefined x-value). There cannot be any local extremum for the function r(x) since there is no change in behavior from increasing to decreasing or vice-versa.***

Based on the graph from your calculator, what is the range of $r(x)$?

Using the graph as a guide, the range of the function will be (- ∞, -4) U (-4, ∞). Visually from the graph it appears that the function only approaches -4 and will never actually equal -

4. Algebraically, we can prove this to be true. By setting r(x) = -4 and solving the resultant equation, we can show it is impossible for the function to equal -4.

Now let's summarize our findings and conclusions:

When is the domain of a rational function not (- ∞, ∞)? So what is your advice on how to determine the domain of a rational function?

The domain of a rational function is not (- ∞, ∞) when the denominator of the function could potentially equal zero for certain x-values. To determine the domain of a rational function, begin by setting the denominator of the function equal to zero and solving for the zeros of the denominator. These solutions should then be excluded from the domain of the rational function since including them would indicate dividing by zero.

Is the range of a rational function difficult to find? Why or why not?

Given the equation and a graph of a rational function, the range is not difficult find, because any extremum or horizontal asymptotes can be taken into account as to whether or not to include or exclude certain y-values. Without the graph, it can be very difficult to determine the range of a rational function.

How do you find the zeros or roots of a rational function?

The zeros of a rational function can be found by setting the numerator of the rational function equal to zero and then solving for the solutions to that equation. The only way for a rational function to equal zero is if the numerator can equal zero.

How do you know where to find vertical asymptotes?

Vertical asymptotes are located at undefined values of a rational function, that is, the x-values where the denominator is equal to zero. These x-values are excluded from the domain of the function and thus, a graph can never cross a vertical asymptote since that particular x-value does not exist in the function's domain.

What does a horizontal asymptote tell you about a rational function? Are they easy to locate? Do you know of any shortcuts to find them?

A horizontal asymptote tells the end behavior of a function, or what happens to the y-values as x approaches positive and negative infinity. Horizontal asymptotes are extremely easy to locate using a comparison of the degrees for the numerator and denominator of the rational function. Further exploration of horizontal asymptotes is a task unto itself in this unit. Many students feel that a function cannot cross a horizontal asymptote because function cannot cross a vertical asymptote. This is a false assumption as evidenced by our first example and

problem #2. A horizontal asymptote only tells us the behavior of the y-values of a rational function at extreme x-values, i.e. as x approaches positive or negative infinity.

How do you know where a rational function will intersect the y-axis? Will a rational function always have a y-intercept? Can you give an example?

You can find the y-intercept of any function by evaluating the function at x = 0. Rational functions do not have to have a y-intercept as there could be a vertical asymptote at x = 0.

What possible things could occur at the x-values where a rational function changes signs?

Rational functions can change signs at the x-values where you find vertical asymptotes and/or zeros of the function. Using these points on a sign chart, one can determine where a rational function has positive or negative y-values. This information along with the knowledge of asymptotes, y-intercepts, and zeros can create a rough sketch of the graph of any rational function.

Do you think it would be possible to use all of our knowledge of rational functions to create sketch without using the graphing calculator? Can you explain how this would work to another classmate?

Answers will vary but should be somewhat similar to the explanation given above.

Characteristics of Rational Functions

Now that we have worked with rational expressions, it is time to look at rational functions themselves. Since a rational function is the quotient of two polynomial functions it is important to first look at the characteristics of the individual polynomials.

Let's investigate $g(x) = x^2 + 3x - 10$. What facts can you write about g(x)?

What is the Domain? How do you determine the Domain?

What is the Range? How do you determine the Range?

Where are the Roots or Zeros found? What are some different ways you know to find them?

What is the End Behavior? How do you know?

Let's investigate $f(x) = x + 1$. What facts can you write about f(x)?

What is the Domain?

What is the Range?

What are the Roots or Zeros?

What is the End Behavior? How do you know?

Now let's consider the case of the rational function $r(x) = \frac{f(x)}{g(x)}$ where f and g are the polynomial functions above. Write the expression for the function f(x).

What is the domain of $r(x)$? Which function, f or g, affects the domain the most? Why?

What do you think the range of $r(x)$ will be? Why is this so difficult to determine?

What are the roots or zeros of $r(x)$? Which function helps you find them?

What do you think the end behavior will be? Why?

Where will $r(x)$ intersect the y-axis? How do you know?

Now let's look at the graph of $r(x)$ using your calculator.

What value does $r(x)$ approach as x approaches infinity? Negative infinity? How could you describe this behavior?

What occurs at the x-values when $g(x) = 0$? Do you think this will happen every time the denominator is equal to zero?

At what x-values does $r(x)$ change signs (either + to – or vice-versa)? What else occurs at these x-values?

Based on the graph from your calculator, what is the range of $r(x)$?

When is the function increasing? decreasing? Are there any local extrema? How can you be certain?

Let's try a few more problems and see if we can discover any patterns…

1. Let $f(x) = 5$ and $g(x) = x^2 - 6x + 8$. Let $r(x) = \dfrac{f(x)}{g(x)}$.

What is the domain of $r(x)$? Which function, f or g, affects the domain the most? Why?

What do you think the range of $r(x)$ will be?

What are the roots or zeros of $r(x)$? Which function helps you find them?

What do you think the end behavior will be? Why?

Where will $r(x)$ intersect the y-axis? How do you know?

Now let's look at the graph of $r(x)$ using your calculator.

What value does $r(x)$ approach as x approaches infinity? Negative infinity? How could you describe this behavior? Why do you think this happens?

What occurs at the x-values when $g(x) = 0$? Do you think this will happen every time the denominator is equal to zero?

At what x-values does $r(x)$ change signs (either + to – or vice-versa)? What else occurs at these x-values?

When is the function increasing? decreasing? Are there any local extrema? How can you be certain?

Based on the graph from your calculator, what is the range of $r(x)$?

2. Let $r(x) = \dfrac{2x^2 + 7x - 4}{x^3 - 1}$.

What is the domain of $r(x)$?

What do you think the range of $r(x)$ will be?

What are the roots or zeros of $r(x)$?

What do you think the end behavior will be?

Where will $r(x)$ intersect the y-axis? How do you know?

Now let's look at the graph of $r(x)$ using your calculator.

What value does $r(x)$ approach as x approaches infinity? Negative infinity? How could you describe this behavior? Why do you think this happens?

What occurs at the x-values when the denominator is equal to zero? Do you think this will happen every time?

At what x-values does $r(x)$ change signs (either + to – or vice-versa)? What else occurs at these x-values?

When is the function increasing? decreasing? Are there any local extrema? How can you be certain?

Based on the graph from your calculator, what is the range of $r(x)$?

3. Let $r(x) = \dfrac{4x+1}{4-x}$.

What is the domain of $r(x)$?

What do you think the range of $r(x)$ will be?

What are the roots or zeros of $r(x)$?

What do you think the end behavior will be?

Where will $r(x)$ intersect the y-axis? How do you know?

Now let's look at the graph of $r(x)$ using your calculator.

What value does $r(x)$ approach as x approaches infinity? Negative infinity? How could you describe this behavior? Why do you think this happens?

What occurs at the x-values when the denominator is equal to zero? Do you think this will happen every time?

At what x-values does $r(x)$ change signs (either + to – or vice-versa)? What else occurs at these x-values?

When is the function increasing? decreasing? Are there any local extrema? How can you be certain?

Based on the graph from your calculator, what is the range of $r(x)$?

Now let's summarize our findings and conclusions.

When is the domain of a rational function not (- ∞, ∞)? So what is your advice on how to determine the domain of a rational function?

Is the range of a rational function difficult to find? Why or why not?

How do you find the zeros or roots of a rational function?

How do you know where to find vertical asymptotes?

What does a horizontal asymptote tell you about a rational function? Are they easy to locate? Do you know of any shortcuts to find them?

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How do you know where a rational function will intersect the y-axis? Will a rational function always have a y-intercept? Can you give an example?

What possible things could occur at the x-values where a rational function changes signs?

Do you think it would be possible to use all of our knowledge of rational functions to create sketch without using the graphing calculator? Can you explain how this would work to another classmate?

Horizontal Asymptotes: How do we find them?

Back to Task Table

Math Goals

- Identify horizontal asymptotes of rational expressions

Georgia Standards of Excellence

MGSE9-12.F.IF.7d Graph rational functions, identifying zeros and asymptotes when suitable factorizations are available, and showing end behavior.

Standards for Mathematical Practice

1. Use appropriate tools strategically
2. Attend to precision
3. Look for and make use of structure
4. Look for and express regularity in repeated reasoning

Introduction

This particular exercise is intended to use exploration with graphing utilities to examine the characteristics of the range values. With the intended goal of generalizing the process of identifying horizontal asymptotes, students will gain a greater understanding of this aspect of rational functions and develop their investigative strategies as well. While we can easily give students the derived algorithms for determining the placement of a horizontal asymptote, the value in the investigation is not to be ignored.

Do not neglect the fact that many students will fail to see a rational function as the division of two functions. Try to drive home the fact that students should look to f(x) and g(x) and their individual characteristics. Also, this will be a good time to revisit any procedural details about using appropriate technology to generate such tables.

Horizontal Asymptotes: How do we find them?

As we discuss the characteristics of rational functions, we know that it is important to consider the properties of the individual functions. Knowing about the individual functions helps us to know about the rational function. But as we discuss the details, let us consider the range values of the rational function. To do this it will be important consider the range values through a table and the graph. But to do this we are going to look at very large values for x.

1. Let $f(x) = 5$ and $g(x) = x^2 - 6x + 8$. Let $r(x) = \dfrac{f(x)}{g(x)}$. Complete the table of values of r(x).

x	r(x)
1000	***0.0000050301***
700	***0.0000102921***
300	***0.0000566842***
200	***0.0001288394***
100	***0.0005314626***
0	***0.6250000000***
-100	***0.0004713424***
-200	***0.0001213357***
-300	***0.0000544615***
-700	***0.0000101172***
-1000	***0.0000049701***

What type of trends do you see from the y-values of this function?

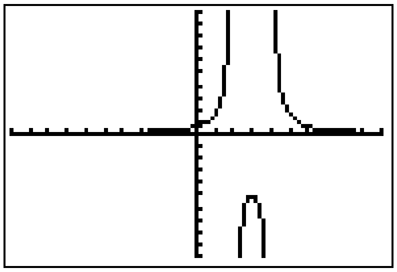
Look for students to discuss the fact that the numbers are getting ridiculously smaller. Will probably need to be ready to address situations about how some calculators will note in scientific notation. Also, depending on the setting, many calculators will not calculate to such decimal places.

As the x-values head toward infinity, is there any significance to the y-values?

Here is a great place to evaluate students' understanding of end behavior.

Examine the graph in your graphing utility to get a better picture.

How does the graph relate to your table?



With the largest y value for this section of the graph being 0.625, the values on decrease either direction. Understanding key concepts of decimal and/or fraction values is critical here. While this seems quite simple, take the time to check with students about understanding. Another hitch that might be encountered here will have to do with a graphing utility that you may be using. Many calculators will revert to scientific notation with these values. Many times students are confused about notation that the calculator will give. We know that the degrees of the numerator and the denominator play into the situation with the horizontal asymptote, so be prepared to encourage student conversation in this direction if needed.

2. Let $r(x) = \dfrac{2x^2 + 7x - 4}{x^3 - 1}$. Complete the table of values for r(x).

x	r(x)
1000	***0.002007***
700	***0.002871***
300	***0.006744***
200	***0.010175***
100	***0.020696***
0	***4***
-100	***-0.0193***
-200	***-0.00982***
-300	***-0.00659***
-700	***-0.00284***
-1000	***-0.00199***

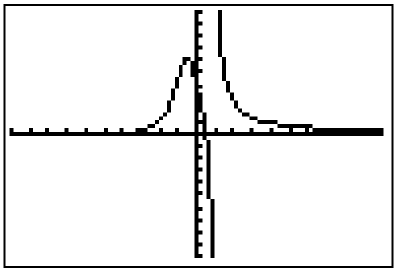
What type of trends do you see from the y-values of this function?

Here we have a similar situation to number 1. Again, it might be helpful to direct students to think about end behavior.

As the x-values head toward infinity, is there any significance to the y-values?

Again, be aware of the notation that the technology being used might show. Do not let technology issues hinder the understanding that is being sought here.

Examine the graph in your graphing utility to get a better picture.
How does the graph relate to your table?



Again with this function, though the degrees of the numerator and denominator are larger, the fact remains that the degree in the denominator is greater. We hope that students might notice this, but understandably this might not be a natural observation that they make. As conversation about these questions continues, interjecting this thought might be necessary. Similar to the last example, the calculator interpretation might need to be addressed. The main difference in this example from the first is to address the varying degrees. Providing other examples for the students with a variety of polynomials with different degrees will be helpful, just make sure that the larger of the two degrees is in the denominator.

3. Let $r(x) = \dfrac{4x+1}{4-x}$. Complete the table of values of r(x).

x	r(x)
1000	***-4.01707***
700	***-4.02443***
300	***-4.05743***
200	***-4.08673***
100	***-4.17708***
0	***0.25***
-100	***-3.83654***
-200	***-3.91667***
-300	***-3.94408***
-700	***-3.97585***
-1000	***-3.98307***

What type of trends do you see from the y-values of this function?

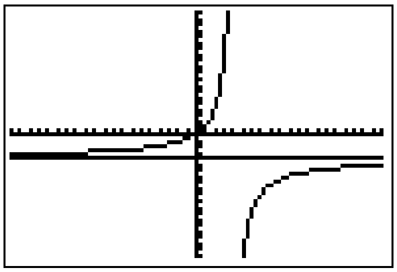
Look for students to observe that the y-values are approaching -4. This is the first chance to see if students are legitimately analyzing the values and connecting it to end behavior.

As the x-values head toward infinity, is there any significance to the y-values?

Again, look for students to express an understanding of the trends.

Examine the graph in your graphing utility to get a better picture.

How does the graph relate to your table?



In this situation, we are now looking at a numerator and denominator that have equal degrees. Again, this may not be a natural thought for the students, but it is the direction that we want them to head towards. The use of the table and the graph will allow students to see the -4 show up as end behavior. In a graphing utility, having students graph f(x) = -4 would enhance the picture. Creating additional examples with equal degrees will give students additional chances to establish the pattern of the ratio of the coefficients with the highest degree.

4. Let $r(x) = \dfrac{3x^2 + 27}{2x^2 - 6x - 8}$.
Complete the table of values of r(x).

x	r(x)
1000	***1.504533***
700	***1.506496***
300	***1.515371***
200	***1.523340***
100	***1.548422***
0	***-3.375000***
-100	***1.458188***
-200	***1.478311***
-300	***1.485362***
-700	***1.493638***
-1000	***1.495533***

What type of trends do you see from the y-values of this function?

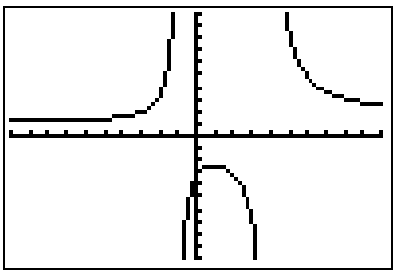
The value at zero really has little consequence to what students are ultimately looking for and it may result in some questions from students as to why they are even considering this value.

As the x-values head toward infinity, is there any significance to the y-values?

Looking for students to connect the 1.5 that is being approached to the ratio found in the function. This will probably take some guidance on the part of the teacher. Be prepared to ask some guiding questions here.

Examine the graph in your graphing utility to get a better picture.

How does the graph relate to your table?



With this example, hopefully it will be easier for students to make the connection that the value the table is leading toward, in both directions, is the same as the ratio of the coefficients from the highest degree variable in the numerator and the denominator. This example shows much more clearly than the last, but it is good to use both to encourage the understanding.

5. Consider the range of the function, $R(x) = \dfrac{x^2 - 2x + 5}{x - 6}$. Complete the table of values of r(x).

x	r(x)
1000	***1004.029175***
700	***704.041787***
300	***304.098639***
200	***204.149485***
100	***104.308511***
0	***-0.833333***
-100	***-96.273585***
-200	***-196.140777***
-300	***-296.094771***
-700	***-696.041076***
-1000	***-996.028827***

What type of trends do you see from the y-values of this function?

It will be interesting to see what students notice here. If the pattern has been established in previous questions to look at the degrees, then many students may be aware that something is different here. If not, the teacher will need to direct here. The value of 1000 and -1000 are

here, but there is not common value as there has been before. This will hopefully lead students to assume that there is no horizontal asymptote.

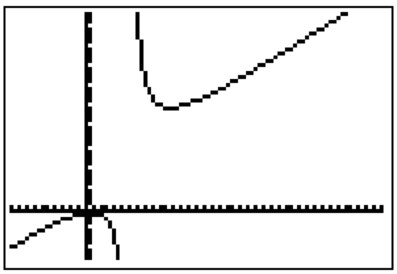
As the x-values head toward infinity, is there any significance to the y-values?

Again, students should note that this function is different than the ones in previous questions.

Examine the graph in your graphing utility to get a better picture.
How does the graph relate to your table?

Again, anticipate that students will see many inconsistencies to previous questions and will note those differences.

Looking at the function how is this one different from the others that we have considered?



Here, students should see that there is no trend developing other than increasing and decreasing values, respectively. With that in mind, students should then use their previous answers to decide how this rational function is different from the rest. The door is now open to introduce slant asymptotes. Looking at the graph is another way to get students to see the stark difference of this function from the others that we have looked at recently. Adjusting the window of this graph will also be critical to seeing a good representation of the picture.

Let's look back at the first task of this unit to help us find the slant asymptote. We can use long division to help us find the equation.

Remind students how to use long division to divide the denominator into the numerator. The quotient of the two is the equation of the slant asymptote. When dealing with end behavior, the x-values are so extreme that it is important to note that the remainder has no effect on the slant asymptote so the equation is only the polynomial portion of the quotient (no remainder).

Now let's summarize our findings and conclusions.

Teacher note: After working through the horizontal asymptote task, it may be beneficial to have students complete the summary again with their new knowledge. In this way, they may

add to and expand upon their previous ideas of piecing together the behavior and characteristics of rational functions.

When is the domain of a rational function not (- ∞, ∞)? So what is your advice on how to determine the domain of a rational function?

Only in extremely limited cases (which we have not discussed) would the domain of a rational function be (- ∞, ∞). It would be good to hear students comment on the fact that the limitation on the domain occurs when the denominator is zero. This would illuminate cases where the denominator is not zero for any real x values (i.e., $x^2 + 3$ as a denominator).

Is the range of a rational function difficult to find? Why or why not?

When we consider the degree of the function in the numerator and the denominator and compare the two, then we can make clear decisions about horizontal asymptotes.

How do you find the zeros or roots of a rational function?

Finding the zeroes of the numerator will also give the zeroes of the rational function. When the numerator is zero and the denominator is not, then the resulting value is zero.

How do you know where to find vertical asymptotes?

Students have factored a great deal by this point. Having them factor the numerator and the denominator is always a good first move. It is too simple of an answer to have students state that the vertical asymptotes are found where the denominator is zero. They must also remember to see if the factors in the denominator cancel with any in the numerator such that it creates a hole in the graph.

What does a horizontal asymptote tell you about a rational function? Are they easy to locate? Do you know of any shortcuts to find them?

Horizontal asymptotes give a graphical description of the range of the function. They are fairly easy to locate as long as we consider the degree of the numerator and the denominator and compare the two. When the degree of the denominator is greater then we have a horizontal asymptote at zero (plus or minus any vertical shifts). If the degree of the denominator is equal to that of the numerator, then we have a horizontal asymptote at the value equal to the ratio of the coefficient of the highest degree term of the numerator and the denominator.

How do you know where a rational function will intersect the y-axis? Will a rational function always have a y-intercept? Can you give an example?

Finding a y-intercept is always as easy as evaluating f(0). No the function will not always have a y-intercept, there may be a vertical asymptote there. Many examples exist.

What possible things could occur at the x-values where a rational function changes signs?

We can anticipate that there is either a vertical asymptote or zero. There are other points of discontinuity that can cause the function to change signs, but these are less common.

Do you think it would be possible to use all of our knowledge of rational functions to create an accurate sketch without using the graphing calculator? Can you explain how this would work to another classmate?

Identifying and drawing vertical, horizontal, and slant asymptotes is more than half way there when it comes to sketching the graph of a function. After that, we will only need to evaluate a few points to see where the curve lies in respect to the asymptotes.

Horizontal Asymptotes: How do we find them?

As we discuss the characteristics of rational functions, we know that it is important to consider the properties of the individual functions. Knowing about the individual functions helps us to know about the rational function. But as we discuss the details, let us consider the range values of the rational function. To do this it will be important consider the range values through a table and the graph. But to do this we are going to look at very large values for x.

1. Let $f(x) = 5$ and $g(x) = x^2 - 6x + 8$. Let $r(x) = \dfrac{f(x)}{g(x)}$. Complete the table of values of r(x).

x	r(x)
1000	
700	
300	
200	
100	
0	
-100	
-200	
-300	
-700	
-1000	

What type of trends do you see from the y-values of this function?

As the x-values head toward infinity, is there any significance to the y-values?

Examine the graph in your graphing utility to get a better picture.

How does the graph relate to your table?

2. Let $r(x) = \dfrac{2x^2 + 7x - 4}{x^3 - 1}$. Complete the table of values of r(x).

x	r(x)
1000	
700	
300	
200	
100	
0	
-100	
-200	
-300	
-700	
-1000	

What type of trends do you see from the y-values of this function?

As the x-values head toward infinity, is there any significance to the y-values?

Examine the graph in your graphing utility to get a better picture.
How does the graph relate to your table?

3. Let $r(x) = \dfrac{4x + 1}{4 - x}$. Complete the table of values of r(x).

x	r(x)
1000	
700	
300	
200	
100	
0	
-100	
-200	
-300	
-700	
-1000	

What type of trends do you see from the y-values of this function?

As the x-values head toward infinity, is there any significance to the y-values?

Examine the graph in your graphing utility to get a better picture.

How does the graph relate to your table?

4. Let $r(x) = \dfrac{3x^2 + 27}{2x^2 - 6x - 8}$.
Complete the table of values of r(x).

x	r(x)
1000	
700	
300	
200	
100	
0	
-100	
-200	
-300	
-700	
-1000	

What type of trends do you see from the y-values of this function?

As the x-values head toward infinity, is there any significance to the y-values?

Examine the graph in your graphing utility to get a better picture.

How does the graph relate to your table?

5. Consider the range of the function, $R(x) = \dfrac{x^2 - 2x + 5}{x - 6}$. Complete the table of values of r(x).

x	r(x)
1000	
700	
300	
200	
100	
0	
-100	
-200	
-300	
-700	
-1000	

What type of trends do you see from the y-values of this function?

As the x-values head toward infinity, is there any significance to the y-values?

Examine the graph in your graphing utility to get a better picture.
How does the graph relate to your table?

Looking at the function how is this one different from the others that we have considered?

Let's look back at the first task of this unit to help us find the slant asymptote. We can use long division to help us find the equation.

Now let's summarize our findings and conclusions.

When is the domain of a rational function not (- ∞, ∞)? So what is your advice on how to determine the domain of a rational function?

Is the range of a rational function difficult to find? Why or why not?

How do you find the zeros or roots of a rational function?

How do you know where to find vertical asymptotes?

What does a horizontal asymptote tell you about a rational function? Are they easy to locate? Do you know of any shortcuts to find them?

How do you know where a rational function will intersect the y-axis? Will a rational function always have a y-intercept? Can you give an example?

What possible things could occur at the x-values where a rational function changes signs?

Do you think it would be possible to use all of our knowledge of rational functions to create an accurate sketch without using the graphing calculator? Can you explain how this would work to another classmate?

Graphing Rational Functions Without a Calculator

Back to Task Table

Math Goals

- Graph rational functions by hand

Georgia Standards of Excellence

MGSE9-12.F.IF.7 Graph functions expressed algebraically and show key features of the graph both by hand and by using technology. *(Limit to radical and rational functions.)*

MGSE9-12.F.IF.7d Graph rational functions, identifying zeros and asymptotes when suitable factorizations are available, and showing end behavior.

MGSE9-12.F.IF.9 Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one function and an algebraic expression for another, say which has the larger maximum.*

Standards for Mathematical Practice
1. Make sense of problems and persevere in solving them
2. Attend to precision
3. Look for and make use of structure
4. Look for and express regularity in repeated reasoning

Introduction
Using the discoveries from the previous examples, the goal is to produce a best guess as to the behavior for each rational function without using a graphing utility. Emphasis is placed on the use of a sign chart to provide the students with the sign of a rational function on either side of its zeros or on either side of points where the function is undefined. Using this information along with knowledge from the previous task, students should be able to formulate a sketch of the rational function. Students will not be able to find extrema or intervals of increasing and decreasing without the aid of the calculator, however, they should be able to determine when it is necessary to look for extrema using their sketch.

In addition, it will be up to each individual teacher to investigate special cases such as rational functions with points of discontinuity (holes) or rational functions with no vertical asymptotes. The same format of questions can be used as can the sign chart, however, students may need more teacher input when trying to piece the information together.

Graphing Rational Functions Without a Calculator

Let $r(x)=\dfrac{3x^2+27}{2x^2-6x-8}$.

What is the domain of $r(x)$?

By factoring the denominator, we see that x = -1 and 4 are zeros for the denominator, so the domain will be (- ∞, -1) U (-1, 4) U (4, ∞).

What are the roots or zeros of $r(x)$?

When we set the numerator equal to zero, there are no x-values that make this equation true. Thus, there are no roots for this rational function, i.e. the function will not cross the x-axis.

What do you think the end behavior will be?

Based on the degrees of both the numerator and denominator, the function's y-values should approach 3/2 as x approaches both positive and negative infinity.

Where will $r(x)$ intersect the y-axis? How do you know?

r(x) should intersect the y-axis when x = 0, so r(0) = $\frac{-27}{8}$.

Are there any vertical asymptotes? If so, where are they located?

The vertical asymptotes are located at x = -1 and x = 4.

Is there a horizontal asymptote? If so, where is it located?

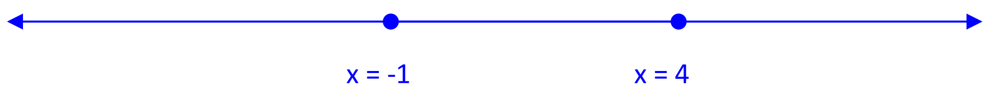
The horizontal asymptote is located at y = 3/2.

At what x-values should $r(x)$ change signs (either + to – or vice-versa)? Why?

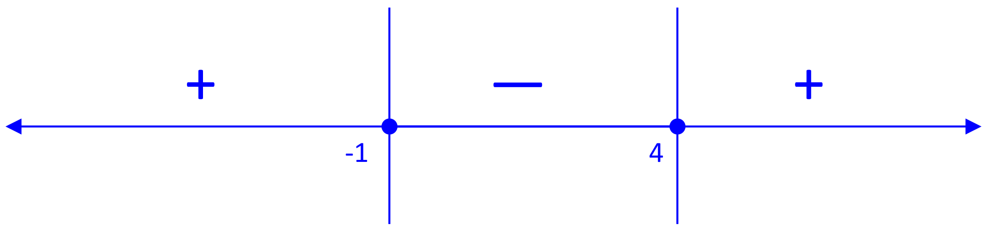
The behavior of all previous rational functions suggest that only at a vertical asymptote or at a zero will a rational function change signs. Since this rational function only has vertical asymptotes, the only sign changes will take place at x = -1 and x = 4.

Where is $r(x) > 0$? Where is $r(x) < 0$? (*Hint: use a sign chart)*

By labeling a number line with our undefined points of x = -1 and x = 4, we can produce a sign chart to help us determine where the rational function is positive and negative. The sign chart will look like this…

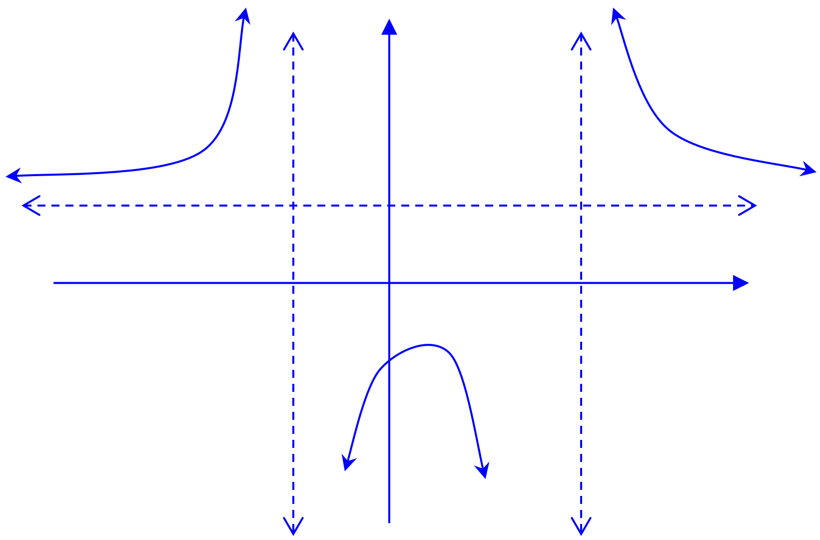


By evaluating the rational function at x-values on both sides and in between these points, we can determine on what side of the x-axis the curve will lay. For instance, we could pick x = -2, evaluate r(-2) , and see that r(-2) > 0. This information tells us that the curve has positive y-values to the left of x = -1. Now let's pick x = 0, and when we evaluate r(0), we know that r(0) $= \frac{-27}{8}$***. Therefore, the part of the curve between the two vertical asymptotes must be below the x-axis. Finally, we could pick x = 6, evaluate r(6), and see that r(6) > 0. Now we know that the curve is once again above the x-axis to the right of x = 4.***



Now let's try to sketch the graph of $r(x)$ without using your calculator.

By combining all of our information on to one graph we should be able to create a rough sketch of the rational function.

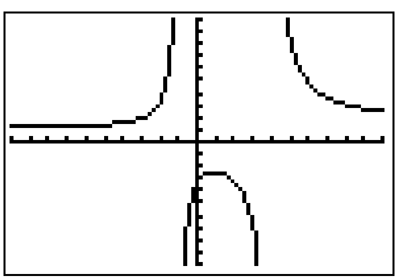


Based on your sketch, what do you think the range of $r(x)$ will be?

From the sketch we found, it appears that the range will be from negative infinity to the local maximum value found between the two vertical asymptotes, and then all y-values greater than y = 3/2 which is the horizontal asymptote.

Now let's compare your sketch to the graph of $r(x)$ using your calculator. How did your sketch match up with the actual curve? What characteristics of the graph were you not able to anticipate?

Solution:



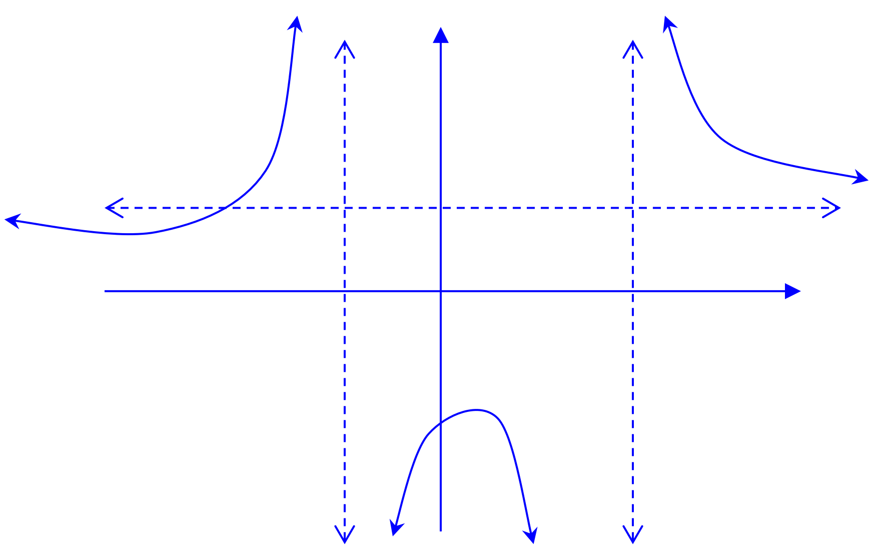
Answers will vary from student to student. Most students will have difficulty in determining the behavior of the function between the two asymptotes as well as determining what y-value to use for their local maximum.

When is the function increasing? When is it decreasing? Are there any local extrema? How can you be certain?

The function appears to be increasing from (- ∞, -1) U (-1, 0.937) and decreasing from (0.937, 4) U (4, ∞). This is the solution most students should be able to determine. There is a local maximum value of y = -2.497 when x = 0.937 because this is the x-value where the curve changes behavior from increasing to decreasing.

Based on the graph from your calculator, what is the range of $r(x)$?

It appears that the range will be (- ∞, -2.497] U (3/2, ∞). Once again, we need to include the maximum value in our interval, however, we do not know whether or not to include the value of the horizontal asymptote since the curve only appears to approach this value and may not actually equal 3/2. By setting r(x) = 3/2, we can check to see if the function does cross the horizontal asymptote, and apparently it does. At x = $\frac{-13}{3}$***, r(x) = 3/2, which indicates that the function not only crosses over the horizontal asymptote, but also must have a change in direction in order to approach y = 3/2 as x approaches negative infinity (see graph below). This means there is also a local minimum value of 1.297 when x = -9.604 for this function. So now we know the function is actually increasing from (-9.604, -1) U (-1, 0.937) and decreasing from (- ∞, -9.604) U (0.937, 4) U (4, ∞). In addition, there is a local minimum at (-9.604, 1.297) and a local maximum at (0.937, -2.497). This will now change the range to be (- ∞, -2.497] U [1.297, ∞). We must include both the local maximum and minimum values since the function actually equals these values.***



Try this one next. Let $r(x) = \dfrac{x^3 + 1}{3x^3 - 27x}$

What is the domain of $r(x)$?

D: (- ∞, -3) U (-3, 0) U (0, 3) U (3, ∞).

What are the roots or zeros of $r(x)$?

There is only one root at x = -1

What do you think the end behavior will be?

The function should approach y = 1/3 as x approaches positive and negative infinity.

Where will $r(x)$ intersect the y-axis? How do you know?

r(x) will not intersect the y-axis because r is undefined when x = 0.

Are there any vertical asymptotes? If so, where are they located?

There are 3 vertical asymptotes at x = -3, x = 0, and x = 3.

Is there a horizontal asymptote? If so, where is it located?

There is a horizontal asymptote at y = 1/3.

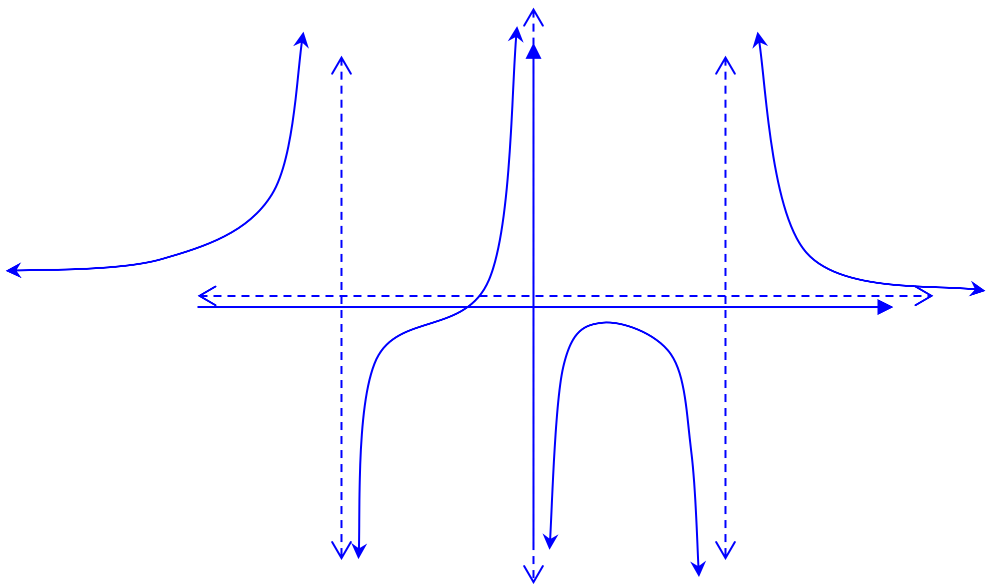
At what x-values should $r(x)$ change signs (either + to – or vice-versa)? Why?

r(x) should change signs at the x-values of -3, -1, 0 and 3 because these values are the critical points (vertical asymptotes and zeros) for r(x).

Where is $r(x) > 0$? Where is $r(x) < 0$?

Using a sign chart, we can see that r(x) > 0 from (- ∞, -3) U (-1, 0) U (3, ∞) and r(x) < 0 from (-3, -1) U (0, 3).

Now let's try to sketch the graph of $r(x)$ without using your calculator.

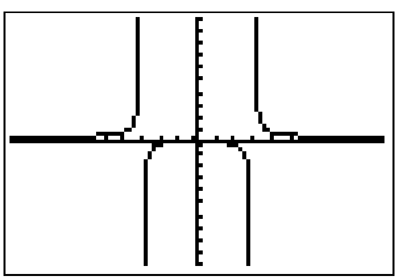


Based on your sketch, what do you think the range of $r(x)$ will be?

The range of r(x) appears to be all y-values from negative infinity to positive infinity or (- ∞, ∞).

Now let's compare your sketch to the graph of $r(x)$ using your calculator. How did your sketch match up with the actual curve? What characteristics of the graph were you not able to anticipate?

Solution:



Answers will vary from student to student. Notice that the calculator graph does not appear to give the same picture, however, by adjusting the window to focus on the behavior between the two asymptotes, students will see that their "guess" was accurate. Most students will have difficulty in determining the behavior of the function between the two asymptotes as well as determining what y-value to use for their local maximum. As long as the max value is less than zero, the student has interpreted the information correctly.

When is the function increasing? decreasing? Are there any local extremum? How can you be certain?

The function is increasing from (- ∞, -3) U (-3, 0) U (0, 0.742) and decreasing from (0.742, 3) U (3, ∞). There is a local maximum value of y = -0.0749 when x = 0.742 because at x = 0.742 there is a change in behavior from increasing to decreasing.

Based on the graph from your calculator, what is the range of $r(x)$?

The range is still (- ∞, ∞). By manipulating the window of our graphing utility, we can produce a more accurate picture of the behavior of the function around the asymptotes.

In your own words, describe the process that you would go through in order to create a sketch of any rational function.

Answers will vary from student to student but hopefully they will all include factoring the denominator to identify the domain and vertical asymptotes, factoring the numerator to identify the roots of the function, creating a sign chart to identify function behavior around the asymptotes, and then applying all this information to a x-y plane along with a good deal of thought.

Graphing Rational Functions without a calculator

Let $r(x) = \dfrac{3x^2 + 27}{2x^2 - 6x - 8}$.

What is the domain of $r(x)$?

What are the roots or zeros of $r(x)$?

What do you think the end behavior will be?

Where will $r(x)$ intersect the y-axis? How do you know?

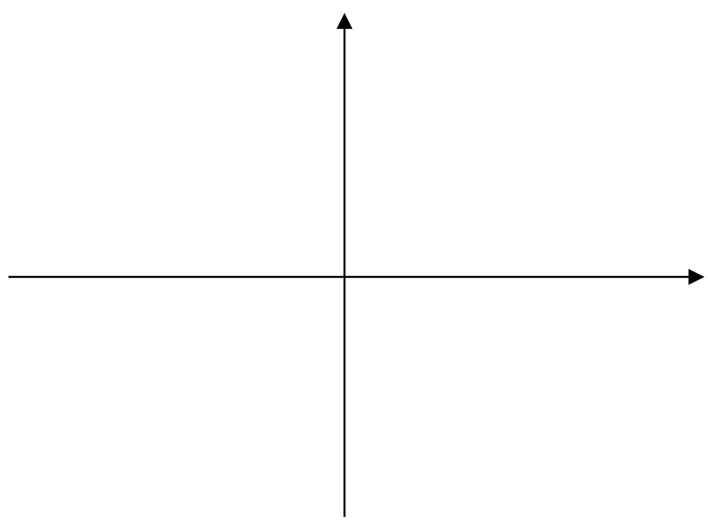
Are there any vertical asymptotes? If so, where are they located?

Is there a horizontal asymptote? If so, where is it located?

At what x-values should $r(x)$ change signs (either + to – or vice-versa)? Why?

Where is $r(x) > 0$? Where is $r(x) < 0$? (*Hint: use a sign chart)*

Now let's try to sketch the graph of $r(x)$ without using your calculator.



Based on your sketch, what do you think the range of $r(x)$ will be?

Now let's compare your sketch to the graph of $r(x)$ using your calculator. How did your sketch match up with the actual curve? What characteristics of the graph were you not able to anticipate?

When is the function increasing? decreasing? Are there any local extremum? How can you be certain?

Based on the graph from your calculator, what is the range of $r(x)$?

Try this one next. Let $r(x) = \dfrac{x^3 + 1}{3x^3 - 27x}$

What is the domain of $r(x)$?

What are the roots or zeros of $r(x)$?

What do you think the end behavior will be?

Where will $r(x)$ intersect the y-axis? How do you know?

Are there any vertical asymptotes? If so, where are they located?

Is there a horizontal asymptote? If so, where is it located?

At what x-values should $r(x)$ change signs (either + to – or vice-versa)? Why?

Where is $r(x) > 0$? Where is $r(x) < 0$?

Now let's try to sketch the graph of $r(x)$ without using your calculator.

Based on your sketch, what do you think the range of $r(x)$ will be?

Now let's compare your sketch to the graph of $r(x)$ using your calculator. How did your sketch match up with the actual curve? What characteristics of the graph were you not able to anticipate?

When is the function increasing? decreasing? Are there any local extremum? How can you be certain?

Based on the graph from your calculator, what is the range of $r(x)$?

In your own words, describe the process that you would go through in order to create a sketch of any rational function.

Jogging into the Wind

Back to Task Table

Mathematical Goals

- Build a function to model a given relationship between two variables.
- Approach the representation and description of the function using a variety of mathematical lenses.

Georgia Standards of Excellence

Build a function that models a relationship between two quantities.

MGSE9-12.F.BF.1a Determine an explicit expression and the recursive process (steps for calculation) from context. *For example, if Jimmy starts out with $15 and earns $2 a day, the explicit expression "2x +15" can be described recursively (either in writing or verbally) as " to find out how much money Jimmy will have tomorrow, you add $2 to his total today."*
$J_n = J_{n-1} + 2, J_0 = 15$

Interpret functions that arise in applications in terms of the context.

MGSE9-12.F.IF.4 Using tables, graphs, and verbal descriptions, interpret the key characteristics of a function which models the relationship between two quantities. Sketch a graph showing key features including: intercepts; interval where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; ~~and periodicity~~. *(Limit to radical and rational functions.)*

MGSE9-12.F.IF.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function. (Limit to radical and rational functions.)*

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Introduction

The purpose of this task is to give students practice constructing functions that represent a quantity of interest in context, and then interpreting features of that function in light of that context. It could either be used as a teaching task or as an assessment. Parts (a) – (e) focus on using function notation as well as basic algebra, whereas parts (f) – (h) focus more on the numerical and graphical behavior of the function near its vertical asymptote. There is a certain amount of redundancy that is noticeable here that is merely meant to reinforce the two different approaches to describing the function.

There are some practical limitations that must be discussed here as well. For example, this task may drive a physicist mad, since the task focuses only on the dimensions of velocity and time, ignoring the absolutely crucial aspects of force, momentum, and friction that would accompany a more sophisticated problem. And thus, by necessity, there are some serious limitations of reality at play here – for example, anything stronger that a tranquil breeze may send Lisa flying backward, not to mention the strong gusts that Chicago is actually known for. However, the realistic limitations do not hinder the mathematical implications present in the task – namely, that functions can be represented and described in a number of ways, and two important methods of representation and description are presented here.

There are also some teachable moments, or some opportunities for students to shine, provided here, because the idea that s must be a value between 0 up to the value of Lisa's velocity is a limitation on reality that can easily be dealt with. A discussion of what a negative value for s would imply is important, as well as the idea that while 0 does not have to be a lower bound for s, there must be a lower bound not too far down the number line if the space-time continuum is to remain intact. The same is true with the obvious issue of Lisa being the world's only jogger that doesn't ever have to cross a street or slow down for an oncoming dog-walker or parent with a stroller or whatever else would regularly meet a jogger in a busy city. Sometimes it's useful to remind students that as scenarios get more complicated (and, thus, more realistic), so does the mathematics involved!

Materials

- Pencil
- Handout
- Calculator

Jogging into the Wind

Lisa is quite an athlete, but sometimes trying to get to work in the windy city of Chicago can be a big challenge. Lisa always jogs from her condominium to her office in downtown Chicago, and this distance is 1.75 miles. Lisa likes to keep a steady pace of 704 feet per minute. Unfortunately, Lisa lives directly west of her office, which means her morning jog to work always puts her directly into the wind coming off of Lake Michigan.

(a) Let s be the speed of the wind in feet per minute. Write an expression for $r(s)$, the speed at which Lisa is moving relative to the total distance of her journey, in terms of s.

The wind is working against Lisa's running, pushing her back, so we subtract the speed of the wind, s, from Lisa's running velocity, 704 feet per minute, to get how fast she is actually going, $r(s)$. So with units of feet per minute, we have $r(s) = 704 - s$.

(b) Lisa wants to know how long it will take her to jog to work. Write an expression for $T(s)$, the time it will take in minutes, in terms of s.

First, convert her total distance to feet: $1.75 \cdot 5280 = 9240$ ft., so
Since we know that Lisa is traveling with constant velocity $r(s)$, measured in feet per minute, and $T(s)$ is the number of minutes it will take Lisa to travel 9240 feet, we know that $r(s) \cdot T(s) = 9240$; the units on the left are (feet per minute) $\times$ (minutes), giving units of feet on the right hand side. Using $r(s) = 704 - s$, we have

$$9240 = (704 - s) \cdot T(s)$$

We solve for $T(s)$ to obtain $T(s) = \frac{9240}{704-s}$

(c) What is the vertical intercept of T? What does this point represent in terms of Lisa's jog to work?

$$T(0) = \frac{9240}{704-0} = 13.125$$

This means that Lisa's jog would take 13.125 minutes from home to work if there was no wind.

(d) At what value of s does the graph have a vertical asymptote? Explain why this makes sense in this situation.

At $s = 704$. This makes sense because at a velocity of 704 feet per second, Lisa could neither make forward progress nor be blown backward if she were running at that velocity. Obviously, this is also the value of s that makes the denominator equal to 0.

(e) For what values of s does $T(s)$ make sense in the context of this problem?

From part (d), we know that for Lisa to be able to make the trip from her home to her office, the wind velocity must be less than 704 feet per second. This creates an implied natural domain of $0 \leq s < 704$, though we do know that if the wind is blowing at Lisa's back (blowing with her, though this violates the scenario outlined in the problem), then s would take on a negative value.

Lisa has been training for a marathon, and now she maintains a constant speed of 720 feet per minute when jogging to work.

(f) On a particular day, Lisa guesses that the wind is blowing at 4.25 miles per hour against her. How long will it take Lisa to get to work?

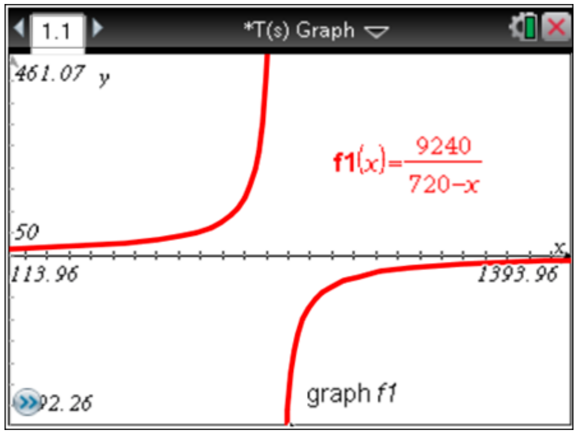
$\frac{4.25 \cdot 5280}{60} = 374$ ***feet per minute for wind resistance, so***

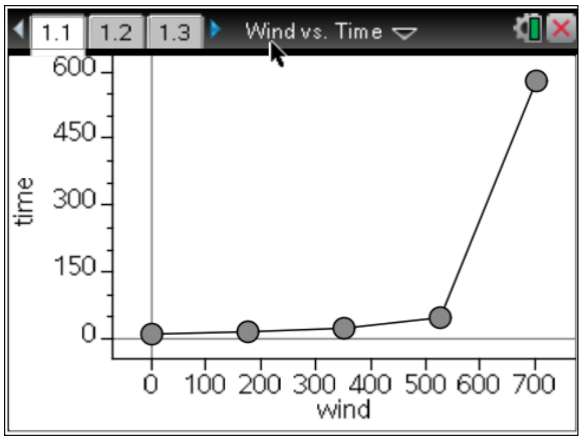
$T(374) = \frac{9240}{720-374} \approx 26.71$ ***minutes***

(g) Obviously, Lisa doesn't really know the speed of the wind. Make a table showing the time it will take her to get to work against the various wind resistances:

Speed of wind (Feet per minute)	**Lisa's speed (Feet per minute)**	**Time for Lisa to travel 1.75 miles to work (Minutes)**
0	***720***	***12.83***
176	***544***	***16.99***
352	***368***	***25.11***
528	***192***	***48.13***
704	***16***	***577.50***
s	$720 - s$	$T(s) = \frac{9240}{720 - s}$

(h) Sketch a graph of the equation from part (g). Explain why $s = 720$ does not make sense for this function, both in terms of the jogging trip and in terms of the equation.





Obviously, at $s = 720$ there is a vertical asymptote because, algebraically, 720 would make the denominator equal to 0, thus making the simple rational function undefined at that value. Contextually, 720 feet per second is the value for wind velocity that would prohibit Lisa from either moving forward or backward at her current constant speed. Also, looking at the scatterplot of wind velocity versus the time needed for the jogging trip, it becomes obvious that an s of over 600 will result in an astronomically high trip time, and thus, would be nonsensical.

Jogging into the Wind

Lisa is quite an athlete, but sometimes trying to get to work in the windy city of Chicago can be a big challenge. Lisa always jogs from her condominium to her office in downtown Chicago, and this distance is 1.75 miles. Lisa likes to keep a steady pace of 704 feet per minute. Unfortunately, Lisa lives directly west of her office, which means her morning jog to work always puts her directly into the wind coming off of Lake Michigan.

(a) Let s be the speed of the wind in feet per minute. Write an expression for $r(s)$, the speed at which Lisa is moving relative to the total distance of her journey, in terms of s.

(b) Lisa wants to know how long it will take her to jog to work. Write an expression for $T(s)$, the time it will take in minutes, in terms of s.

(c) What is the vertical intercept of T? What does this point represent in terms of Lisa's jog to work?

(d) At what value of s does the graph have a vertical asymptote? Explain why this makes sense in this situation.

(e) For what value of s does $T(s)$ make sense in the context of this problem?

Lisa has been training for a marathon, and now she maintains a constant speed of 720 feet per minute when jogging to work.

(f) On a particular day, Lisa guesses that the wind is blowing at 4.25 miles per hour against her. How long will it take Lisa to get to work?

(g) Obviously, Lisa doesn't really know the speed of the wind. Make a table showing the time it will take her to get to work against the various wind resistances:

Speed of wind (Feet per minute)	**Lisa's speed (Feet per minute)**	**Time for Lisa to travel 1.75 miles to work (Minutes)**
0		
176		
352		
528		
704		
s		

(h) Sketch a graph of the equation from part (g). Explain why $s = 720$ does not make sense for this function, both in terms of the jogging trip and in terms of the equation.

Hank's Hot Dog Stand

Back to Task Table

Mathematical Goals

- Interpret a simple rational function, and restrict its domain based on the context of the problem.
- Look at a function from two perspectives, both from a table/numerical standpoint, and from a graphical/function standpoint.

Georgia Standards of Excellence

Interpret functions that arise in applications in terms of the context.

MGSE9-12.F.IF.4 Using tables, graphs, and verbal descriptions, interpret the key characteristics of a function which models the relationship between two quantities. Sketch a graph showing key features including: intercepts; interval where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; ~~and periodicity~~. *(Limit to radical and rational functions.)*

MGSE9-12.F.IF.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function. (Limit to radical and rational functions.)*

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Introduction

The purpose of this task is to give students practice investigating functions that represent a quantity of interest in context, and then interpreting features of that function in light of that context. The goal here is for students to see how a function can be investigated by a numerical/spreadsheet approach, as well as from the traditional graphing and function notation approach. One of the benefits of part (b) is that it introduces students to simple business decision making. For example, would anyone ever sell a $10 hot dog on a street corner? While selling 100,000 or 1,000,000 hot dogs is very desirable in terms of the profit that could be made, what sort of logistical nightmares would this produce? And while there would certainly be a lower bound for the number of hot dogs that would have to be sold for someone to continue in this business, is there an upper limit to that number based on logistical comfort, and what would it be to suit individual preferences? At what point would someone expand the business to more than one cart? These are interesting discussions that support the practical purposes of the task.

Obviously, domain restrictions have to be included when looking at the average cost function. A discussion of why graphs are more than just pictures produced from technology, but are important tools in decision making should probably take place here, just as it should anywhere that domain restrictions are appropriate.

Materials

- Pencil
- Handout
- Calculator

Hank's Hot Dog Stand

Hank runs a successful hot dog stand right across from the arch at the University of Georgia in downtown Athens. Hank has to order his hot dogs, buns, mustard, relish, and all other condiments in bulk, as well as pay taxes, licensing fees, and other small business expenses. Therefore, Hank has a relatively large "sunk" cost associated with his business – it averages out to $950 per week just to keep the cart open. The cost of producing h hot dogs is given by

$$C(h) = 950 + 0.45h$$

(a) Hank wants to figure out how much to charge a customer for a hot dog if he wishes to make a $0.25 profit on each hot dog sold. Suppose Hank sold 100 hot dogs in a week. What is the cost of making this many hot dogs? How much is this per hot dog? What should Hank charge per hot dog?

$\boldsymbol{C(100) = 950 + 0.45(100) = 995}$ → ***Cost of making 100 hot dogs***

$\boldsymbol{\frac{995}{100} = 9.95}$ → ***Cost per hot dog when selling 100 per week***

Hank would have to charge $\boldsymbol{9.95 + 0.25 = \$10.20}$ ***just to turn a (very) small profit!***

(b) Hank wants to analyze what his cost per hot dog would be for different levels of sales. Complete the table below showing his costs at these different levels.

Number of Hot Dogs Sold	**0**	**10**	**100**	**1000**	**10,000**	**100,000**	**1,000,000**
Total Cost	***950***	***954.50***	***995***	***1400***	***5450***	***45950***	***450950***
Cost per Hot Dog	**-**	***95.45***	***9.95***	***1.40***	***0.55***	***0.46***	***0.45***
Hank Should Charge?	**-**	***95.70***	***10.20***	***1.65***	***0.80***	***0.71***	***0.70***

(c) Explain why the average cost per hot dog levels off.

The reason the average cost is changing is because of the fixed cost of $950. As more hot dogs are sold, this fixed cost is shared by so many hot dogs that it barely adds anything to the cost of each hot dog. The average cost levels off at $0.45, which is what each additional hot dog adds to the cost function.

(d) Find an equation for the average cost per hot dog of producing h hot dogs.

Just divide the total cost function from part (a) by the average number of hot dogs sold:

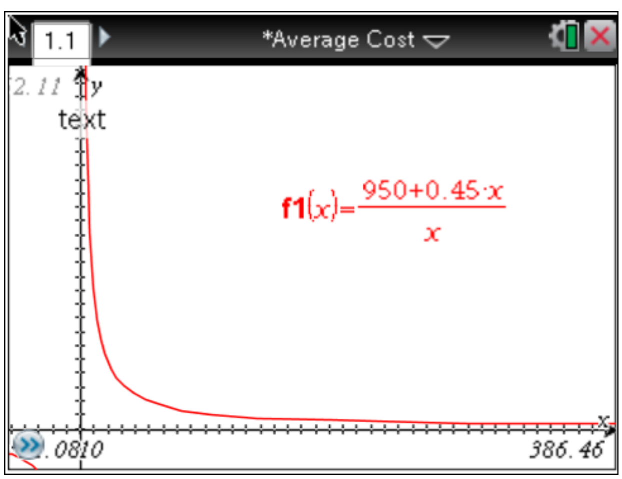
$$\frac{950+0.45x}{x}$$

(e) Find the domain of the average cost function.

The domain for the average cost function is the set of positive integers. It obviously makes no sense to consider producing a negative or fractional number of hot dogs, and you cannot compute an average cost if there are no hot dogs sold, so the domain cannot include 0 either.

(f) Using the data points from your table above, sketch the average cost function. How does the graph reflect that the average cost levels off?

The graph of the average cost function is shown below (for $x > 0$). As you follow the graph to the right (i.e., as the number of hot dogs sold increases), the graph gets closer and closer to a horizontal asymptote, which must necessarily be $y = 0.45$, since as is explained in the solution for part (c), this is the long-term average cost of selling each additional hot dog.



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$$C(h) = 950 + 0.45h$$

(a) Hank wants to figure out how much to charge a customer for a hot dog if he wishes to make a $0.25 profit on each hot dog sold. Suppose Hank sold 100 hot dogs in an afternoon. What is the cost of making this many hot dogs? How much is this per hot dog? What should Hank charge per hot dog?

(b) Hank wants to analyze what his cost per hot dog would be for different levels of sales. Complete the table below showing his costs at these different levels.

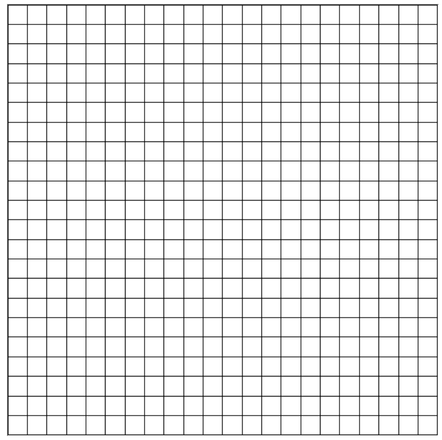
Number of Hot Dogs Sold	**0**	**10**	**100**	**1000**	**10,000**	**100,000**	**1,000,000**
Total Cost							
Cost per Hot Dog							
Hank Should Charge?							

(c) Explain why the average cost per hot dog levels off.

(d) Find an equation for the average cost per hot dog of producing h hot dogs.

(e) Find the domain of the average cost function.

(f) Using the data points from your table above, sketch the average cost function. How does the graph reflect that the average cost levels off?



That's Radical Dude

Back to Task Table

Math Goals

- Graph square root function
- Graph cube root functions
- Graph transformations of radical functions
- Understand the domain of radical function

Georgia Standards of Excellence

MGSE9-12.F.IF.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function. (Limit to radical and rational functions.)*

MGSE9-12.F.IF.7 Graph functions expressed algebraically and show key features of the graph both by hand and by using technology. *(Limit to radical and rational functions.)*

MGSE9-12.F.IF.7b Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions.

MGSE9-12.F.IF.9 Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one function and an algebraic expression for another, say which has the larger maximum.*

Standards for Mathematical Practice

1. Attend to precision
2. Look for and make use of structure
3. Look for and express regularity in repeated reasoning

Introduction

Students must understand that the radicand of a radical must be non-negative only if the index is even; if the index is odd (for example, you have a cube root or a 5^{th} root), then the radicand can be negative and that this affects the domain of the function. This task encourages students to graph using a table which will produce some error messages because the x-values are outside of the domain. If students are struggling to see the entire function, then encourage them to expand their table to include more x-values.

That's Radical Dude:

Let's explore radical functions. By definition, a radical function is one that contains any sort of radical. We are going to explore two of the more common radical functions, the square root and the cube root.

Complete the table of value for the function, $f(x) = \sqrt{x}$. This is the square root function.

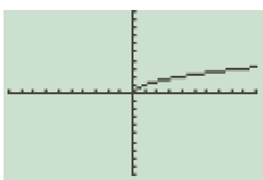
x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	*Error*	*Error*	*Error*	*Error*	*0*	*1*	≈ 1.41	≈ 1.73	*2*

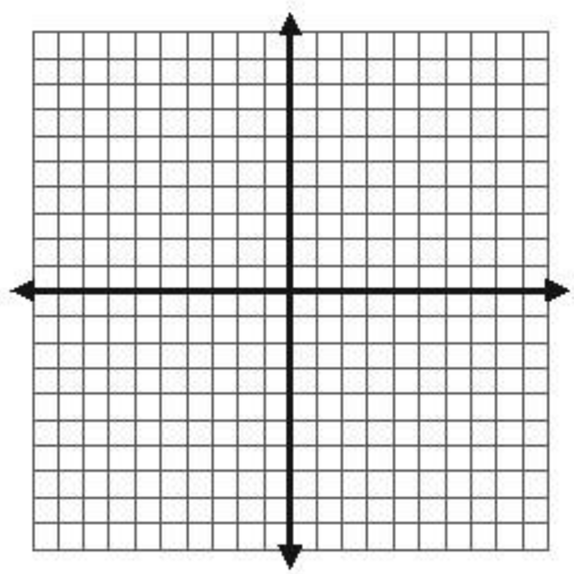
What did you notice about some of the values? If you typed the function into a calculator and tried to evaluate it for some of the x-values, what message appeared? Why?

Not every x-value will produce a y-value. You can't take the square root of negative numbers so some of the values will yield an error message on the calculator.

Graph the function in the grid provided below.

Solution:





What is the domain of this function?

D: $[0, \infty)$

What is the range of the function?

R: $[0, \infty)$

Complete the table of values for the function, $f(x) = \sqrt{x+2}$

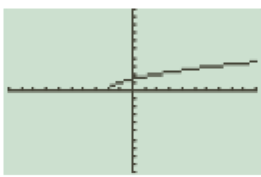
x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	***Error***	***Error***	***0***	***1***	$\approx \mathbf{1.41}$	$\approx \mathbf{1.73}$	***2***	$\approx \mathbf{2.24}$	$\approx \mathbf{2.45}$

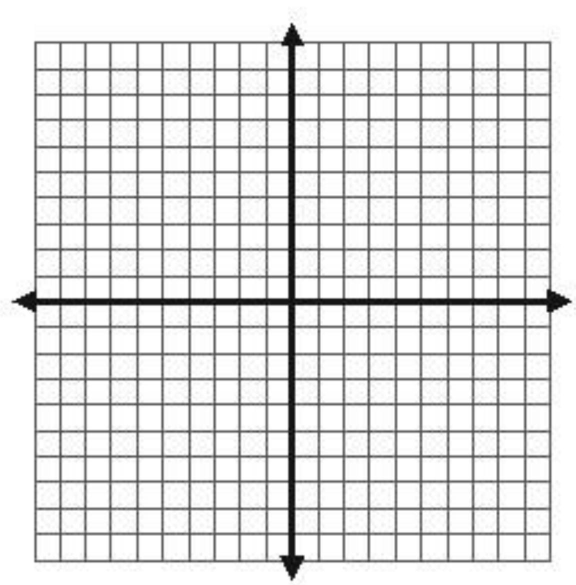
What did you notice about some of the values? If you typed the function into a calculator and tried to evaluate it for some of the x-values, what message appeared? Why?

Not every x-value will produce a y-value. You can't take the square root of negative numbers so some of the values will yield an error message on the calculator. This time though, not all the negative x-values produced an error message.

Graph the function in the grid provided below.

Solution:





What is the domain of this function?

D: $[-2, \infty)$

What is the range of the function?

R: $[0, \infty)$

Complete the table of values for the function, $f(x) = \sqrt{9 - x^2}$.

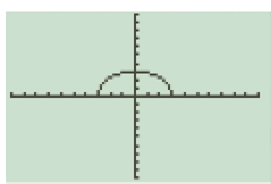
x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	***Error***	***0***	$\approx \mathbf{2.24}$	$\approx \mathbf{2.83}$	***3***	$\approx \mathbf{2.83}$	$\approx \mathbf{2.24}$	***0***	***Error***

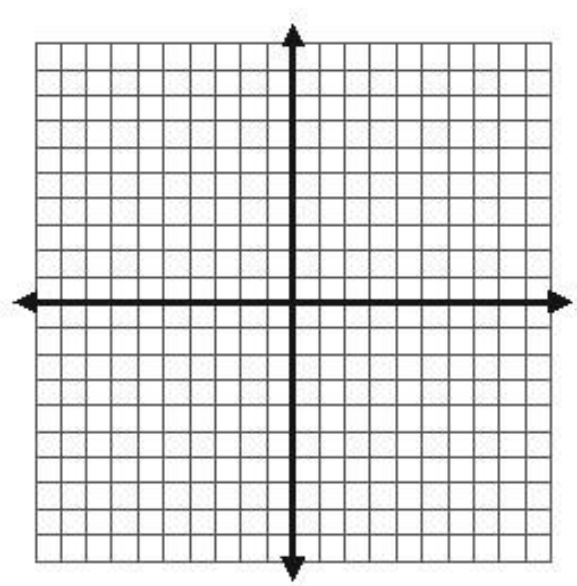
What did you notice about some of the values? If you typed the function into a calculator and tried to evaluate it for some of the x-values, what message appeared? Why?

Not every x-value will produce a y-value. You can't take the square root of negative numbers so some of the values will yield an error message on the calculator. This time though, even some of the positive numbers produced an error message.

Graph the function in the grid provided below.

Solution:





What is the domain of this function?

D: $[\mathbf{-3, 3})$

What is the range of the function?

R: $[\mathbf{0, 3})$

Using the three examples above, make a conjecture about the domain of a radical function.

Answers may vary, but you want to guide students until they understand that the radicand of a radical function cannot be a number less than 0.

Use your conjecture to determine the domain of this function, $f(x) = \sqrt{2x + 5}$, without graphing it. Check your solution by graphing it on a graphing calculator.

Students should see that setting up the inequality $2x+5\geq 0$ will help them determine the domain. Once they solve the inequality they will see that the domain must be $x\geq\frac{-5}{2}$. If they graph this function, they will see that the graph starts at (-2.5,0) and extends in the positive direction.

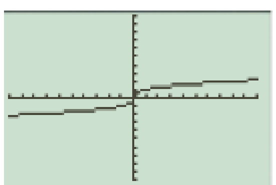
Now let's look at another common radical function, the cube root.
Complete the table of values for the function, $f(x)=\sqrt[3]{x}$.

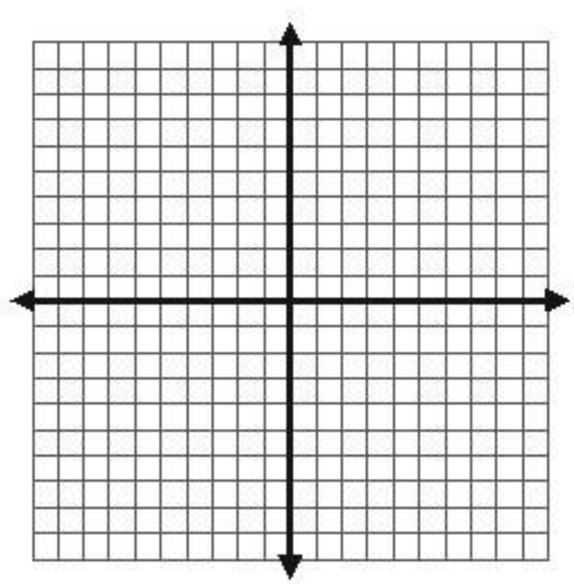
x	-8	-6	-2	-1	0	1	2	6	8
$f(x)$	***-2***	≈ -1.82	≈ -1.26	***-1***	***0***	***1***	≈ 1.26	≈ 1.82	***2***

Do you get any of the same error messages for this function that you did in the table of values for the square root function? Why do you think that is so?

There are no error messages this time. This is because you can take a cube root of a negative number so the domain of this function is all real numbers.

Graph the function in the grid provided below.
Solution:





What is the domain of this function?

D: $(-\infty,\infty)$

What is the range of the function?

R: $(-\infty,\infty)$

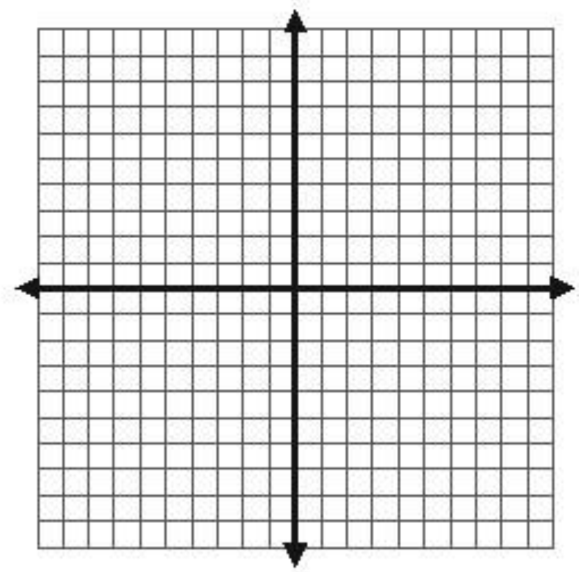
That's Radical Dude:

Let's explore radical functions. By definition, a radical function is one that contains any sort of radical. We are going to explore two of the more common radical functions, the square root and the cube root.

Complete the table of value for the function, $f(x) = \sqrt{x}$. This is the square root function.

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$									

What did you notice about some of the values? If you typed the function into a calculator and tried to evaluate it for some of the x-values, what message appeared? Why?
Graph the function in the grid provided below.



What is the domain of this function?

What is the range of the function?

Complete the table of values for the function, $f(x) = \sqrt{x+2}$

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$f(x)$									

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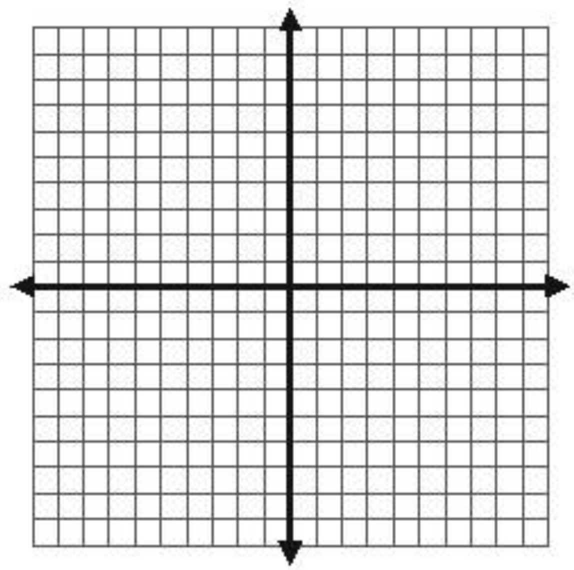


What is the domain of this function?

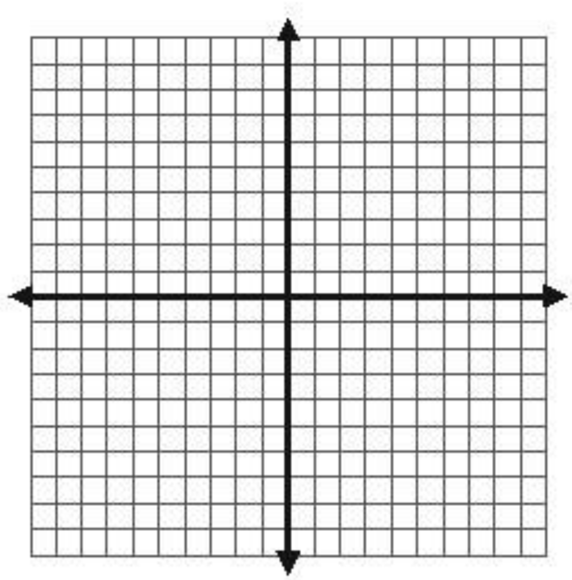
What is the range of the function?

Complete the table of values for the function, $f(x) = \sqrt{9 - x^2}$.

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$									

What did you notice about some of the values? If you typed the function into a calculator and tried to evaluate it for some of the x-values, what message appeared? Why?

Graph the function in the grid provided below.



What is the domain of this function?

What is the range of the function?

Using the three examples above, make a conjecture about the domain of a radical function.

Use your conjecture to determine the domain of this function, $f(x) = \sqrt{2x + 5}$, without graphing it. Check your solution by graphing it on a graphing calculator.

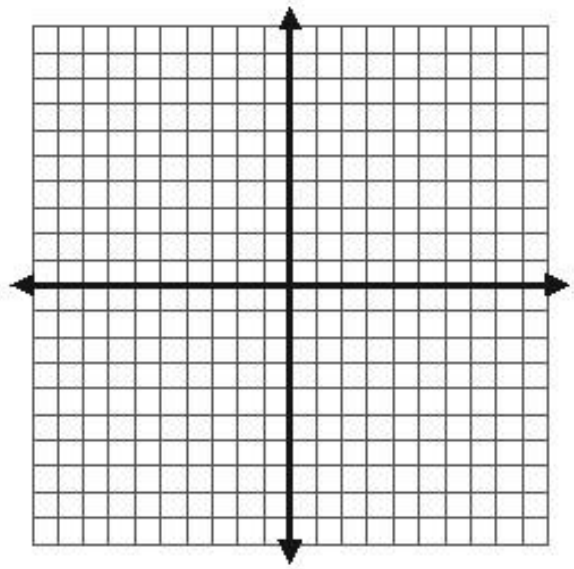
Now let's look at another common radical function, the cube root.

Complete the table of values for the function, $f(x) = \sqrt[3]{x}$.

x	-8	-6	-2	-1	0	1	2	6	8
$f(x)$									

Do you get any of the same error messages for this function that you did in the table of values for the square root function? Why do you think that is so?

Graph the function in the grid provided below.



What is the domain of this function?

What is the range of the function?

Let's Get to "Work"

Back to Task Table

Math Goals

- Write equations for rational and radical functions given a problem situation
- Solve rational equations
- Solve radical equations

Georgia Standards of Excellence

MGSE9-12.A.CED.1 Create equations and inequalities in one variable and use them to solve problems. Include equations arising from ~~linear, quadratic,~~ simple rational, and exponential functions ~~(integer inputs only)~~.

MGSE9-12.A.REI.2 Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them
2. Reason abstractly and quantitatively
3. Model with mathematics

Introduction

Work problems are great examples of simple rational equations in real life. Help students to see that if a person can complete a job in 3 hours then they will complete $\frac{1}{3}$ of the job in one hour. This will be important to helping them set up the equation. Average cost is also a real world example of rational equations. The pendulum problem is a practical example of using radical equations. There are actually many equations in physics that involve radicals.

Let's Get to "Work":

Rational Equations can be used to model some interesting real life phenomena. Distance, rate, and time problems as well as multi-person work problems are particularly suited to be modeled with a rational equation.

"Work" Problems

Two are better than one when it comes to completing a job. We can use rational equations to help us figure just how much better two can do a job than one person acting alone.

Jonah can paint a house by himself in 12 hours. Steve can do the same job in eight hours. How long will it take them to complete the job together? To help us answer this question we first need to think about how much of the job Jonah and Steve can do in one hour.

Let's let t = hours of work it takes to do job together. So to find out how much of the job they complete together in one hour, we will use this equation: $\frac{1}{12}+\frac{1}{8}=\frac{1}{t}$

Now we have to think about how to solve this equation. There are actually a few different ways to approach it. Talk to your partner to see what ways you can come up with together.

One way to approach the problem would be to add the fractions on the left and then solve the resulting proportion. Students might also suggest finding a LCD for all fractions and then setting the numerators equal to each other. Make sure to illustrate and discuss both ways. The method that this task is going to suggest that students try to solve the problem is by multiplying each term of the equation by the LCD. This will in effect "clear the fraction" from the equation and allow students to solve the resulting linear or quadratic equation using previously taught methods.

Equations are balanced statements and can be easily changed as long as you make sure to perform the same operation to every piece of the equation. This idea lets us multiply the equation by the LCD in order to create a simpler equation to solve. Try it with the equation above to see how long it will take Jonah and Steve to paint the house together.

Solution: $\boldsymbol{24t\left(\frac{1}{12}+\frac{1}{8}=\frac{1}{t}\right)\rightarrow 2t+3t=24\rightarrow 5t=24\rightarrow t=\frac{24}{5}}$ ***or*** $\boldsymbol{4\frac{4}{5}}$ ***hours. It is important for students to see that even though we often just deal with improper fractions in algebra, this is a time when mixed numbers make the most sense contextually.***

Let's try another problem. Paul can paint a room two times as fast as Jamie. Working together they can paint the room in three hours. How long would it take each of them to paint the room alone?

Solution: Students might struggle with setting up this problem. Let them see that if p = number of hours it takes Paul to paint a room, then 2p = the number of hours it takes Jamie to paint the same room. Therefore the resulting equation is $\boldsymbol{\frac{1}{p}+\frac{1}{2p}=\frac{1}{3}}$***. It would take Paul 4.5 hours to paint the room and it would take Jamie 9 hours to paint the room by himself.***

Average Cost

Have you ever wondered why large retailers such as Wal-Mart can offer products at such low costs? The secret is in the quantity that they purchase. To understand this idea we are going to explore average cost.

At the Stir Mix-A-Lot blender company, the weekly cost to run the factory is $1400 and the cost of producing each blender is an additional $4 per blender.

a. Write a function rule representing the weekly cost in dollars, *C(x)*, of producing x blenders.
C(x) = 4x + 1400

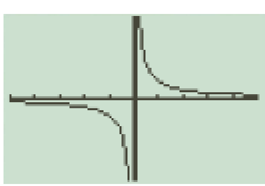
b. What is the total cost of producing 100 blenders in one week?
C(100) = 4(100) + 1400 = 1800. It will cost $1800 to produce 100 blenders in one week.

c. If you produce 100 blenders in one week, what is the total production cost per blender?
$\frac{\$1800}{100} = \$18$ ***If 100 blenders are produced the total production cost per blender is $18.***

d. Will the total production cost per blender always be the same? Justify your answer.
No. Justifications may vary. If 200 blenders are produced in one week, the cost is C(200) = 4(200) + 1400 = 2200. The total production cost is $\frac{\$2200}{200} = \$11$ ***per blender. Since $11 does not equal $18, the total production cost per blender is not the same when the number of blenders produced varies.***

e. Write a function rule representing the total production cost per blender *P(x)* for producing *x* blenders.
$P(x) = \frac{C(x)}{x}$ *so* $P(x) = \frac{4x+1400}{x}$ *or* $P(x) = 4 + \frac{1400}{x}$

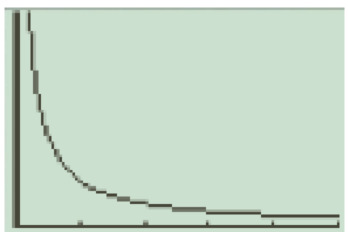
f. Using your graphing calculator, create a graph of your function rule from part e. Does the entire graph make sense for this situation? If not, what part does?

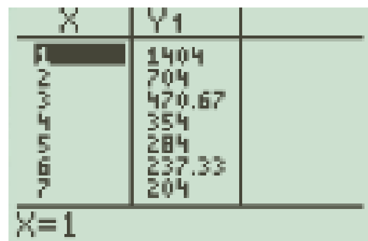


It is important for students to note that the whole graph is not applicable to this situation. Since our input value represents the number of blenders made in a week the domain for this situation is when x is a positive integer.

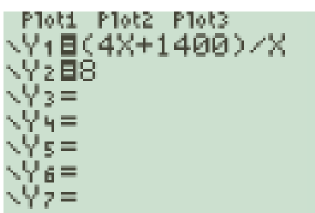
g. What is the production cost per blender if 300 blenders are produced in one week? If 500 blenders are produced in one week?
If 300 blenders are produced, the total production cost per blender is $8.67. If 500 blenders are produced, the total production cost per blender is $6.80.

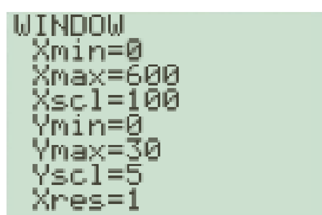
h. What happens to the total production cost per blenders as the number of blenders produced increases? Explain your answer.
As the number of blenders produced increases, the total production cost per blender decreases. Looking at the graph will show that as x increases, y decreases. In addition, looking at the table of values in the graphing calculator you can see that as the x values increase, the y values decrease.





i. How many blenders must be produced to have a total production cost per blender of $8?





The viewing window above shows that the intersection of y = 4 + (1400/x) and y = 8 is the point (350, 8). Therefore, when 350 blenders are produced in one week, the total production cost per blender is $8. It is important for students to also verify this solution algebraically. $4 + \frac{1400}{x} = 8$.

j. The function for the production cost of the blenders is a rational function. What other information can we gather about this situation based on the characteristics of rational functions?
This function $P(x) = \frac{4x+1400}{x}$ ***has a horizontal asymptote at y = 4 which means that the lowest production cost possible is $4 and that will be reached when extreme numbers of blenders are produced each week. This function also has a vertical***

asymptote at x = 0. This tells us that there is no production cost associated with making zero blenders which makes sense in this real world situation.

Pendulum

Tommy visited the Museum of History and Technology with his class. They saw Focault's Pendulum in Pendulum Hall and it was fascinating to Tommy. He knew from science class that the time it takes a pendulum to complete a full cycle or swing depends upon the length of the pendulum. The formula is given by $T = 2\pi\sqrt{\frac{L}{32}}$ where T represents the time in seconds and L represents the length of the pendulum in feet. He timed the swing of the pendulum with his watch and found that it took about 8 seconds for the pendulum to complete a full cycle. Help him figure out the length of the pendulum in feet.

Solution: $8 = 2\pi\sqrt{\frac{L}{32}}$ *therefore the length is approximately* 52 *feet.*

Tommy thought that a pendulum that took a full 20 seconds to complete a full cycle would be very dramatic for a museum. How long must that pendulum be? If ceilings in the museum are about 20 feet high, would this pendulum be possible?

Copyright Digital Image Smithsonian Institution, 1998

Solution:

$$20 = 2\pi\sqrt{\frac{L}{32}}$$

therefore the length is approximately 324 *feet.*

The building would have to be over 16 stories tall to accommodate this pendulum!

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b. What is the total cost of producing 100 blenders in one week?
c. If you produce 100 blenders in one week, what is the total production cost per blender?
d. Will the total production cost per blender always be the same? Justify your answer.
e. Write a function rule representing the total production cost per blender *P(x)* for producing *x* blenders.
f. Using your graphing calculator, create a graph of your function rule from part e. Does the entire graph make sense for this situation? If not, what part does?
g. What is the production cost per blender if 300 blenders are produced in one week? If 500 blenders are produced in one week?
h. What happens to the total production cost per blenders as the number of blenders produced increases? Explain your answer.
i. How many blenders must be produced to have a total production cost per blender of $8?
j. The function for the production cost of the blenders is a rational function. What other information can we gather about this situation based on the characteristics of rational functions?

Pendulum

Tommy visited the Museum of History and Technology with his class. They saw Focault's Pendulum in Pendulum Hall and it was fascinating to Tommy. He knew from science class that the time it takes a pendulum to complete a full cycle or swing depends upon the length of the pendulum. The formula is given by $T = 2\pi\sqrt{\frac{L}{32}}$ where T represents the time in seconds and L represents the length of the pendulum in feet. He timed the swing of the pendulum with his watch and found that it took about 8 seconds for the pendulum to complete a full cycle. Help him figure out the length of the pendulum in feet.

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Tommy thought that a pendulum that took a full 20 seconds to complete a full cycle would be very dramatic for a museum. How long must that pendulum be? If ceilings in the museum are about 20 feet high, would this pendulum be possible?

Sailing Into the Wind (Spotlight Task)

Back to Task Table

Georgia Standards of Excellence

Create equations that describe numbers or relationships

MGSE9-12.A.CED.1 Create equations and inequalities in one variable and use them to solve problems. Include equations arising from ~~linear, quadratic,~~ simple rational, ~~and exponential~~ functions ~~(integer inputs only)~~.

MGSE9-12.A.CED.2 Create ~~linear, quadratic, and exponential~~ equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. *(Limit to rational and radical functions. The phrase "in two or more variables" refers to formulas like the compound interest formula, in which $A = P(1 + r/n)^{nt}$ has multiple variables.)*

Understand solving equations as a process of reasoning and explain the reasoning

MGSE9-12.A.REI.2 Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.

STANDARDS FOR MATHEMATICAL PRACTICE

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

ESSENTIAL QUESTIONS

- What information do you need to make sense of this problem?
- How can you use estimation strategies to find out possible solutions to the questions you generated based on the video provided?

MATERIALS REQUIRED

- Access to videos for each Act
- Student Recording Sheet
- Pencil

TIME NEEDED

- 1 day

TEACHER NOTES

Task Description

In this task, students will watch the video, generate questions that they would like to answer, make reasonable estimates, and then justify their estimates mathematically. This is a student-centered task that is designed to engage learners at the highest level in learning the mathematics content. During Act 1, students will be asked to discuss what they wonder or are curious about after watching the quick video. These questions should be recorded on a class chart or on the board. Students will then use mathematics, collaboration, and prior knowledge to answer their own questions. Students will be given additional information needed to solve the problem based on need. When they realize they don't have a piece of information they need to help address the problem and ask for it, it will be given to them.

More information along with guidelines for 3-Act Tasks may be found in the Comprehensive Course Guide.

ACT 1:
Watch the video:

http://real.doe.k12.ga.us/vod/gso/math/videos/Act-1-sailboat.wmv

Ask students what they want to know.

The students may say the following:

- How fast is the sailboat traveling?
- What is the wind speed?
- How is the wind affecting the boat's balance?

Give students adequate "think time" between the two acts to discuss what they want to know. Focus in on one of the questions generated by the students, i.e. Ask students to use the information from the video in the first act to figure it out the following question: What is the wind speed causing the boat to tilt over?

.
Circulate throughout the classroom and ask probing questions, as needed.

ACT 2:
Reveal the following information as requested:

Reveal #1: Share information about The Beaufort Wind Scale.
BEAUFORT WIND SCALE

The Beaufort wind scale was devised to measure wind speed. The Beaufort numbers B, which range from 0 to 12, can be modeled by $B = 1.69\sqrt{(s+4.45)} - 3.49$***, where s is the speed (in miles per hour) of the wind.***

Beaufort Wind Scale		
Beaufort number	Force of wind	Effects of wind
0	Calm	Smoke rises vertically.
1	Light air	Direction shown by smoke.
2	Light breeze	Leaves rustle; wind felt on face.
3	Gentle breeze	Leaves move; flags extend.
4	Moderate breeze	Small branches sway; paper blown about.
5	Fresh breeze	Small trees sway.
6	Strong breeze	Large branches sway; umbrellas difficult to use.
7	Moderate gale	Large trees sway; walking difficult.
8	Fresh gale	Twigs break; walking hindered.
9	Strong gale	Branches scattered about; slight damage to buildings.
10	Whole gale	Trees uprooted; severe damage to buildings.
11	Storm	Widespread damage.
12	Hurricane	Devastation.

Ask students how they would use this information to solve the problem to find the wind speed.

Reveal #2: Show the Act 2 video. http://real.doe.k12.ga.us/vod/gso/math/videos/Act-2-sailboat.wmv

Give students time to work in groups to figure it out.
Circulate throughout the classroom and ask probing questions, as needed.

ACT 3

Show the Act 3 video reveal.

http://real.doe.k12.ga.us/vod/gso/math/videos/Act-3-sailboat-the-reveal.wmv

Students will compare and share solution strategies.

- Reveal the answer. Discuss the theoretical math versus the practical outcome.
- How appropriate was your initial estimate?
- Share student solution paths. Start with most common strategy.
- Revisit any initial student questions that weren't answered.

ACT 4: Extension/ Assessment Task

IN AUGUST OF 1883, a volcano erupted on the island of Krakatau, Indonesia. The eruption caused a tsunami to form and travel into the Indian Ocean and into the Java Sea. The speed, s, (in kilometers per hour) that a tsunami travels can be modeled by $s = 356\sqrt{d}$ where d is the depth (in kilometers) of the water.

1. A tsunami from Krakatau hit Jakarta traveling about 60 kilometers per hour. What is the average depth of the water between Krakatau and Jakarta?
2. After 15 hours and 12 minutes a tsunami from Krakatau hit Port Elizabeth, South Africa, 7546 kilometers away. Find the average speed of the tsunami.

Intervention:

Ask specific, probing questions during Act 2, such as:

- ***What do you need to know about the original problem to help you find your solution?***
- ***What information is given?***
- ***How do you determine the wind speed given the Beaufort number?***

Formative Assessment Check

Students should investigate the following questions providing adequate justification for their reasoning:

How will the equation should change if the Beaufort number scale included 20 numbers on the scale instead of 12. What would need to increase or decrease in the formula? Explain your reasoning.

Consider a different sailboat that has a wind speed of 69. How would you determine the Beaufort number? Explain your reasoning and justify your answer.

Student Recording Sheet

Task Title:________________________ Name:________________________

ACT 1

What did/do you notice?

What questions come to your mind?

Main Question:____

Estimate the result of the main question? Explain?
Place an estimate that is too high and too low on the number line
Low estimate *Place an "x" where your estimate belongs* High estimate

ACT 2

What information would you like to know or do you need to solve the question posed by the class?

Record the given information you have from Act 1 and any new information provided in Act 2.

If possible, give a better estimate using this information:_______________________________

Act 2 (*continued*)

Use this area for your work, tables, calculations, sketches, and final solution.

ACT 3

What was the result?

Which Standards for Mathematical Practice did you use?	
□ Make sense of problems & persevere in solving them	□ Use appropriate tools strategically.
□ Reason abstractly & quantitatively	□ Attend to precision.
□ Construct viable arguments & critique the reasoning of others.	□ Look for and make use of structure.
□ Model with mathematics.	□ Look for and express regularity in repeated reasoning.

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Extraneous Solutions

Back to Task Table

Math Goals

- Solve rational and radical equations while identifying any extraneous solutions

Georgia Standards of Excellence

MGSE9-12.A.REI.2 Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.

Standards for Mathematical Practice

1. Make sense of problems and persevere to solve them
2. Attend to precision
3. Look for and make use of structure
4. Look for and express regularity in repeated reasoning

Introduction
It is important for students to understand that sometimes not all solutions of an equation are valid. Rational and Radical equations can often produce extraneous solutions and students must check their solutions to see if they need to make adjustments to the final solution of the problem. It is important to not only solve these problems algebraically, but to also look at the graphical solution too. This often helps students to see when a solution will not work for a certain problem.

Extraneous Solutions

Rational Equations are useful to help you solve some real-life problems like work and average cost. Because those problems have real-life context, issues such as zeros in the denominator can be avoided because it doesn't make sense to solve for 0 hours of work or for 0 blenders. When you look at a rational equation algebraically, though, you have to watch out for such issues. Solutions that create a zero in the denominator are known as extraneous solutions. Let's work an example.
Solve $\frac{2m}{m-1}+\frac{m-5}{m^2-1}=1$

Solution: You want students to first identify the common denominator as $(m+1)(m-1)$. Once they multiply the entire equation by this common denominator, the result will be a quadratic that they can solve by factoring. This will produce two solutions: $m=-4$ and $m=1$. If they substitute both of these answers back into the original equation they will see that $m=1$ will produce a zero in the denominator and must be an extraneous solution. Therefore the only solution to this problem is $m=-4$.

You should get two solutions. Do both solutions give you a balanced equation? Check each one by substituting the value back into the original equation. Do you see any problems?

Students should see that $m = 1$ will create an undefined fraction in the equation.

It is important to always check rational equations for extraneous solutions or restrictions on your domain. Not every solution that you find works!

Let's practice a few more:

a. $\frac{3}{5x} + \frac{7}{2x} = 1$

Solution: $x = \frac{41}{10}$

b. $\frac{4}{k^2-8k+12} = \frac{k}{k-2} + \frac{1}{k-6}$

Solution: $k = 3$

c. $\frac{2b-5}{b-2} - 2 = \frac{3}{b+2}$

Solution: $b = 1$

d. $\frac{1}{x-2} = \frac{x}{2x-4} + 1$

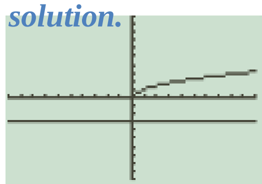
Solution: no solution. The algebraically generated solution of $x = 2$ is extraneous.

e. $\frac{1}{x} + \frac{1}{x-3} = \frac{x-2}{x-3}$

Solution: $x = 1$

Rational equations aren't the only type of equations that can create extraneous solutions. Radical equations can also produce extraneous solutions. Let's look at a simple example first. Take the equation $\sqrt{x} = -3$. One way to find solutions is to graph both sides of the equation to find the point of intersection. Do that now. What do you notice?

Solution: Students should see that the functions $y = \sqrt{x}$ and $y = -3$ do not intersect graphically which means that there is no solution.



Now let's solve this equation algebraically. The key to solving a radical equation is to isolate the radical on one side and then square both sides of the equation in order to eliminate the radical. Do that now for the equation above. What appears to be the answer? Is it really the answer? How can we check other than looking at the graph?

Solution: If you try to solve the equation above by squaring both sides you end up with the solution $x = 9$. Students must remember that the radical sign denotes only the positive root of the number. If we substitute $x = 9$ into the equation the result will be 3 and not -3, so $x = 9$ is not a solution.

Some radical equations do have solutions. Try solving $x = 1 + \sqrt{5x - 9}$. Make sure to check your answers for any extraneous solutions.

Solution: $x = 5$ and $x = 2$. It is important for students to see that they must isolate the radical first before squaring each side of the equation. You can show them if you square both sides before subtracting the one, the result will still produce a radical which gets you no closer to a solution.

Let's practice a few more:

The following problems all illustrate a different type of radical problem. Please make sure to point out these variations as your students solve the problems.

f. $\sqrt{2x - 1} + 5 = 2$

Solution: No solution. To solve this problem, you must first subtract 5 from both sides which will result in the equation $\sqrt{2x - 1} = -3$. Some students might see that there is no solution as this point because the radical sign always denotes the positive solution. Others might keep going algebraically which will result in $x = 5$. It is important that they see that if they substitute their solution back into the problem, they do NOT get balanced equations.

g. $x - 3 = \sqrt{30 - 2x}$

Solution: $x = 7$ and $x = -3$. In this problem you must square the binomial $(x - 3)$ on the left side of the equation and not just a number.

h. $\sqrt{5x + 3} = \sqrt{3x + 7}$

Solution: $x = 2$. In this problem when a radical equals another radical, you can simply set each radicand equal to the other and solve.

i. $2\sqrt{x + 8} = 3\sqrt{x - 2}$

Solution: $x = 10$. Students need to notice the numbers in front of the radicals and take those into consideration when squaring each side.

Extraneous Solutions:

Rational Equations are useful to help you solve some real-life problems like work and average cost. Because those problems have real-life context, issues such as zeros in the denominator can be avoided because it doesn't make sense to solve for 0 hours of work or for 0 blenders. When you look at a rational equation algebraically, though, you have to watch out for such issues. Solutions that create a zero in the denominator are known as extraneous solutions. Let's work an example.

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a. $\frac{3}{5x}+\frac{7}{2x}=1$

b. $\frac{4}{k^2-8k+12}=\frac{k}{k-2}+\frac{1}{k-6}$

c. $\frac{2b-5}{b-2}-2=\frac{3}{b+2}$

d. $\frac{1}{x-2}=\frac{x}{2x-4}+1$

e. $\frac{1}{x}+\frac{1}{x-3}=\frac{x-2}{x-3}$

Rational equations aren't the only type of equations that can create extraneous solutions. Radical equations can also produce extraneous solutions. Let's look at a simple example first. Take the equation $\sqrt{x}=-3$. One way to find solutions is to graph both sides of the equation to find the point of intersection. Do that now. What do you notice?

Now let's solve this equation algebraically. The key to solving a radical equation is to isolate the radical on one side and then square both sides of the equation in order to eliminate the radical. Do that now for the equation above. What appears to be the answer? Is it really the answer? How can we check other than looking at the graph?

Some radical equations do have solutions. Try solving $x=1+\sqrt{5x-9}$. Make sure to check your answers for any extraneous solutions.

Let's practice a few more:

f. $\sqrt{2x-1}+5=2$

g. $x-3=\sqrt{30-2x}$

h. $\sqrt{5x+3}=\sqrt{3x+7}$

i. $2\sqrt{x+8}=3\sqrt{x-2}$

To Bracket or Not To Bracket

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Math Goals

- Simplify rational expressions
- Add and subtract rational expressions
- Multiply and divide rational expressions

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3. Look for and express regularity in repeated reasoning

Introduction

This task serves an introduction to rational inequalities. Students have to learn what critical values are and how they relate to helping you identify the positive and negative intervals of the function. It is important for students to realize that not all critical values can be included in the solution using brackets. The last problem in this task serves to make them think about this idea. Rigor can be added by creating more challenging inequalities or by using complicated rational functions that are not factored. Likewise, the problems can be simplified by giving students inequalities that already have 0 on one side and using simple rational functions. Students should work more problems than what are presented in this task in order to be proficient at solving rational inequalities.

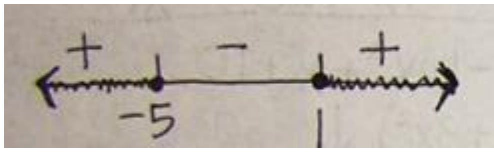
To Bracket or Not To Bracket…That is the Question

Rational Inequalities

Solving rational inequalities is very similar to solving polynomial inequalities. This means that we are trying to figure out when the function itself has a value greater than or less than 0. But because rational expressions have denominators (and therefore may have places where they're not defined), you have to be a little more careful in finding your solutions.

To solve a rational inequality, you first find the zeroes (from the numerator) and the undefined points (from the denominator). You use these zeroes and vertical asymptotes to divide the number line into intervals. Then you find the sign of the rational function on each interval. The final solution of the inequality should be written using interval notation.

Let's try one: $\frac{x-1}{x+5} \geq 0$



So $\frac{x-1}{x+5} \geq 0$ on the interval $(-\infty, -5) \cup [1, \infty)$. Make sure that students are aware that -5 is not included in the solution because it is a vertical asymptote.

Now that we've worked through an entire problem, think about this type of problem: $\frac{5x+1}{x} < 1$. How is this problem different from our first problem? What should we do so that we can solve it?

Students should notice that the inequality does not have 0 on one side. It is important that they realize that when solving inequalities we are talking about when the value of the function is greater than or less than 0. In order to make this happen, we must subtract one from both sides so the result is $\frac{5x+1}{x} - 1 < 0$. Then we must combine the functions on the left side of the equation using subtraction of rational expressions into one rational expression. The result is $\frac{4x+1}{x} < 0$. Students can now follow the steps outlined in the problem above.

Now complete the problem.

Solution: $\left(-\frac{1}{4}, 0\right)$. Students must be reminded to choose test points that are not zeros nor points that give vertical asymptotes.

John is struggling with the same problem that you just completed. His idea was to multiply both sides by *x* so he could "clear the fraction". His answer doesn't match the one we found above. Explain why his method doesn't work.

Solution: If you try to solve the problem the way John does, then you will eliminate one of the vertical asymptotes, $x = 0$. It is important for students to see that you must find all the zeros of both the numerator and denominator. It is possible that some students will have solved the problem exactly like John did. Encourage them to graph the function on their graphing calculator. They need to change the scale of the x-axis to see that the graph indeed changes sign between $x = -\frac{1}{4}$ and $x = 0$.

Jasmine is struggling with interval notation. She remembers from polynomials that the symbols $\geq$ *and* $\leq$ require different notation than the symbols > and < do, but she isn't sure how that will affect rational inequalities. She is working on the problem $\frac{x+3}{x-5} \leq 0$ and she thinks the final answer should be $[-3, 5]$. Kendra comes by and looks at Jasmine's paper. Kendra thinks the answer should be $[-3, 5)$. Who is right and why?

Solution: Kendra is right. 5 is actually a vertical asymptote and is not included in the domain of the function $f(x) = \frac{x+3}{x-5}$ so you can not include it in the solution using brackets. You must use parenthesis around 5 even though the problem is a "less than or equal to" inequality. This is an important distinction to be made to students. Vertical asymptotes should never be included using brackets in any solution. They will always be denoted with parenthesis.

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Rational Inequalities

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Now that we’ve worked through an entire problem, think about this type of problem: $\frac{5x+1}{x} < 1$. How is this problem different from our first problem? What should we do so that we can solve it?

Now complete the problem.

John is struggling with the same problem that you just completed. His idea was to multiply both sides by x so he could “clear the fraction”. His answer doesn’t match the one we found above. Explain why his method doesn’t work.

Jasmine is struggling with interval notation. She remembers from polynomials that the symbols $\geq and \leq$ require different notation than the symbols $> and <$ do, but she isn’t sure how that will affect rational inequalities. She is working on the problem $\frac{x+3}{x-5} \leq 0$ and she thinks the final answer should be $[-3, 5]$. Kendra comes by and looks at Jasmine’s paper. Kendra thinks the answer should be $[-3, 5)$. Who is right and why?

Culminating Task: NFL Passer Rating: Applications of Rational Functions

Back to Task Table

Math Goals

- Use rational equations and inequalities to solve problem situations

Georgia Standards of Excellence

MGSE9-12.A.CED.1 Create equations and inequalities in one variable and use them to solve problems. Include equations arising from ~~linear, quadratic,~~ simple rational, ~~and exponential~~ functions ~~(integer inputs only)~~.

MGSE9-12.A.REI.2 Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them
2. Reason abstractly and quantitatively
3. Model with mathematics

Introduction

This culminating task asks students to apply their knowledge of rational functions to a real world situation that can be expressed using a rational function. **Important Note: Some students may not understand the terminology of the problem however this can be easily explained in the first 5 to 10 minutes of a class period.** The first part of the task is simply familiarizing students with how the formula works and its different components. Then, by using the totals from a previous season, the students are asked to solve for a specific variable A (resulting in a rational function R in terms of A). Then the students will be asked to solve inequalities using their new formula and interpret the results in this real world setting. In addition, the students will be analyzing the characteristics of their new rational function in a different light, not as a mathematical relation, but as a real life scenario where each characteristic has a corresponding meaning in the real world. This task will give students the opportunity to discover and explain some of the limitations of mathematical models in the real world since as we approach various asymptotes the results of the model no longer are applicable.

NFL Passer Rating: Application of Rational Functions

The National Football League has developed a rating system for quarterbacks based upon a number of different factors including passing attempts, completions, yards earned, touchdowns, and interceptions. These factors can be combined into one rational function which will give the Quarterback Passer Rating or QB Rating based upon passing attempts in a game. The QB Rating function is

$$R = \frac{25A + 1000C + 50Y + 4000T - 5000I}{12A}$$

where R is the QB Rating, A are attempts per game, C are completions per game, Y are the total passing yards per game, T are the touchdowns per game, and I are the interceptions per game. These statistics can also be looked at using yearly totals to calculate a season long QB Rating.

In the 2008 NFL season, Peyton Manning, quarterback of the Indianapolis Colts, completed 371 out of 555 passes he threw during a 16 game season. He threw for 4002 yards, 27 touchdowns, and 12 interceptions. Using his complete season statistics, calculate Peyton Manning's QB Rating for the 2008 season.

Solution: R is the unknown, A = 555 passes, C = 371 completions, Y = 4002 yards, T = 27 touchdowns, and I = 12 interceptions. By substitution, we see that

$$R = \frac{25(555) + 1000(371) + 50(4002) + 4000(27) - 5000(12)}{12(555)}$$

R = 95.0

Now, find the average number of completions, attempts, yards, touchdowns, and interceptions per game in the 2008 season. Calculate his QB Rating using these average values for the season and compare to the previous rating. Analyze your findings as to any similarities or differences.

Solution: To find the average game statistics, simply divide the total of each category by the number of games played. So Peyton's average number of completions per game was 23.1875 out of an average number of attempts of 34.6875. He averaged 250.125 total yards passing per game, with an average of 1.6875 touchdowns per game and .75 interceptions per game. By using these statistics in the same formula, we can calculate his QB rating again…

$$R = \frac{25(34.6875) + 1000(23.1875) + 50(250.125) + 4000(1.6875) - 5000(.75)}{12(34.6875)}$$

R = 95.0

Looking at the solutions for each calculation of *R*, determine if the solutions were similar or different. Explain why the ratings were the same or different.

Solution: The solutions for both values of R came out to be exactly the same. This is because we are using the entire season's stats in the first formula and in the second formula we are using an average of those same stats. A difference might be found if we were trying to compare a particular game stats to the average stats of the entire season.

Peyton is looking to improve his QB Rating for next season, and he feels very confident in his ability to repeat most of his statistics. He knows that if he can improve his completion percentage (passes completed / passes attempted) then he should be able to improve his QB Rating.

Suppose we let all of Peyton's statistics for 2009 be the same as in 2008 except for the number of attempts, which we will leave as *A*.

Write the formula for *R* in terms of *A*.

Solution:

$$R(A)=\frac{25(A)+1000(371)+50(4002)+4000(27)-5000(12)}{12(A)}=\frac{25A+619100}{12A}$$

How many attempts will Peyton need in 2009 to generate a Rating > 100? Is this more or less attempts than in 2008? Is this feasible or actually possible? Why or why not?

Solution: This question can be solved a couple of different ways. One way is to graph R(A) along with y = 100. Then find the intersection point of the two curves. This x-value (or the rounded-up one) should be the number of pass attempts A needed to score a QB rating greater than 100. The x-value needed is 526.894 or 527 pass attempts. This is less attempts than in 2008, and it should be possible. The number of completions needs to stay the same as well as all other statistics, but it is completely possible. Another way to solve this inequality is by using algebra to clear fractions.

$$\frac{25A+619100}{12A}>100$$

$$25A+619100>100(12A)$$

$$619100>1175A$$

$$526.894>A$$

How many attempts will generate a Rating < 75? Is this more or less attempts than in 2008?
Solution: This question can be answered by either of the two methods mentioned above, although algebraic manipulation will be quicker.

$$\frac{\mathbf{25A + 619100}}{\mathbf{12A}} < 75$$

$$\mathbf{25A + 619100} < 75(12A)$$

$$619100 < 875A$$

$$707.543 < A$$

So if Peyton throws 708 passes and still only complete the same number as in 2008 (with all other statistics remaining constant) his QB rating will drop below 75.

What kind of a relationship is there between Peyton's QB Rating and the number of attempts that he makes during the season?

Solution: It certainly seems that if all other statistics remain constant, then there is an inverse relationship between Peyton's QB rating and the number of attempts that he makes during a season. Of course, this makes sense because if he throws fewer passes, but still has the same number of completions, then his accuracy is improving as should his rating.

How many attempts will generate a Rating < 50? What do you think the absolute minimum QB Rating could be for Peyton in 2009? Is this feasible? Why or why not?

Solution: Peyton would have to throw more than 1076 passes to generate a QB rating less than 50. This is probably not feasible simply because most teams will not allow their quarterback to complete less than 50% of their passes and remain on the field. Peyton's completion percentage in this case would be 34.4%. The absolute minimum rating Peyton could have in 2009 would be 2.5 because of the horizontal asymptote of y = 2.5.
(FYI: The NFL Quarterback Passing Rating Scale has a maximum value of 158.3 and a minimum value of 0.)

Based on our formula, is there a maximum QB Rating for Peyton in 2009? How did you determine this number?

Solution: By examining the rational function to generate the QB rating, a student might suggest that there is not a max because as x approaches 0, the y-values will be approaching positive infinity. However, this is not the case since it would be impossible to complete more passes than you actually attempt. The max rating would only be possible if the number of completions equaled the number of attempts. In this case, A = 371 and thus R(371) = 141.1.

Find the inverse function for *R*(*A*). What information would the inverse function give us that the original does not? Would this be helpful in answering the questions above?

Solution: The inverse of R(A) would be A(R) = $\frac{619100}{12R-25}$***. This formula would be able to tell us how many attempts, A, a quarterback would need in order to achieve a known QB rating, R. So assuming all other statistics are constant, this would be a great formula to find out just how many less attempts would be needed in order to improve one's QB rating to a certain level. The statistic that is really being affected here is completion percentage, since we are assuming the number of completions does not change, but the number of attempts is changing. Using this formula and one other (completion percentage), a QB could determine what completion percentage he would need to average for an entire season in order to boost his QB rating by any number of points.***

What would be the completion percentage needed in order for Peyton to have a QB rating of 125 in 2008 (assuming all other statistics remain constant)?

Solution: By using the A(R) formula, we can calculate the number of attempts Peyton would need to make, completing 371 of them, in order to have a 125 QB rating. This turns out to be 420 attempts. That would be a completion percentage of 88% which is very difficult to accomplish, especially since Peyton's highest completion percentage of his career was 67.6% in 2004.

All statistics are courtesy of Yahoo! Sports:
http://sports.yahoo.com/nfl/players/4256/career;_ylt=AnqjkbY3U8GcO7mk_cXjLOD.uLYF

Culminating Task: NFL Passer Rating: Application of Rational Functions

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Now, find the average number of completions, attempts, yards, touchdowns, and interceptions per game in the 2008 season. Calculate his QB Rating using these average values for the season and compare to the previous rating. Analyze your findings as to any similarities or differences.

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Suppose we let all of Peyton's statistics for 2009 be the same as in 2008 except for the number of attempts, which we will leave as A.

Write the formula for R in terms of A.

How many attempts will Peyton need in 2009 to generate a Rating > 100? Is this more or less attempts than in 2008? Is this feasible or actually possible? Why or why not?

How many attempts will generate a Rating < 75? Is this more or less attempts than in 2008?

What kind of a relationship is there between Peyton's QB Rating and the number of attempts that he makes during the season?

How many attempts will generate a Rating < 50? What do you think the absolute minimum QB Rating could be for Peyton in 2009? Is this feasible? Why or why not?

Based on our formula, is there a maximum QB Rating for Peyton in 2009? How did you determine this number?

Find the inverse function for $R(A)$. What information would the inverse function give us that the original does not? Would this be helpful in answering the questions above?

What would be the completion percentage needed in order for Peyton to have a QB rating of 125 in 2008 (assuming all other statistics remain constant)?

Culminating Task: Create A Can

Back to Task Table

Math Goals

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MGSE9-12.A.APR.7 Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a nonzero rational expression; add, subtract, multiply, and divide rational expressions.

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them
2. Reason abstractly and quantitatively
3. Model with mathematics

Introduction

This task could be used as part of a culminating assessment or as an alternative to the NFL task. In this task, students will be asked to create a cost-effective cylindrical container using recycled materials. The combination of the volume formula of the can and a cost formula for the net of the can yields a rational function that can be solved for specific results and also interpreted by students showing an understanding of the function as a whole.

The second part of the activity is very open-ended. Students may wish to consider shipping constraints, marketing qualities (golden ratio for appearances), and the potential for the use of other containers to distribute the product, as well as many other factors to explain why a company would pick a different size other than the one with the minimum cost. The task could even be extended so that students design their own can and give mathematical explanations for the dimensions they choose. The instructor can vary the constraints (cost of materials) as needed in order to individualize the assignment.

Create A Can

The 10 O'clock coffee company is creating a can to market their coffee. They are concerned with Going Green in their production as well as being cost efficient. They are using recycled aluminum and need it to hold 135 cubic inches. The top and bottom of the can is made with a thicker aluminum that cost 6 cents per square inch, while the sides are manufactured with a thinner grade that costs 4 cents per square inch.

(This task was modified from the Pearson – Addison Wesley textbook seventh edition.)

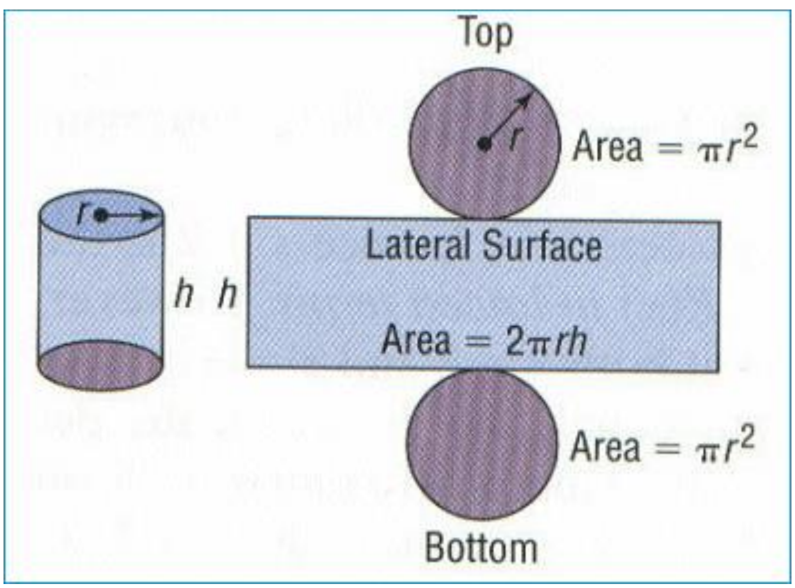
Part I:

A. Determine a formula for the volume of the can in terms of h, r, and π.

B. Create a net diagram for the coffee can. Determine a formula for the cost of the can based upon your net diagram.

Solution for A and B:

Volume = $\boldsymbol{\pi r^2 h}$



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$C =$ cost of top and bottom $+$ cost of side
$C = 2(\pi r^2) *$ cost of lids $+ (2\pi rh) *$ cost of lateral surface
$C = 2(\pi r^2) * (.06) + (2\pi rh) * (.04)$
$C = .12\pi r^2 + .08\pi rh$

C. Find an equation for the cost of the coffee can that includes all the requirements mentioned in the problem scenario.

This problem has one more requirement. The radius and height must be chosen so that the volume of the can is 135 cubic inches.

Volume $= \pi r^2 h$

$135 = \pi r^2 h$ **or** $h = \dfrac{135}{\pi r^2}$

$C = F(r) = .12\pi r^2 + .08\pi r\left(\dfrac{135}{\pi r^2}\right)$

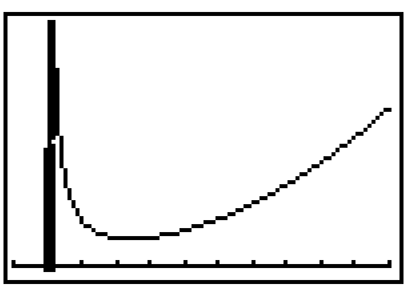
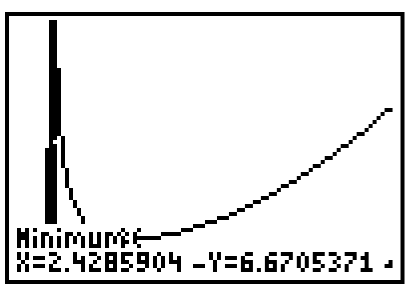
Reduce:

$F(r) = .12\pi r^2 + \dfrac{10.8}{r}$

$F(r) = \dfrac{.12\pi r^3 + 10.8}{r}$

D. What is the minimum cost of the coffee can? What are the dimensions of this coffee can?

This results in the following graphs (the second graph shows the minimum):

When the radius is about 2.4 inches and the height is 7.5 inches, the coffee can has a minimum cost. However, the minimum cost is about 6.7 dollars! Hopefully, students will realize the ineffectiveness of using this particular material to package their coffee can.

Part II:

Pick a cylindrical can product of your own and determine its radius and height. Let's assume the same 3 to 2 cost ratio for the lids and side of your can as in Part I, but let's make the material much cheaper (.6 cents for a lid and .4 cents for the side). Investigate how the dimensions of your product fit the minimum cost model you created in Part I. Obviously, the volume of your can may be different so your model will need to be adjusted to fit the new information. In addition, explain why there may be differences between your results and the actual product.

Sample Solution:

1 Can of Del Monte Green Beans
radius = 1.5 in., height = 4.5 in.
Volume = 31.8 cubic inches

So our can needs to hold 31.8 cubic inches, so

$$Volume = 31.8 = \pi r^2 h$$

$$h = \frac{31.8}{\pi r^2}$$

$$C = F(r) = .012\,\pi r^2 + .008\,\pi r\left(\frac{31.8}{\pi r^2}\right)$$

$$F(r) = \frac{(.012\,\pi r^3 + .2544)}{r}$$

When the minimum of this rational function is found (r = 1.5 inches), this is the exact dimension of our can, and thus, the Del Monte company is minimizing the cost (about 25 cents a can) of their containers of canned vegetables. However, we assumed the cost of the top and bottom differs from the cost of the sides of the can and this may or may not be true in this case.

Culminating Task: Create A Can

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C. Find an equation for the cost of the coffee can that includes all the requirements mentioned in the problem scenario.

D. What is the minimum cost of the coffee can? What are the dimensions of this coffee can?

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