Georgia
Standards of Excellence
Curriculum Frameworks

Mathematics

Accelerated GSE Analytic Geometry B/
Advanced Algebra

Unit 9: Mathematical Modeling

Georgia Department of Education

Richard Woods, Georgia's School Superintendent
"Educating Georgia's Future"

Unit 9
Mathematical Modeling

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OVERVIEW

In this unit students will:

- Synthesize and generalize what they have learned about a variety of function families
- derive the formula for the sum of a finite geometric series and use it to solve problems
- Explore the effects of transformations on graphs of diverse functions, including functions arising in an application, in order to abstract the general principle that transformations on a graph always have the same effect regardless of the type of the underlying functions
- Identify appropriate types of functions to model a situation,
- Adjust parameters to improve the model,
- Compare models by analyzing appropriateness of fit and making judgments about the domain over which a model is a good fit
- Determine whether it is best to model with multiple functions creating a piecewise function.

The description of modeling as "the process of choosing and using mathematics and statistics to analyze empirical situations, to understand them better, and to make decisions" is at the heart of this unit. The narrative discussion and diagram of the modeling cycle should be considered when knowledge of functions, statistics, and geometry is applied in a modeling context.

Although the units in this instructional framework emphasize key standards and big ideas at specific times of the year, routine topics such as estimation, mental computation, and basic computation facts should be addressed on an ongoing basis. Ideas related to the eight process standards should be addressed constantly as well. To assure that this unit is taught with the appropriate emphasis, depth, and rigor, it is important that the tasks listed under "Evidence of Learning" be reviewed early in the planning process. A variety of resources should be utilized to supplement this unit. This unit provides much needed content information, but excellent learning activities as well. The tasks in this unit illustrate the types of learning activities that should be utilized from a variety of sources.

STANDARDS ADDRESSED IN THIS UNIT

Mathematical standards are interwoven and should be addressed throughout the year in as many different units and activities as possible in order to emphasize the natural connections that exist among mathematical topics.

Write expressions in equivalent forms to solve problems.

MGSE9-12.A.SSE.4 Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems. *For example, calculate mortgage payments*

Create equations that describe numbers or relationships

MGSE9-12.A.CED.1 Create equations and inequalities in one variable and use them to solve problems. Include equations arising from linear, quadratic, simple rational, and exponential functions ~~(integer inputs only)~~.

MGSE9-12.A.CED.2 Create linear, quadratic, and exponential equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. (The phrase "in two or more variables" refers to formulas like the compound interest formula, in which $A = P(1 + r/n)^{nt}$ has multiple variables.)

MGSE9-12.A.CED.3 Represent constraints by equations or inequalities, and by systems of equation and/or inequalities, and interpret data points as possible (i.e. a solution) or not possible (i.e. a non-solution) under the established constraints.

MGSE9-12.A.CED.4 Rearrange formulas to highlight a quantity of interest using the same reasoning as in solving equations. *Examples: Rearrange Ohm's law $V = IR$ to highlight resistance R; Rearrange area of a circle formula $A = \pi r^2$ to highlight the radius r.*

Represent and solve equations and inequalities graphically

MGSE9-12.A.REI.11 Using graphs, tables, or successive approximations, show that the solution to the equation f(x) = g(x) is the x-value where the y-values of f(x) and g(x) are the same.

Interpret functions that arise in applications in terms of the context

MGSE9-12.F.IF.6 Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.

MGSE9-12.F.IF.9 Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one function and an algebraic expression for another, say which has the larger maximum.*

Build new functions from existing functions

MGSE9-12.F.BF.3 Identify the effect on the graph of replacing f(x) by f(x) + k, k f(x), f(kx), and f(x + k) for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them.

RELATED STANDARDS

Interpret the structure of expressions

MGSE9-12.A.SSE.1 Interpret expressions that represent a quantity in terms of its context.

MGSE9-12.A.SSE.1a Interpret parts of an expression, such as terms, factors, and coefficients, in context.

MGSE9-12.A.SSE.1b Given situations which utilize formulas or expressions with multiple terms and/or factors, interpret the meaning (in context) of individual terms or factors.

MGSE9-12.A.SSE.2 Use the structure of an expression to rewrite it in different equivalent forms. For example, see $x^4 – y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 – y^2) (x^2 + y^2)$.

Interpret functions that arise in applications in terms of the context.

MGSE9-12.F.IF.4 Using tables, graphs, and verbal descriptions, interpret the key characteristics of a function which models the relationship between two quantities. Sketch a graph showing key features including: intercepts; interval where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; ~~and periodicity~~.

MGSE9-12.F.IF.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*

Analyze functions using different representations.

MGSE9-12.F.IF.7 Graph functions expressed algebraically and show key features of the graph both by hand and by using technology.

MGSE9-12.F.IF.7a Graph linear and quadratic functions and show intercepts, maxima, and minima (as determined by the function or by context).

MGSE9-12.F.IF.7b Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions.

MGSE9-12.F.IF.7c Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior.

MGSE9-12.F.IF.7d Graph rational functions, identifying zeros and asymptotes when suitable factorizations are available and showing end behavior.

MGSE9-12.F.IF.7e Graph exponential and logarithmic functions, showing intercepts and end behavior, ~~and trigonometric functions, showing period, midline, and amplitude.~~

MGSE9-12.F.IF.8 Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.

MGSE9-12.F.IF.8a Use the process of factoring and completing the square in a quadratic function to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context. *For example, compare and contrast quadratic functions in standard, vertex, and intercept forms.*

Build a function that models a relationship between two quantities

MGSE9-12.F.BF.1 Write a function that describes a relationship between two quantities.

MGSE9-12.F.BF.1b Combine standard function types using arithmetic operations in contextual situations (Adding, subtracting, and multiplying functions of different types).

MGSE9-12.F.BF.1c Compose functions. *For example, if T(y) is the temperature in the atmosphere as a function of height, and h(t) is the height of a weather balloon as a function of time, then T(h(t)) is the temperature at the location of the weather balloon as a function of time.*

MGSE9-12.F.BF.4 Find inverse functions.

MGSE9-12.F.BF.4a Solve an equation of the form f(x) = c for a simple function f that has an inverse and write an expression for the inverse. For example, $f(x) = 2(x^3)$ or $f(x) = (x+1)/(x-1)$ for $x \neq 1$.

MGSE9-12.F.BF.4b Verify by composition that one function is the inverse of another.

MGSE9-12.F.BF.4c Read values of an inverse function from a graph or a table, given that the function has an inverse.

STANDARDS FOR MATHEMATICAL PRACTICE

Refer to the Comprehensive Course Overview for more detailed information about the Standards for Mathematical Practice.

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.

6. Attend to precision.

7. Look for and make use of structure.

8. Look for and express regularity in repeated reasoning.

ENDURING UNDERSTANDINGS

- Equations are mathematical tools that arise from attention to specific patterns found in a data set, a graph, a table of values, or some other representation of a mathematical relationship.
- Model building is the result of attention to mathematical patterns and often involves revision based on new data or unexpected changes in pattern.
- Models form the lifeblood of mathematics, and the use of mathematics to solve real-world problems is the result of modeling.
- Modeling is built on the need for accuracy and efficiency.
- Any mathematical function can be used to model natural phenomena, and the key to choosing among the diversity of functions available is to examine long-term patterns in the data being considered.
- Functions can be described, combined, and transformed using multiple lenses, including using algebraic, graphical, numerical, and contextual perspectives.
- The combination and composition of current functions can be used to build new functions to model natural phenomena.
- Mathematical precision is of the utmost importance in using mathematics in other disciplines.
- An extensive knowledge and understanding of mathematical vocabulary is essential in effective modeling.

ESSENTIAL QUESTIONS

- How can an appropriate equation be built by looking at a mathematical pattern?
- How can prior knowledge of functions be used to build precise and efficient models?
- How do the multiple representation of functions aid in building more efficient and more accurate models?
- How can technology be employed to help build mathematical models, and how can it be used to assess the appropriateness of a specific model?
- How can we derive and apply the formula for the sum of a finite geometric series?
- How can both algebraic and geometric models optimize particular important values?
- How can systems of equations and inequalities be used to define feasible regions of solutions to solve problems?

- What is the purpose of building constraints for a model, including using constraints to define feasible solutions and using domain restrictions when analyzing graphs to ensure validity of a function?
- Why is revision necessary in model building?
- Why is a deep knowledge of the various types of basic mathematical functions absolutely necessary in order to build models for real-world phenomena?
- Why is building functions, including combining and composing functions, important in the process of mathematical modeling?

CONCEPTS/SKILLS TO MAINTAIN

It is expected that students will have prior knowledge/experience related to the concepts and skills identified below. It may be necessary to pre-assess in order to determine if time needs to be spent on conceptual activities that help students develop a deeper understanding of these ideas.

- quantitative reasoning
- solving various functions (finding zeros) through factoring, using other algebraic processes, using geometry, or by graphing
- properties of exponents and the associated properties of logarithms
- a working knowledge of geometric vocabulary
- writing explicit and recursive formulas for geometric sequences
- the ability to recall and apply basic algebraic and geometric processes
- an ability to understand mathematics through a variety of representations
- familiarity with technology, particularly the graphing calculator
- prior knowledge and understanding of functions learned earlier in the course, as this is the culminating unit

SELECTED TERMS AND SYMBOLS

The following terms and symbols are often misunderstood. These concepts are not an inclusive list and should not be taught in isolation. However, due to evidence of frequent difficulty and misunderstanding associated with these concepts, instructors should pay particular attention to them and how their students are able to explain and apply them.

The definitions below are for teacher reference only and are not to be memorized by the students. Students should explore these concepts using models and real life examples. Students should understand the concepts involved and be able to recognize and/or demonstrate them with words, models, pictures, or numbers.

http://intermath.coe.uga.edu/dictnary/homepg.asp
Definitions and activities for these and other terms can be found on the Intermath website. Because Intermath is geared towards middle and high school, grade 3-5 students should be directed to specific information and activities.

- **Absolute Value:** The absolute value of a number is the distance the number is from zero on the number line.
- **Base (of a Power)**: The number or expression used as a factor for repeated multiplication
- **Geometric Sequence:** is a sequence with a constant ratio between successive terms
- **Geometric Series:** the expression formed by adding the terms of a geometric sequence
- **Degree**: The exponent of a number or expression
- **Degree of a Polynomial**: The largest exponent of x which appears in the polynomial
- **Domain**: The set of x-coordinates of the set of points on a graph; the set of x-coordinates of a given set of ordered pairs. The value that is the input in a function or relation.
- **Estimate**: A guess about the size, cost, or quantity of something.
- **Exponential**: A number written with an exponent. For example, $6,^3$ is called an exponential expression.
- **Factor**: When two or more integers are multiplied, each integer is a factor of the product. "To factor" means to write the number or term as a product of its factors.
- **Function**: A rule of matching elements of two sets of numbers in which an input value from the first set has only one output value in the second set.
- **Graph of a Function**: The set of all the points on a coordinate plane whose coordinates make the rule of function true.
- **Integer**: The set of numbers ...,-3,-2,-1,0,1,2,3,…
- **Interest**: The percent of the money on deposit (the principal) paid to a lender for the use of the principle
- **Interval**: A regular distance or space between values. The set of points between two numbers.
- **Pattern**: A set of numbers or objects that are generated by following a specific rule.
- **Power**: The exponent of a number or expression, which indicates the number of times the number or expression is used as a factor.

- **Polynomial**: An algebraic expression involving variable with nonnegative integer exponents with one or more unlike terms.

- **Quadratic Function**: A function of degree 2 whose graph is a parabola.

- **Range**: The y-coordinates of the set of points on a graph. Also, the y-coordinates of a given set of ordered pairs. The range is the output in a function or a relation.

- **Rate**: A comparison of two quantities that have different units of measure.

- **Recursive**: A type of sequence in which successive terms are generated by preceding terms in the sequence.

- **Scatterplot**: The graph of a collection of ordered pairs that allows an exploration of the relationship between the points.

- **Substitute**: To replace one element of a mathematical equation or expression with another.

- **Sum of a finite geometric series**: The sum, S_n, of the first *n* terms of a geometric sequence is given by $S_n = \frac{a_1 - a_1 r^n}{1-r} = \frac{a_1(1-r^n)}{1-r}$, where a_1 is the first term and *r* is the common ratio ($r \neq 1$).

- **Sum of an infinite geometric series:** The general formula for the sum *S* of an infinite geometric series $a_1 + a_2 + a_3 + ...$ with common ratio *r* where $|r| < 1$ is $S = \frac{a_1}{1-r}$. If an infinite geometric series has a sum, i.e. if $|r| < 1$, then the series is called a **convergent** geometric series. All other geometric (and arithmetic) series are **divergent**.

- **Symmetry**: The property of a figure or expression that allows for parts of it to be interchanged without forcing a change in the whole

- **Three-Dimensional Figure**: Figures that have length, width, and height.

- **Two-Dimensional Figure**: Figures that have length and width (no height).

- **Unit**: A fixed amount that is used as a standard of measurement.

- **Variable**: A letter or symbol used to represent a number.

- **x-intercept**: The value on the x-axis where a graph crosses the x-axis.

- **y-intercept**: The value on the y-axis where a graph crosses the y-axis.

EVIDENCE OF LEARNING

By the conclusion of this unit, students should be able to demonstrate the following competencies:

- build equations from data sets, graphs, tables of values, and written or verbal descriptions
- analyze functions algebraically, graphically, contextually, and numerically
- graph functions by hand and by using appropriate technology
- find solutions (zeros) of functions using factoring, other algebraic processes, and geometric processes
- build functions using data from a variety of representations
- build functions using a process of combining and composing functions which already exist
- apply geometric reasoning and formulae in order to build appropriate models for given constraints or to fit data
- optimize and constrain specific values algebraically, geometrically, and graphically
- assess the validity of a model and make decisions in regards to model revision
- reason both critically and quantitatively, demonstrating a sophisticated conceptual understanding of modeling for real-world purposes

FORMATIVE ASSESSMENT LESSONS (FAL)

Formative Assessment Lessons are intended to support teachers in formative assessment. They reveal and develop students' understanding of key mathematical ideas and applications. These lessons enable teachers and students to monitor in more detail their progress towards the targets of the standards. They assess students' understanding of important concepts and problem solving performance, and help teachers and their students to work effectively together to move each student's mathematical reasoning forward.

More information on Formative Assessment Lessons may be found in the Comprehensive Course Overview.

SPOTLIGHT TASKS

A Spotlight Task has been added to each GSE mathematics unit in the Georgia resources for middle and high school. The Spotlight Tasks serve as exemplars for the use of the Standards for Mathematical Practice, appropriate unit-level Georgia Standards of Excellence, and research-based pedagogical strategies for instruction and engagement. Each task includes teacher commentary and support for classroom implementation. Some of the Spotlight Tasks are revisions of existing Georgia tasks and some are newly created. Additionally, some of the Spotlight Tasks are 3-Act Tasks based on 3-Act Problems from Dan Meyer and Problem-Based Learning from Robert Kaplinsky.

3-ACT TASK

A Three-Act Task is a whole group mathematics task consisting of 3 distinct parts: an engaging and perplexing Act One, an information and solution seeking Act Two, and a solution discussion and solution revealing Act Three.

More information along with guidelines for 3-Act Tasks may be found in the Comprehensive Course Overview.

TASKS

The following tasks represent the level of depth, rigor, and complexity expected of all students in Advanced Algebra/Algebra II. These tasks or tasks of similar depth and rigor should be used to demonstrate evidence of learning. It is important that all elements of a task be addressed throughout the learning process so that students understand what is expected of them. While some tasks are identified as a performance task, they may also be used for teaching and learning (learning task). Because of the very large size of this unit, and because of the many diverse standards that it encompasses, learning tasks, performance tasks, and culminating tasks are interspersed throughout the unit. Many tasks involve more than one and even several standards, so teachers will have to make decisions as how and when to best utilize some of the more involved tasks.

Task Name	Task Type *Grouping Strategy*	Content Addressed
Fascinating Fractals Learning Task	Learning Task *Partner/Small Group*	Derive the formula for the sum of a geometric series and use it to solve problems
Ted's Quest for a Tablet	Learning/ Performance *Individual/Partner*	Creation of equations in multiple variables in order to model realistic phenomena.
Will I Hit the Hoop? (Spotlight Task)	3-Act Performance Task *Individual/Small Group*	Analyzing functions using different representations
Writing Constraints	Learning *Individual/Partner*	Writing constraint equations for a given context.
A Trip to the Sugar Bowl	Performance *Individual*	Writing a system of constraints for a given context.
Harvesting Fields	Performance/Culminating *Partner/Small Group*	Setting up equations in order to solve problems.
A Gaggle of Graphs	Performance *Individual*	Compare characteristics of function graphs and apply the difference to the context of the problem.
A Game at Cameron Indoor Stadium	Performance *Individual/Partner*	Interpret features of functions within a specific context.
Polynomial Potpourri	Learning *Individual/Partner*	Review of important features of polynomial functions and their graphs.
Combining and Describing Functions	Learning/Performance *Individual/Partner*	Compositions and combining different functions.
Say Yes to the Dress! …or, A Model Marriage	Performance/Culminating *Partner/Small Group*	Create and analyze models of realistic phenomena using various types of functions.

Fascinating Fractals Learning Task

[Back to Task Table]

Mathematical Goals

- Derive the formula for the sum of a finite geometric series
- Use the formula to solve problems

Georgia Standards of Excellence

Interpret the structure of expressions

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MGSE9-12.A.SSE.1a Interpret parts of an expression, such as terms, factors, and coefficients, in context.

MGSE9-12.A.SSE.1b Given situations which utilize formulas or expressions with multiple terms and/or factors, interpret the meaning (in context) of individual terms or factors.

MGSE9-12.A.SSE.2 Use the structure of an expression to rewrite it in different equivalent forms. For example, see $x^4 – y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 – y^2) (x^2 + y^2)$.

Write expressions in equivalent forms to solve problems.

MGSE9-12.A.SSE.4 Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems. *For example, calculate mortgage payments.*

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
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Introduction

In previous courses, students learned about arithmetic and geometric sequences and their relationships to linear and exponential functions, respectively. This unit builds on students' understandings of those sequences and extends students' knowledge to include geometric series, both finite and infinite. Summation notation and properties of sums are also introduced. Additionally, students will examine other types of sequences and, if appropriate, proof by induction. They will use their knowledge of the characteristics of the types of sequences and the corresponding functions to compare scenarios involving different sequences.

Materials

- Pencil
- Handout
- Scissors
- Cardboard or large paper
- Rulers
- Calculators

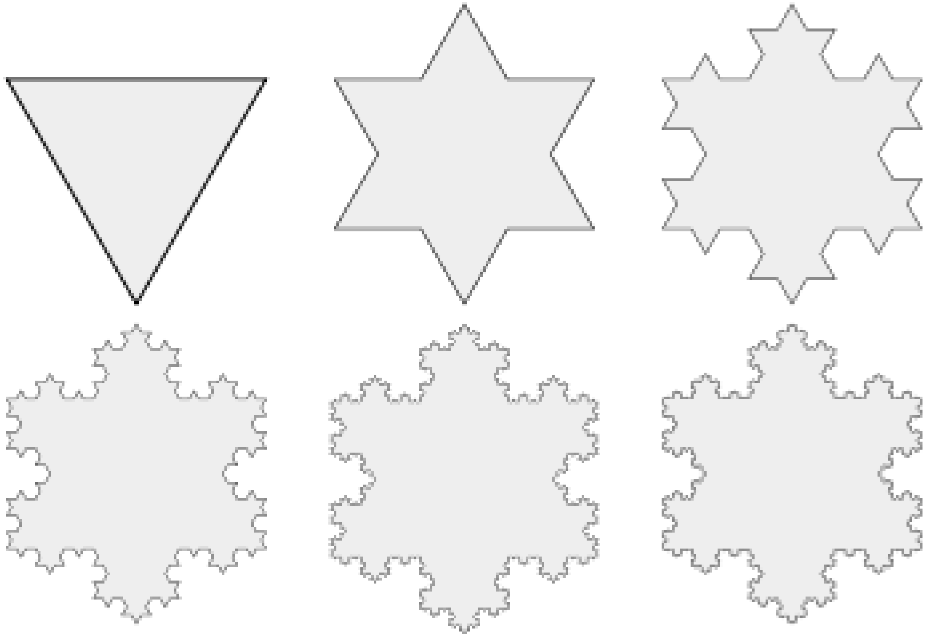
Notes on Fascinating Fractals Learning Task

This task reviews students' understanding of geometric sequences and builds on that knowledge to examine geometric series. Students will examine a variety of geometric sequences that arise in fractals. They will investigate the number of triangles in different iterations of fractals, lengths of segments, perimeters, and areas. Divergence and converge will be discussed, along with when series diverge or converge. Students will also practice computing geometric sums. Links are also made to summing series and horizontal asymptotes.

FASCINATING FRACTALS

Sequences and series arise in many classical mathematics problems as well as in more recently investigated mathematics, such as fractals. The task below investigates some of the interesting patterns that arise when investigating manipulating different figures.

Part One: Koch Snowflake[1]



(Images obtained from Wikimedia Commons at http://commons.wikimedia.org/wiki/Koch_snowflake)

This shape is called a fractal. Fractals are geometric patterns that are repeated at ever smaller increments. The fractal in this problem is called the Koch snowflake. At each stage, the middle third of each side is replaced with an equilateral triangle. (See the diagram.)

To better understand how this fractal is formed, let's create one!

On a large piece of paper, construct an equilateral triangle with side lengths of 9 inches.

Now, on each side, locate the middle third. (How many inches will this be?) Construct a new equilateral triangle in that spot and erase the original part of the triangle that now forms the base of the new, smaller equilateral triangle.

How many sides are there to the snowflake at this point? (Double-check with a partner before continuing.)

Now consider each of the sides of the snowflake. How long is each side? Locate the middle third of each of these sides. How long would one-third of the side be? Construct new equilateral triangles at the middle of each of the sides.

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[1] Often the first picture is called stage 0. For this problem, it is called stage 1. The Sierpinski Triangle, the next problem, presents the initial picture as Stage 0.

A number of excellent applets are available on the web for viewing iterations of fractals.

How many sides are there to the snowflake now? Note that every side should be the same length. Continue the process a few more times, if time permits.

1. Now complete the first three columns of the following chart.

	Number of Segments	**Length of each Segment (in)**	**Perimeter (in)**
Stage 1	3	9	27
Stage 2	***12***	***3***	***36***
Stage 3	***48***	***1***	***48***

2. Consider the **number of segments** in the successive stages.

a. Does the sequence of number of segments in each successive stage represent an arithmetic or a geometric sequence (or neither)? Explain.

Solution: The number of segments is 4 times the number in the previous stage, so this is a geometric sequence. In a geometric sequence, consecutive terms differ by a common ratio. In this case, that ratio is 4.

b. What type of graph does this sequence produce? Make a plot of the stage number and number of segments in the figure to help you determine what type of function you will use to model this situation.

Solution: The plot will produce an exponential graph. So the explicit formula will be the equation of an exponential function.

c. Write a recursive and explicit formula for the number of segments at each stage.

Solution: $\begin{cases} a_1 = 3 \\ a_n = 4a_{n-1} \end{cases}$ ***and*** $a_n = 3(4^{n-1})$

Teaching Note: Students may need some guidance and a direct refresher about exponential functions to come up with the explicit formula.

d. Find the 7th term of the sequence. Find 12th term of the sequence. Now find the 16th. Do the numbers surprise you? Why or why not?

Solutions: $a_7 = 3(4^6) = 12{,}288$; $a_{12} = 3(4^{11}) = 12{,}582{,}912$; $a_{16} = 3{,}221{,}225{,}472$. ***Some students may claim that they expect the numbers to go up so fast because it's exponential. Others may be surprised.***

3. Consider the **length of each segment** in the successive stages.

a. Does this sequence of lengths represent an arithmetic or a geometric sequence (or neither)? Explain.

Solution: The number of segments is 1/3 the number in the previous stage, so this is a geometric sequence. In a geometric sequence, consecutive terms differ by a common ratio. In this case, that ratio is 1/3.

b. Write a recursive and explicit formula for the length of each segment at each stage.

Solution: $\begin{cases} a_1 = 9 \\ a_n = \frac{1}{3}a_{n-1} \end{cases}$ ***and*** $a_n = 9\left(\frac{1}{3}\right)^{n-1}$

c. Find the 7th term of the sequence. Find the 12th term of the sequence. Now find the 16th. How is what is happening to these numbers similar or different to what happened to the sequence of the number of segments at each stage? Why are these similarities or differences occurring?

Solutions: $a_7 = 9(1/3)^6 = 1/81 \approx .012345679$; $a_{12} = 9(1/3)^{11} = 1/(3^9) = 1/19683 \approx 5.08 \times 10^{-5}$; and $a_{16} = 9(1/3)^{15} = 1/(3^{13}) \approx 6.27 \times 10^{-7}$. In the other problem, the numbers were getting bigger. In this problem, they are getting smaller. This is because the common ratio was bigger than one in the first problem and less than one in this problem.

4. Consider the **perimeter** of the Koch snowflake.

a. How did you determine the perimeter for each of the stages in the table?

Solution: To find the perimeter, multiply the number of segments by the length of each segment.

b. Using this idea and your answers in the last two problems, find the approximate perimeters for the Koch snowflake at the 7th, 12th, and 16th stages.

Solutions: Perimeter of the 7th stage: 151.70 inches. Perimeter of the 12th stage: 639.28 inches. Perimeter of the 16th stage: 2020.43 inches.

c. What do you notice about how the perimeter changes as the stage increases?

Solution: As we increase the stage, the perimeter gets much larger.

d. **Extension:** B. B. Mandelbrot used the ideas above, i.e. the length of segments and the associated perimeters, in his discussion of fractal dimension and to answer the question, "How long is the coast of Britain?" Research Mandelbrot's argument and explain why some might argue that the coast of Britain is infinitely long.

Solutions will vary depending on how this is used in the classroom.

5. Up to this point, we have not considered the **area** of the Koch snowflake.

a. Using whatever method you know, determine the exact area of the original triangle.

Solution: The area of the original triangle is $\frac{81\sqrt{3}}{4}$ ***square inches.***

Teaching Note: It will be helpful for the students to recall or derive the formula for the area of an equilateral triangle: $A = s^2\frac{\sqrt{3}}{4}$ ***. This would be a good place to review the derivation of the***

formula to help students remember their right triangle trig in preparation for the upcoming trig units.

b. How do you think we might find the area of the second stage of the snowflake? What about the third stage? The 7th stage? Are we adding area or subtracting area?

Solution: Students can discuss conjectures here. Ideally, someone will suggest that we find the area of the "new" triangles that are added at each stage, multiply by the number of new triangles, and then add to the previous area. Students may also need to be guided to this idea.

c. To help us determine the area of the snowflake, complete the first two columns of the following chart. Note: The sequence of the number of "new" triangles is represented by a geometric sequence. Consider how the number of segments might help you determine how many new triangles are created at each stage.

Stage	**Number of Segments**	**"New" triangles created**	**Area of each of the "new" triangles**	**Total Area of the New Triangles**
1	3	--	--	--
2	***12***	3		
3	***48***	***12***		
4	***192***	***48***		
5	***768***	***192***		
…	***…***	***…***		
n	$3(4^{n-1})$	$3(4^{n-2}), n > 2$		

d. Determine the exact areas of the "new" triangles and the total area added by their creation for Stages 1 – 4. Fill in the chart above. (You may need to refer back to problem 1 for the segment lengths.)

Solutions: *Again, using the formula for the equilateral triangle, we get the following:*

Stage	Number of Segments	"New" triangles created	Area of each of the "new" triangles	Total Area of the New Triangles
1	3	--	--	--
2	*12*	3	$\frac{9\sqrt{3}}{4}$	$\frac{27\sqrt{3}}{4}$
3	*48*	*12*	$\frac{\sqrt{3}}{4}$	$\frac{12\sqrt{3}}{4} = 3\sqrt{3}$
4	*192*	*48*	$\left(\frac{1}{3}\right)^2 \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{36}$	$\frac{48\sqrt{3}}{36} = \frac{4\sqrt{3}}{3}$
5	*768*	*192*	$\left(\frac{1}{9}\right)^2 \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{324}$	$\frac{192\sqrt{3}}{324} = \frac{16\sqrt{3}}{27}$
…	…	…		
n	*$3(4^{n-1})$*	*$3(4^{n-2}), n > 2$*		

e. Because we are primarily interested in the total area of the snowflake, let's look at the last column of the table. The values form a sequence. Determine if it is arithmetic or geometric. Then write the recursive and explicit formulas for the total area added by the new triangles at the *nth* stage.

Solution: *This is a geometric sequence with a common ratio of 4/9.*

$$\begin{cases} a_1 = \frac{81\sqrt{3}}{4}, a_2 = \frac{27\sqrt{3}}{4} \\ a_n = \frac{4}{9} a_{n-1}, n > 2 \end{cases} \quad \textit{and} \quad a_n = \left(\frac{4\sqrt{3}}{3}\right)\left(\frac{4}{9}\right)^{n-4}$$

f. Determine how much area would be added at the 10th stage.

Solution: $a_{10} = \left(\frac{4\sqrt{3}}{3}\right)\left(\frac{4}{9}\right)^6 = \frac{4^7}{3^{13}}\sqrt{3} = \frac{16384\sqrt{3}}{1594323}$

6. Rather than looking at the area at a specific stage, we are more interested in the **TOTAL area** of the snowflake. So we need to sum the areas. However, these are not necessarily numbers that we want to try to add up. Instead, we can use our rules of exponents and properties of summations to help us find the sum.

a. Write an expression using summation notation for the sum of the areas in the snowflake.

Solution: There may be a variety of different summations given. Based on the answer above, one possibility is $\frac{81\sqrt{3}}{4} + \sum_{i=2}^{n}\left(\frac{4\sqrt{3}}{3}\right)\left(\frac{4}{9}\right)^{i-4}$ ***. The important things to remember are that the first***

stage is not included in the summation but needs to be included in the overall expression. Because the first stage is not included, the index of summation must start at 2.

b. Explain how the expression you wrote in part a is equivalent to

$$\frac{81\sqrt{3}}{4}+\left(\frac{9}{4}\right)^4\left(\frac{4\sqrt{3}}{3}\right)\sum_{i=2}^{n}\left(\frac{4}{9}\right)^i.$$

Solutions: Answers should address the rules of exponents and the summation properties previously learned.

Now, the only part left to determine is how to find the **sum of a finite geometric series**. Let's take a step back and think about how we form a finite geometric series:

$S_n = a_1 + a_1r + a_1r^2 + a_1r^3 + \ldots + a_1r^{n-1}$

Multiplying both sides by r, we get $rS_n = a_1r + a_1r^2 + a_1r^3 ++ a_1r^4 \ldots + a_1r^n$

Subtracting these two equations: $S_n - rS_n = a_1 - a_1r^n$

Factoring: $S_n(1 - r) = a_1(1 - r^n)$

And finally, $S_n = \dfrac{a_1\left(1-r^n\right)}{1-r}$

c. Let's use this formula to find the total area of only the new additions through the 5th stage. (What is a_1 in this case? What is n?) Check your answer by summing the values in your table.

Solution: Because we are only summing 4 numbers, n = 4. Our a_1 is the area added in the second stage. $S_4=\dfrac{\frac{27\sqrt{3}}{4}\left(1-\left(\frac{4}{9}\right)^4\right)}{1-\frac{4}{9}}=\dfrac{1261\sqrt{3}}{108}$ ***inches²***.

d. Now, add in the area of the original triangle. What is the total area of the Koch snowflake at the fifth stage?

Solution: $\dfrac{81\sqrt{3}}{4}+\dfrac{1261\sqrt{3}}{108}=\dfrac{862\sqrt{3}}{27}$ ***inches²***.

Do you think it's possible to find the area of the snowflake for a value of n equal to infinity? This is equivalent to finding the **sum of an infinite geometric series**. You've already learned that we cannot find the sum of an infinite arithmetic series, but what about a geometric one?

e. Let's look at an easier series: 1 + ½ + ¼ + … Make a table of the first 10 sums of this series. What do you notice?

Terms	1	2	3	4	5	6	7	8	9	10
Sum	1	3/2	7/4	***15/8***	***31/16***	***63/32***	***127/64***	***255/128***	***511/256***	***1023/512***

Students should notice that the sum is approaching the value 2.

f. Now, let's look at a similar series: 1 + 2 + 4 + …. Again, make a table. How is this table similar or different from the one above? Why do think this is so?

Terms	1	2	3	4	5	6	7	8	9	10
Sum	1	3	7	***15***	***31***	***63***	***127***	***255***	***511***	***1023***

Students should notice that the sum keeps getting larger. Because the ratio is "small" in the first problem, the sum approaches a specific number.

Recall that any real number -1 < r < 1 gets smaller when it is raised to a positive power; whereas numbers less than -1 and greater than 1, i.e. $|r| > 1$, get larger when they are raised to a positive power.

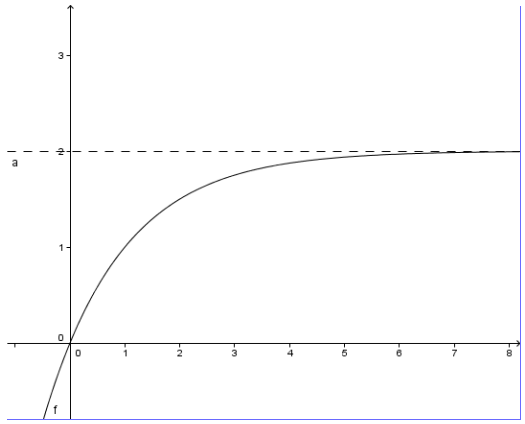
Thinking back to our sum formula, $S_n = \dfrac{a_1\left(1-r^n\right)}{1-r}$, this means that if $|r| < 1$, as *n* gets larger, r^n approaches 0. If we want the sum of an infinite geometric series, we now have $S_\infty = \dfrac{a_1(1-0)}{1-r} = \dfrac{a_1}{1-r}$. We say that if a sum of an infinite series exists—in this case, the sum of an infinite geometric series only exists if $|r| < 1$--- then the series **converges to a sum.** If an infinite series does not have a sum, we say that it **diverges**.

g. Of the series in parts e and f, which would have an infinite sum? Explain. Find, using the formula above, the sum of the infinite geometric series.

Solution: Part e would have a sum because the ratio is ½, which is less than 1.
S = 1/(1-1/2) = 2.

h. Write out the formula for the sum of the first *n* terms of the sequence you summed in part g. Graph the corresponding function. What do you notice about the graph and the sum you found?

Solution: $S_n = \frac{1(1-.5^n)}{1-.5} = 2(1-.5^n)$. ***The students should notice that the sum is the same as the asymptote. This should make sense because an asymptote describes the end behavior of graphs and the infinite sum tells the sum as the number of terms approaches infinity.***



i. *Graphs and infinite series*. Write each of the following series using sigma notation. Then find the sum of the first 20 terms of the series; write out the formula. Finally, graph the function corresponding to the sum formula for the first *n* terms. What do you notice about the numbers in the series, the function, the sum, and the graph?

1. $2+2\left(\frac{1}{3}\right)+2\left(\frac{1}{3}\right)^2+2\left(\frac{1}{3}\right)^3+\ldots$
2. $4+4(0.6)+4(0.6)^2+4(0.6)^3+\ldots$

Solutions:

1. $\sum_{i=1}^{n} 2\left(\frac{1}{3}\right)^{i-1}$; $S_{20} = \frac{2\left[1-\left(\frac{1}{3}\right)^{20}\right]}{1-\frac{1}{3}} \approx 3$; ***The graph of the function*** $f(x) = \frac{2\left[1-\left(\frac{1}{3}\right)^{x}\right]}{1-\frac{1}{3}}$ ***has an asymptote at y = 3. The ratio, r = 1/3 < 1, so the geometric series converges.***

2. $\sum_{i=1}^{n} 4(0.6)^{i-1}$; $S_{20} = \frac{4\left[1-(0.6)^{20}\right]}{1-0.6} \approx 10$; ***The graph of the function*** $f(x) = \frac{4\left[1-(0.6)^{x}\right]}{1-0.6}$ ***has an asymptote at y = 10. The ratio, r = 0.6 < 1, so the geometric series converges.***

j. Let's return to the area of the Koch Snowflake. If we continued the process of creating new triangles infinitely, could we find the area of the entire snowflake? Explain.

Solution: Yes! The ratio of the areas is less than one so the series converges. We would find the area using the infinite sum formula and then add on the original triangle.

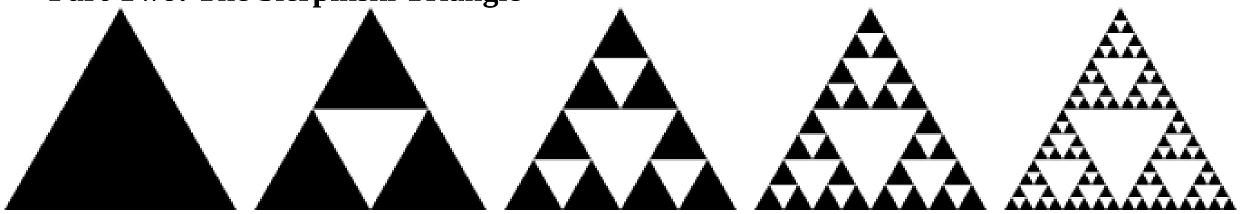
k. If it is possible, find the total area of the snowflake if the iterations were carried out an infinite number of times.

Solution: $S = \dfrac{\frac{27\sqrt{3}}{4}}{1-\frac{4}{9}} = \dfrac{27}{4}\dfrac{9}{5}\sqrt{3} = \dfrac{3^5\sqrt{3}}{20}$ ***then S + original =***

$$Total = \frac{81\sqrt{3}}{4} + \frac{3^5\sqrt{3}}{20} = \frac{(405+243)\sqrt{3}}{20} = \frac{162\sqrt{3}}{5}\ \textit{inches}^2$$

This problem is quite interesting: We have a finite area but an infinite perimeter!

Part Two: The Sierpinski Triangle



(Images taken from Wikimedia Commons at
http://en.wikipedia.org/wiki/File:Sierpinski_triangle_evolution.svg.)

Another example of a fractal is the Sierpinski triangle. Start with a triangle of side length 1. This time, we will consider the original picture as Stage 0. In Stage 1, divide the triangle into 4 congruent triangles by connecting the midpoints of the sides, and remove the center triangle. In Stage 2, repeat Stage 1 with the three remaining triangles, removing the centers in each case. This process repeats at each stage.

1. *Mathematical Questions*: Make a list of questions you have about this fractal, the Sierpinski triangle. What types of things might you want to investigate?

Solutions will vary. The point of this question is to invite students to ask mathematical questions. Hopefully, they will investigate some of their questions in the task.

2. *Number of Triangles in the Stages of the Sierpinski Fractal*

   a. How many shaded triangles are there at each stage? How many removed triangles are there? Use the table to help organize your answers.

Stage	Number of Shaded Triangles	Number of Newly Removed Triangles
0	1	0
1	***3***	***1***
2	***9***	***3***
3	***27***	***9***
4	***81***	***27***
5	***243***	***81***

   b. Are the sequences above—the number of shaded triangles and the number of newly removed triangles—arithmetic or geometric sequences? How do you know?

Solution: The two sequences are essentially the same. They are both geometric. The common ratio is 3.

   c. How many shaded triangles would there be in the *nth* stage? Write both the recursive and explicit formulas for the number of shaded triangles at the *nth* stage.

Solution: $\begin{cases} a_0 = 1 \\ a_n = 3a_{n-1} \end{cases}$ ***and*** $a_n = 3^n$

   d. How many newly removed triangles would there be in the *nth* stage? Write both the recursive and explicit formulas for the number of newly removed triangles at the *nth* stage.

Solution: $\begin{cases} a_0 = 0, a_1 = 3 \\ a_n = 3a_{n-1}, n \geq 2 \end{cases}$ ***and*** $a_n = 3^{n-1}$***,*** $n>0$

e. We can also find out how many removed triangles are in each evolution of the fractal. Write an expression for the total number of removed triangles at the *nth* stage. Try a few examples to make sure that your expression is correct.

Solution: $\sum_{i=1}^{n} 3^{i-1} = \frac{1}{3}\sum_{i=1}^{n} 3^i$ ***or*** $\sum_{i=0}^{n-1} 3^i$

Teaching Note: Students may present a number of different correct and incorrect conjectures. Class discussion can help reveal ideas and misconceptions as well as different ways of thinking. If it is not brought up by the students, this would be a good place for the teacher to discuss how using different indices can change the expression being summed and vice versa.

f. Find the total number of removed triangles at the 10th stage.

Solution: $S_{10} = 1(1-3^{10})/(1-3) = 29{,}524$

g. If we were to continue iterating the Sierpinski triangle infinitely, could we find the total number of removed triangles? Why or why not. If it is possible, find the sum.

Solution: No, we cannot find the sum. This is a divergent geometric series with r > 1.

3. *Perimeters of the Triangles in the Sierpinski Fractal*

a. Assume that the sides in the original triangle are one unit long. Find the perimeters of the shaded triangles. Complete the table below.

Stage	**Length of a Side of a Shaded Triangle**	**Perimeter of each Shaded Triangle**	**Number of Shaded Triangles**	**Total Perimeter of the Shaded Triangles**
0	1	3	1	3
1	½	*3/2*	3	*9/2*
2	¼	¾	*9*	*27/4*
3	*1/8*	*3/8*	*27*	*81/8*
4	*1/16*	*3/16*	*81*	*243/16*
5	*1/32*	*3/32*	*243*	*729/32*
n	$1/(2^n)$	$3/(2^n)$	3^n	$(3^{n+1})/(2^n)$

b. Find the perimeter of the shaded triangles in the 10th stage.

Solution: ***Using the expression above,*** $(3^{11})/(2^{10}) = 177147/1024$.

c. Is the sequence of values for the total perimeter arithmetic, geometric, or neither? Explain how you know.

Solution: ***The sequence is geometric. The common ratio is 3/2.***

d. Write recursive and explicit formulas for this sequence. In both forms, the common ratio should be clear.

Solution: $\begin{cases} a_0 = 3 \\ a_n = \left(\frac{3}{2}\right) a_{n-1} \end{cases}$ ***and*** $a_n = 3\left(\frac{3}{2}\right)^n$

4. *Areas in the Sierpinski Fractal*

a. Assume that the length of the side of the original triangle is 1. Determine the exact area of each shaded triangle at each stage. Use this to determine the total area of the shaded triangles at each stage. (Hint: How are the shaded triangles at stage alike or different?)

Stage	**Length of a Side of a Shaded Triangle (in)**	**Area of each Shaded Triangle (in^2)**	**Number of Shaded Triangles**	**Total Area of the Shaded Triangles (in^2)**
0	1	$\frac{\sqrt{3}}{4}$	1	$\frac{\sqrt{3}}{4}$
1	½	$\left(\frac{1}{2}\right)^2 \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{16}$	3	$\frac{3\sqrt{3}}{16}$
2	¼	$\left(\frac{1}{4}\right)^2 \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{64}$	*9*	$\frac{9\sqrt{3}}{64}$
3	*1/8*	$\left(\frac{1}{8}\right)^2 \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{256}$	*27*	$\frac{27\sqrt{3}}{256}$
4	*1/16*	$\left(\frac{1}{16}\right)^2 \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{1024}$	*81*	$\frac{81\sqrt{3}}{1024}$
5	*1/32*	$\left(\frac{1}{32}\right)^2 \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{4096}$	*243*	$\frac{243\sqrt{3}}{4096}$
n	*$1/(2^n)$*	$\frac{\sqrt{3}}{2^{2n+2}} = \frac{\sqrt{3}}{4}\left(\frac{1}{4}\right)^n$	*3^n*	$\frac{3^n\sqrt{3}}{2^{2n+2}} = \frac{\sqrt{3}}{4}\left(\frac{3}{4}\right)^n$

Differentiation Note: If the numbers seem too cumbersome, an alternate approach is to assume that the original area is 1 in^2.
Solution note: Students may identify a number of different expressions that can be used in the last row of this column. These should be explored and discussed. This exercise could serve to reinforce students' skill with the properties of exponents and arithmetic.

a. Explain why both the sequence of the area of each shaded triangle and the sequence of the total area of the shaded triangles are geometric sequences. What is the common ratio in each? Explain why the common ratio makes sense in each case.

Solution: For the sequence of the area of each shaded triangle, the ratio is ¼. That is, to get from one stage to the next, we multiply the previous area by ¼. This makes sense because each side length is ½ of the previous side length, and to find the area, we square the side length. So at each stage, the area is $(1/2)^2$ = ¼ of the area at the previous stage.

For the sequence of the total area of the shaded triangles, the ratio is ¾. The ¼ piece is the same as described above. Because each stage has 3 times as many shaded triangles as the previous stage, we must also multiply by 3 times the total area from the previous stage. Combing these leads to the common ratio of ¾.

b. Write the recursive and explicit formulas for the sequence of the area of each shaded triangle. Make sure that the common ratio is clear in each form.

Solution: $\begin{cases} a_0 = \dfrac{\sqrt{3}}{4} \\ a_n = \dfrac{1}{4}a_{n-1} \end{cases}$ ***and*** $a_n = \dfrac{\sqrt{3}}{4}\left(\dfrac{1}{4}\right)^n$

c. Write the recursive and explicit formulas for the sequence of the total area of the shaded triangles at each stage. Make sure that the common ratio is clear in each form.

Solution: $\begin{cases} a_0 = \dfrac{\sqrt{3}}{4} \\ a_n = \dfrac{3}{4}a_{n-1} \end{cases}$ ***and*** $a_n = \dfrac{\sqrt{3}}{4}\left(\dfrac{3}{4}\right)^n$

d. Propose one way to find the sum of the areas of the removed triangles using the results above. Find the sum of the areas of the removed triangles in stage 5.

Solution: We can use the original area, $\dfrac{\sqrt{3}}{4}$***, and subtract off the area of the shaded triangles,*** $a_n = \dfrac{\sqrt{3}}{4}\left(\dfrac{3}{4}\right)^n$***. So we get*** $\dfrac{\sqrt{3}}{4} - \dfrac{\sqrt{3}}{4}\left(\dfrac{3}{4}\right)^n = \dfrac{\sqrt{3}}{4}\left(1-\left(\dfrac{3}{4}\right)^n\right)$***. For the 5th stage, we get the sum of the areas of the removed triangles to be*** $\dfrac{\sqrt{3}}{4}\left(1-\left(\dfrac{3}{4}\right)^5\right) = \dfrac{\sqrt{3}}{4}\dfrac{781}{1024} = \dfrac{781\sqrt{3}}{4096}$.

e. Another way to find the sum of the areas of the removed triangles is to find the areas of the *newly removed* triangles at each stage and sum them. Use the following table to help you organize your work.

Stage	Length of a Side of a Removed Triangle (in)	Area of each Newly Removed Triangle (in²)	Number of Newly Removed Triangles	Total Area of the Newly Removed Triangles (in²)
0	0	0	0	0
1	½	$\left(\frac{1}{2}\right)^2\frac{\sqrt{3}}{4}=\frac{\sqrt{3}}{16}$	1	$\frac{\sqrt{3}}{16}$
2	¼	$\left(\frac{1}{4}\right)^2\frac{\sqrt{3}}{4}=\frac{\sqrt{3}}{64}$	3	$\frac{3\sqrt{3}}{64}$
3	1/8	$\left(\frac{1}{8}\right)^2\frac{\sqrt{3}}{4}=\frac{\sqrt{3}}{256}$	9	$\frac{9\sqrt{3}}{256}$
4	1/16	$\left(\frac{1}{16}\right)^2\frac{\sqrt{3}}{4}=\frac{\sqrt{3}}{1024}$	27	$\frac{27\sqrt{3}}{1024}$
5	1/32	$\left(\frac{1}{32}\right)^2\frac{\sqrt{3}}{4}=\frac{\sqrt{3}}{4096}$	81	$\frac{81\sqrt{3}}{4096}$
n	$1/(2^n) = (1/2)^n$	$\frac{\sqrt{3}}{2^{2n+2}}=\frac{\sqrt{3}}{4}\left(\frac{1}{4}\right)^n$	3^{n-1}	$\frac{3^{n-1}\sqrt{3}}{2^{2n+2}}=\left(\frac{1}{3}\right)\frac{\sqrt{3}}{4}\left(\frac{3}{4}\right)^n$

f. Write the explicit and recursive formulas for the area of the removed triangles at stage *n*.

Solution: $\begin{cases} a_0 = 0, a_1 = \frac{\sqrt{3}}{16} \\ a_n = \frac{3}{4}a_{n-1} \end{cases}$ ***and*** $a_n = \left(\frac{1}{3}\right)\frac{\sqrt{3}}{4}\left(\frac{3}{4}\right)^n$

g. Write an expression using summation notation for the sum of the areas of the removed triangles at each stage. Then use this formula to find the sum of the areas of the removed triangles in stage 5.

Solution: $\sum_{i=1}^{n}\left(\frac{1}{3}\right)\frac{\sqrt{3}}{4}\left(\frac{3}{4}\right)^i = \frac{\sqrt{3}}{12}\sum_{i=1}^{n}\left(\frac{3}{4}\right)^i$;

$$\frac{\sqrt{3}}{12}\sum_{i=1}^{5}\left(\frac{3}{4}\right)^i = \frac{\sqrt{3}}{12}\left[\frac{\frac{3}{4}\left(1-\left(\frac{3}{4}\right)^5\right)}{1-\frac{3}{4}}\right] = \frac{\sqrt{3}}{12}\left(\frac{2343}{1024}\right) = \frac{781\sqrt{3}}{4096}$$

h. Find the sum of the areas of the removed triangles at stage 20. What does this tell you about the area of the shaded triangles at stage 20? (Hint: What is the area of the original triangle?)

Solution: $\frac{\sqrt{3}}{12}\sum_{i=1}^{20}\left(\frac{3}{4}\right)^{i}=\frac{\sqrt{3}}{12}\left[\frac{\frac{3}{4}\left(1-\left(\frac{3}{4}\right)^{20}\right)}{1-\frac{3}{4}}\right]\approx\frac{\sqrt{3}}{3\cdot 4}(2.990...)\approx\frac{\sqrt{3}}{4}$***. The sum of the areas of the removed triangles at stage 20 is almost the entire area of the original triangle. So the area of the shaded triangles will eventually be 0.***

i. If we were to continue iterating the fractal, would the sum of the areas of the removed triangles converge or diverge? How do you know? If it converges, to what value does it converge? Explain in at least two ways.

Solution: The sum of the areas of the removed triangles would converge. For one, the area can never be greater than the area of the original triangle. Secondly, the common ratio is ¾ and because $|r|<1$***, the geometric sequence converges. The sum converges to the original area of the triangle. This could be seen by graphing the summation formula and looking at the asymptote of the graph. (Other reasonable explanations are possible.)***

Part Three: More with Geometric Sequences and Series

Up to this point, we have only investigated geometric sequences and series with a positive common ratio. We will look at some additional sequences and series to better understand how the common ratio impacts the terms of the sequence and the sum.

1. For each of the following sequences, determine if the sequence is arithmetic, geometric, or neither. If arithmetic, determine the common difference *d*. If geometric, determine the common ratio *r*. If neither, explain why not.

a. 2, 4, 6, 8, 10, …
b. 2, -4, 6, -8, 10, …
c. -2, -4, -6, -8, -10, …
d. 2, 4, 8, 16, 32, …
e. 2, -4, 8, -16, 32, …
f. -2, -4, -8, -16, -32, …

Solutions:

a. arithmetic; d = 2

b. neither; although the numbers themselves increase by two, the signs alternate. The differences are -6, 10, -14, etc., and is therefore not common.

c. arithmetic; d = -2

d. geometric; r = 2 ***e. geometric; r = -2*** ***f. geometric; r = 2***

2. Can the signs of the terms of an arithmetic or geometric sequence alternate between positive and negative? Explain.

Solution: The signs of an arithmetic sequence cannot alternate. If the signs alternate, the difference could not be common. The signs of a geometric sequence can alternate if the common ratio is negative.

3. Write out the first 6 terms of the series $\sum_{i=1}^{n}(-2)^i$, $\sum_{i=0}^{n}(-2)^i$, and $\sum_{i=1}^{n}(-1)^{i-1}(2^i)$. What do you notice?

Solutions: $\sum_{i=1}^{n}(-2)^i$ ***: -2 + 4 – 8 + 16 – 32 + 64 + …;*** $\sum_{i=0}^{n}(-2)^i$ ***: 1 – 2 + 4 – 8 + 16 – 32 + … ;*** $\sum_{i=1}^{n}(-1)^{i-1}(2^i)$***: 2 – 4 + 8 – 16 + 32 – 64 + …; Some of the things students might notice include that all three of these geometric series have alternating signs. Comparing the first and third ones, the students should see that they can make the first or second term negative if they write the explicit formula correctly.***

4. Write each series in summation notation and find sum of first 10 terms.
   a. 1 – ½ + ¼ - 1/8 +…
   b. 3 + ¾ + 3/16 + 3/64 + …
   c. - 4 + 12 – 36 + 108 - …

Solutions: ***a.*** $\sum_{i=1}^{n}\left(-\frac{1}{2}\right)^{i-1}$ ***;*** $\sum_{i=1}^{10}\left(-\frac{1}{2}\right)^{i-1}=\frac{1-\left(-\frac{1}{2}\right)^{10}}{1-\left(-\frac{1}{2}\right)}=\frac{341}{512}$

b. $\sum_{i=1}^{n}3\left(\frac{1}{4}\right)^{i-1}$ ***;*** $\sum_{i=1}^{10}3\left(\frac{1}{4}\right)^{i-1}=\frac{3\left(1-\left(\frac{1}{4}\right)^{10}\right)}{1-\left(\frac{1}{4}\right)}\approx 4$

c. $\sum_{i=1}^{n}4(-1)^i(3)^{i-1}$***;*** $\sum_{i=1}^{10}4(-1)^i(3)^{i-1}=\frac{-4\left(1-(-3)^{10}\right)}{1-(-3)}=59048$

Students may write different correct summation formulas. These could provide for profitable class discourse.

5. Which of the series above would converge? Which would diverge? How do you know? For the series that will converge, find the sum of the infinite series.

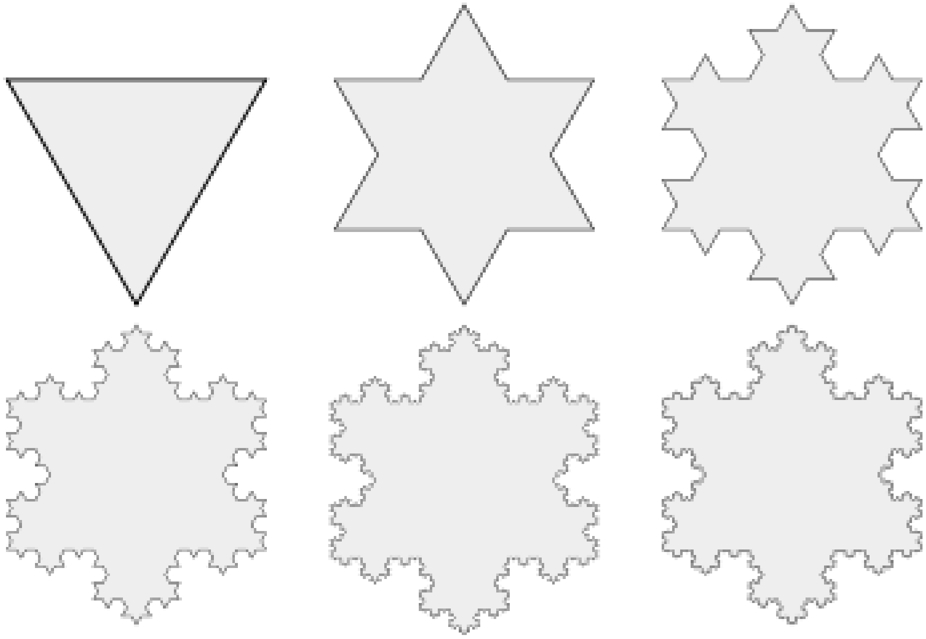
Solution: Parts a and b would converge. In both cases, we have geometric series with $|r|<1$***, so they converge. Part c is a geometric series with*** $|r|>1$***, so it diverges.***

$\sum_{i=1}^{\infty}\left(-\frac{1}{2}\right)^{i-1}=\frac{1}{1-\left(-\frac{1}{2}\right)}=\frac{2}{3}$ ***and*** $\sum_{i=1}^{\infty}3\left(\frac{1}{4}\right)^{i-1}=\frac{3}{1-\frac{1}{4}}=4$

FASCINATING FRACTALS LEARNING TASK:

Sequences and series arise in many classical mathematics problems as well as in more recently investigated mathematics, such as fractals. The task below investigates some of the interesting patterns that arise when investigating manipulating different figures.

Part One: Koch Snowflake[2]



(Images obtained from Wikimedia Commons at http://commons.wikimedia.org/wiki/Koch_snowflake)

This shape is called a fractal. Fractals are geometric patterns that are repeated at ever smaller increments. The fractal in this problem is called the Koch snowflake. At each stage, the middle third of each side is replaced with an equilateral triangle. (See the diagram.)

To better understand how this fractal is formed, let's create one!

On a large piece of paper, construct an equilateral triangle with side lengths of 9 inches.

Now, on each side, locate the middle third. (How many inches will this be?) Construct a new equilateral triangle in that spot and erase the original part of the triangle that now forms the base of the new, smaller equilateral triangle.

How many sides are there to the snowflake at this point? (Double-check with a partner before continuing.)

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[2] Often the first picture is called stage 0. For this problem, it is called stage 1. The Sierpinski Triangle, the next problem, presents the initial picture as Stage 0.

A number of excellent applets are available on the web for viewing iterations of fractals.

Now consider each of the sides of the snowflake. How long is each side? Locate the middle third of each of these sides. How long would one-third of the side be? Construct new equilateral triangles at the middle of each of the sides.

How many sides are there to the snowflake now? Note that every side should be the same length. Continue the process a few more times, if time permits.

1. Now complete the first three columns of the following chart.

	Number of Segments	**Length of each Segment (in)**	**Perimeter (in)**
Stage 1	3	9	27
Stage 2			
Stage 3			

2. Consider the **number of segments** in the successive stages.
   a. Does the sequence of number of segments in each successive stage represent an arithmetic or a geometric sequence (or neither)? Explain.

   b. What type of graph does this sequence produce? Make a plot of the stage number and number of segments in the figure to help you determine what type of function you will use to model this situation.

   c. Write a recursive and explicit formula for the number of segments at each stage.

   d. Find the 7^{th} term of the sequence. Find 12^{th} term of the sequence. Now find the 16^{th}. Do the numbers surprise you? Why or why not?

3. Consider the **length of each segment** in the successive stages.
   a. Does this sequence of lengths represent an arithmetic or a geometric sequence (or neither)? Explain.

   b. Write a recursive and explicit formula for the length of each segment at each stage.

   c. Find the 7^{th} term of the sequence. Find the 12^{th} term of the sequence. Now find the 16^{th}.

   d. How is what is happening to these numbers similar or different to what happened to the sequence of the number of segments at each stage? Why are these similarities or differences occurring?

4. Consider the **perimeter** of the Koch snowflake.
   a. How did you determine the perimeter for each of the stages in the table?

   b. Using this idea and your answers in the last two problems, find the approximate perimeters for the Koch snowflake at the 7^{th}, 12^{th}, and 16^{th} stages.

   c. What do you notice about how the perimeter changes as the stage increases?

   d. **Extension:** B. B. Mandelbrot used the ideas above, i.e. the length of segments and the associated perimeters, in his discussion of fractal dimension and to answer the question, "How long is the coast of Britain?" Research Mandelbrot's argument and explain why some might argue that the coast of Britain is infinitely long.

5. Up to this point, we have not considered the **area** of the Koch snowflake.

   a. Using whatever method you know, determine the exact area of the original triangle.

   b. How do you think we might find the area of the second stage of the snowflake? What about the third stage? The 7^{th} stage? Are we adding area or subtracting area?

   c. To help us determine the area of the snowflake, complete the first two columns of the following chart. Note: The sequence of the number of "new" triangles is represented by a geometric sequence. Consider how the number of segments might help you determine how many new triangles are created at each stage.

Stage	**Number of Segments**	**"New" triangles created**	**Area of each of the "new" triangles**	**Total Area of the New Triangles**
1	3	--	--	--
2		3		
3				
4				
5				
…				
n				

   d. Determine the exact areas of the "new" triangles and the total area added by their creation for Stages 1 – 4. Fill in the chart above. (You may need to refer back to problem 1 for the segment lengths.)

   e. Because we are primarily interested in the total area of the snowflake, let's look at the last column of the table. The values form a sequence. Determine if it is arithmetic or geometric. Then write the recursive and explicit formulas for the total area added by the new triangles at the *nth* stage.

   f. Determine how much area would be added at the 10^{th} stage.

6. Rather than looking at the area at a specific stage, we are more interested in the **TOTAL area** of the snowflake. So we need to sum the areas. However, these are not necessarily numbers that we want to try to add up. Instead, we can use our rules of exponents and properties of summations to help us find the sum.
   a. Write an expression using summation notation for the sum of the areas in the snowflake.

   b. Explain how the expression you wrote in part a is equivalent to $\frac{81\sqrt{3}}{4}+\left(\frac{9}{4}\right)^4\left(\frac{4\sqrt{3}}{3}\right)\sum_{i=2}^{n}\left(\frac{4}{9}\right)^i$.

Now, the only part left to determine is how to find the **sum of a finite geometric series**. Let's take a step back and think about how we form a finite geometric series:

$$S_n = a_1 + a_1r + a_1r^2 + a_1r^3 + \ldots + a_1r^{n-1}$$

Multiplying both sides by r, we get $rS_n = a_1r + a_1r^2 + a_1r^3 ++ a_1r^4 \ldots + a_1r^n$

Subtracting these two equations: $S_n - rS_n = a_1 - a_1r^n$

Factoring: $S_n(1 - r) = a_1(1 - r^n)$

And finally, $S_n = \frac{a_1\left(1-r^n\right)}{1-r}$

   c. Let's use this formula to find the total area of only the new additions through the 5th stage. (What is a_1 in this case? What is n?) Check your answer by summing the values in your table.

   d. Now, add in the area of the original triangle. What is the total area of the Koch snowflake at the fifth stage?

Do you think it's possible to find the area of the snowflake for a value of n equal to infinity? This is equivalent to finding the **sum of an infinite geometric series**. You've already learned that we cannot find the sum of an infinite arithmetic series, but what about a geometric one?

   e. Let's look at an easier series: 1 + ½ + ¼ + … Make a table of the first 10 sums of this series. What do you notice?

Terms	1	2	3	4	5	6	7	8	9	10
Sum	1	3/2	7/4							

f. Now, let's look at a similar series: 1 + 2 + 4 + …. Again, make a table. How is this table similar or different from the one above? Why do think this is so?

Terms	1	2	3	4	5	6	7	8	9	10
Sum	1	3	7							

Recall that any real number -1 < r < 1 gets smaller when it is raised to a positive power; whereas numbers less than -1 and greater than 1, i.e. $|r| > 1$, get larger when they are raised to a positive power.

Thinking back to our sum formula, $S_n = \dfrac{a_1\left(1-r^n\right)}{1-r}$, this means that if $|r| < 1$, as *n* gets larger, r^n approaches 0. If we want the sum of an infinite geometric series, we now have $S_\infty = \dfrac{a_1\left(1-0\right)}{1-r} = \dfrac{a_1}{1-r}$. We say that if a sum of an infinite series exists—in this case, the sum of an infinite geometric series only exists if $|r| < 1$--- then the series **converges to a sum.** If an infinite series does not have a sum, we say that it **diverges**.

g. Of the series in parts e and f, which would have an infinite sum? Explain. Find, using the formula above, the sum of the infinite geometric series.

h. Write out the formula for the sum of the first *n* terms of the sequence you summed in part g. Graph the corresponding function. What do you notice about the graph and the sum you found?

i. *Graphs and infinite series*. Write each of the following series using sigma notation. Then find the sum of the first 20 terms of the series; write out the formula. Finally, graph the function corresponding to the sum formula for the first *nth* terms. What do you notice about the numbers in the series, the function, the sum, and the graph?

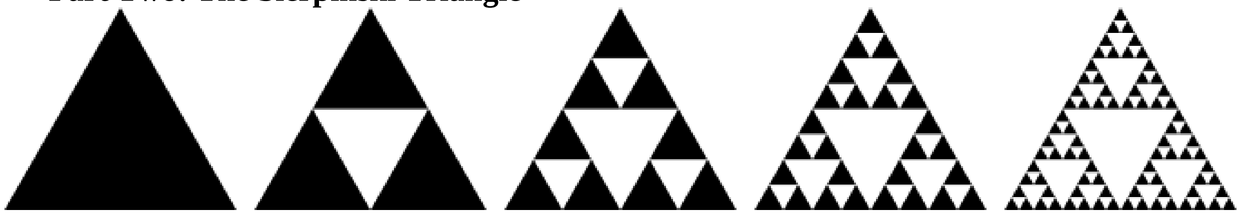
1. $2 + 2\left(\frac{1}{3}\right) + 2\left(\frac{1}{3}\right)^2 + 2\left(\frac{1}{3}\right)^3 + \ldots$
2. $4 + 4\left(0.6\right) + 4\left(0.6\right)^2 + 4\left(0.6\right)^3 + \ldots$

j. Let's return to the area of the Koch Snowflake. If we continued the process of creating new triangles infinitely, could we find the area of the entire snowflake? Explain.

k. If it is possible, find the total area of the snowflake if the iterations were carried out an infinite number of times.

This problem is quite interesting: We have a finite area but an infinite perimeter!

Part Two: The Sierpinski Triangle



(Images taken from Wikimedia Commons at http://en.wikipedia.org/wiki/File:Sierpinski_triangle_evolution.svg.)

Another example of a fractal is the Sierpinski triangle. Start with a triangle of side length 1. This time, we will consider the original picture as Stage 0. In Stage 1, divide the triangle into 4 congruent triangles by connecting the midpoints of the sides, and remove the center triangle. In Stage 2, repeat Stage 1 with the three remaining triangles, removing the centers in each case. This process repeats at each stage.

1. *Mathematical Questions*: Make a list of questions you have about this fractal, the Sierpinski triangle. What types of things might you want to investigate?
2. *Number of Triangles in the Stages of the Sierpinski Fractal*
   a. How many shaded triangles are there at each stage? How many removed triangles are there? Use the table to help organize your answers.

Stage	Number of Shaded Triangles	Number of Newly Removed Triangles
0	1	0
1		
2		
3		
4		
5		

   b. Are the sequences above—the number of shaded triangles and the number of newly removed triangles—arithmetic or geometric sequences? How do you know?

   c. How many shaded triangles would there be in the *nth* stage? Write both the recursive and explicit formulas for the number of shaded triangles at the *nth* stage.

   d. How many newly removed triangles would there be in the *nth* stage? Write both the recursive and explicit formulas for the number of newly removed triangles at the *nth* stage.

   e. We can also find out how many removed triangles are in each evolution of the fractal. Write an expression for the total number of removed triangles at the *nth* stage. Try a few examples to make sure that your expression is correct.

   f. Find the total number of removed triangles at the 10^{th} stage.

g. If we were to continue iterating the Sierpinski triangle infinitely, could we find the total number of removed triangles? Why or why not. If it is possible, find the sum.

3. *Perimeters of the Triangles in the Sierpinski Fractal*

a. Assume that the sides in the original triangle are one unit long. Find the perimeters of the shaded triangles. Complete the table below.

Stage	**Length of a Side of a Shaded Triangle**	**Perimeter of each Shaded Triangle**	**Number of Shaded Triangles**	**Total Perimeter of the Shaded Triangles**
0	1	3	1	3
1			3	
2				
3				
4				
5				
n				

a. Find the perimeter of the shaded triangles in the 10^{th} stage.

b. Is the sequence of values for the total perimeter arithmetic, geometric, or neither? Explain how you know.

c. Write recursive and explicit formulas for this sequence. In both forms, the common ratio should be clear.

4. *Areas in the Sierpinski Fractal*

Assume that the length of the side of the original triangle is 1. Determine the exact area of each shaded triangle at each stage. Use this to determine the total area of the shaded triangles at each stage. (Hint: How are the shaded triangles at stage alike or different?)

Stage	Length of a Side of a Shaded Triangle (in)	Area of each Shaded Triangle (in^2)	Number of Shaded Triangles	Total Area of the Shaded Triangles (in^2)
0	1		1	
1			3	
2				
3				
4				
5				
n				

a. Explain why both the sequence of the area of each shaded triangle and the sequence of the total area of the shaded triangles are geometric sequences. What is the common ratio in each? Explain why the common ratio makes sense in each case.

b. Write the recursive and explicit formulas for the sequence of the area of each shaded triangle. Make sure that the common ratio is clear in each form.

c. Write the recursive and explicit formulas for the sequence of the total area of the shaded triangles at each stage. Make sure that the common ratio is clear in each form.

d. Propose one way to find the sum of the areas of the removed triangles using the results above. Find the sum of the areas of the removed triangles in stage 5.

e. Another way to find the sum of the areas of the removed triangles is to find the areas of the *newly removed* triangles at each stage and sum them. Use the following table to help you organize your work.

Stage	**Length of a Side of a Removed Triangle (in)**	**Area of each Newly Removed Triangle (in^2)**	**Number of Newly Removed Triangles**	**Total Area of the Newly Removed Triangles (in^2)**
0	0	0	0	0
1				
2				
3				
4				
5				
n				

f. Write the explicit and recursive formulas for the area of the removed triangles at stage *n*.

g. Write an expression using summation notation for the sum of the areas of the removed triangles at each stage. Then use this formula to find the sum of the areas of the removed triangles in stage 5.

h. Find the sum of the areas of the removed triangles at stage 20. What does this tell you about the area of the shaded triangles at stage 20? (Hint: What is the area of the original triangle?)

i. If we were to continue iterating the fractal, would the sum of the areas of the removed triangles converge or diverge? How do you know? If it converges, to what value does it converge? Explain in at least two ways.

Part Three: More with Geometric Sequences and Series

Up to this point, we have only investigated geometric sequences and series with a positive common ratio. We will look at some additional sequences and series to better understand how the common ratio impacts the terms of the sequence and the sum.

1. For each of the following sequences, determine if the sequence is arithmetic, geometric, or neither. If arithmetic, determine the common difference *d*. If geometric, determine the common ratio *r*. If neither, explain why not.
   a. 2, 4, 6, 8, 10, …
   b. 2, -4, 6, -8, 10, …
   c. -2, -4, -6, -8, -10, …
   d. 2, 4, 8, 16, 32, …
   e. 2, -4, 8, -16, 32, …
   f. -2, -4, -8, -16, -32, …

2. Can the signs of the terms of an arithmetic or geometric sequence alternate between positive and negative? Explain.

3. Write out the first 6 terms of the series $\sum_{i=1}^{n}(-2)^i$, $\sum_{i=0}^{n}(-2)^i$, and $\sum_{i=1}^{n}(-1)^{i-1}(2^i)$. What do you notice?

4. Write each series in summation notation and find sum of first 10 terms.
   a. 1 – ½ + ¼ - 1/8 +…

   b. 3 + ¾ + 3/16 + 3/64 + …

   c. - 4 + 12 – 36 + 108 - …

5. Which of the series above would converge? Which would diverge? How do you know? For the series that will converge, find the sum of the infinite series.

Ted's Quest for a Tablet

Back to Task Table

Mathematical Goals

- Create and use linear, quadratic, and exponential models to represent realistic scenarios and to solve for variables in those scenarios.

Georgia Standards of Excellence

Create equations that describe numbers or relationships

MGSE9-12.A.CED.1 Create equations and inequalities in one variable and use them to solve problems. Include equations arising from linear, quadratic, simple rational, and exponential functions ~~(integer inputs only)~~.

MGSE9-12.A.CED.2 Create linear, quadratic, and exponential equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. (The phrase "in two or more variables" refers to formulas like the compound interest formula, in which $A = P(1 + r/n)^{nt}$ has multiple variables.)

MGSE9-12.A.CED.3 Represent constraints by equations or inequalities, and by systems of equation and/or inequalities, and interpret data points as possible (i.e. a solution) or not possible (i.e. a non-solution) under the established constraints.

MGSE9-12.A.CED.4 Rearrange formulas to highlight a quantity of interest using the same reasoning as in solving equations. *Examples: Rearrange Ohm's law $V = IR$ to highlight resistance R; Rearrange area of a circle formula $A = \pi r^2$ to highlight the radius r.*

Represent and solve equations and inequalities graphically

MGSE9-12.A.REI.11 Using graphs, tables, or successive approximations, show that the solution to the equation f(x) = g(x) is the x-value where the y-values of f(x) and g(x) are the same.

Analyze functions using different representations.

MGSE9-12.F.IF.7 Graph functions expressed algebraically and show key features of the graph both by hand and by using technology.

MGSE9-12.F.IF.7a Graph linear and quadratic functions and show intercepts, maxima, and minima (as determined by the function or by context).

MGSE9-12.F.IF.7e Graph exponential and logarithmic functions, showing intercepts and end behavior, ~~and trigonometric functions, showing period, midline, and amplitude.~~

MGSE9-12.F.IF.8 Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.

MGSE9-12.F.IF.8a Use the process of factoring and completing the square in a quadratic function to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context. *For example, compare and contrast quadratic functions in standard, vertex, and intercept forms.*

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Introduction

In this task, we create equations (of one to three variables) in order to model some realistic phenomena. The structure of this task could, of course, be manipulated to suit the preferences of the teacher. The reasoning for the structure of this task is based on two needs: First, the obvious need to give a narrative structure to the context of the task. Second, and more importantly, while teachers (or students) may question why the task begins with a system of equations utilizing three variables, the task begins here to present (and ask students to evaluate) the types of equations that students will be required to create on their own later. The task covers a number of relationships between variables, including linear, exponential, and quadratic.

Materials

- Pencil
- Handout
- Calculator

Ted's Quest for a Tablet

Ted has had his eye on a tablet computer for several months, and he's trying to figure out a way to save enough money so that he can buy one using cash instead of credit. Ted is trying every possible method to build his computer fund – he's looking for change in the couch, he's drawn up a savings plan, he's budgeting in other areas of his life, and so on. But Ted is growing impatient, and he's afraid that he will resort to using a credit card so that he can get one quickly, but of course that means Ted will have to pay interest on his purchase.

Let's first look at a few ways that Ted has attempted to build his fund.

1. Change under the Sofa Cushion

Ted first tries the easiest way to find money – he looks under the cushions of his sofa! And he finds, to his amazement, enough coins to equal $7.75! If Ted has

- found only nickels, dimes, and quarters,
- a total of 65 coins,
- 5 more nickels than the total number of dimes and quarters together,

Which of the following can be used to find the number of nickels, n, the number of dimes, d, and the number of quarters, q, that Ted has?

(a) $\begin{cases} n + d + q = 7.75 \\ 65n + 65d + 65q = 7.75 \\ n - 5 = d + q \end{cases}$

(b) $\begin{cases} n + d + q = 65 \\ 0.05n + 0.1d + 0.25q = 65 \\ n - (d + q) = 5 \end{cases}$

(c) $\begin{cases} n + d + q = 65 \\ 0.05n + 0.1d + 0.25q = 7.75 \\ n - d - q = 5 \end{cases}$

(d) $\begin{cases} n + d + q = 7.75 \\ 0.05n + 0.1d + 0.25q = 7.75 \\ (d + q) + 5 = n \end{cases}$

Explain your choice.

The correct choice is (c). The quantity of coins in relation to the total, the monetary value of coins, and the comparative number of coins in relation to one another are all correct and set equal to their appropriate total.

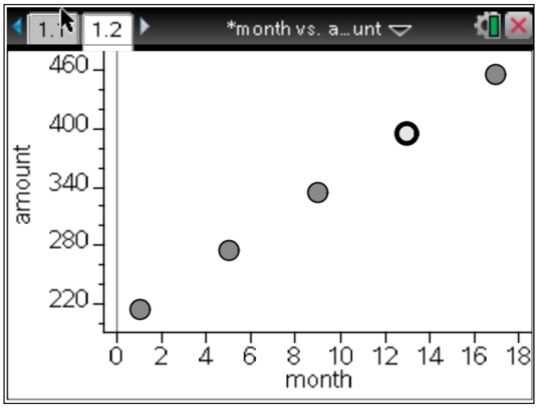
2. Money in the Bank

Ted's grandmother, Miss Tedrina Mae, really likes to spoil her grandson and knows how much he wants a tablet computer. She sets up a checking account (with no interest) for Ted with an initial amount of $200 and she will add $15 each month afterward.

(a) Write an equation whose solution is the number of months, m, it takes for the account balance to reach $395.

$$\mathbf{200 + 15m = 395}$$

(b) Make a plot of the balance after m months for $m = 1, 5, 9, 13, 17$ and indicate on the plot the solution to your equation in part (a).



Ted will have $395 after 13 months.

Ted loves his grandmother and gladly accepts her $200 gift, but he doesn't want her to open up a checking account and put extra money in it (she's done too much already!) – He believes he has a better idea, anyway. Ted remembered that he took some notes in a finance course he was enrolled in at the local community college regarding interest-bearing accounts:

<u>Compound Interest (2 Types)</u>

1. n compounding periods: $A = P\left(1+\frac{r}{n}\right)^{nt}$

2. Continuous compounding: $A = Pe^{rt}$

<u>Values of n:</u>
Annually = Once per year
Semiannually = Twice per year
Quarterly = Four times per year
Monthly = Twelve times per year
Weekly = Fifty-two times per year

A = Final amount

P = Principal (starting or initial amount)

r = interest rate (decimal form)

n = compounding periods per year

t = time in years

There are two banks that Ted will decide between to open a savings account (which is one type of interest-bearing account) – Bernoulli Bank or Euler Federal Bank. Bernoulli Bank offers a 6.75% quarterly-compounded interest rate, and Euler Federal offers a 6.75% continuously-compounded interest rate.

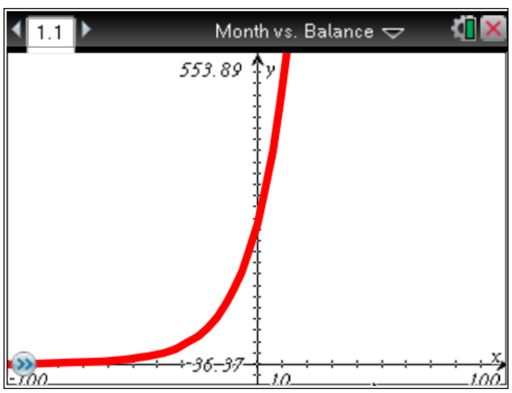
Ted plans on investing his $200 gift – plus an additional $50 bill that he found in the sofa cushion when he was looking for change – in a savings account for one year.

(c) Give the simplified form of the equation representing the amount earned at any given month at Bernoulli Bank.

$$A = 250\left(1+\frac{0.0675}{4}\right)^{4t} \text{ so } A = 250(1.016875)^{4t}$$

While this is a simple one-step process, it is an important transformation so that the student can see that this is, indeed, an exponential function is a format that is recognizable.

(d) Graph the function from part (c). What type of function is this?



The students should recognize that this is, indeed, an exponential growth function. At this point, it is helpful to have the students identify important information about and from the graph. For example, it is necessary for students to recognize this as an exponential growth function, not only because of the graph, but because the exponential base is greater than one. A comparison with a decay function would be useful here. As for the essential information from the graph, the meaning of the y-intercept is important, as is the presence of the horizontal asymptote (which isn't relevant to the context, but is useful in helping to identify exponential functions in general). These characteristics are not only important in identifying an exponential, but are also some of the key tools to compare with the inverse, a logarithmic function, which occurs later in the unit.

(e) Fill in the table below and explain which bank you would choose.

Year	Bernoulli Bank	Euler Federal Bank
1	*267.31*	*267.46*
2	*285.81*	*286.13*
3	*305.60*	*306.12*
4	*326.75*	*327.49*
5	*349.37*	*350.36*
6	*373.56*	*374.83*
t	$250(1.016875)^{4t}$	$250e^{0.0675t}$

Obviously, Euler Federal wins by a very small margin. The driving factor here is the small starting amount and relatively low interest rate – otherwise, larger differences would be observable over time. Still, the lesson remains the same – when interest is compounded at more frequent intervals (assuming equal interest rates), the owner of the corresponding account will make more money over time.

(f) If Ted chooses the savings account with Euler Federal, how long will it take for Ted to save the same amount that he would have had in six months with his grandmother's original plan?

With his grandmother's original plan, Ted would have had the following after six months:

$$200 + (6 \cdot 15) = 290$$

Using the formula associated with Ted's savings account at Euler Federal, we have the following:

$$290 = 250e^{0.0675t}$$

$$1.16 = e^{0.0675t}$$

$$\ln 1.16 = \ln e^{0.0675t}$$

$$\frac{\ln 1.16}{0.0675} = t \;\; \text{so } t \approx 2.2$$

Or, using graphing calculator: let $Y_1=250e^{0.0675x}$ and $Y_2=290$, chose appropriate viewing window then select 2nd CALC #5 intersect which gives the solution x=2.1988

Therefore, even though Ted starts with a \$50 higher initial amount with his savings account, it would still take him over two years to earn the same amount that he would in six months using his grandmother's generosity.

It's very important here to remind students that the need for using logarithms can pop up anywhere you find exponential functions!

3. Ted's Totally Terrific Guitars

This is a simple exercise in creating equations from a situation with many variables. Within the different scenarios, the problem requires students to keep going back to the definitions of the variables, thus emphasizing the importance of defining variables when you write an equation. In order to reinforce this aspect of the problem, the variables have not been given names that remind the student of what they stand for. The emphasis here is on setting up equations, not solving them.

Ted just happens to be a really talented craftsman, and is known throughout his hometown as being a very good guitar builder. Ted hopes to make some money from this business for his tablet computer purchase. Ted builds three types of guitars: archtops, electrics, and acoustics. Ted builds 1 archtop guitar per month, 2 acoustic guitars per month, and 3 electric guitars per month. Suppose that it takes Ted x hours to build an archtop guitar, y hours to build an electric guitar, and z hours to build an acoustic guitar.

(a) Write an equation relating x, y, and z if Ted spends a total of 134 hours per month building guitars.

$$\boldsymbol{x + 3y + 2z = 134}$$

(b) If Ted charges $90 per hour for an archtop guitar, $45 per hour for an electric guitar, and $65 per hour for an acoustic guitar, write an equation relating x, y, and z if Ted builds $9240 worth of guitars.

$$\boldsymbol{90x + 45y + 65z = 9240}$$

Ted buys large blocks of specific varieties of expensive woods to build his guitars. Ted has found that the best varieties of tone woods for instruments are spruce and mahogany. Ted buys w blocks of spruce for y dollars each, and x blocks of mahogany for z dollars each. In a given month, Ted spends a total of Cdollars, where $C = wy + xz$. For the following, write an equation whose solution is the given quantity.

(c) The number of blocks of mahogany that Ted can afford to buy if he wishes to spend a total of $5000 this month, mahogany costs $450 per block, and he has already bought 7 blocks of spruce at $200 each.

$$\boldsymbol{5000 = 450x + 7(200)}$$

$$\boldsymbol{3600 = 450x}$$

(d) The price of spruce blocks if Ted bought 9 of them, in addition to the 10 blocks of mahogany that Ted bought at $425 a block, for a total of $6275.

$$6275 = 9y + 10(425)$$

$$2025 = 9y$$

(e) The price of a block of mahogany, given that a block of spruce costs $\frac{5}{9}$ as much as a block of mahogany, and Ted has bought 12 blocks of spruce and 15 blocks of mahogany for a total cost of $9750.

$$y = \frac{5}{9}z$$

$$12 \cdot \frac{5}{9}z + 15z = 9750$$

$$\frac{65}{3}z = 9750$$

4. Ted's Toss

This activity deals with quadratic functions, both in simple general terms, and in more complicated practical application. In the middle of the problem, there is a short and straightforward quadratic review so that students are reminded of some basic skills before moving on to applying them back to the original context of the problem. There are a few goals at work here – students should obviously be able to manipulate variables to solve equations in higher-order functions and they should finish that problem-solving process in an application-oriented way.

Ted is still trying to find ways to make money, and since the county fair is in town, he's decided to try his skill in the ball-throwing competition (this is a fairly simple county fair!) with a $50 grand prize. The winner is the thrower that produces the longest time in the air for the ball. When Ted throws this particular type of ball, it moves vertically upward at a speed of v feet/second and rises a distance of d feet in t seconds, given by

$$d = 4 + vt - 10t^2$$

Write an equation whose solution is
(a) the time it takes a ball thrown at a speed of 88 ft/sec to rise 20 feet;

$$\mathbf{20 = 4 + 88}\boldsymbol{t}\mathbf{ - 10}\boldsymbol{t}^{\mathbf{2}}$$

(b) the speed with which the ball must be thrown to rise 20 feet in 2 seconds.

$$\mathbf{20 = 4 + 2}\boldsymbol{v}\mathbf{ - 10(2^2)}$$

$$\mathbf{20 = 2}\boldsymbol{v}\mathbf{ - 36}$$

Before we move on to the second part of "Ted's Toss", let's review quadratics.

Quick Quadratic Review

This exploration can be done in class near the beginning of a unit on graphing parabolas. Students need to be familiar with intercepts, and need to know what the vertex is. It is effective after students have graphed parabolas in vertex form [$\boldsymbol{y = a(x - h)^2 + k}$], but have not yet explored graphing other forms. Part (a) is not obvious to them; they are excited to realize that equivalent expressions produce the same graph. Parts (b) and (c) lead to important discussions about the value of different forms of equations, culminating in a discussion of how we can convert between forms and when we might want to do so.

A natural extension of this task is to have the students share some of the different equations that they found for a given condition and have them graph two or more simultaneously. For example, students could graph three different equations that all have the same x-intercepts and discuss the effect that the different constant factors have on the graph.

Graph these equations on your graphing calculator at the same time. What happens? Why?

Y1: $(x-3)(x+1)$

Y2: x^2-2x-3

Y3: $(x-1)^2-4$

Y4: x^2-2x+1

The first three equations produce the graph of the same parabola – these equations represent the same quadratic function simply written in different forms (intercept, standard, and vertex). The fourth equation produces a different parabola.

Below are the first three equations from the previous problem.

Y1: $(x-3)(x+1)$

Y2: x^2-2x-3

Y3: $(x-1)^2-4$

These three equations all describe the same function. What are the coordinates of the following points on the graph of the function? From which equation is each point most easily determined? Explain.

vertex: _____

y-intercept: _____

x-intercept(s): _____

Vertex @ (1, 4) from Y3 (vertex form). The y-intercept is (0, 3), which is visible as the constant in Y2 (standard form) since the other terms are 0 when you plug in $\boldsymbol{x=0}$***. The x-intercepts are (3, 0) and (-1, 0) from Y1 (intercept form).***

Make up an equation for a quadratic function whose graph satisfies the given condition. Use whatever form is most convenient.

(a) Has a vertex at (−2,−5).

Answers vary. Possible equation: $\boldsymbol{y=3(x+2)^2-5}$

(b) Has a y-intercept at (0,6)

Answers vary. Possible equation: $\boldsymbol{y = 2x^2 + 6}$

(c) Has x-intercepts at (3,0) and (5,0)

Answers vary. Possible equation: $\boldsymbol{y = -\frac{1}{2}(x-3)(x-5)}$

(d) Has x-intercepts at the origin and (4, 0)

Answers vary. Possible equation: $\boldsymbol{y = 7x(x+4)}$

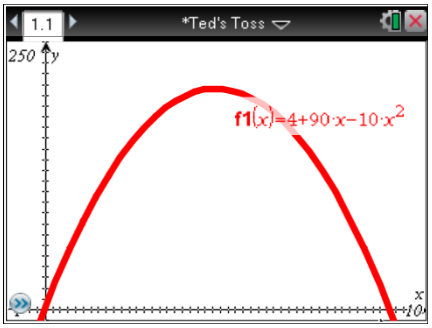
(e) Goes through the points (4,-10) and (1, 2)

Answers vary. Possible equation: $\boldsymbol{y = -\frac{1}{2}(x+4)(x-1) + 2}$

Now let's say that Ted has a pretty good throwing arm (he has entered a throwing competition, after all!). In fact, on his particular throw for this competition, Ted throws the ball at a speed of 90 ft/sec. Using $d = 4 + vt - 10t^2$, answer the following.

(c) Write the equation that will be used to model Ted's throw, and then graph his throw, from the time it leaves his hand to the time it falls back to the ground. At what height above the ground does the ball leave Ted's hand?

$$\boldsymbol{d = 4 + 90t - 10t^2}$$



It would be useful here to discuss the domain restrictions as they relate to the practical range for the trajectory of the ball.

The ball would leave Ted's hand at $t = 0$, so the height of the ball at release would be 4 feet.

(d) For Ted's throw, what is the maximum height of the ball? Show how you would find this using the equation you created in part (c), and then check your answer with your graph.

Quadratic functions are symmetric about their vertex, which means that the maximum height (vertex) should occur right in the middle of the domain if the ball is being released from a height of zero and falling back to the ground to a height of zero. After graphing, some students may want to simply take the middle x-value and plug it back in to get the associated y. However, the ball is being released at a height of 4 feet and falling back to the ground to a height of zero – and therefore the scenario presented isn't exactly symmetric. Four isn't far from zero, so it would be conceivable that the maximum height should still occur very near the center of the domain of the graph. This may not be a judgment call that teachers feel their students mathematically aware or mature enough to make, so of course algebraic transition from standard form to vertex form will easily yield the vertex of the graph, and thus, give us the true maximum.

$$d = 4 + 90t - 10t^2$$

$$d - 4 - 10(\quad) = -10(t^2 - 9t + \quad)$$

$$d - 4 - 10\left(\frac{81}{4}\right) = -10\left(t^2 - 9t + \frac{81}{4}\right)$$

$$d - 4 - \frac{810}{4} = -10\left(t - \frac{9}{2}\right)^2$$

$$d = -10\left(t - \frac{9}{2}\right)^2 + \frac{16}{4} + \frac{810}{4}$$

$$d = -10\left(t - \frac{9}{2}\right)^2 + 206.5$$

Therefore, the vertex occurs at ($t = 4.5$ seconds, $d = 206.5$ feet), so the maximum height is 206.5 feet.

Or, using graphing calculator: let Y_1=4+90x-10x^2 then choose appropriate viewing window then select 2nd CALC–4: maximum which gives the solution after 4.5 sec, ball has height =206.5

Ted was the last person to throw in the competition. Unfortunately, up to the time of Ted's throw, his arch-nemesis Billy Bob Bigglesby (who already owns the most expensive tablet computer on the market) had the longest throw with the ball having stayed in the air for 8.8 seconds. The model for Billy Bob's throw is the same as for Ted [$d = 4 + vt - 10t^2$]. Obviously, they are the same height and have roughly the same throwing position.

(e) Who won the $50 prize, Ted or Billy Bob? How must the speed of Billy Bob's throw compare to Ted's?

We know from our graph above that the time in the air for Ted's throw was approximately 9.04 seconds, which is longer than Billy Bob's throw. Obviously, Ted won the $50 prize. Also, since the same model represents both their throws, Ted's throw must have possessed greater speed than Billy Bob's.

(f) What was the speed of Billy Bob's throw if the maximum height for his throw was 197.6 feet?

First, the student must recognize that you cannot simply plug in 8.8 seconds to solve for the speed of the throw because the highest point should be somewhere near the center of the model's domain. And just as above, it is important for the student to realize that this is not a perfectly symmetric quadratic model because of the difference in the starting height and ending height of the ball. But just as above, because 4 feet is relatively close to zero, substituting 4.4 seconds into the equation will produce an accurate answer in this instance, and it does show that the student is thinking about where the function's maximum will occur. Using this method, we have

$$\mathbf{197.6 = 4 + 4.4v - 10(4.4^2)}$$

$$\mathbf{197.6 = -189.6 + 4.4v}$$

$$\mathbf{88 = v}$$

For a (very slightly) more accurate solution – and a truly accurate process – using proportions, we know that the vertical distance from the starting position of 4 feet to

the maximum is 193.6 feet, and the vertical distance from the maximum back to the ground is 197.6 feet. Therefore we have

$$\frac{193.6}{193.6 + 197.6} \approx 0.495$$

So we can now use this information to find the number of seconds it takes to get about 49.5% of the way through the domain of the model

$$0.495(8.8) \approx 4.355 \textit{ seconds}$$

$$197.6 = 4 + 4.355v - 10(4.355^2)$$

$$88.005 \approx v$$

So we now know that Billy Bob threw the ball at a speed of approximately 88.005 feet per second.

5. Ted's Test

Ted also happens to be a student at the local college, where he is taking a physics course. Ted has a special scholarship arrangement – instead of having his tuition paid, the foundation that pays for his scholarship gives him $250 for every course in which he makes an A. Ted is taking a physics final exam, and he is being asked to re-arrange important equations in order to solve for a desired variable.

Help Ted out by using inverse operations to solve the equations for the unknown variable, or for the designated variable if there is more than one.

(a) $-3 = \frac{x}{-27}$

$\boldsymbol{x = 81}$

(b) $16z = \frac{1}{4}$

$\boldsymbol{z = \frac{1}{64}}$

(c) $\frac{1}{3}w + 7 = \frac{9}{5}$

$\boldsymbol{w = -\frac{78}{5}}$

(d) $a^2 + b^2 = c^2$ for b

$\boldsymbol{b = \pm\sqrt{c^2 - a^2}}$

(e) $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ for c

$\boldsymbol{c = \frac{-(2ax+b)^2 + b}{4a}}$

(f) $F = qE$ for E

$\boldsymbol{E = \frac{F}{q}}$

(g) $E_k = \frac{1}{2}mv^2$ for v

$\boldsymbol{v = \pm\sqrt{\frac{2E_k}{m}}}$

(h) $ax + c = R$ for x

$\boldsymbol{x = \frac{R-c}{a}}$

(i) $x = x_0 + v_0t + \frac{1}{2}at^2$ for a

$\boldsymbol{a = \frac{2(x-x_0-v_0t)}{t^2}}$

(j) $F = \frac{GMm}{r^2}$ for Mm

$\boldsymbol{Mm = \frac{Fr^2}{G}}$

6. Ted's Tablet

The emphasis in this task is not on complex solution procedures. Rather, the progression of equations, from two that involve different values of the sales tax, to one that involves the sales tax as a parameter, is designed to foster the habit of looking for regularity in solution procedures, so that students don't approach every equation as a new problem but learn to notice familiar types.

Ted finally made enough money (by checking sofa cushions, building guitars, using his savings account, throwing balls in competitions, and doing well in college) to buy himself the tablet computer of his dreams! Ted bought the computer for exactly \$817.53, which is the list price plus sales tax. Find the list price of the tablet if Ted bought the computer in

(a) Vidalia, where the sales tax is 5%;

$$\boldsymbol{1.05p = 817.53}$$
$$\boldsymbol{p = 778.60}$$

The list price in Vidalia would be \$778.60.

(b) Marietta, where the sales tax is 7.55%;

$$\boldsymbol{1.0755p = 817.53}$$
$$\boldsymbol{p = 760.14}$$

The list price in Marietta would be \$760.14.

(c) a city in Georgia where the sales tax is r.

$$\boldsymbol{(1+r)p = 817.53}$$
$$\boldsymbol{p = \frac{817.53}{1+r}}$$

The list price in a city with a sales tax rate of $\boldsymbol{r}$ is $\boldsymbol{\frac{817.53}{1+r}}$.

Ted's Quest for a Tablet

Ted has had his eye on a tablet computer for several months, and he's trying to figure out a way to save enough money so that he can buy one using cash instead of credit. Ted is trying every possible method to build his computer fund – he's looking for change in the couch, he's drawn up a savings plan, he's budgeting in other areas of his life, and so on. But Ted is growing impatient, and he's afraid that he will resort to using a credit card so that he can get one quickly, but of course that means Ted will have to pay interest on his purchase.

Let's first look at a few ways that Ted has attempted to build his fund.

1. Change under the Sofa Cushion

Ted first tries the easiest way to find money – he looks under the cushions of his sofa! And he finds, to his amazement, enough coins to equal $7.75! If Ted has

- found only nickels, dimes, and quarters,
- a total of 65 coins,
- 5 more nickels than the total number of dimes and quarters together,

which of the following can be used to find the number of nickels, n, the number of dimes, d, and the number of quarters, q, that Ted has?

(a) $\begin{cases} n + d + q = 7.75 \\ 65n + 65d + 65q = 7.75 \\ n - 5 = d + q \end{cases}$

(b) $\begin{cases} n + d + q = 65 \\ 0.05n + 0.1d + 0.25q = 65 \\ n - (d + q) = 5 \end{cases}$

(c) $\begin{cases} n + d + q = 65 \\ 0.05n + 0.1d + 0.25q = 7.75 \\ n - d - q = 5 \end{cases}$

(d) $\begin{cases} n + d + q = 7.75 \\ 0.05n + 0.1d + 0.25q = 7.75 \\ (d + q) + 5 = n \end{cases}$

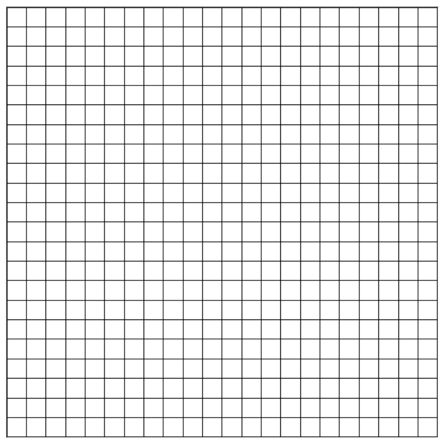
Explain your choice.

2. Money in the Bank

Ted's grandmother, Miss Tedrina Mae, really likes to spoil her grandson and knows how much he wants a tablet computer. She sets up a checking account (with no interest) for Ted with an initial amount of \$200 and will she add \$15 each month afterward.

(a) Write an equation whose solution is the number of months, m, it takes for the account balance to reach \$395.

(b) Make a plot of the balance after m months for $m = 1, 5, 9, 13, 17$ and indicate on the plot the solution to your equation in part (a).



Ted loves his grandmother and gladly accepts her \$200 gift, but he doesn't want her to open up a checking account and put extra money in it (she's done too much already!) – he believes he has a better idea, anyway. Ted remembered that he took some notes in a finance course he was enrolled in at the local community college regarding interest-bearing accounts:

Compound Interest (2 Types)

1. n compounding periods: $A = P\left(1+\frac{r}{n}\right)^{nt}$

2. Continuous compounding: $A = Pe^{rt}$

Values of n:
Annually = Once per year
Semiannually = Twice per year
Quarterly = Four times per year
Monthly = Twelve times per year
Weekly = Fifty-two times per year

A = Final amount

P = Principal (starting or initial amount)

r = interest rate (decimal form)

n = compounding periods per year

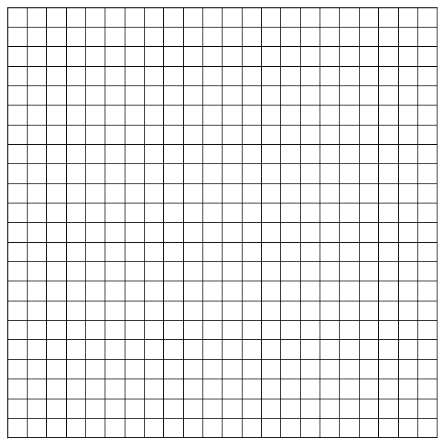
t = time in years

There are two banks that Ted will decide between to open a savings account (which is one type of interest-bearing account) – Bernoulli Bank or Euler Federal Bank. Bernoulli Bank offers a 6.75% quarterly-compounded interest rate, and Euler Federal offers a 6.75% continuously-compounded interest rate.

Ted plans on investing his \$200 gift – plus an additional \$50 bill that he found in the sofa cushion when he was looking for change – in a savings account for one year.

(c) Give the simplified form of the equation representing the amount earned at any given month at Bernoulli Bank.

(d) Graph the function from part (c). What type of function is this?



(e) Fill in the table below and explain which bank you would choose.

Year	**Bernoulli Bank**	**Euler Federal Bank**
1		
2		
3		
4		
5		
6		
t		

(f) If Ted chooses the savings account with Euler Federal, how long will it take for Ted to save the same amount that he would have had in six months with his grandmother's original plan?

3. Ted's Totally Terrific Guitars

Ted just happens to be a really talented craftsman, and is known throughout his hometown as being a very good guitar builder. Ted hopes to make some money from this business for his tablet computer purchase. Ted builds three types of guitars: archtops, electrics, and acoustics. Ted builds 1 archtop guitar per month, 2 acoustic guitars per month, and 3 electric guitars per month. Suppose that it takes Ted x hours to build an archtop guitar, y hours to build an electric guitar, and z hours to build an acoustic guitar.

(a) Write an equation relating x, y, and z if Ted spends a total of 134 hours per month building guitars.

(b) If Ted charges \$90 per hour for an archtop guitar, \$45 per hour for an electric guitar, and \$65 per hour for an acoustic guitar, write an equation relating x, y, and z if Ted builds \$9240 worth of guitars.

Ted buys large blocks of specific varieties of expensive woods to build his guitars. Ted has found that the best varieties of tone woods for instruments are spruce and mahogany. Ted buys w blocks of spruce for y dollars each, and x blocks of mahogany for z dollars each. In a given month, Ted spends a total of C dollars, where $C = wy + xz$. For the following, write an equation whose solution is the given quantity.

(c) The number of blocks of mahogany that Ted can afford to buy if he wishes to spend a total of \$5000 this month, mahogany costs \$450 per block, and he has already bought 7 blocks of spruce at \$200 each.

(d) The price of spruce blocks if Ted bought 9 of them, in addition to the 10 blocks of mahogany that Ted bought at \$425 a block, for a total of \$6275.

(e) The price of a block of mahogany, given that a block of spruce costs $\frac{5}{9}$ as much as a block of mahogany, and Ted has bought 12 blocks of spruce and 15 blocks of mahogany for a total cost of \$9750.

4. Ted's Toss

Ted is still trying to find ways to make money, and since the county fair is in town, he's decided to try his skill in the ball-throwing competition (this is a fairly simple county fair!) with a $50 grand prize. The winner is the thrower that produces the longest time in the air for the ball. When Ted throws this particular type of ball, it moves vertically upward at a speed of v feet/second and rises a distance of d feet in t seconds, given by

$$d = 4 + vt - 10t^2$$

Write an equation whose solution is
(a) the time it takes a ball thrown at a speed of 88 ft/sec to rise 20 feet;

(b) the speed with which the ball must be thrown to rise 20 feet in 2 seconds.

Before we move on to the second part of "Ted's Toss", let's look at a little quadratic review.

Quick Quadratic Review

Graph these equations on your graphing calculator at the same time. What happens? Why?

Y1: $(x-3)(x+1)$

Y2: x^2-2x-3

Y3: $(x-1)^2-4$

Y4: x^2-2x+1

Below are the first three equations from the previous problem.

Y1: $(x-3)(x+1)$

Y2: x^2-2x-3

Y3: $(x-1)^2-4$

These three equations all describe the same function. What are the coordinates of the following points on the graph of the function? From which equation is each point most easily determined? Explain.

Vertex: _____

y-intercept: _____

x-intercept(s): _____

Make up an equation for a quadratic function whose graph satisfies the given condition. Use whatever form is most convenient.

(a) Has a vertex at (−2,−5).

(b) Has a *y*-intercept at (0,6)

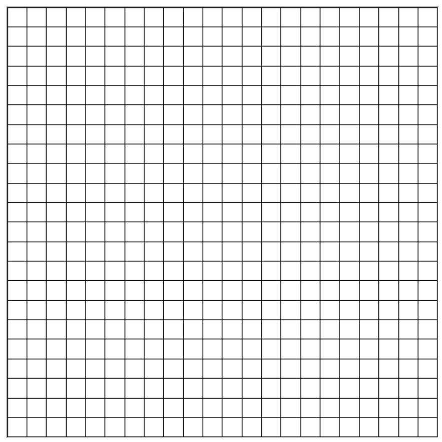
(c) Has *x*-intercepts at (3,0) and (5,0)

(d) Has *x*-intercepts at the origin and (4,0)

(e) Goes through the points (4,-10) and (1,2)

Now let's say that Ted has a pretty good throwing arm (he has entered a throwing competition, after all!). In fact, on his particular throw for this competition, Ted throws the ball at a speed of 90 ft/sec. Using $d = 4 + vt - 10t^2$, answer the following.

(c) Write the equation that will be used to model Ted's throw, and then graph his throw, from the time it leaves his hand to the time it falls back to the ground. Label all axes and units carefully. At what height does the ball leave Ted's hand?



(d) For Ted's throw, what is the maximum height of the ball? Show how you would find this using the equation you created in part (c), and then check your answer with your graph.

Ted was the last person to throw in the competition. Unfortunately, up to the time of Ted's throw, his arch-nemesis Billy Bob Bigglesby (who already owns the most expensive tablet computer on the market) had the longest throw with the ball having stayed in the air for 8.8 seconds. The model for Billy Bob's throw is the same as for Ted [$d = 4 + vt - 10t^2$]. Obviously, they are the same height and have roughly the same throwing position.

(e) Who won the $50 prize, Ted or Billy Bob? How must the speed of Billy Bob's throw compare to Ted's?

(f) What was the speed of Billy Bob's throw if the maximum height for his throw was 197.6 feet?

5. Ted's Test

Ted also happens to be a student at the local college, where he is taking a physics course. Ted has a special scholarship arrangement – instead of having his tuition paid, the foundation that pays for his scholarship gives him $250 for every course in which he makes an A. Ted is taking a physics final exam, and he is being asked to re-arrange important equations in order to solve for a desired variable.

Help Ted out by using inverse operations to solve the equations for the unknown variable, or for the designated variable if there is more than one.

(a) $-3 = \frac{x}{-27}$

(b) $16z = \frac{1}{4}$

(c) $\frac{1}{3}w + 7 = \frac{9}{5}$

(d) $a^2 + b^2 = c^2$ for b

(e) $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ for c

(f) $F = qE$ for E

(g) $E_k = \frac{1}{2}mv^2$ for v

(h) $ax + c = R$ for x

(i) $x = x_0 + v_0 t + \frac{1}{2}at^2$ for a

(j) $F = \frac{GMm}{r^2}$

6. Ted's Tablet

Ted finally made enough money (by checking sofa cushions, building guitars, using his savings account, throwing balls in competitions, and doing well in college) to buy himself the tablet computer of his dreams! Ted bought the computer for exactly $817.53, which is the list price plus sales tax. Find the list price of the tablet if Ted bought the computer in

(a) Vidalia, where the sales tax is 5%;

(b) Marietta, where the sales tax is 7.55%;

(c) a city in Georgia where the sales tax is r.

Will I Hit The Hoop? (Spotlight Task)

Back to Task Table

Taken from Dan Meyer *http://blog.mrmeyer.com/2010/wcydwt-will-it-hit-the-hoop/*

Georgia Standards of Excellence

Build new functions from existing functions

MGSE9-12.F.BF.3 Identify the effect on the graph of replacing f(x) by f(x) + k, k f(x), f(kx), and f(x + k) for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them.

STANDARDS FOR MATHEMATICAL PRACTICE

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

ESSENTIAL QUESTIONS

- What information do you need to make sense of this problem?
- How can you use estimation strategies to find out possible solutions to the questions you generated based on the video provided?

MATERIALS REQUIRED

- Access to videos for each Act: http://www.mrmeyer.com/wcydwt/hitthehoop/take1.zip
- Student Recording Sheet
- Pencil

TIME NEEDED

- 1 day

TEACHER NOTES

Task Description
In this task, students will watch the video, generate questions that they would like to answer, make reasonable estimates, and then justify their estimates mathematically. This is a student-centered task that is designed to engage learners at the highest level in learning the mathematics content. During Act 1, students will be asked to discuss what they wonder or are curious about after watching the quick video. These questions should be recorded on a class chart or on the board. Students will then use mathematics, collaboration, and prior knowledge to answer their own questions. Students will be given additional information needed to solve the problem based on need. When they realize they don't have a piece of information they need to help address the problem and ask for it, it will be given to them.

ACT 1:
Watch the video:
This is the video from the zip file titled act1-willithitthehoop

Ask students what they want to know.

The students may say the following:

- Will the ball go in the hoop?
- How long before the ball lands in the basket?
- How hard did he throw the ball?
- How tall is the thrower?
- How tall is the basketball goal?

Give students adequate "think time" between the two acts to discuss what they want to know. Focus in on one of the questions generated by the students, i.e. Will the ball go in the hoop?, and ask students to use the information from the video in the first act to figure it out.

Circulate throughout the classroom and ask probing questions, as needed.

ACT 2:
Reveal the following information as requested:
Information is provided in the GeoGebra Sketch, act2-willithitthehoop from the zip file.

Ask students how they would use this information to further refine their answer to the original question.

Give students time to work in groups to figure it out.
Circulate throughout the classroom and ask probing questions, as needed.

ACT 3

Show the Act 3 video reveal.
This is the video titled act3-willithitthehoop from the zip file

Students will compare and share solution strategies.

- Reveal the answer. Discuss the theoretical math versus the practical outcome.
- How appropriate was your initial estimate?
- Share student solution paths. Start with most common strategy.
- Revisit any initial student questions that weren't answered.

ACT 4

Sequel: There is a Take 2, Take 3, Take 4, Take 5, Take 6, and Take 7 series of 3-Act videos. You could divide the class into groups and have them look at a sequel to solidify understanding. This could be done in small groups or via stations in the classroom.

Student Recording Sheet

Task Title:________________________ Name:________________________

ACT 1

What did/do you notice?

What questions come to your mind?

Main Question:____

Estimate the result of the main question? Explain?
Place an estimate that is too high and too low on the number line
Low estimate *Place an "x" where your estimate belongs* High estimate

ACT 2

What information would you like to know or do you need to solve the question posed by the class?

Record the given information you have from Act 1 and any new information provided in Act 2.

If possible, give a better estimate using this information:_______________________________

Act 2 (*continued*)
Use this area for your work, tables, calculations, sketches, and final solution.

ACT 3

What was the result?

Which Standards for Mathematical Practice did you use?	
□ Make sense of problems & persevere in solving them	□ Use appropriate tools strategically.
□ Reason abstractly & quantitatively	□ Attend to precision.
□ Construct viable arguments & critique the reasoning of others.	□ Look for and make use of structure.
□ Model with mathematics.	□ Look for and express regularity in repeated reasoning.

Writing Constraints

Back to Task Table

(This task is adapted from *Algebra: Form and Function, McCallum et al., Wiley 2010*)

Mathematical Goals

- Develop quantitative reasoning in modeling realistic scenarios.
- Use constraints to build a valid mathematical model or system of models.

Georgia Standards of Excellence

Create equations that describe numbers or relationships.

MGSE9-12.A.CED.3 Represent constraints by equations or inequalities, and by systems of equation and/or inequalities, and interpret data points as possible (i.e. a solution) or not possible (i.e. a non-solution) under the established constraints.

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Introduction

The purpose of this task is to give students practice writing a constraint equation for a given context. Instruction accompanying this task should introduce the notion of a constraint equation as an equation governing the possible values of the variables in question (i.e., "constraining" said values). In particular, it is worth differentiating the role of constraint equations from more functional equations, e.g., formulas to convert from degrees Celsius to degree Fahrenheit. The task has students interpret the context and choose variables to represent the quantities, which are governed by the constraint equation and the fact that they are non-negative (allowing us to restrict the graphs to points in the first quadrant only).

Materials

- Pencil
- Handout
- Calculator

Writing Constraints

In parts a, b, c, and d below, (i) write a constraint equation, (ii) determine two solutions, and (iii) graph the equation and mark your solutions.

(a) The relation between quantity of chicken and quantity of steak if chicken costs $1.29/lb and steak costs $3.49/lb, and you have $100 to spend on a barbecue.

(i) ***Let c be the number of pounds of chicken you buy and s the number of pounds of steak. Then*** $\mathbf{1.29c + 3.49s = 100}$***.***

(ii) ***Many combinations are reasonable. For example, you could buy 10 lbs of chicken, so that*** $\mathbf{c = 10}$***. This gives***

$$\mathbf{1.29 \cdot 10 + 3.49s = 100}$$

$$\mathbf{s = \frac{100 - 12.9}{3.49} = 24.957 \approx 25}$$

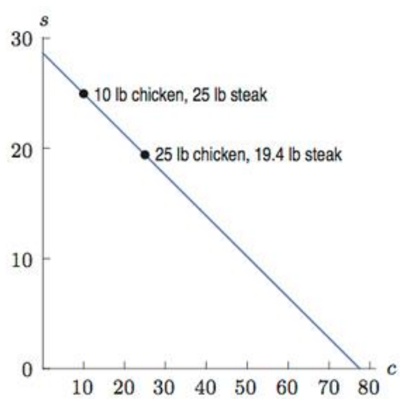
So you would buy approximately 25 lb of steak. Thus (10, 25) is one reasonable solution. Alternatively, you could buy 25 lb of chicken, so that $\mathbf{c = 25}$***, and compute:***

$$\mathbf{1.29 \cdot 25 + 3.49s = 100}$$

$$\mathbf{s = \frac{100 - 1.29 \cdot 25}{3.49} = 19.412 \approx 19.4}$$

So you would buy about 19.4 lb of steak. Thus (25, 19.4) is another reasonable solution.

(iii)



(b) The relation between the time spent walking and driving if you walk at 3 mph then get picked up by your friend in her car and ride at 75 mph, covering a total distance of 60 miles.

(i) ***If you walk for w hours and drive for d hours, then*** $3w + 75d = 60$.

(ii) ***If you walk for 2 hours, then*** $w = 2$.

$$3 \cdot 2 + 75d = 60$$

$$d = \frac{60 - 6}{75} = 0.72$$

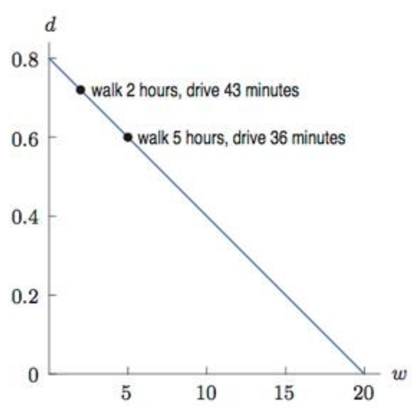
so you ride for 0.72 hours, $0.72 \cdot 60 \approx 43$ ***minutes. So (2, 0.72) is one reasonable solution. If you walk for 5 hours, then*** $w = 5$***, so***

$$3 \cdot 5 + 75d = 60$$

$$d = \frac{60 - 15}{75} = 0.6$$

So you ride for 0.6 hours, or $0.6 \cdot 60 = 36$ ***minutes. So another reasonable solution is (5, 0.6).***

(iii)



(c) The relation between the volume of titanium and iron in a bicycle weighing 5 kg, if titanium has a density of 4.5g/cm and iron has a density of 7.87 g/cm (ignore other materials).

(i) ***If t is the volume of titanium and i is the volume of iron, in (cm^3), then $4.5t + 7.87i = 5000$. Note that the density is given in grams and that the total weight of the bicycle is given in kilograms, so we must convert 5 kg to 5000 g.***

(ii) ***If you use $600\ cm^3$ of titanium, then $t = 600$, and***

$$4.5 \cdot 600 + 7.87i = 5000$$

$$i = \frac{5000 - 4.5 \cdot 600}{7.87} = 292.25$$

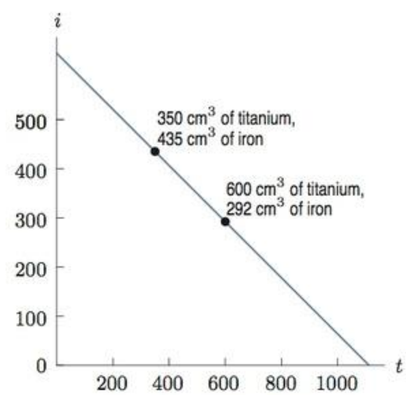
so you would use about $292\ cm^3$ of iron. So a possible solution is (600, 292). Or you could use $350\ cm^3$ of titanium, so

$$4.5 \cdot 350 + 7.87i = 5000$$

$$i = \frac{5000 - 4.5 \cdot 350}{7.87} = 435.20$$

so you would use about $435\ cm^3$ of iron. So a possible solution is (350, 435).

(iii)



(d) The relation between the time spent walking and the time spent canoeing on a 30 mile trip if you walk at 4 mph and canoe at 7 mph.

(i) ***If w is the time spent walking and c is the time spent canoeing, both in hours, then*** $4w + 7c = 30$.

(ii) ***If you walk for 3 hours, then*** $w = 3$, ***so***

$$4 \cdot 3 + 7c = 30$$

$$c = \frac{30 - 12}{7} = 2.57$$

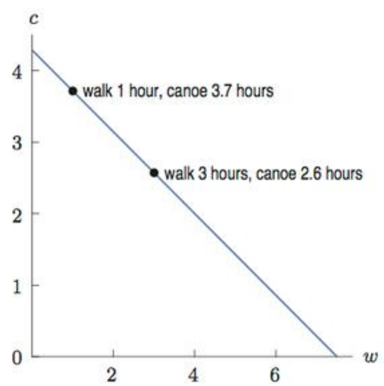
so you canoe for about 2.6 hours. So one possible solution is (3, 2.6). If you walk for 1 hour, then $w = 1$, ***so***

$$4 \cdot 1 + 7c = 30$$

$$c = \frac{30 - 4}{7} = 3.71$$

so you canoe about 3.7 hours. So another possible solution is (1, 3.7).

(iii)



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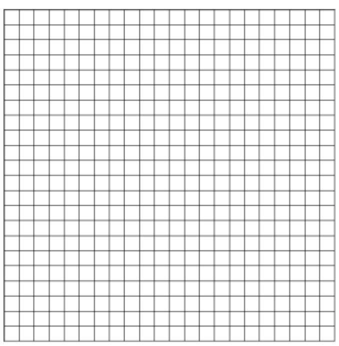
(i)

(ii)

(iii)

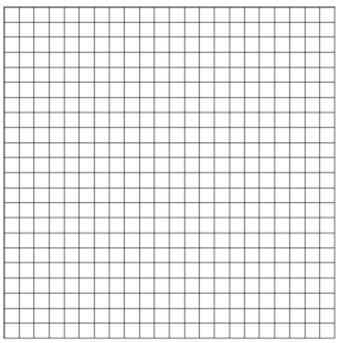


(b) The relation between the time spent walking and driving if you walk at 3 mph then get picked up by your friend in her car and ride at 75 mph, covering a total distance of 60 miles.

(i)

(ii)

(iii)

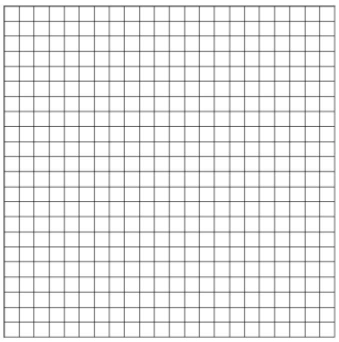


(c) The relation between the volume of titanium and iron in a bicycle weighing 5 kg, if titanium has a density of 4.5g/cm and iron has a density of 7.87 g/cm (ignore other materials).

(i)

(ii)

(iii)

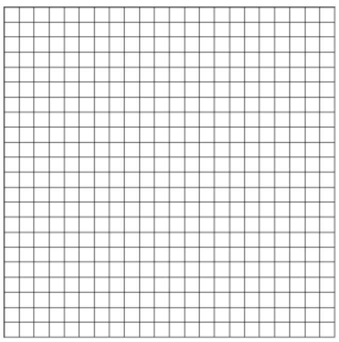


(d) The relation between the time spent walking and the time spent canoeing on a 30 mile trip if you walk at 4 mph and canoe at 7 mph.

(i)

(ii)

(iii)



A Trip to the Sugar Bowl

Back to Task Table

Mathematical Goals

- Develop quantitative reasoning in building constraining inequalities.
- Use constraints to develop and optimize valid solutions to systems of inequalities.

Georgia Standards of Excellence

Create equations that describe numbers or relationships.

MGSE9-12.A.CED.2 Create linear, quadratic, and exponential equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. (The phrase "in two or more variables" refers to formulas like the compound interest formula, in which $A = P(1 + r/n)^{nt}$ has multiple variables.)

MGSE9-12.A.CED.3 Represent constraints by equations or inequalities, and by systems of equation and/or inequalities, and interpret data points as possible (i.e. a solution) or not possible (i.e. a non-solution) under the established constraints.

Standards for Mathemat**ical Practice**

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Introduction

The purpose of this task is to give students practice writing a system of constraints for a given context. The task has students interpret the context and choose variables to represent the quantities, which are governed by the constraint inequalities and the fact that they are non-negative (allowing us to restrict the graphs to points in the first quadrant only). Ultimately, this relatively short task is a basic linear programming problem that brings the idea of creating equations and inequalities to a useful culmination in optimizing the conditions associated with a scenario.

Materials

- Pencil
- Handout

A Trip to the Sugar Bowl

A tourism agency can sell up to 1200 travel packages for the Sugar Bowl college football postseason game in New Orleans. The package includes airfare, weekend accommodations, and the choice of two types of flights: a nonstop flight or a two-stop flight. The nonstop flight can carry up to 150 passengers, and the two-stop flight can carry up to 100 passengers. The agency can locate no more than 10 planes for the travel packages. Each package with a nonstop flight sells for \$1200, and each package with a two-stop flight sells for \$900. Assume that each plane will carry the maximum number of passengers.

(a) Define the variables for this situation.

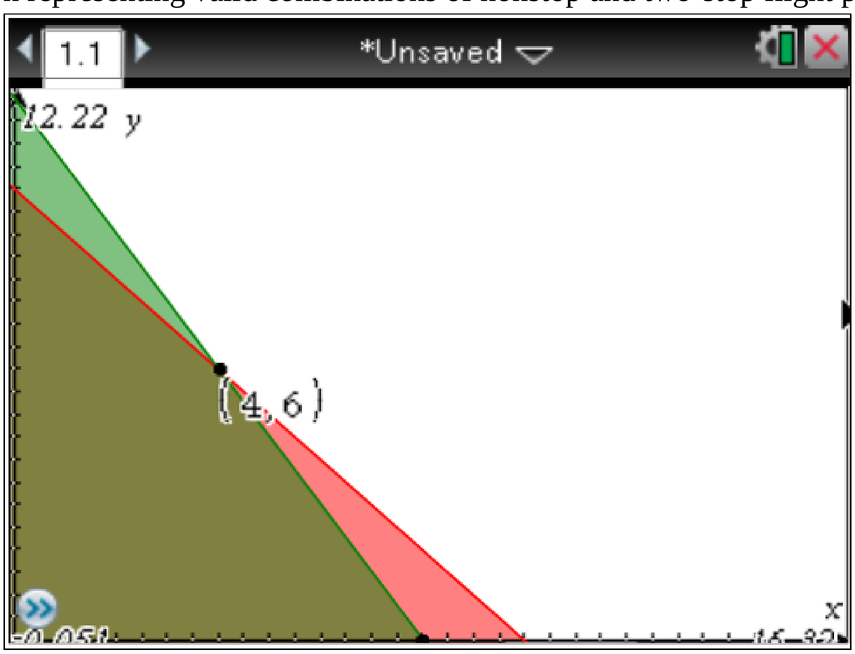
Answers vary; possible variables are x = nonstop flight and y = two-stop flight

(b) Write a system of linear inequalities to represent the constraints.

Using the variable assignment from above

$x \geq 0$ $\quad$ $y \geq 0$ $\quad$ $x + y \leq 10$ $\quad$ $150x + 100y \leq 1200$

(c) Graph the system of linear inequalities below, and shade the feasible region that shows the area of the graph representing valid combinations of nonstop and two-stop flight packages.



(d) Write an objective function that maximizes the revenue for the tourism agency.

$$\mathbf{1200}\boldsymbol{x}+\mathbf{900}\boldsymbol{y}=\boldsymbol{R}$$

(e) Find the maximum revenue for the given constraints and give the combination of flights that achieves this maximum.

$$\mathbf{1200(0)+900(0)=0}$$

$$\mathbf{1200(0)+900(10)=9000}$$

$$\mathbf{1200(4)+900(6)=10,200}$$

$$\mathbf{1200(8)+900(0)=9,600}$$

The maximum revenue would be $10,200 and would be achieved using 4 nonstop flights and 6 two-stop flights.

<u>A Trip to the Sugar Bowl</u>

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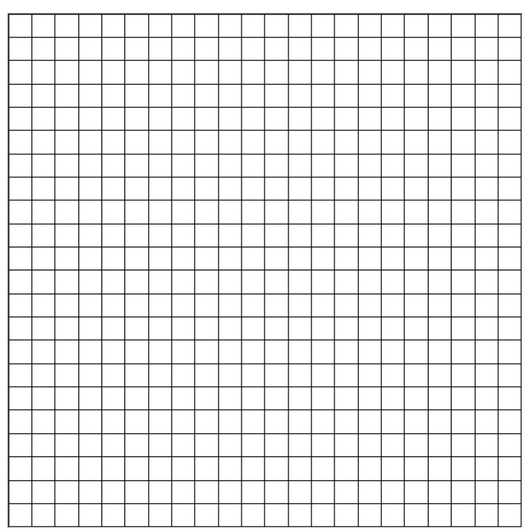

(d) Write an objective function that maximizes the revenue for the tourism agency.

(e) Find the maximum revenue for the given constraints and give the combination of flights that achieves this maximum.

Harvesting Fields

Back to Task Table

Mathematical Goals

- Develop quantitative reasoning in modeling realistic scenarios.
- Develop and enhance a sophisticated use of number sense.

Georgia Standards of Excellence

Create equations that describe numbers or relationships.

MGSE9-12.A.CED.1 Create equations and inequalities in one variable and use them to solve problems. Include equations arising from linear, quadratic, simple rational, and exponential functions ~~(integer inputs only)~~.

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Introduction

This is a challenging task, suitable for extended work, and reaching into a deep understanding of units. An algebraic solution is possible but complicated; a numerical solution is both simpler and more sophisticated, requiring skilled use of units and quantitative reasoning. Thus the task aligns with either A-CED.1 or N-Q.1, depending on the approach. Students who believe that they have found solutions should be encouraged to check their solutions, to see if they work, because by checking their solutions they will understand the problem more clearly.

Although it is not stated explicitly, it is assumed that the farm-workers all work at the same rate, harvesting the same area in any given period of time, and that for any period of time, the area cleared by a group of farm-workers is proportional to the number of farm-workers working. A flexible understanding of units simplifies some of the solutions. For example, the second solution is simpler if $R = 1$, which is achieved by using the rate of a single farm-worker as the unit. And the third solution can be understood in a more sophisticated way as setting the unit for area to the size of one small field.

Materials

- Pencil
- Handout
- Calculator

Harvesting Fields

A team of farm workers was assigned the task of harvesting two fields, one twice the size of the other. They worked for the first half of the day on the larger field and then the team split into two groups of equal number. The first group continued working in the larger field and finished it by evening. The second group harvested the smaller field, but did not finish by evening. The next day one farm worker finished the smaller field in a single day's work. How many farm workers were on the team?

Solution: Harvesting the Fields, Method 1 – Setting Up an Equation

Let x be the number of farm workers on the team (in units of $\frac{person}{team}$), and let R be the area one farm worker harvests in a day (in units of acres per person per day or $\frac{acres}{person \cdot day}$). The area harvested in that first half day was

$$\frac{1}{2} day \cdot R \frac{acres}{day} \cdot x \frac{persons}{team} \cdot 1\ team = \frac{1}{2} xR\ acres$$

In the second day, the area scythed was

$$\frac{1}{2} day \cdot R \frac{acres}{person \cdot day} \cdot x \frac{persons}{team} \cdot \frac{1}{2} team = \frac{1}{4} xR\ acres.$$

So the area of the larger field is

$$A = \frac{1}{2} xR + \frac{1}{4} xR = \frac{3}{4} xR.$$

The area of the smaller field is the area harvested by half the team in half a day added to the area harvested by one person in 1 day.

$$\frac{1}{2} day \cdot R \frac{acres}{person \cdot day} \cdot x \frac{persons}{team} \cdot \frac{1}{2} team + R \frac{acres}{person \cdot day} \cdot 1\ person \cdot 1 day = \frac{1}{4} xR + R\ acres.$$

This is also half of the area of the larger meadow, so

$$\frac{1}{4} xR + R = \frac{1}{2} \cdot \frac{3}{4} xR.$$

We can divide both sides of this equation by R since it is not zero, and solve the resulting equation for x.

$$\frac{1}{4} x + 1 = \frac{3}{8} x$$

$$x = 8$$

Solution: Harvesting the Fields, Method 2 – Reasoning with Rates

First notice that the larger field would have taken the entire team $\frac{3}{4}$ of a day to harvest. This is because they worked on it for half a day and then half a team worked on it for another half a day. If the whole team had worked on it for the second half-day, they would have finished it in half the time, that is, $\frac{1}{4}$ day. Adding this to the first half-day we get $\frac{3}{4}$ day for the whole team.

Since the smaller field is half the size of the larger field, it would have taken the whole team $\frac{3}{8}$ of a day to harvest the smaller field. As it was, only half the team worked on the smaller field, so it would have taken them $\frac{3}{4}$ of a day to finish. They only worked for $\frac{1}{2}$ of a day, so they still had $\frac{1}{4}$ of a day's work left, which would have been $\frac{1}{8}$ of a day's work for the entire team.

It took one worker a whole day to finish up, so it took one worker a day to do what the entire team could have done in $\frac{1}{8}$ day. Therefore the team had 8 workers.

Solution: Harvesting the Fields, Method 3 – Choosing a Size for the Fields

Let's assume that the smaller field is 1 acre and the larger one is 2 acres. Assuming these particular sizes does not change our final answer; if the sizes are different, then the rate at which the farm workers can harvest is different, but the total number of farm workers stays the same. [A more sophisticated way of saying this would be to say that we are working in units of size one small field. In working with this solution it might make sense to make up a name for this unit.]

Then the larger field would be cleared by the whole team in $\frac{3}{4}$ of a day, which means that working together, they clear $\frac{8}{3}$ of an acre per day. Then half of them clear $\frac{2}{3}$ of an acre in a half day, so for the 1 acre field, this would mean that there is $\frac{1}{3}$ of an acre left for the single farm-worker, so he or she clears $\frac{1}{3}$ of an acre per day. This is $\frac{1}{8}$ of what the entire team can clear, so the entire team is 8 farm workers.

Harvesting Fields

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A Gaggle of Graphs

Back to Task Table

Mathematical Goals

- Formulate a verbal description of a graph of a function using key features from that graph.
- Compare characteristics of function graphs and apply the differences to the context of the problem.

Georgia Standards of Excellence

Interpret functions that arise in applications in terms of the context.

MGSE9-12.F.IF.4 Using tables, graphs, and verbal descriptions, interpret the key characteristics of a function which models the relationship between two quantities. Sketch a graph showing key features including: intercepts; interval where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; ~~and periodicity~~.

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
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7. Look for and make use of structure.
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Introduction

This task is actually a conglomeration of several smaller mini-tasks, with the goal of reinforcing students' graphical and function literacy. A number of approaches are taken in this task to having students describe the important features of graphs, and more importantly, to use those features to describe various natural phenomena.

Each part of the task includes a commentary tailored to that particular section.

Materials

- Pencil
- Handout

A Gaggle of Graphs

1. How is the weather?

This task can be used as a quick assessment to see if students can make sense of a graph in the context of a real world situation. Students also have to pay attention to the scale on the vertical axis to find the correct match. The first and third graphs look very similar at first glance, but the function values are very different since the scales on the vertical axis are very different. The task could also be used to generate a group discussion on interpreting functions given by graphs.

The follow-up question could lead into a discussion about the seasons, hours of daylight as a function of latitude, etc. There are many interesting questions that can be investigated in this area mathematically.

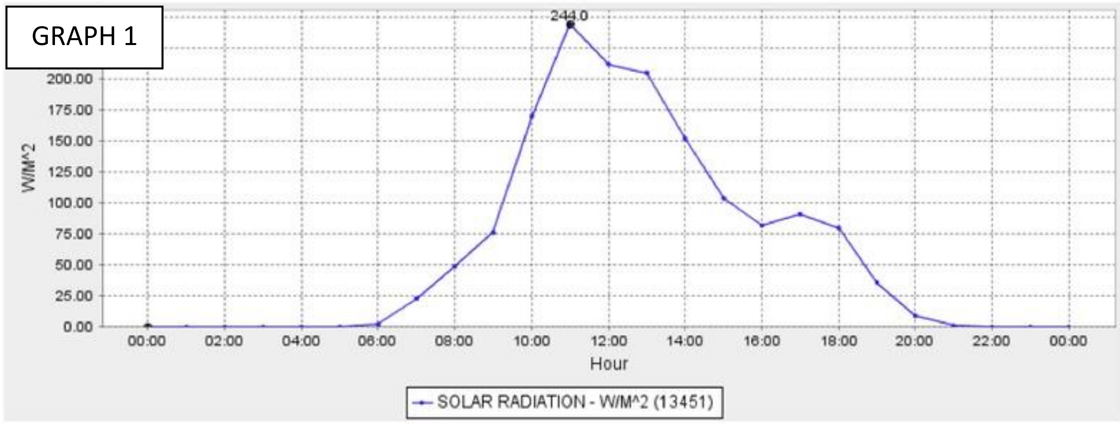
Technically, the graphs only show some values of the functions they are meant to represent. A bivariate data plot is a representation of a function in the same way that a table is a representation of a function; while it has some gaps in information, there is an underlying function that the bivariate data plot is assumed to sample. (In this case, the data points are joined by lines which means we are interpolating between our given values.) So the tasks implicitly expect students to answer the question about the solar radiation as a function of time based on the sampled data alone. Given the qualitative nature of the tasks, this does not present a problem.

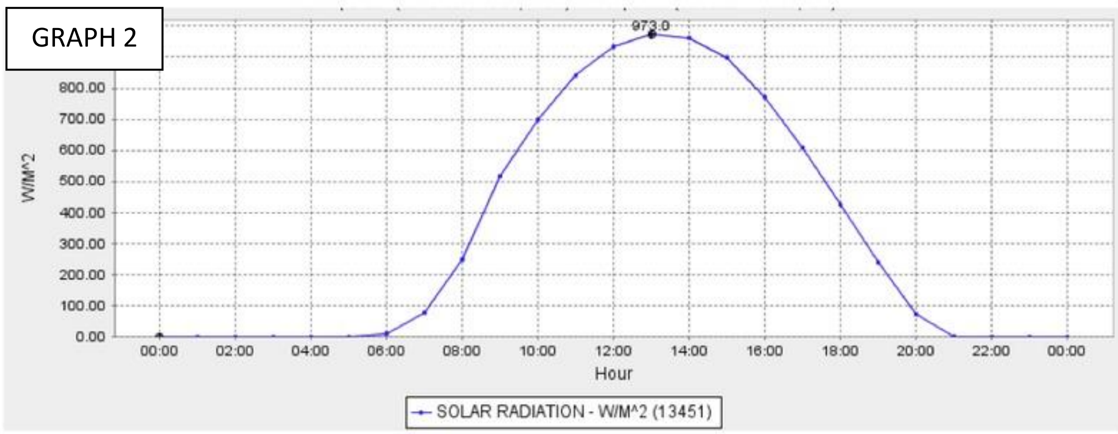
Given below are three graphs that show solar radiation, S, in watts per square meter, as a function of time, t, in hours since midnight. We can think about this quantity as the maximum amount of power that a solar panel can absorb, which tells us how intense the sunshine is at any given time. Match each graph to the corresponding description of the weather during the day.

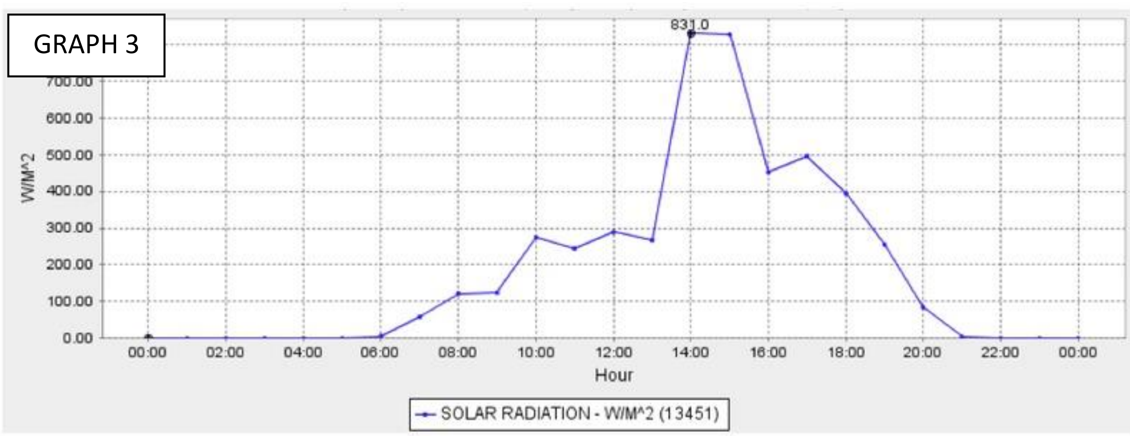
(a) It was a beautifully sunny day from sunrise to sunset – not a cloud in the sky.

(b) The day started off foggy but eventually the fog lifted and it was sunny the rest of the day.

(c) It was a pretty gloomy day. The morning fog never really lifted.







The graphs come from the website of the California Department of Water Resources at *http://cdec.water.ca.gov/*.

All three graphs show solar radiation measured in Santa Rosa, a city in northern California. What other information can you get from the graph?

(a) Graph (2): Graph (2) is the most striking by its symmetry. Once the sun starts rising shortly after 6 a.m., the solar radiation increases steeply until it reaches a maximum of 973 watts per square meter around 1 pm, at $t = 13$. Then the solar radiation decreases until the sun sets around 9 p.m.

(b) Graph (3): We can see again that sunrise is around 6 a.m., but this time the solar radiation does not increase as fast as in graph (2). Solar radiation stays below 300 watts per square meter (consistent with foggy weather) until 1 pm and then increases very quickly to 833 watts per square meter (consistent with the fog clearing away). This value is close to the solar radiation on the day that started out sunny at the same time of day, indicating sunny skies in the afternoon.

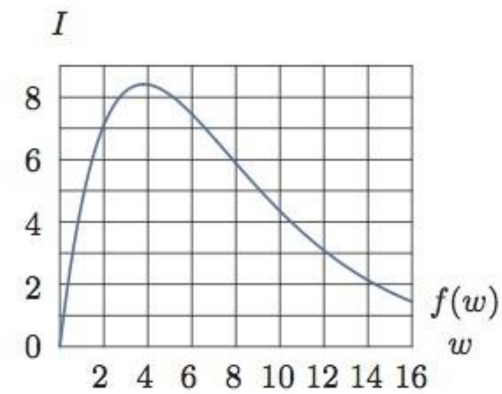
(c) Graph (1): Aside from using the process of elimination we observe that for this graph the solar radiation never gets above 244 watts per square meter. Even though the shape is similar to graph (3), the function values are much lower for the majority of the day, especially the afternoon. This suggests that the sky was overcast for the entire day.

We already mentioned that sunrise is at 6 a.m. and sunset is around 9 p.m. for graph (2). We can quickly check that this is actually true for all three graphs. Even though the weather was very different on those three days, since the location is the same and sunrise and sunset are the same, the three days must have happened in the same season. We can also conclude that the season was summer, since we have 15 hours of daylight and 9 hours of darkness. (Some research would reveal an even closer estimate of the date (June 17) if we use the latitude of Santa Rosa.)

It would also be interesting to investigate how much power a solar panel produces on a sunny day versus a cloudy day.

2. Influenza Epidemic
(Task from *Functions Modeling Change: A Preparation for Calculus, Connally et al., Wiley 2010.*)

An epidemic of influenza spreads through a city. The figure below is the graph of $I = f(w)$, where I is the number of individuals (in thousands) infected w weeks after the epidemic begins.



(a) Estimate $f(2)$ and explain its meaning in terms of the epidemic.

To evaluate $f(2)$, we determine which value of I corresponds to $w = 2$. Looking at the graph, we see that $I \approx 7$ when $w = 2$. This means that approximately 7000 people were infected two weeks after the epidemic began.

(b) Approximately how many people were infected at the height of the epidemic? When did that occur? Write your answer in the form $f(a) = b$.

The height of the epidemic occurred when the largest number of people were infected. To find this, we look on the graph to find the largest value of I, which seems to be approximately 8.5, or 8500 people. This seems to have occurred when $w = 4$, or four weeks after the epidemic began. We can say that the height of the epidemic corresponds to the evaluation $f(4) = 8.5$.

(c) For approximately which w is $f(w) = 4.5$; explain what the estimates mean in terms of the epidemic.

To find a solution to $f(w) = 4.5$, we must find the value of w for which $I = 4.5$, or 4500 people were infected. We see from the graph that there are actually two values of w at which $I = 4.5$, namely $w \approx 1$ and $w \approx 10$. This means that 4500 people were infected after the first week when the epidemic was on the rise, and that after the tenth week, when the epidemic was slowing, 4500 people remained infected.

(d) An equation for the function used to plot the image above is $f(w) = 6w(1.3)^{-10}$. Use the graph to estimate the solution of the inequality $6w(1.3)^{-w} \geq 6$. Explain what the solution means in terms of the epidemic.

We are looking for all the values of w for which $f(w) \geq 6$. Looking at the graph, this seems to happen for all values of $w \geq 1.5$ and $w \leq 8$. This means that more than 6000 people were infected starting in the middle of the second week and lasting until the end of the eighth week, after which time the number of infected people fell below 6000.

3. Telling a Story with a Graph

In this task students are given graphs of quantities related to weather. The purpose of the task is to show that graphs are more than a collection of coordinate points, that they can tell a story about the variables that are involved and together they can paint a very complete picture of a situation, in this case the weather. Features in one graph, like maximum and minimum points, correspond to features in another graph, for example, on a rainy day the solar radiation is very low and the cumulative rainfall graph is increasing with a large slope.

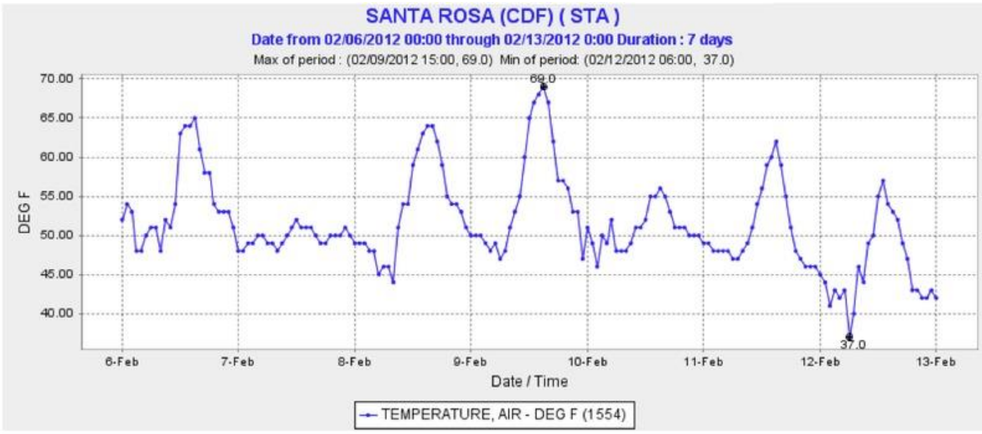
Some of the quantities shown are very familiar to students, such as temperature, where others might be less familiar, such as solar radiation. We can think about this quantity as the maximum amount of power that a solar panel can absorb. Depending on the experience of the students, teachers might want to discuss the idea of cumulative rainfall, i.e., the total amount of rain that has fallen since the beginning of the season.

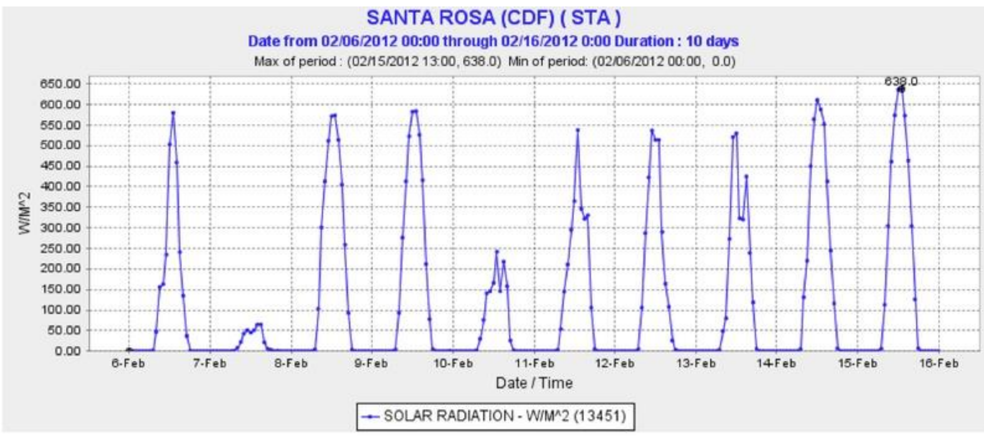
Technically, the graphs only show some of the values of the functions they are meant to represent. A bivariate data plot is a representation of a function in the same way that a table is a representation of a function; while it has some gaps in information, there is an underlying function that the bivariate data plot is assumed to sample. (In this case, the data points are joined by lines which means we are interpolating between our given values.) So the tasks implicitly expect students to answer the question about the temperature (or solar radiation, or precipitation) that is a function of time based on the information about it provided by the sampled data. Given the qualitative nature of the tasks, this does not present a problem.

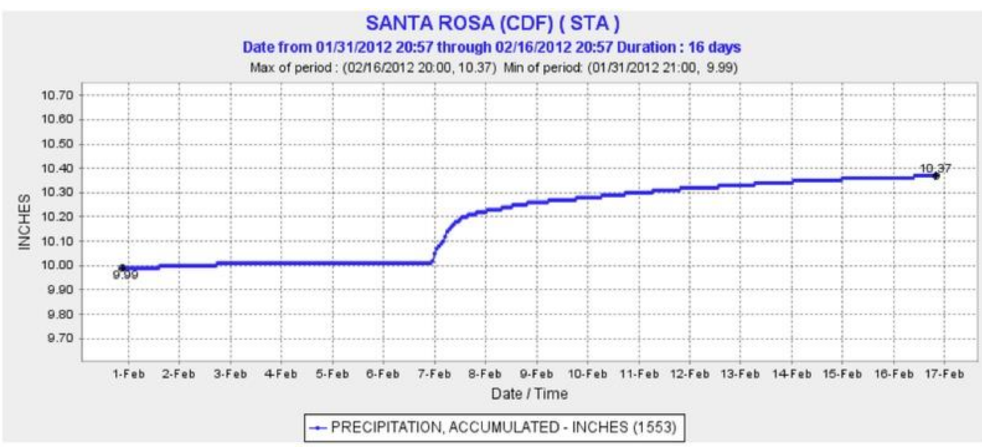
Each of the following graphs tells a story about some aspect of the weather: temperature (in degrees Fahrenheit), solar radiation (in watts per square meters), and cumulative rainfall (in inches) measured by sensors in Santa Rosa, California in February 2012. Note that the vertical gridlines represent the start of the day whose date is given.

(a) Give a verbal description of the function represented in each graph. What does each function tell you about the weather in Santa Rosa?

(b) Tell a more detailed story using information across several graphs. What are the connections between the graphs?







All the presented data come from the website of the California Department of Water Resources and can be found at http://cdec.water.ca.gov/

(a) ***All graphs show functions that have the same independent variable, namely the time t, measured in days. All graphs have different domains, but they do overlap. They all show domains for different time periods in February of 2012. The independent variables are different in each graph. All graphs show a different weather feature in Santa Rosa, CA.***

The first graph shows temperature, T, in degrees Fahrenheit, as a function of time (by date and time) over a one-week period starting at midnight on February 6, 2012. On five days the temperature rose into the high 50s to low 60s during the day and fell to the high 40s to low 50s during the night. The maximum temperature during the given time period was $69°F$ and it occurred in the early afternoon of February 9. The minimum temperature was $37°F$ and it occurred in the early morning of February 12. February 7 and 10 were special insofar that the temperature did not change much throughout the entire day. Particularly on February 7, the temperature stayed in the low 50s all day long.

The second graph shows solar radiation in watts per square meter, as a function of time for 10 days starting on February 6, 2012. We can think of solar radiation as the power that a square meter of solar panel produces. This function shows some definite regularity. Every day the function values are zero for a certain time interval. This corresponds to the hours when it is dark and a solar panel would not produce any power. The function increases in the morning, reaches a peak in the middle of the day and decreases in the evening. On most days the maximum value is between 550 and 650 watts per square meter. Again, February 7 and February 10 are the exception. During those two days the maximum solar radiation was just over 50 and just below 250 watts per square meter, respectively.

The third graph shows the cumulative amount of rainfall in inches as a function of time; this is the total amount of rain that has fallen since the season started. With the information given, we don't really know when the beginning of the season was. This function is increasing on the entire time interval shown (February 1 through February 17, 2012), which makes sense, since we are keeping track of the total amount of rainfall. We can see that the function is increasing slowly from February 1 until February 7 and then the graph becomes much steeper. The cumulative amount of rain increased much more on February 7 than on any other day.

(b) ***After analyzing all the graphs, it becomes clear what the weather was like in early February of 2012 in Santa Rosa. Most days it was sunny with temperatures reaching the mid 60 during the day and the mid 40 during the night. On February 7 it rained, but not very hard. We see that the cumulative rain graph is steeper during that time, but it only increases by 0.2 inches, so the rain can't have been very heavy. Also, since the solar radiation numbers were very low, this shows that there was not much sunshine during the day, which we would expect for a rainy day.***

On February 10 it was cloudy and cooler during the day but not especially rainy. A cooler air system moved into the area after February 10 since daytime temperatures reach highs in the low 60s to high 50s and the nighttime low temperatures even drop into the 30s. We can't say anything about the temperatures after February 13, but the solar radiation and rain graphs suggest continuing sunny days.

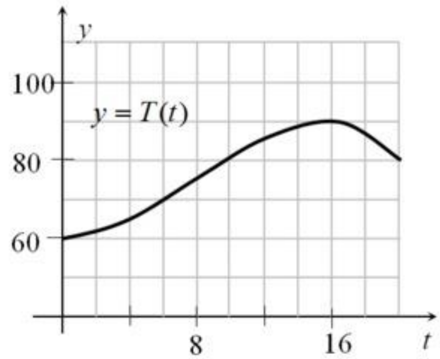
There must have been a little bit of rain after February 7 as the cumulative rainfall continues to increase slightly, but it wasn't very much and seems to have been spread

out over a number of days which is consistent with the information about solar radiation and temperature on those days.

4. Warming and Cooling

This task is meant to be a straightforward assessment task of graph reading and interpreting skills. This task helps reinforce the idea that when a variable represents time, $t = 0$ is chosen as an arbitrary point in time and positive times are interpreted as times that happen after that.

The figure shows the graph of T, the temperature (in degrees Fahrenheit) over one particular 20-hour period in Santa Elena as a function of time t.



(a) Estimate $T(14)$.

$T(14)$ is a little less than 90 degrees Fahrenheit; maybe 88 or 89 degrees.

(b) If $t = 0$ corresponds to midnight, interpret what we mean by $T(14)$ in words.

The temperature was almost 90 degrees at 2:00 in the afternoon.

(c) Estimate the highest temperature during this period from the graph.

The highest temperature was about 90 degrees.

(d) When was the temperature decreasing?

The temperature was decreasing between 4:00 p.m. and 8:00 p.m. It might have continued to decrease after that, but there is no information about the temperature after 8:00 p.m.

(e) If Anya wants to go for a two-hour hike and return before the temperature gets over 80 degrees, when should she leave?

The temperature reaches 80 degrees just before 10:00 a.m. If Anya wants to go for a two-hour hike and return before the temperature gets over 80 degrees, then she should start her hike before 8:00 a.m.

A Gaggle of Graphs

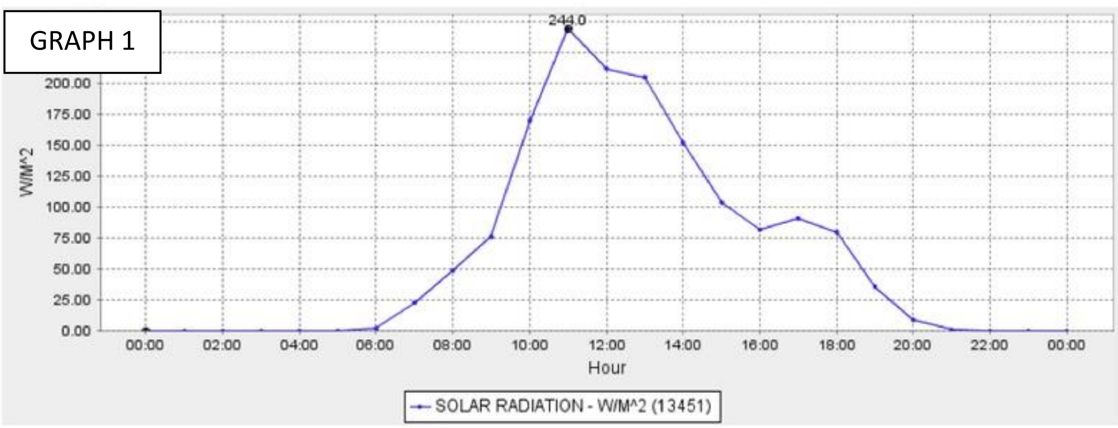
1. How is the weather?

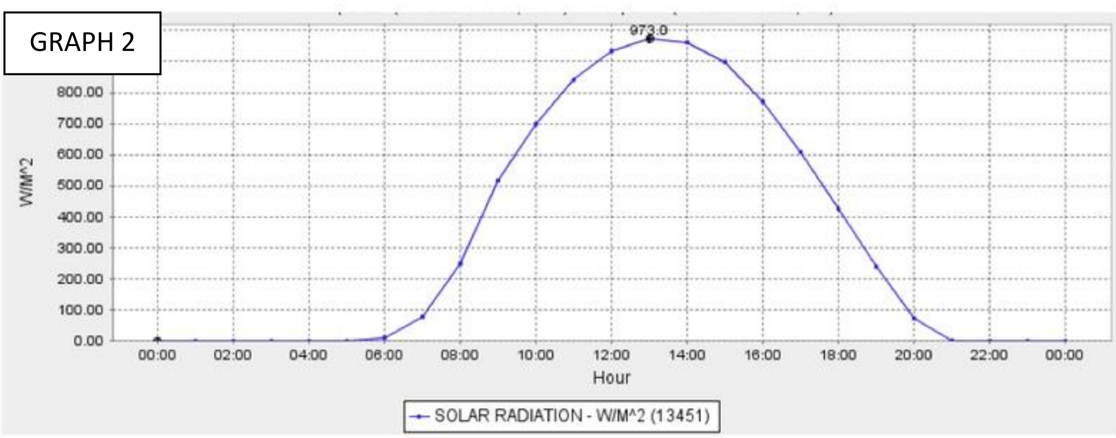
Given below are three graphs that show solar radiation, S, in watts per square meter, as a function of time, t, in hours since midnight. We can think about this quantity as the maximum amount of power that a solar panel can absorb, which tells us how intense the sunshine is at any given time. Match each graph to the corresponding description of the weather during the day.

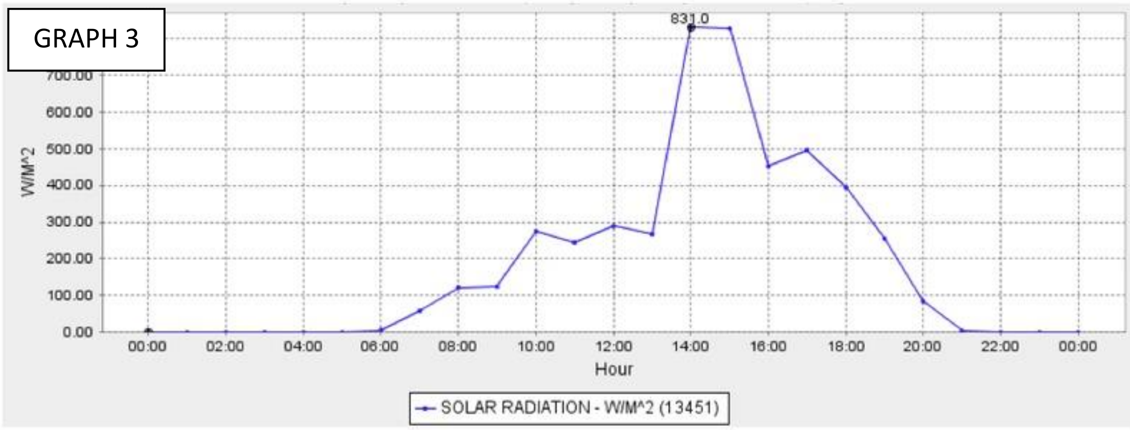
(a) It was a beautifully sunny day from sunrise to sunset – not a cloud in the sky.

(b) The day started off foggy but eventually the fog lifted and it was sunny the rest of the day.

(c) It was a pretty gloomy day. The morning fog never really lifted.





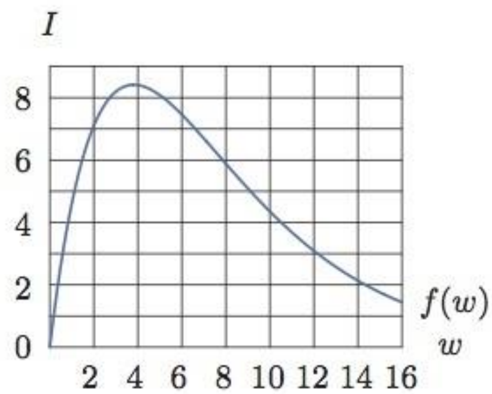


The graphs come from the website of the California Department of Water Resources at *http://cdec.water.ca.gov/*.

All three graphs show solar radiation measured in Santa Rosa, a city in northern California. What other information can you get from the graph?

2. Influenza Epidemic
(Task from *Functions Modeling Change: A Preparation for Calculus, Connally et al., Wiley 2010.*)

An epidemic of influenza spreads through a city. The figure below is the graph of $I = f(w)$, where I is the number of individuals (in thousands) infected w weeks after the epidemic begins.



(a) Estimate $f(2)$ and explain its meaning in terms of the epidemic.

(b) Approximately how many people were infected at the height of the epidemic? When did that occur? Write your answer in the form $f(a) = b$.

(c) For approximately which w is $f(w) = 4.5$; explain what the estimates mean in terms of the epidemic.

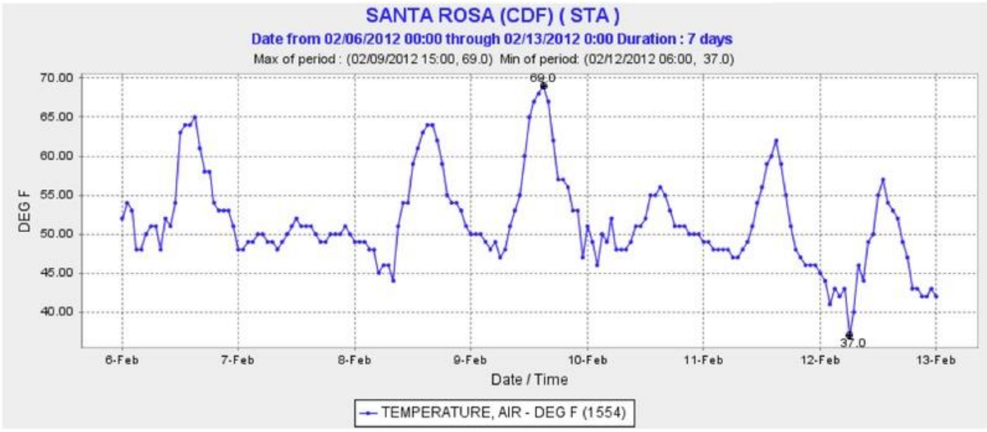
(d) An equation for the function used to plot the image above is $f(w) = 6w(1.3)^{-10}$. Use the graph to estimate the solution of the inequality $6w(1.3)^{-w} \geq 6$. Explain what the solution means in terms of the epidemic.

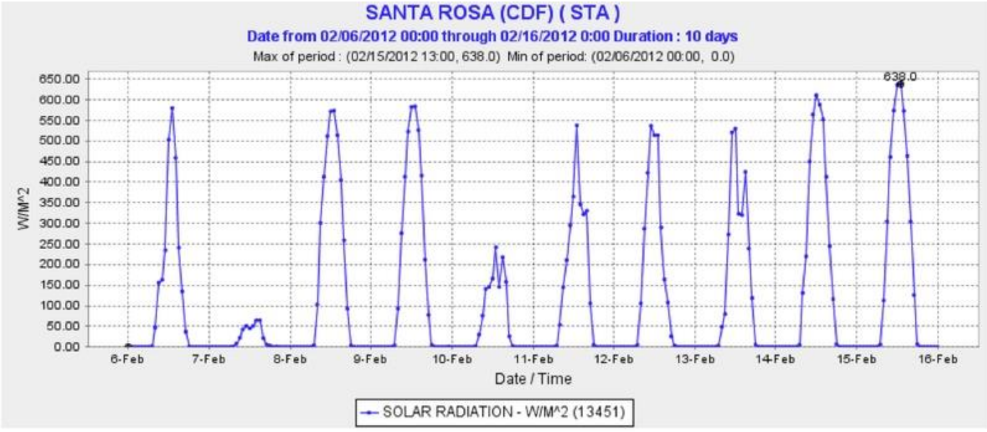
3. Telling a Story with a Graph

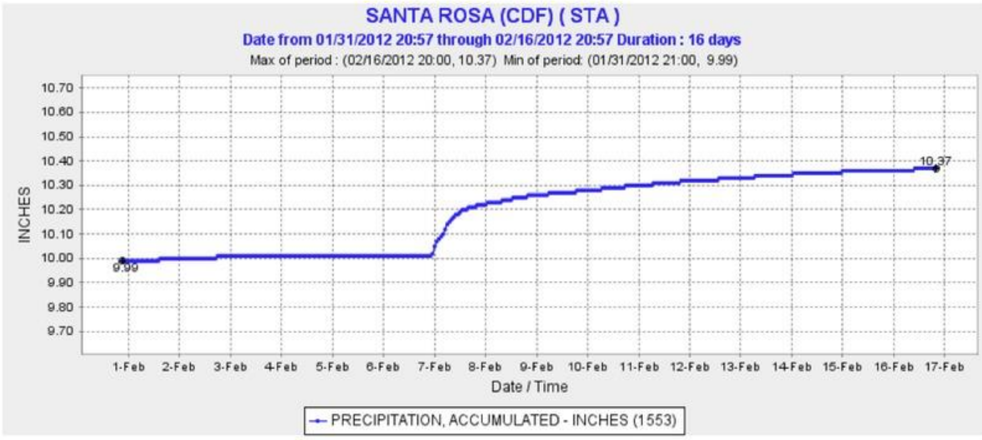
Each of the following graphs tells a story about some aspect of the weather: temperature (in degrees Fahrenheit), solar radiation (in watts per square meters), and cumulative rainfall (in inches) measured by sensors in Santa Rosa, California in February 2012. Note that the vertical gridlines represent the start of the day whose date is given.

(a) Give a verbal description of the function represented in each graph. What does each function tell you about the weather in Santa Rosa?

(b) Tell a more detailed story using information across several graphs. What are the connections between the graphs?







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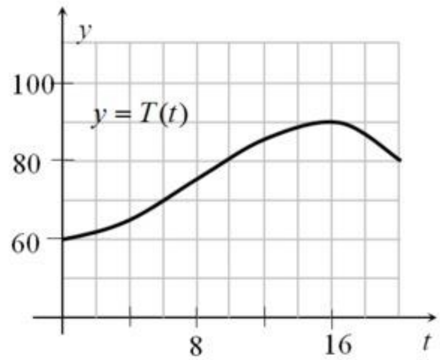

(a) Estimate $T(14)$.

(b) If $t = 0$ corresponds to midnight, interpret what we mean by $T(14)$ in words.

(c) Estimate the highest temperature during this period from the graph.

(d) When was the temperature decreasing?

(e) If Anya wants to go for a two-hour hike and return before the temperature gets over 80 degrees, when should she leave?

A Game at Cameron Indoor Stadium

Back to Task Table

Mathematical Goals

- Relate the domain and range of a function to its context rather than merely to its equation.
- Calculate and interpret the average rate of change of a function, and evaluate the validity of possible options for this value.

Georgia Standards of Excellence

Interpret functions that arise in applications in terms of the context.

MGSE9-12.F.IF.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*

MGSE9-12.F.IF.6 Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Introduction

The first part of the task is deceptively simple and asks students to find the domain and range of a function from a given context. The function is linear and if simply looked at from a formulaic point of view, students might find the formula for the line and say that the domain and range are all real numbers. However, in the context of this problem, this answer does not make sense, as the context requires that all input and output values are non-negative integers, and imposes additional restrictions. This problem could serve different purposes. Its primary purpose is to illustrate that the domain of a function is a property of the function in a specific context and not a property of the formula that represents the function. Similarly, the range of a function arises from the domain by applying the function rule to the input values in the domain. A second purpose would be to illicit and clarify a common misconception, that the domain and range are properties of the formula that represent a function. Finally, the context of the task as written could be used to transition into a more involved modeling problem, finding the Duke Blue Devils' profit after one takes into account overhead costs, costs per attendee, etc.

The second part of the task is a basic exercise in finding and interpreting the average rate of change for a function. One of the issues that may arise in this part of the task is the use of function notation in terms of minutes from a starting point to correlate to specific times given in the problem. This task is excellent in evaluating students' quantitative reasoning as related to the basic concept of average rate of change.

Materials

- Pencil
- Handout
- Calculator (only to find the upper endpoint of the range in part 1)

A Game at Cameron Indoor Stadium

1. Blue Devil Revenue

Cameron Indoor Stadium at Duke University is one of the most revered sites in all of college basketball, as well as in all of sports period. Duke's men's and women's basketball programs have attained quite a few wins in the building over the last seventy years. Cameron Indoor Stadium is capable of seating 9,450 people. For each game, the amount of money that the Duke Blue Devils' athletic program brings in as revenue is a function of the number of people, n, in attendance. If each ticket costs $45.50, find the domain and range of this function.

The domain is all <u>integer</u> values in the interval $[\mathbf{0}, \mathbf{9450}]$.

The range is all <u>multiples of 45.5</u> in the interval $[\mathbf{0}, \mathbf{429975}]$.

2. My, it's hot in here!

While Cameron Indoor Stadium is a sports icon, it is also one of the oldest indoor arenas in the United States. The place is known to be extremely loud and extremely crowded during Duke basketball games, but unfortunately, it is also known to be extremely hot (the arena wasn't really designed for indoor air conditioning in the 1930's!).

A game is scheduled for 3:00 pm on a Saturday afternoon. The crowd begins to file in about an hour before the game. About 20 minutes before the game ends (4:40 pm), the temperature in the arena remains a steady 85 degrees Fahrenheit for a few minutes. Later that afternoon, after the game is over (at 5:00 pm), the fans begin to leave the arena and it begins to slowly cool down.

Let T denote the temperature of the arena in degrees Fahrenheit and M denote the time, in minutes, since 2:00 (the time that the doors open and the fans begin to file in the arena).

(a) Is M a function of T? Explain why or why not.

$\boldsymbol{M}$ is not a function of $\boldsymbol{T}$. The problem states that the arena stayed a constant temperature of 85 degrees for a "few" minutes, meaning that the definition of a function is violated – for every one input (temperature) there is more than one output (minutes).

(b) Explain why T is a function of M, and consider the function $T = g(M)$. Interpret the meaning of $g(0)$ in the context of the problem.

In contrast to part (a), it is the case that for each number of minutes M after 2:00, there is one and only one temperature T in the room, so T is a function of M. The quantity $g(0)$ is the temperature in the arena when $M = 0$, which is the temperature in degrees Fahrenheit at 2:00 pm, when the doors to the arena open.

(c) Your friend Roy, a North Carolina fan who has come to the game with you, says: "The temperature increased 5 degrees in the first half hour after the game began. Cameron is an embarrassment!" Which of the following equations best represents this statement? Explain your choice.

(i) $g(30) = 5$ (ii) $g(90) = 5$

(iii) $\frac{g(90)-g(0)}{90} = 5$ (iv) $\frac{g(90)-g(60)}{30} = 5$

(v) $g(90) - g(60) = 5$ (vi) $T = g(90) - 5$

***The answer is (v).** This equation is a simple but specific representation of what Roy stated. Students may need to be reminded that $g(90)$ represents a time of 3:30 pm, which is 30 minutes after the game begins, at $g(60)$, which is 3:00 pm.*

(d) Which of the following represents the most reasonable quantity for $\frac{g(195)-g(180)}{15}$? Explain your choice.

(i) 4 (ii) 0.3

(iii) 0 (iv) -0.2

(v) -5

***The answer is (iv).** The time at $g(195)$ is 5:15 pm, which is 15 minutes after the game has ended. The time at $g(180)$ is at 5:00 pm, which is when the game ends. We know that the quantity described by the expression must be negative, since we are told that the arena slowly cools as fans leave. We are able to ignore the first three choices, and (v) is much too large of a temperature decrease, since the arena should be "slowly" cooling, and even if the decrease remained constant for fifteen minutes, the drop of 5 degrees per minute would mean that the arena would reach a temperature below freezing!*

A Game at Cameron Indoor Stadium

1. Blue Devil Revenue

Cameron Indoor Stadium at Duke University is one of the most revered sites in all of college basketball, as well as in all of sports period. Duke's men's and women's basketball programs have attained quite a few wins in the building over the last seventy years. Cameron Indoor Stadium is capable of seating 9,450 people. For each game, the amount of money that the Duke Blue Devils' athletic program brings in as revenue is a function of the number of people, n, in attendance. If each ticket costs $45.50, find the domain and range of this function.

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(a) Is M a function of T? Explain why or why not.

(b) Explain why T is a function of M, and consider the function $T = g(M)$. Interpret the meaning of $g(0)$ in the context of the problem.

(c) Your friend Roy, a North Carolina fan who has come to the game with you, says: "The temperature increased 5 degrees in the first half hour after the game began. Cameron is an embarrassment!" Which of the following equations best represents this statement? Explain your choice.

(i) $g(30) = 5$

(ii) $g(90) = 5$

(iii) $\frac{g(90)-g(0)}{90} = 5$

(iv) $\frac{g(90)-g(60)}{30} = 5$

(v) $g(90) - g(60) = 5$

(vi) $T = g(90) - 5$

(d) Which of the following represents the most reasonable quantity for $\frac{g(195)-g(180)}{15}$? Explain your choice.

(i) 4

(ii) 0.3

(iii) 0

(iv) -0.2

(v) -5

Polynomial Potpourri

Back to Task Table

Mathematical Goals

- Review important features of polynomial functions and their graphs, and discuss the ability to generalize some of these features to all functions
- Prepare students for modeling applications using polynomials and other functions.

Georgia Standards of Excellence

Analyze functions using different representations.

MGSE9-12.F.IF.7c Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior.

Build new functions from existing functions.

MGSE9-12.F.BF.3 Identify the effect on the graph of replacing f(x) by f(x) + k, k f(x), f(kx), and f(x + k) for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them.

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Introduction

Some teachers will look at this task and think that its lack of application-oriented activities is too test- or worksheet-like and not suitable as a practical task. However, the idea behind this task is to prepare students for the variety of modeling situations that arise from polynomial and other types of functions. Some general ideas need to be covered – even versus odd functions, end behavior, etc., that are not only applicable to all functions, but that students must understand in order to adequately apply characteristics of functions to practical scenarios. This task is not meant to be a substitute for the *Polynomial Functions* unit that should occur earlier in the course, but is instead meant to function as a review and summation of that important unit in order to prepare students for further application of various types of functions.

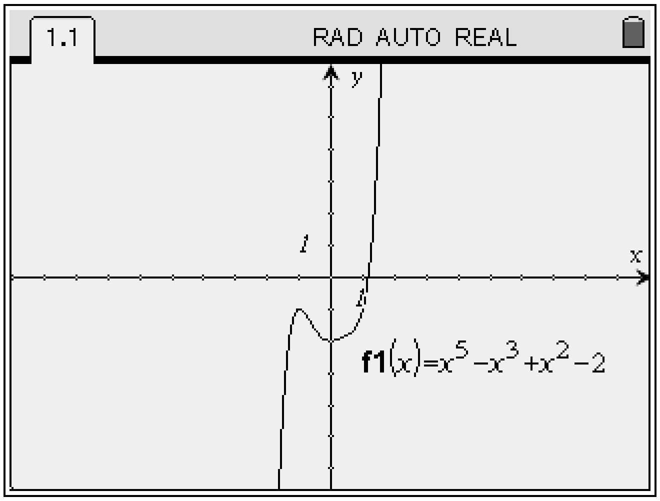
The task is split up between a general discussion of even/odd functions and end behavior, a more sophisticated graphing section, and finally an all-around polynomial review. The focus of the entire task is really on graphing, even when students are being asked to tackle problems algebraically. It is imperative that students see the algebra "moving" with the graph (and vice versa!), so to speak.

Materials

- Pencil
- Handout
- Calculator

Polynomial Potpourri

Part I: End Behavior & Even/Odd Functions



Degree: Odd

Lead Coefficient: Positive

How does $f(x)$ behave as $x$approaches positive infinity? In other words, what is y doing as you look to the right? How does $f(x)$ behave as x approaches negative infinity (or what is y doing as you look to the left)?

End Behavior Statement:
As $x \to -\infty, f(x) \to -\infty$
As $x \to \infty, f(x) \to \infty$

Is this function even, odd, or neither? It's not symmetric with respect to either the y-axis (even) or the origin (odd), so the answer is neither. We can also see this algebraically.

If a function is even, $f(-x) = f(x)$, meaning if a negative x-value is plugged into the function, it will yield the same y-value (or function value) as its positive counterpart.

Even Functions:
Contain both (x, y) and $(-x, y)$

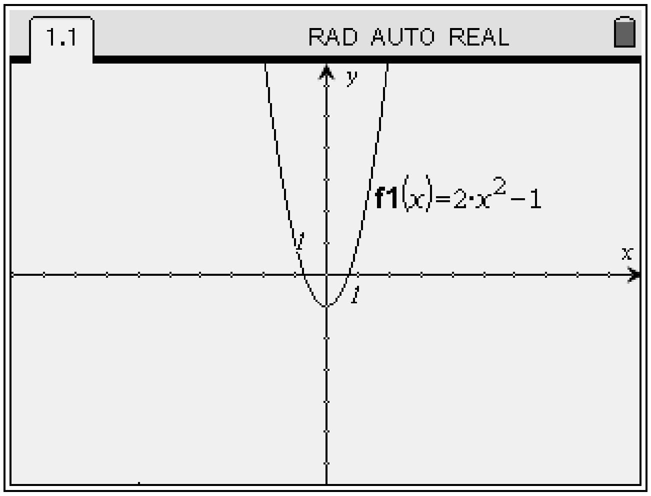
If a function is odd, $f(-x) = -f(x)$, meaning if a negative x-value is plugged into the function, it will yield the opposite (in terms of sign) y-value as its positive counterpart.

Odd Functions:
Contain both (x, y) and $(-x, -y)$

When we look at $f(-x)$ for this function:

$$f(-x) = (-x)^5 - (-x)^3 + (-x)^2 - 2$$
$$f(-x) = -x^5 + x^3 + x^2 - 2$$

In this case, $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$. Therefore, this function is neither even nor odd.



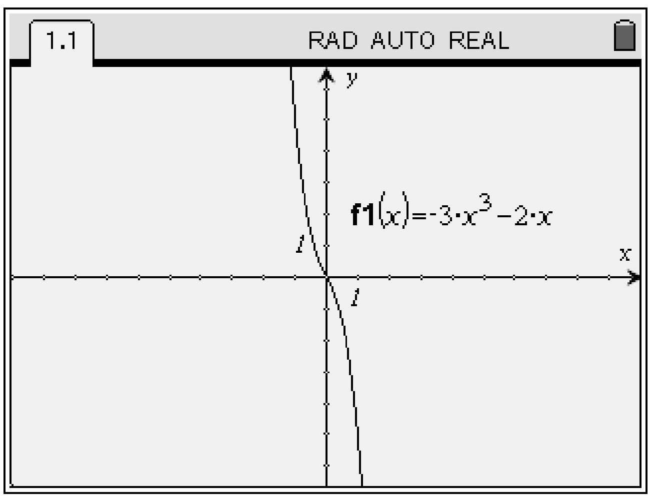
Degree: ***Even***

Lead Coefficient: ***Positive***

End Behavior:

As $\boldsymbol{x \to -\infty, f(x) \to \infty}$

As $\boldsymbol{x \to \infty, f(x) \to \infty}$



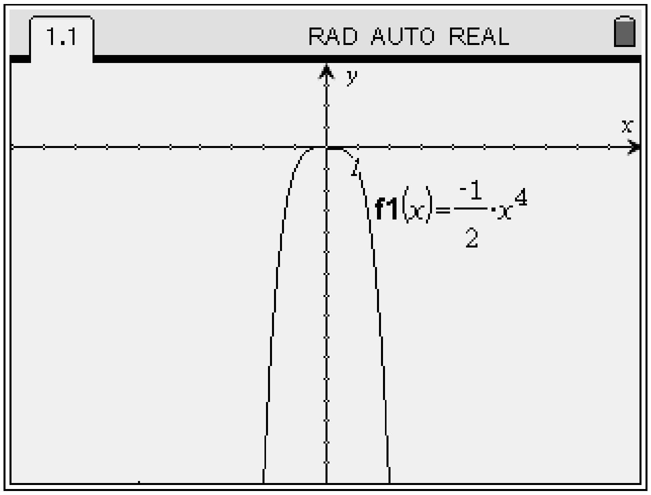
Degree: ***Odd***

Lead Coefficient: ***Negative***

End Behavior:

As $\boldsymbol{x \to -\infty, f(x) \to \infty}$

As $\boldsymbol{x \to \infty, f(x) \to -\infty}$



Degree: ***Even***

Lead Coefficient: ***Negative***

End Behavior:

As $\boldsymbol{x \to -\infty, f(x) \to -\infty}$

As $\boldsymbol{x \to \infty, f(x) \to -\infty}$

Without graphing, give the end behavior of each of the following polynomial functions, and then determine whether the function is even, odd, or neither algebraically.

1. $f(x) = -x^3 + 2x$

As $\boldsymbol{x \to -\infty, f(x) \to \infty}$
As $\boldsymbol{x \to \infty, f(x) \to -\infty}$

2. $f(x) = x^2 + 1$

As $\boldsymbol{x \to -\infty, f(x) \to \infty}$
As $\boldsymbol{x \to \infty, f(x) \to \infty}$

3. $f(x) = -2x^7 + x^5 - 6x^3 + x$

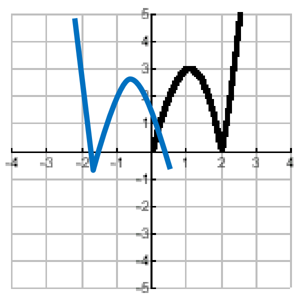
As $\boldsymbol{x \to -\infty, f(x) \to \infty}$
As $\boldsymbol{x \to \infty, f(x) \to -\infty}$

4. $f(x) = 8x^3 + 2x^2 - 7x + 1$

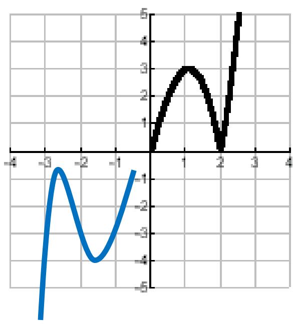
As $\boldsymbol{x \to -\infty, f(x) \to -\infty}$
As $\boldsymbol{x \to \infty, f(x) \to \infty}$

5. A function has domain [-4, 4] and a portion of its graph is shown.

Complete the graph of $f(x)$ if it is an even function.



Complete the graph of $f(x)$ if it is an odd function.



6. A cubic function contains the

points $(2,3)$ and $(-2,-3)$. Is the function even, odd, or neither? Why?

This cubic function would be odd, because $f(-x) = -f(x)$. We know this because the function contains both (x, y) and $(-x, -y)$, which means that the cubic function would have to be symmetric about the origin.

7. A quadratic function contains the points $(0,4)$ and $(4,4)$. Is the function even, odd, or neither? Why?

This quadratic function is neither even nor odd. The axis of symmetry would have to be at $x = 2$, which means that the function cannot be symmetric about the y-axis.

Table for End Behavior

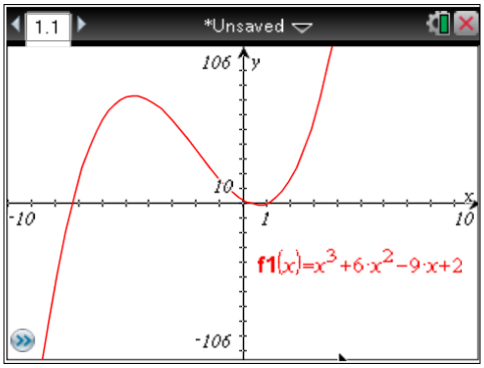
		Left End Behavior		Right End Behavior
Even Degree Positive Lead Coefficient		As x→-∞, f(x)→+∞		As x→+∞, f(x)→+∞
Even Degree Negative Lead Coefficient		As x→-∞, f(x)→-∞		As x→+∞, f(x)→-∞
Odd Degree Positive Lead Coefficient		As x→-∞, f(x)→-∞		As x→+∞, f(x)→+∞
Odd Degree Negative Lead Coefficient		As x→-∞, f(x)→+∞		As x→+∞, f(x)→-∞

Remember, you don't need this table! Just *think of all even degree functions as parabolas* and *all odd degree functions as lines*. Then, figuring out end behaviors based on the signs of lead coefficients is easy!

Part II: Graphing Polynomials

Use the calculator to help you find all roots – decide which ones are rational and write them as reduced fractions, decide which ones are irrational and write those as simplified radicals, and decide which ones are not real and write those in complex number form. Then find all relative maximum and minimum points of the function.

1. $f(x) = x^3 + 6x^2 - 9x + 2$



Rational roots: $\boldsymbol{x = 1}$

Irrational roots: $\boldsymbol{x = \frac{-7 \pm \sqrt{57}}{2}}$

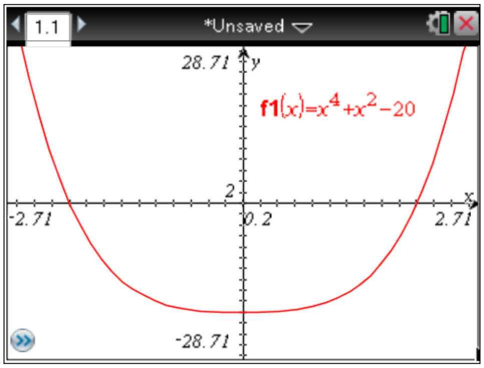
Non-real roots: ***None***

Relative maximum points: ***(–4.646, 73.04)***

Relative minimum points: ***(.6458, –1.041)***

End behavior: $\boldsymbol{x \to -\infty, f(x) \to -\infty; \quad x \to \infty, f(x) \to \infty}$

2. $f(x) = x^4 + x^2 - 20$



Rational roots: $\boldsymbol{x = -2, 2}$

Irrational roots: ***None***

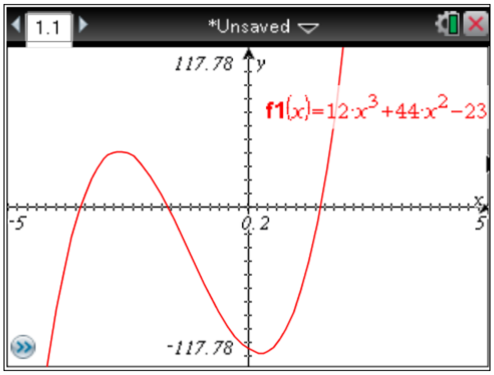
Non-real roots: $\boldsymbol{x = -\sqrt{5}i, \sqrt{5}i}$

Relative maximum points: ***None***

Relative minimum points: ***(0, -20)***

End behavior: $\boldsymbol{x \to -\infty, f(x) \to \infty;\ \ x \to \infty, f(x) \to \infty}$

3. $f(x) = 12x^3 + 44x^2 - 23x - 105$



Rational roots: $\boldsymbol{x = \frac{3}{2}, -\frac{5}{3}, -\frac{7}{2}}$

Irrational roots: ***None***

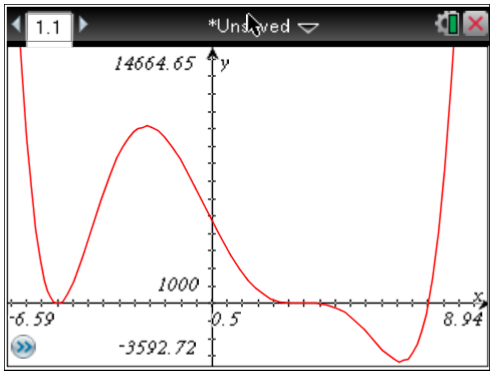
Non-real roots: ***None***

Relative maximum points: ***(–2.683, 41.68)***

Relative minimum points: ***(.2382, –107.8)***

End behavior: $\boldsymbol{x \to -\infty, f(x) \to -\infty; \;\; x \to \infty, f(x) \to \infty}$

4. $f(x) = x^6 - 6x^5 - 45x^4 + 284x^3 + 279x^2 - 3510x + 4725$



Rational roots: $\boldsymbol{x = 3\ (\times 3), 7, -5\ (\times 2)}$

Irrational roots: ***None***

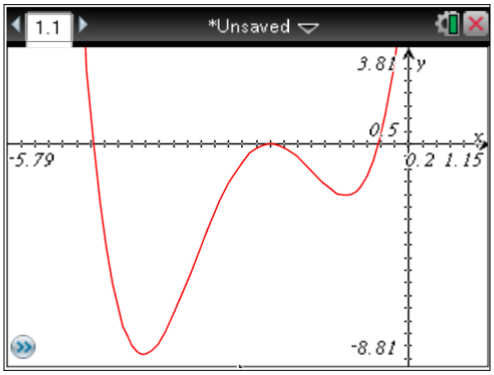
Non-real roots: ***None***

Relative maximum points: ***(–2.123, 10152)***

Relative minimum points: ***(–5, 0), (6.123, –3305)***

End behavior: $\boldsymbol{x \to -\infty, f(x) \to \infty;\ \ x \to \infty, f(x) \to \infty}$

5. $f(x) = x^4 + 9x^3 + 26x^2 + 28x + 8$



Rational roots: $\boldsymbol{x = -2}\ (\times\ \boldsymbol{2})$

Irrational roots: $\boldsymbol{x = \frac{-5 \pm \sqrt{17}}{2}}$

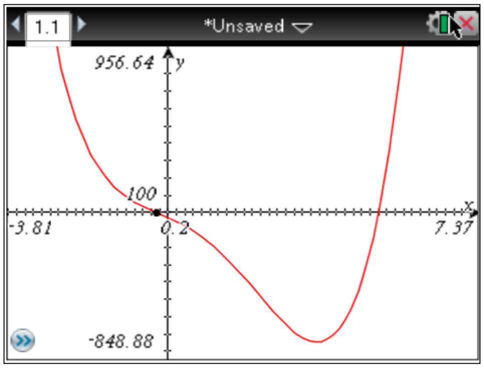
Non-real roots: ***None***

Relative maximum points: ***(–2, 0)***

Relative minimum points: ***(–3.838, –8.310), (–.9119, –2.046)***

End behavior: $\boldsymbol{x} \to -\infty, \boldsymbol{f(x)} \to \infty;\ \ \boldsymbol{x} \to \infty, \boldsymbol{f(x)} \to \infty$

6. $f(x) = 7x^4 - 26x^3 - 22x^2 - 109x - 30$



Rational roots: $\boldsymbol{x = 5, -\frac{2}{7}}$

Irrational roots: ***None***

Non-real roots: $\boldsymbol{x = \frac{-1 \pm \sqrt{11}i}{2}}$

Relative maximum points: ***None***

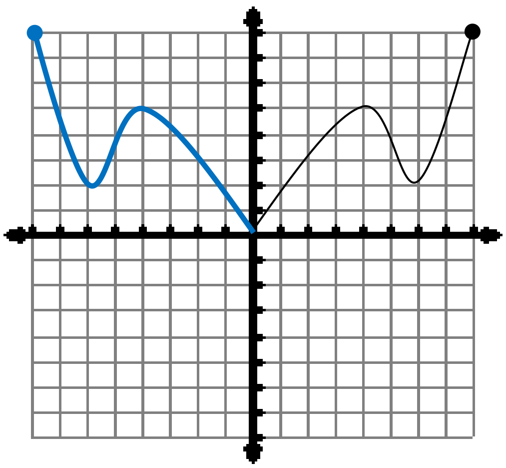
Relative minimum points: ***(3.540, –745.7)***

End behavior: $\boldsymbol{x \to -\infty, f(x) \to \infty; \quad x \to \infty, f(x) \to \infty}$

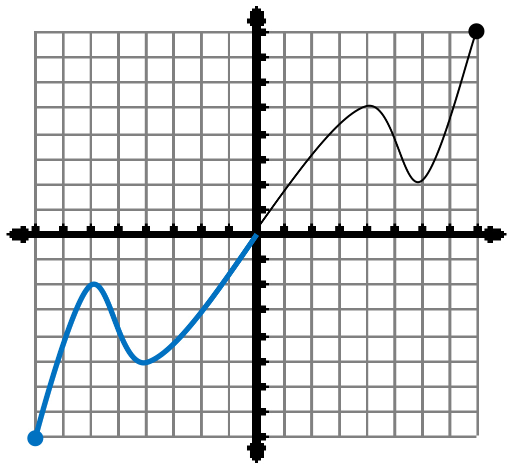
Part III: Polynomial Practice

1. Given the following portions of a graph of a function defined over the domain [-8,8]

(a) Finish the graph below if the function is **even**.



(b) Finish the graph below if the function is **odd**.



2. Write the polynomial with the following zeros: $1+\sqrt{2}$, $2i$

$$\boldsymbol{x = 1+\sqrt{2}, x = 1-\sqrt{2}, x = 2i, x = -2i}$$

$$\boldsymbol{\left(x-1-\sqrt{2}\right)\left(x-1+\sqrt{2}\right)(x-2i)(x+2i)}$$

$$\boldsymbol{\left(x-1-\sqrt{2}\right)\left(x-1+\sqrt{2}\right)(x^2+2)}$$

$$\boldsymbol{\left(x^2-x+x\sqrt{2}-x+1-\sqrt{2}-x\sqrt{2}+\sqrt{2}-2\right)(x^2+2)}$$

$$\boldsymbol{(x^2-2x-1)(x^2+2)}$$

$$\boldsymbol{x^4+2x^2-2x^3-4x-x^2-2}$$

$$\boldsymbol{f(x) = x^4-2x^3+x^2-4x-2}$$

3. What is the remainder when $f(x) = 2x^{24} - 7x^{12} + 2$ is divided by $x+1$? From this information, give one point on the graph (without graphing).

If the divisor is $x+1$, then some students will want to try synthetic division with $x=-1$, which takes too long and leaves too many placeholder errors possible. Students should realize that we can use substitution, which is much quicker here:

$$f(-1)=2(-1)^{24}-7(-1)^{12}+2=-3$$

Using this method allows students to see the connection between either polynomial long division or synthetic division and substituting an input (or x-value) into the function. They produce the same results! More importantly, students can see the relationship between divisor and remainder – they are actual points on the graph of the function. Therefore, to answer the second part of the question, we know a point on the function's graph must be $(-1,-3)$.

4. Find k so that -3 is a zero of x^3-4x^2-kx+9

Students may use the substitution method above or synthetic division to come up with the value for k. It is important for students to remember that either a function value of zero or a remainder of zero represents an x-intercept. Working with vocabulary here is very important.

Using either method, $k=18$, so that $(-3)^3-4(-3)^2-(18\cdot -3)+9=0$

5. Given the function $f(x)=x^4-6x^3+8x^2+6x-9$

a. Give the possible rational zeros.

When we use the $\frac{p}{q}$ method, we get the following: $\pm\left(\frac{1,3,9}{1}\right)$

Therefore, the possible rational roots are $1,-1,3,-3,9,-9$

b. Find the roots. Show all of your work.

The actual roots are $x=-1,1,3\ (\times 2)$

6. The function $f(x) = x^4 - 6x^3 + 6x^2 + 24x - 40$ has $3 - i$ as a zero. Find the remaining zeros.

If $3 - i$ is a zero, then by the conjugate rule, $3 + i$ is also a zero. We can find the product of these factors to attain a polynomial divisor without imaginary numbers, and then use the divisor to solve for the remaining zeros.

$$x = 3 + i, x = 3 - i$$

$$(x - 3 - i)(x - 3 + i) = x^2 - 6x + 10$$

$$(x^4 - 6x^3 + 6x^2 + 24x - 40) \div (x^2 - 6x + 10) = x^2 - 4$$

Therefore, the remaining zeros are $x = 2$ and $x = -2$.

7. Describe the end behavior of the function $f(x) = -2x^{11} - 9x^7 + 8x - 9$

It is important for students to know that the actual shape of this higher-order polynomial doesn't matter, because like all polynomials, its end behavior can be described according to the function's degree and the sign of its lead coefficient.

The function has an odd degree and a negative lead coefficient, so it shares the same end behavior as a downward-sloping line. Therefore
As $x \to -\infty, f(x) \to \infty$ and as $x \to \infty, f(x) \to -\infty$.

Given a polynomial $g(x)$ with $g(-2) = -1$, $g(0) = 2$, $g(3) = 0$, $g(4) = -1$, answer the following.

The goal here is to get students thinking about graphs without them looking at a graph. By having students piece together limited information, it puts a needed focus on both the use of vocabulary and the real meaning of the terms (including different terms meaning the same thing mathematically), as well as a deeper understanding of function notation.

8. What is a root of $g(x)$? $x = 3$

9. What is a factor of $g(x)$? $x - 3$

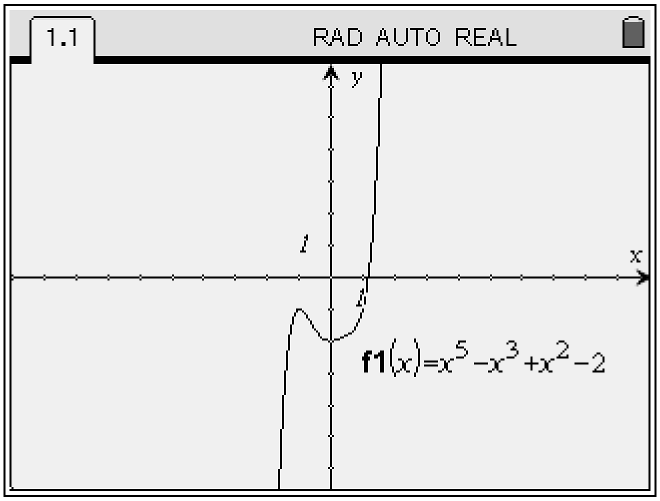
10. What is the remainder when $g(x)$ is divided by $x - 4$? -1

11. What is the y-intercept? $(0, 2)$

12. Name a point on the graph in quadrant III. $(-2, -1)$

Polynomial Potpourri

Part I: End Behavior & Even/Odd Functions



Degree: Odd

Lead Coefficient: Positive

How does $f(x)$ behave as $x$approaches positive infinity? In other words, what is y doing as you look to the right? How does $f(x)$ behave as x approaches negative infinity (or what is y doing as you look to the left)?

End Behavior Statement:
As $x \to -\infty, f(x) \to -\infty$
As $x \to \infty, f(x) \to \infty$

Is this function even, odd, or neither? It's not symmetric with respect to either the y-axis (even) or the origin (odd), so the answer is neither. We can also see this algebraically.

If a function is even, $f(-x) = f(x)$, meaning if a negative x-value is plugged into the function, it will yield the same y-value (or function value) as its positive counterpart.

Even Functions: Contain both (x, y) and $(-x, y)$

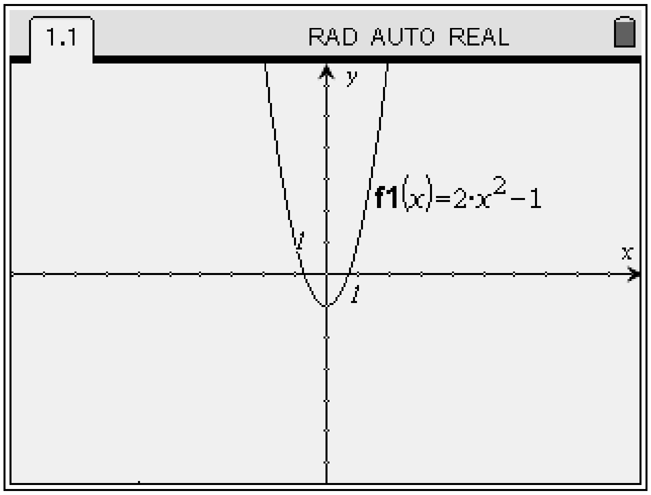
If a function is odd, $f(-x) = -f(x)$, meaning if a negative x-value is plugged into the function, it will yield the opposite (in terms of sign) y-value as its positive counterpart.

Odd Functions: Contain both (x, y) and $(-x, -y)$

When we look at $f(-x)$ for this function:

$$f(-x) = (-x)^5 - (-x)^3 + (-x)^2 - 2$$
$$f(-x) = -x^5 + x^3 + x^2 - 2$$

In this case, $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$. Therefore, this function is neither even nor odd.



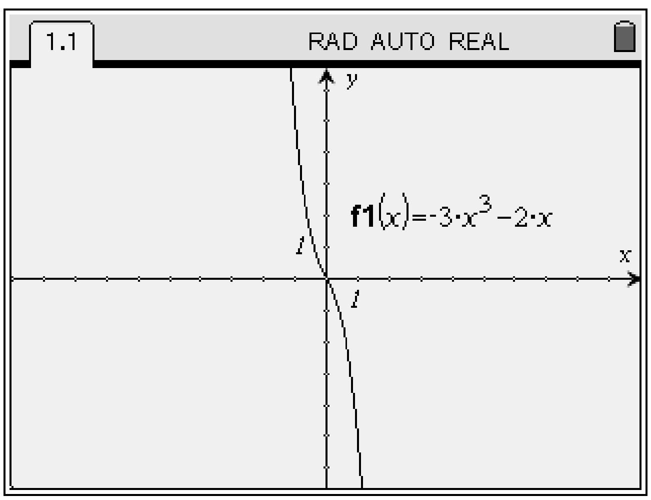
Degree:

Lead Coefficient:

End Behavior:

As $x \to$___, $f(x) \to$___

As $x \to$___, $f(x) \to$___



Degree:

Lead Coefficient:

End Behavior:

As $x \to$___, $f(x) \to$___

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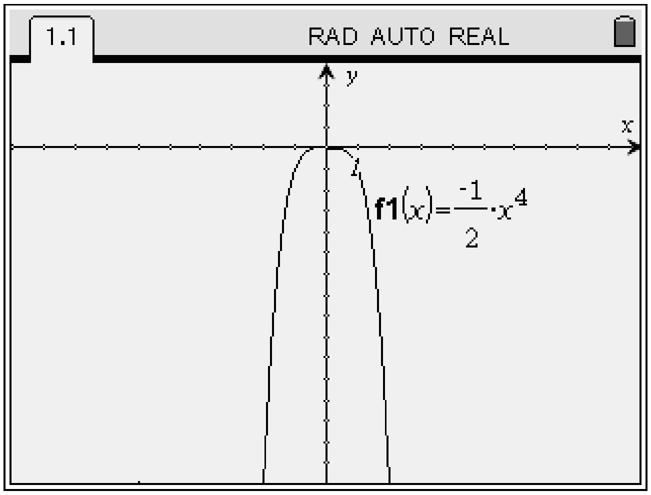


Degree:

Lead Coefficient:

End Behavior:

As $x \to$___, $f(x) \to$___

As $x \to$___, $f(x) \to$___

Without graphing, give the end behavior of each of the following polynomial functions, and then determine whether the function is even, odd, or neither algebraically.

1. $f(x) = -x^3 + 2x$

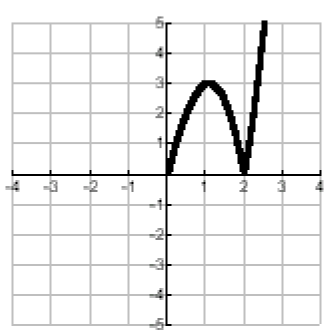
2. $f(x) = x^2 + 1$

3. $f(x) = -2x^7 + x^5 - 6x^3 + x$

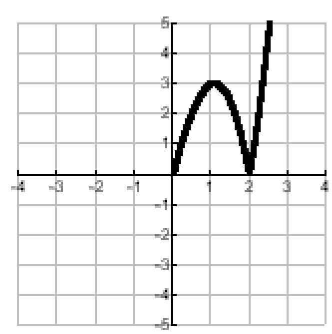
4. $f(x) = 8x^3 + 2x^2 - 7x + 1$

5. A function has domain [-4, 4] and a portion of its graph is shown.

Complete the graph of $f(x)$ if it is an even function.



Complete the graph of $f(x)$ if it is an odd function.



6. A cubic function contains the points $(2,3)$ and $(-2,-3)$. Is the function even, odd, or neither? Why?

7. A quadratic function contains the points $(0,4)$ and $(4,4)$. Is the function even, odd, or neither? Why?

Table for End Behavior

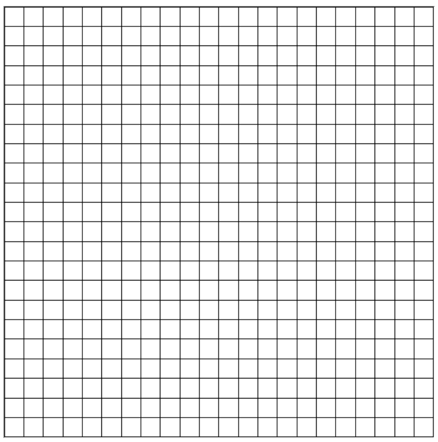
		Left End Behavior		Right End Behavior
Even Degree Positive Lead Coefficient		As x→-∞, f(x)→+∞		As x→+∞, f(x)→+∞
Even Degree Negative Lead Coefficient		As x→-∞, f(x)→-∞		As x→+∞, f(x)→-∞
Odd Degree Positive Lead Coefficient		As x→-∞, f(x)→-∞		As x→+∞, f(x)→+∞
Odd Degree Negative Lead Coefficient		As x→-∞, f(x)→+∞		As x→+∞, f(x)→-∞

Remember, you don't need this table! Just *think of all even degree functions as parabolas* and *all odd degree functions as lines*. Then, figuring out end behaviors based on the signs of lead coefficients is easy!

Part II: Graphing Polynomials

Use the calculator to help you find all roots – decide which ones are rational and write them as reduced fractions, decide which ones are irrational and write those as simplified radicals, and decide which ones are not real and write those in complex number form. Then find all relative maximum and minimum points of the function.

1. $f(x) = x^3 + 6x^2 - 9x + 2$



Rational roots: _______________

Irrational roots: _______________

Non-real roots: _______________

Relative maximum points: _____________

Relative minimum points: ______________

End behavior: _______________

2. $f(x) = x^4 + x^2 - 20$

Rational roots: _______________

Irrational roots: _______________

Non-real roots: _______________

Relative maximum points: _____________

Relative minimum points: ______________

End behavior: _______________

3. $f(x) = 12x^3 + 44x^2 - 23x - 105$

Rational roots: _______________

Irrational roots: _______________

Non-real roots: _______________

Relative maximum points: _____________

Relative minimum points: ______________

End behavior: _______________

4. $f(x) = x^6 - 6x^5 - 45x^4 + 284x^3 + 279x^2 - 3510x + 4725$

Rational roots: _______________

Irrational roots: _______________

Non-real roots: _______________

Relative maximum points: _____________

Relative minimum points: ______________

End behavior: _______________

5. $f(x) = x^4 + 9x^3 + 26x^2 + 28x + 8$

Rational roots: _______________

Irrational roots: _______________

Non-real roots: _______________

Relative maximum points: _____________

Relative minimum points: ______________

End behavior: _______________

6. $f(x) = 7x^4 - 26x^3 - 22x^2 - 109x - 30$

Rational roots: _______________

Irrational roots: _______________

Non-real roots: _______________

Relative maximum points: _____________

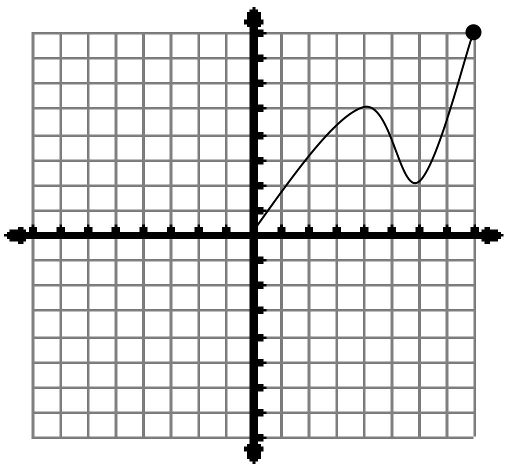
Relative minimum points: ______________

End behavior: _______________

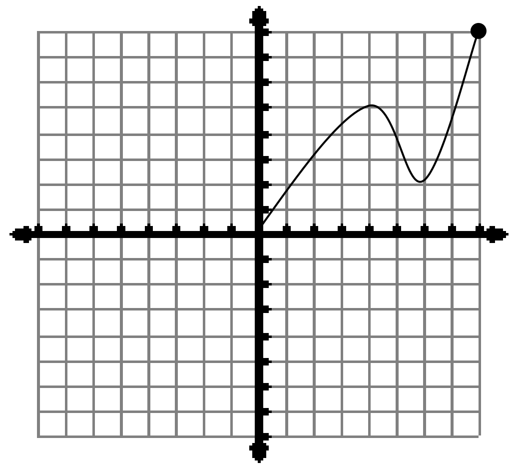
Part III: Polynomial Practice

1. Given the following portions of a graph of a function defined over the domain [-8,8]

(a) Finish the graph below if the function is **even**.



(b) Finish the graph below if the function is **odd**.



2. Write the polynomial with the following zeros: $1 + \sqrt{2}$, $2i$

3. What is the remainder when $f(x) = 2x^{24} - 7x^{12} + 2$ is divided by $x + 1$? From this information, give one point on the graph (without graphing).

4. Find k so that -3 is a zero of $x^3 - 4x^2 - kx + 9$

5. Given the function $f(x) = x^4 - 6x^3 + 8x^2 + 6x - 9$

a. Give the possible rational zeros.

b. Find the roots. Show all of your work.

6. The function $f(x) = x^4 - 6x^3 + 6x^2 + 24x - 40$ has $3 - i$ as a zero. Find the remaining zeros.

7. Describe the end behavior of the function $f(x) = -2x^{11} - 9x^7 + 8x - 9$

Given a polynomial $g(x)$ with $g(-2) = -1$, $g(0) = 2$, $g(3) = 0$, $g(4) = -1$, answer the following.

8. What is a root of $g(x)$? __________

9. What is a factor of $g(x)$? __________

10. What is the remainder when $g(x)$ is divided by $x - 4$? __________

11. What is the y-intercept? __________

12. Name a point on the graph in quadrant III. __________

Combining and Describing Functions

Back to Task Table

Mathematical Goals

- Students will explore ways to compose and combine different functions, not only arising from different contexts, but also from different function types.

Georgia Standards of Excellence

Analyze functions using different representations.

MGSE9-12.F.IF.9 Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one function and an algebraic expression for another, say which has the larger maximum.*

Build a function that models a relationship between two quantities.

MGSE9-12.F.BF.1b Combine standard function types using arithmetic operations in contextual situations (Adding, subtracting, and multiplying functions of different types).

MGSE9-12.F.BF.1c Compose functions. *For example, if T(y) is the temperature in the atmosphere as a function of height, and h(t) is the height of a weather balloon as a function of time, then T(h(t)) is the temperature at the location of the weather balloon as a function of time.*

Build new functions from existing functions.

MGSE9-12.F.BF.3 Identify the effect on the graph of replacing f(x) by f(x) + k, k f(x), f(kx), and f(x + k) for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them.

MGSE9-12.F.BF.4 Find inverse functions.

MGSE9-12.F.BF.4a Solve an equation of the form f(x) = c for a simple function f that has an inverse and write an expression for the inverse. For example, $f(x) = 2(x^3)$ or f(x) = (x+1)/(x-1) for $x \neq 1$.

MGSE9-12.F.BF.4b Verify by composition that one function is the inverse of another.

MGSE9-12.F.BF.4c Read values of an inverse function from a graph or a table, given that the function has an inverse.

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Introduction

This task is designed to get students to think deeply about what combining and composing functions really means. The task takes an algebraic, a graphical, and a numerical perspective in trying to help students understand how a combination of functions can be used to model a variety of important applications. The ultimate goal is to for students to use more sophisticated reasoning related to functions, graphs, and even down to the "atomic" level of a function – a simple coordinate.

Materials

- Pencil
- Handout
- Calculator (only for part 1)

Combining & Describing Functions

1. Inverse Functions

You already know that exponential functions and logarithmic functions are inverses of one another. Now, let's see what is implied by the term "inverse".

Given $f(x) = 2e^{3x} + 1$

(a) Find the inverse, $f^{-1}(x)$.

$$x = 2e^{3y} + 1 \rightarrow x - 1 = 2e^{3y} \rightarrow ln\left(\frac{x-1}{2}\right) = 3y \rightarrow so\ f^{-1}(x) = \frac{ln\left(\frac{x-1}{2}\right)}{3}$$

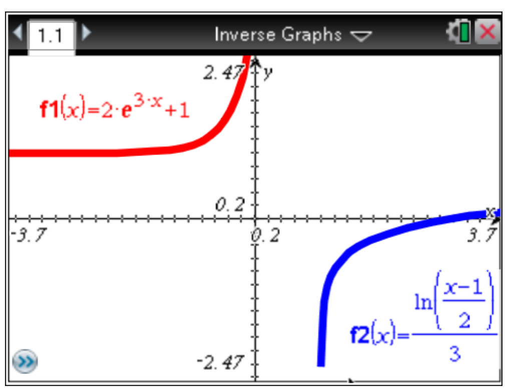
(b) Now fill in the table below for $f(x)$ and $f^{-1}(x)$.

x	0	1	2	3	4
$f(x)$	*3*	*41.171*	*807.86*	*16207*	*325511*
$f^{-1}(x)$	*Undef.*	*Undef.*	*-0.231*	*0*	*0.13516*

Is there a specific pair of points that stand out to you?

$f(x)$ contains the point (0, 3) and $f^{-1}(x)$ contains the point (3, 0)

(c) Sketch the graph of both $f(x)$ and $f^{-1}(x)$ on the same axes below.



(d) How do the graphs compare to one another?

They reflect across the line $y = x$, meaning that they are inverses of one another.

(e) For $f(x)$, give the domain, range, and the equation (and type) of the asymptote.

D: $(-\infty, \infty)$ R: $(\mathbf{1}, \infty]$ A: ***horizontal @ $y = 1$***

(f) For $f^{-1}(x)$, give the domain, range, and the equation (and type) of the asymptote.

D: $(\mathbf{1}, \infty]$ R: $(-\infty, \infty)$ A: ***vertical @ $x = 1$***

Now, let's look at an anonymous function represented by a table of values.

$\boldsymbol{x}$	0	1	2	3	4
$\boldsymbol{f(x)}$	0	1	1	5	3

(a) Using the table below, give a table of values for the inverse of the above function.

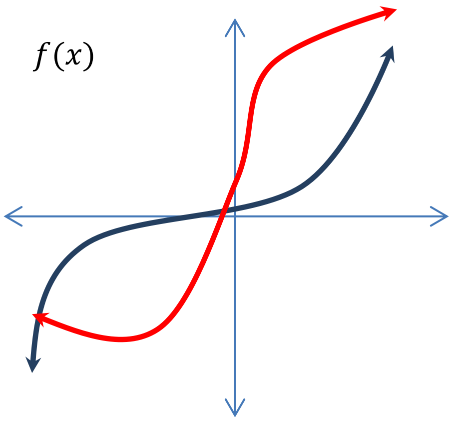
$\boldsymbol{x}$	***0***	***1***	***1***	***5***	***3***
$\boldsymbol{f^{-1}(x)}$	***0***	***1***	***2***	***3***	***4***

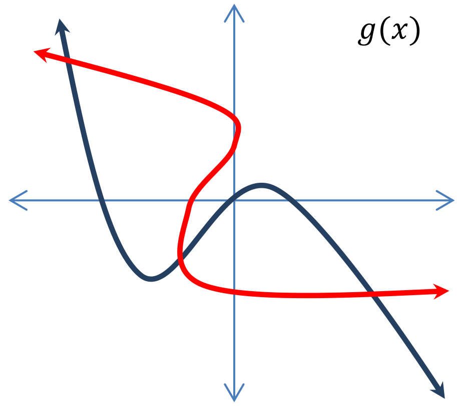
(b) Is the inverse a function? How can you tell?

No; the definition of a function requires that for every one x-value, there is only one y-value. In this case, $x = 1$ has two different y-values.

One-to-one functions are functions that have an inverse that is also a function. You can tell graphically if a function is one-to-one without graphing the inverse – it must pass the horizontal line test.

Consider the following functions:





(a) Is either function one-to-one?

$f(x)$ is a one-to-one function because it passes the horizontal line test; $g(x)$ does not.

(b) Draw the inverse function on each graph.

Find the inverse of the following and give their domain.

1. $g(x) = \frac{3}{x-1}$

$g^{-1}(x) = \frac{3}{x} + 1$ ***and D:*** $x \neq 0$

2. $f(x) = x^2 - 1$

$f^{-1}(x) = \pm\sqrt{x+1}$ ***and D:*** $x \geq -1$

3. $f(x) = \sqrt[3]{\frac{x-7}{3}}$

$f^{-1}(x) = 3x^3 + 7$ ***and D:*** $(-\infty, \infty)$

4. $h(x) = log_3(x^2 + 2)$

$h^{-1}(x) = \pm\sqrt{3^x - 2}$ ***and D:*** $x \geq \frac{log\,2}{log\,3}$

We'll revisit inverse functions in a moment.

2. Combining and Compositions

I. Basic Combinations and Compositions

For 1 – 8 below, use the following information. Give the domain of each.

$\boldsymbol{f(x) = -2x^2 - 2x + 1}$ and $\boldsymbol{g(x) = x + 1}$

1. Find $f(x) + g(x)$

$(f+g)(x) = -2x^2 - x + 2 \quad D: (-\infty, \infty)$

2. Find $f(x) - g(x)$

$(f-g)(x) = -2x^3 - 3x \quad D: (-\infty, \infty)$

3. Find $f(x) \cdot g(x)$

$(f \cdot g)(x) = -2x^3 - 4x^2 - x + 1 \quad D: (-\infty, \infty)$

4. Find $\frac{f(x)}{g(x)}$

$\left(\frac{f}{g}\right)(x) = \frac{-2x^2-2x+1}{x+1} \quad D: x \neq -1$

5. Find $f(g(x))$

$f(g(x)) = -2x^2 - 6x - 3 \quad D: (-\infty, \infty)$

6. Find $g(f(x))$

$g\big(f(x)\big) = -2x^2 - 2x + 2 \quad D: (-\infty, \infty)$

7. Find $g(g(x))$

$g\big(g(x)\big) = x + 2 \quad D: (-\infty, \infty)$

8. Find $f(g(-1))$

$f\big(g(-1)\big) = 1$

No domain – this is a function value

II. Function Compositions Using Sets of Points

f={(–2, 3), (–1, 1), (0, 0), (1, –1), (2, –3)}

g={(–3, 1), (–1, –2), (0, 2), (2, 2), (3, 1)}

Using the information above, find the following:

1. f(1)

-1

2. g(-1)

-2

3. g(f(1))

-2

4. f(g(0))

-3

5. f(g(-1))

3

6. g(f(-1))

undefined

Given two functions, $f(x)$ and $g(x)$, evaluate the following given that:

For $f(x)$: $f(-2) = 5,\ f(-1) = 2,\ f(0) = -1,\ f(1) = -3,\ f(2) = 3$

For $g(x)$: $g(-2) = -1,\ g(-1) = -2,\ g(0) = 0,\ g(1) = 2,\ g(2) = 3$

1. $(f + f)(0)$

-2

2. $(f - g)(-2)$

6

3. $f(g(-1))$

5

4. $g(f(0))$

-2

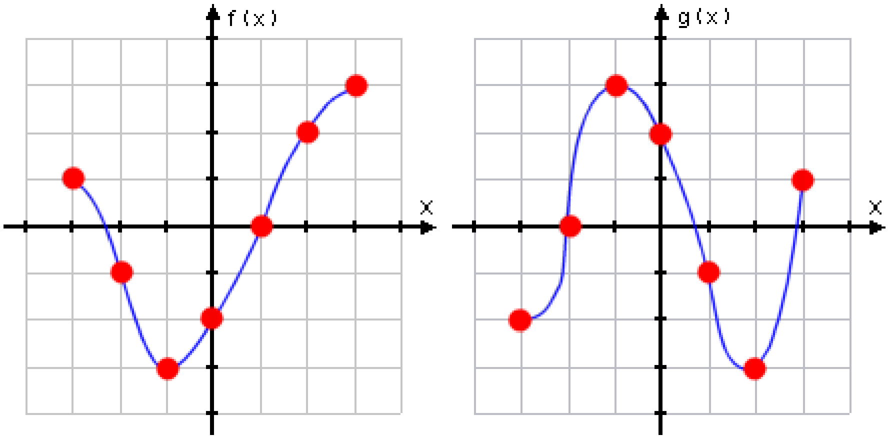
5. $f\big(g(1)\big) - g(f(-1))$

0

6. $f^{-1}(f(2))$

2

III. Function Compositions Using Graphs



Given $f(x)$ and $g(x)$ as shown in the graphs above, find the following:

1. $f(g(1))$

-3

2. $g(f(-2))$

3

3. $f(f(0))$

-1

4. $f^{-1}(g(2))$

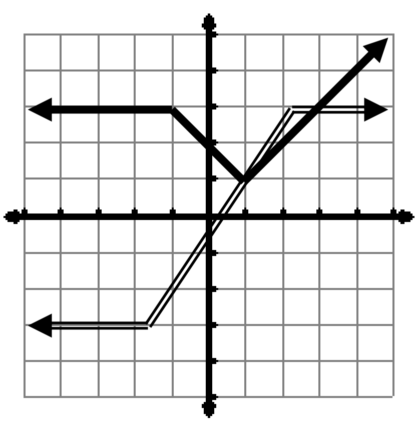
-1

5. $g(f^{-1}(2))$

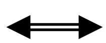
-3

6. $g^{-1}(g^{-1}(1))$

-1



$f(x) =$ 

$g(x) =$ 

Use the graph to the left for the following:

1. $(f + f)(2)$ ***4***
2. $f(g(1))$ ***1***
3. $g(f(-1))$ ***3***
4. $f^{-1}(f^{-1}(1))$ ***1***
5. $(f + g)(3)$ ***6***
6. $f(4) - g(-1)$ ***6***
7. $(f - g)(-3) + f(f(2))$ ***8***

IV. Composition Extensions and Applications

1. Show that $f(x) = 2x^2 - 1$ and $g(x) = \sqrt{\frac{x+1}{2}}$ are inverse functions using compositions.

$$f(g(x)) = 2\left(\sqrt{\frac{x+1}{2}}\right)^2 - 1 = x$$

$$g(f(x)) = \sqrt{\frac{(2x^2 - 1) + 1}{2}} = x$$

2. Verify that $f(x) = \sqrt{\frac{x-2}{3}}$ and $g(x) = 3x^2 + 2$ are inverses.

$$f(g(x)) = \sqrt{\frac{(3x^2 + 2) - 2}{3}} = x$$

$$g(f(x)) = 3\left(\sqrt{\frac{x-2}{3}}\right)^2 + 2 = x$$

3. Given $f(x) = \sqrt{x}$ and $g(x) = x - 2$, find the domains of $f(g(x))$ and $g(f(x))$.

$$f(g(x)) = \sqrt{x-2} \quad D: x \geq 2$$

$$g(f(x)) = \sqrt{x} - 2 \quad D: x \geq 0$$

4. Given $h(x) = (x+1)^2 + 2(x+1) - 3$, determine two functions $f(x)$ and $g(x)$ which, when composed, generate $h(x)$.

$$\boldsymbol{f(x) = x^2 + 2x - 3}$$

$$\boldsymbol{g(x) = x + 1}$$

5. Given $h(x) = \sqrt{4x+1}$, determine two functions $f(x)$ and $g(x)$ which, when composed, generate $h(x)$.

$$\boldsymbol{f(x) = \sqrt{x}}$$

$$\boldsymbol{g(x) = 4x + 1}$$

6. Given $h(x) = \frac{(3x-1)^2}{5}$, determine two functions $f(x)$ and $g(x)$ which, when composed, generate $h(x)$.

$$\boldsymbol{f(x) = \frac{x^2}{5}}$$

$$\boldsymbol{g(x) = 3x - 1}$$

7. You work forty hours a week at a furniture store. You receive a $220 weekly salary, plus a 3% commission on sales over $5000. Assume that you sell enough this week to get the commission. Given the functions $f(x) = 0.03x$ and $g(x) = x - 5000$, which composed function, $f(g(x))$ or $g(f(x))$, represents your commission?

$\boldsymbol{f\big(g(x)\big) = 0.03(x - 5000)}$ ***represents commission***

8. You make a purchase at a local hardware store, but what you've bought is too big to take home in your car. For a small fee, you arrange to have the hardware store deliver your purchase for you. You pay for your purchase, plus the sales taxes, plus the fee. The taxes are 7.5% and the fee is $20.

(a) Write a function $t(x)$ for the total, after taxes, on purchase amount x. Write another function $f(x)$ for the total, including the delivery fee, on purchase amount x.

$$t(x) = 1.075x$$

$$f(x) = x + 20$$

(b) Calculate and interpret $f(t(x))$ and $t(f(x))$. Which results in a lower cost to you?

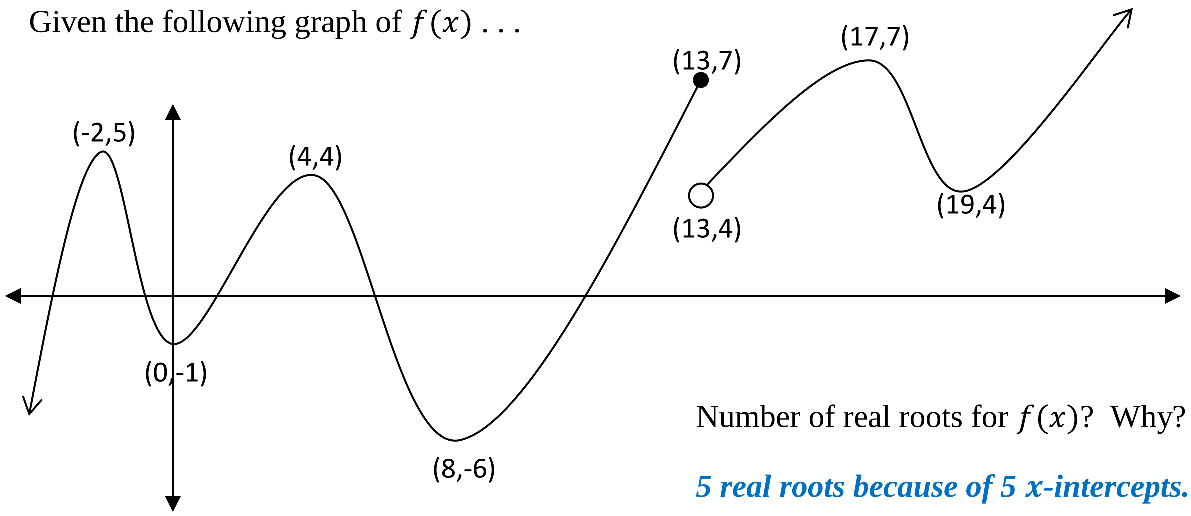
> $\boldsymbol{f(t(x)) = 1.075x + 20}$ ***represents the amount you would pay if taxes were not charged on the delivery fee.***
>
> $\boldsymbol{t(f(x)) = 1.075x + 21.50}$ ***represents the amount you would pay if taxes were charged on the delivery fee.***
>
> $\boldsymbol{f(t(x))}$ ***would result in a lower cost to you.***

(c) Suppose taxes, by law, are not to be charged on delivery fees. Which composite function must then be used?

> $\boldsymbol{f(t(x))}$ ***would have to be used since taxes are not applied to delivery fees in this composite function.***

3. Describing Functions

Given the following graph of $f(x)$ . . .



Number of real roots for $f(x)$? Why?

5 real roots because of 5 x-intercepts.

Complete the table below.

x	f(x)	f(x)+2	f(x)-1	-f(x)	2f(x)	$-\frac{1}{2}f(x)$	$\lvert f(x) \rvert$
-2	*5*	*7*	*4*	*-5*	*10*	*-2.5*	*5*
0	*-1*	*1*	*-2*	*1*	*-2*	*0.5*	*1*
4	*4*	*6*	*3*	*-4*	*8*	*-2*	*4*
8	*-6*	*-4*	*-7*	*6*	*-12*	*3*	*6*
13	*7*	*9*	*6*	*-7*	*14*	*-3.5*	*7*
17	*7*	*9*	*6*	*-7*	*14*	*-3.5*	*7*
19	*4*	*6*	*3*	*-4*	*8*	*-2*	*4*

- How can you tell that $f(x)$ is a function? Explain.

  Every input has only one corresponding output.

- What is the domain and range of $f(x)$?

  $\boldsymbol{D}: (-\infty, \infty)$ ***and*** $\boldsymbol{R}: (-\infty, \infty)$

- Is $f(x)$ a continuous function? How can you tell?

  No; $\boldsymbol{f(x)}$ ***is not a polynomial function because there is a jump discontinuity at*** $\boldsymbol{x = 13.}$

- What is the end behavior of $f(x)$?

  As $\boldsymbol{x \to -\infty, f(x) \to -\infty}$; ***as*** $\boldsymbol{x \to \infty, f(x) \to \infty}$

- Give the intervals of increase and decrease and local maximums and minimums for $f(x)$.

Intervals of increase: $(-\infty, -2) \cup (0, 4) \cup (8, 13] \cup (13, 17) \cup (19, \infty)$

Intervals of decrease: $(-2, 0) \cup (4, 8) \cup (17, 19)$

- What is $f(2x)$ if $x = 4$?

$2x = 8$, *so* $f(8) = -6$

Combining & Describing Functions

1. Inverse Functions

You already know that exponential functions and logarithmic functions are inverses of one another. Now, let's see what is implied by the term "inverse".

Given $f(x) = 2e^{3x} + 1$

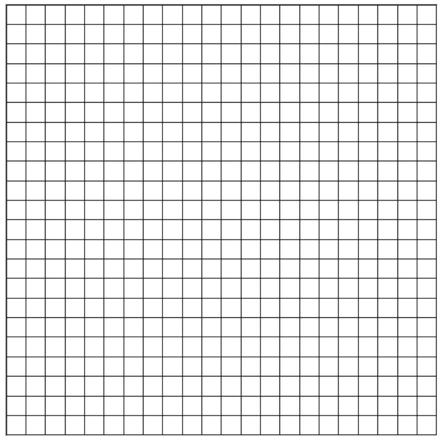
(a) Find the inverse, $f^{-1}(x)$.

(b) Now fill in the table below for $f(x)$ and $f^{-1}(x)$.

x	0	1	2	3	4
$f(x)$					
$f^{-1}(x)$					

Is there a specific pair of points that stand out to you?

(c) Sketch the graph of both $f(x)$ and $f^{-1}(x)$ on the same axes below.



(d) How do the graphs compare to one another?

(e) For $f(x)$, give the domain, range, and the equation (and type) of the asymptote.

D: R: A:

(f) For $f^{-1}(x)$, give the domain, range, and the equation (and type) of the asymptote.

D: R: A:

Now, let's look at an anonymous function represented by a table of values.

$\boldsymbol{x}$	0	1	2	3	4
$\boldsymbol{f(x)}$	0	1	1	5	3

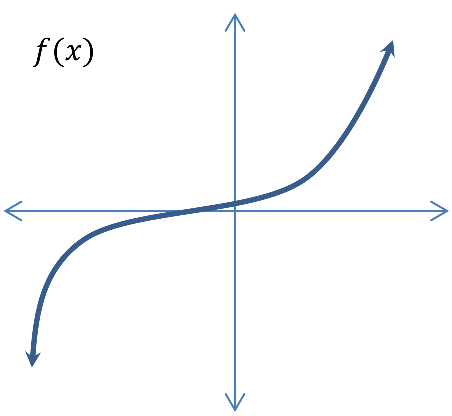
(a) Using the table below, give a table of values for the inverse of the above function.

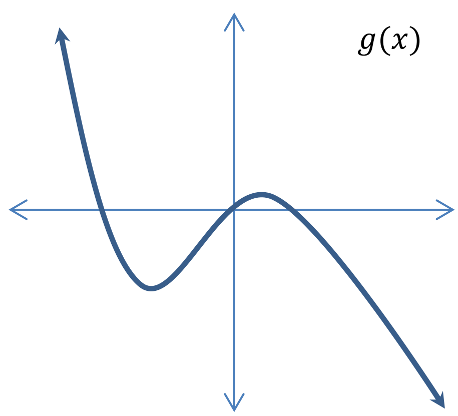
x					
$f^{-1}(x)$					

(b) Is the inverse a function? How can you tell?

One-to-one functions are functions that have an inverse that is also a function. You can tell graphically if a function is one-to-one without graphing the inverse – it must pass the horizontal line test.

Consider the following functions:





(a) Is either function one-to-one?

(b) Draw the inverse function on each graph.

Find the inverse of the following and give their domain.

1. $g(x) = \frac{3}{x-1}$

2. $f(x) = x^2 - 1$

3. $f(x) = \sqrt[3]{\frac{x-7}{3}}$

4. $h(x) = log_3(x^2 + 2)$

We'll revisit inverse functions in a moment.

2. Combining and Compositions

I. Basic Combinations and Compositions

For 1 – 8 below, use the following information. Give the domain of each.

$\boldsymbol{f(x) = -2x^2 - 2x + 1}$ and $\boldsymbol{g(x) = x + 1}$

1. Find $f(x) + g(x)$

2. Find $f(x) - g(x)$

3. Find $f(x) \cdot g(x)$

4. Find $\frac{f(x)}{g(x)}$

5. Find $f(g(x))$

6. Find $g(f(x))$

7. Find $g(g(x))$

8. Find $f(g(-1))$

II. Function Compositions Using Sets of Points

f={(–2, 3), (–1, 1), (0, 0), (1, –1), (2, –3)}

g={(–3, 1), (–1, –2), (0, 2), (2, 2), (3, 1)}

Using the information above, find the following:

1. f(1)   2. g(-1)   3. g(f(1))

4. f(g(0))   5. f(g(-1))   6. g(f(-1))

Given two functions, $f(x)$ and $g(x)$, evaluate the following given that:

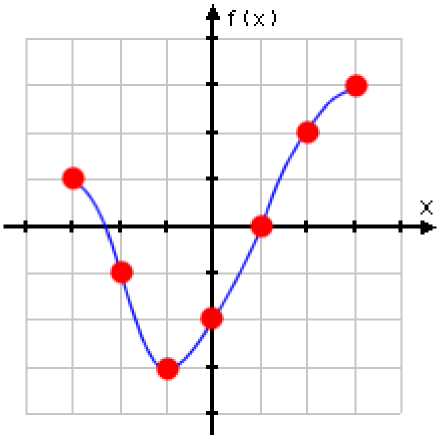
For $f(x)$: $f(-2) = 5,\ f(-1) = 2,\ f(0) = -1,\ f(1) = -3,\ f(2) = 3$

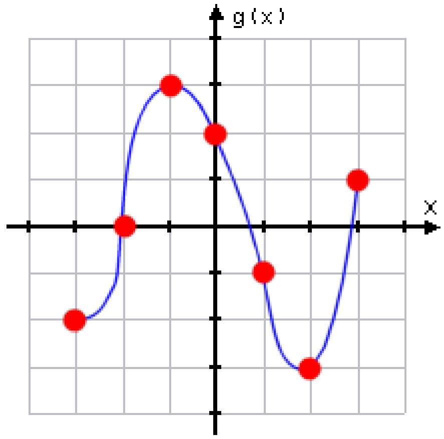
For $g(x)$: $g(-2) = -1,\ g(-1) = -2,\ g(0) = 0,\ g(1) = 2,\ g(2) = 3$

1. $(f + f)(0)$ 2. $(f - g)(-2)$ 3. $f(g(-1))$

4. $g(f(0))$ 5. $f\big(g(1)\big) - g(f(-1))$ 6. $f^{-1}(f(2))$

III. Function Compositions Using Graphs

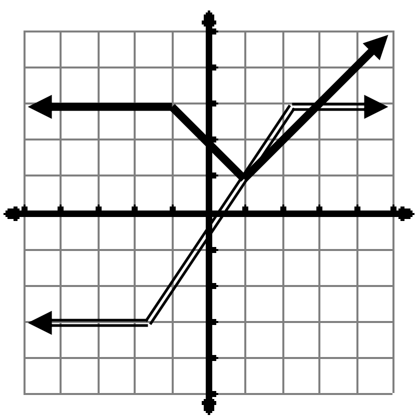




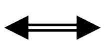
Given $f(x)$ and $g(x)$ as shown in the graphs above, find the following:

1. $f(g(1))$
2. $g(f(-2))$
3. $f(f(0))$

4. $f^{-1}(g(2))$
5. $g(f^{-1}(2))$
6. $g^{-1}(g^{-1}(1))$



$f(x) =$ 

$g(x) =$ 

Use the graph to the left for the following:

1. $(f + f)(2)$
2. $f(g(1))$
3. $g(f(-1))$
4. $f^{-1}(f^{-1}(1))$
5. $(f + g)(3)$
6. $f(4) - g(-1)$
7. $(f - g)(-3) + f(f(2))$

IV. Composition Extensions and Applications

1. Show that $f(x) = 2x^2 - 1$ and $g(x) = \sqrt{\frac{x+1}{2}}$ are inverse functions using compositions.

2. Verify that $f(x) = \sqrt{\frac{x-2}{3}}$ and $g(x) = 3x^2 + 2$ are inverses.

3. Given $f(x) = \sqrt{x}$ and $g(x) = x - 2$, find the domains of $f(g(x))$ and $g(f(x))$.

4. Given $h(x) = (x+1)^2 + 2(x+1) - 3$, determine two functions $f(x)$ and $g(x)$ which, when composed, generate $h(x)$.

5. Given $h(x) = \sqrt{4x+1}$, determine two functions $f(x)$ and $g(x)$ which, when composed, generate $h(x)$.

6. Given $h(x) = \frac{(3x-1)^2}{5}$, determine two functions $f(x)$ and $g(x)$ which, when composed, generate $h(x)$.

7. You work forty hours a week at a furniture store. You receive a $220 weekly salary, plus a 3% commission on sales over $5000. Assume that you sell enough this week to get the commission. Given the functions $f(x) = 0.03x$ and $g(x) = x - 5000$, which composed function, $f(g(x))$ or $g(f(x))$, represents your commission?

8. You make a purchase at a local hardware store, but what you've bought is too big to take home in your car. For a small fee, you arrange to have the hardware store deliver your purchase for you. You pay for your purchase, plus the sales taxes, plus the fee. The taxes are 7.5% and the fee is $20.

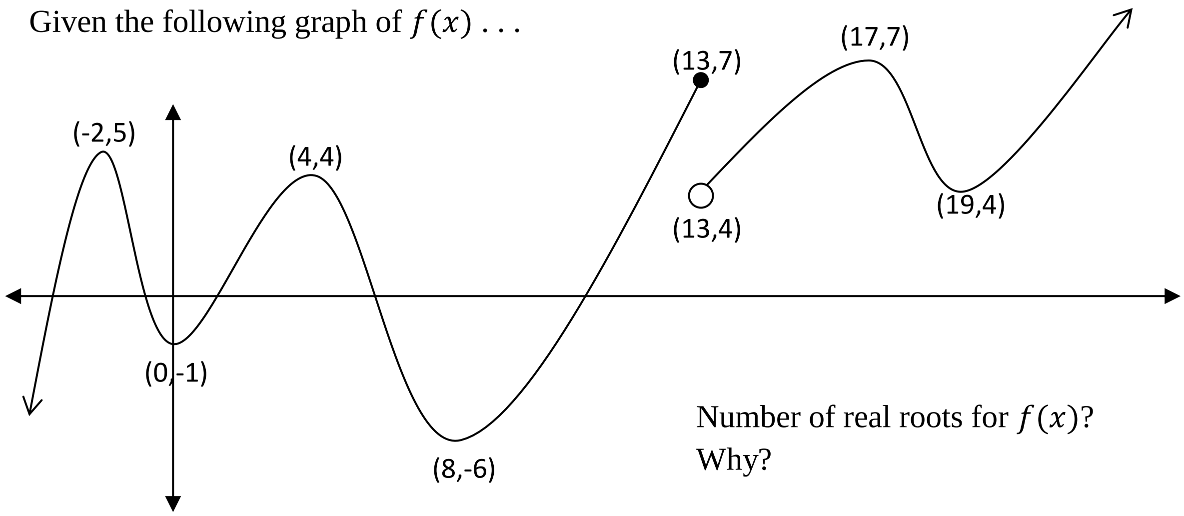
(a) Write a function $t(x)$ for the total, after taxes, on purchase amount x. Write another function $f(x)$ for the total, including the delivery fee, on purchase amount x.

(b) Calculate and interpret $f(t(x))$ and $t(f(x))$. Which results in a lower cost to you?

(c) Suppose taxes, by law, are not to be charged on delivery fees. Which composite function must then be used?

3. Describing Functions

Given the following graph of $f(x)$ . . .



Number of real roots for $f(x)$?
Why?

Complete the table below.

x	f(x)	f(x)+2	f(x)-1	-f(x)	2f(x)	$-\frac{1}{2}$f(x)	| f(x) |
-2							
0							
4							
8							
13							
17							
19							

- How can you tell that $f(x)$ is a function? Explain.
- What is the domain and range of $f(x)$?
- Is $f(x)$ a continuous function? How can you tell?
- What is the end behavior of $f(x)$?
- Give the intervals of increase and decrease and local maximums and minimums for $f(x)$.
- What is $f(2x)$ if $x = 4$?

Say Yes to the Dress! ...or, A Model Marriage

Back to Task Table

Mathematical Goals

- Students will use what they already know about various types of functions to create and analyze models for realistic phenomena.

Georgia Standards of Excellence

Analyze functions using different representations.

MGSE9-12.F.IF.7b Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions.

MGSE9-12.F.IF.7c Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior.

MGSE9-12.F.IF.7d Graph rational functions, identifying zeros and asymptotes when suitable factorizations are available, and showing end behavior.

MGSE9-12.F.IF.7e Graph exponential and logarithmic functions, showing intercepts and end behavior, ~~and trigonometric functions, showing period, midline, and amplitude.~~

Build a function that models a relationship between two quantities.

MGSE9-12.F.BF.1 Write a function that describes a relationship between two quantities.

Apply geometric concepts in modeling situations.

MGSE9-12.G.MG.3 Apply geometric methods to solve design problems

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Introduction

"A Model Marriage" is an appropriate name for this task because students will use a variety of models to represent the scenarios described in the problem, and sometimes will use more than one type of model or modeling logic to solve the same problem. The task puts the responsibility on the student to take learning that should have occurred previously in the course (trigonometric functions, polynomials, rational functions, etc.) and apply them to a modeling situation, sometimes without being prompted by anything other than the behavior of the data or a loose description of a context.

While the standards for creating equations are not included for this task, there are several points where students are asked to synthesize the relationship between variables and, with their background in various functions, create an appropriate equation for the data. Because of the nature of some parts of this task, it could be a good small-group activity and would be an effective culminating task in modeling with various functions.

Technology is an important part of this task, and students should be encouraged to use graphing calculators to test their assumptions about a scenario. The task also provides further learning opportunities in using technology efficiently to model phenomena (i.e., sinusoidal regression).

Materials

- Pencil
- Handout
- Calculator

Say Yes to the Dress! ...or, A Model Marriage

1. Dress Sales

Mary Knupp-Shull runs a very posh (and therefore, very expensive) wedding dress boutique in Atlanta's Buckhead neighborhood. A lot of people think that Mary's life is all fabulous dresses and glamorous customers, but what makes Mary very successful is that she keeps a close watch on her sales and inventory.

Mary does not keep dresses in the store long, and she usually doesn't do repeat orders on dresses because she likes to be on the cutting edge. The average time it takes for Mary to sell out of a dress is 13 months. The sales of one wedding dress model designed by Fabio Fabulisi is modeled by the function

$$f(x) = -0.000795x^4 + 0.0256x^3 - 0.2834x^2 + 1.161x,$$

where x is the number of months since the release of the dress and $f(x)$ is the number of dresses sold in multiples of ten.

(a) About how long did the dress stay on the shelves? How did the sales of this dress perform compared to the average dress model in Mary's boutique?

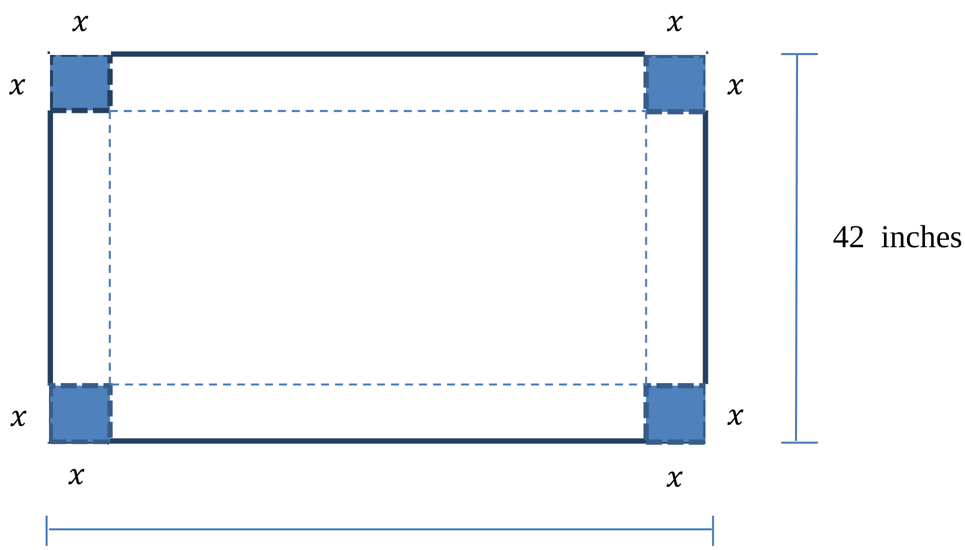
The goal here is to find the rightmost zero of the polynomial, which occurs between month 14 and 15 (approximately 14.734 months). This dress model took a bit longer to sell out than the average dress model at the boutique.

(b) What was the most dresses sold in any one month?

The most dresses sold were $\mathbf{1.57(10) \approx 157}$ during the 3rd month.

2. Making Boxes

Mary needs open-topped boxes to store her excess inventory at year's end. Mary purchases large rectangles of thick cardboard with a length of 78 inches and width of 42 inches to make the boxes. Mary is interested in maximizing the volume of the boxes and wants to know what size squares to cut out at each corner of the cardboard (which will allow the corners to be folded up to form the box) in order to do this.



(a) Volume is a three-dimensional measure. What is the third dimension that the value x represents?

***x* represents the height of the box**

(b) Using the table below, choose five values of x and find the corresponding volumes.

Answers vary

x	**Length**	**Width**	**Volume**

You tested several different values of x above, and calculated five different volumes. There is a way to guarantee that you use dimensions that will maximize volume, and now we're going to work through that process.

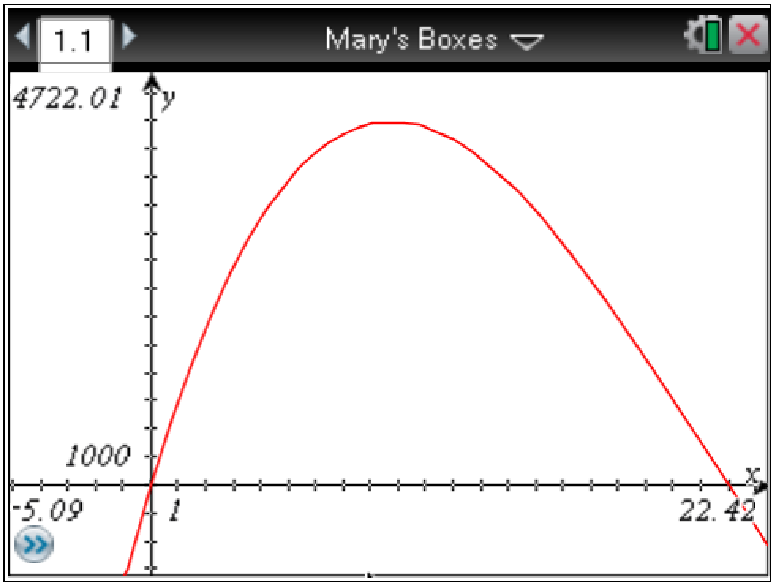
(c) Write an equation for volume in terms of the three dimensions of the box.

$$V = x(78 - 2x)(42 - 2x)$$

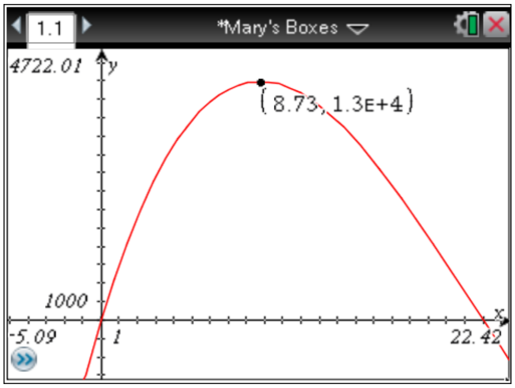
$$V = x(3276 - 240x + 4x^2)$$

$$V = 3276x - 240x^2 + 4x^3$$

(d) Graph the equation from part (c).



(e) From your graph, what are the values of the three dimensions that maximize the volume of the box? What is the maximum volume of the box?



From the graph, it appears that the maximum occurs at approximately $(\mathbf{8.73}, \mathbf{13000})$, so the maximum volume would be 13000 cubic inches with a height of 8.73 inches, a length of 60.54 inches, and a width of 24.54 inches.

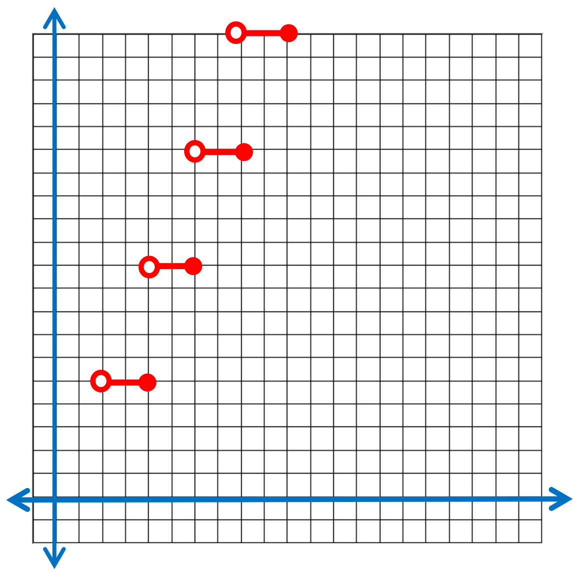
3. Mary's Money

Mary has a whole team of bridal consultants who help customers pick out the perfect dress. Some customers find the perfect dress quickly, and some have to spend the entire day. Because this is such an involved process, Mary charges for the use of a consultant on an hourly scale. The first hour is free, and every hour after that is $25.00 per hour. A customer's time is rounded up to the nearest whole hour.

(a) Graph the function that represents the fee structure for Mary's bridal consultants.

Consultant Fees

y-scale: 1 unit = $5



Appointment Time

x-scale: 2 units = 1 hour

(b) How much would a customer be charged if she stayed

(i) 59 minutes? ***$0***

(ii) 61 minutes? ***$25***

(iii) 180 minutes? ***$50***

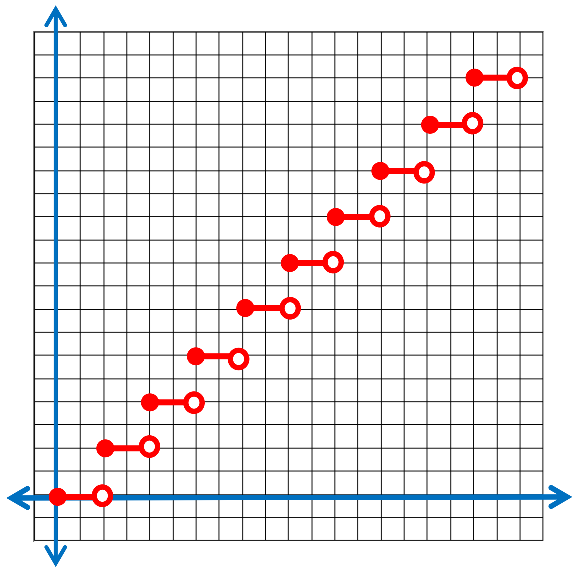
(iv) 493 minutes? ***$200***

Mary's tailoring department also has a fee schedule where labor for tailors is charged at $32.50 per hour, and the time taken on a dress is always rounded down to the next whole hour.

(c) Graph the function that represents the fee structure for Mary's tailors.

Tailoring Fees

y-scale: 1 unit = $16.25



Alterations Time

x-scale: 2 units = 1 hour

(d) How much would a customer be charged if the tailoring for her dress took

(i) 30 minutes? ***$0***

(ii) 60 minutes? ***$32.50***

(iii) 119 minutes? ***$32.50***

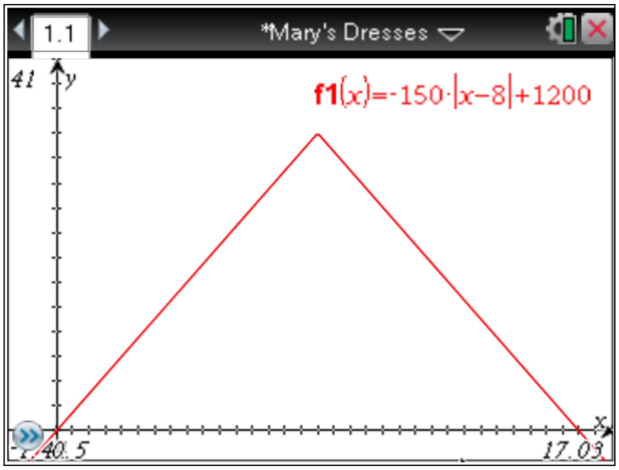
(iv) 121 minutes? ***$65.00***

Mary has found that, to maximize her profits, she should sell 8 dresses per day for an expected daily profit of $1200. Each additional dress that she either sells or does not sell costs her $150 per dress. Mary will never schedule more than 12 appointments in a single day.

(e) What is the domain and range of the function that represents Mary's daily profits?

The domain of this function is $[0, 12]$ ***and the range is*** $[0, 1200]$**.**

(f) Graph the function that represents Mary's daily profits.



(g) Is this function a polynomial? Why or why not?

No; polynomial functions should produce smooth, continuous graphs with no sharp points or discontinuities.

(h) Write a single equation for this function.

Hopefully, students can see from the graph that, due to the shape and symmetry, this is an absolute value function. Also, the domain restriction cannot be forgotten. Therefore, $\boldsymbol{f(x) = -150|x-8| + 1200 \text{ if } 0 \le x \le 12.}$

(i) Write a piecewise equation for this function.

$$f(x) = \begin{cases} 150x \text{ if } 0 \le x \le 8 \\ 2400 - 150x \text{ if } 8 < x \le 12 \end{cases}$$

4. Daisy Mae's Wedding

Daisy Mae has purchased a beautiful dress from Mary's boutique and is now getting ready for her wedding day. Daisy Mae has figured out that the "sunk" cost of the wedding reception (including the dress, venue, permits, food and drink, etc.) is $24,000. Daisy Mae also knows that for every 30 minutes that her wedding reception lasts, the variable costs increase by $150.

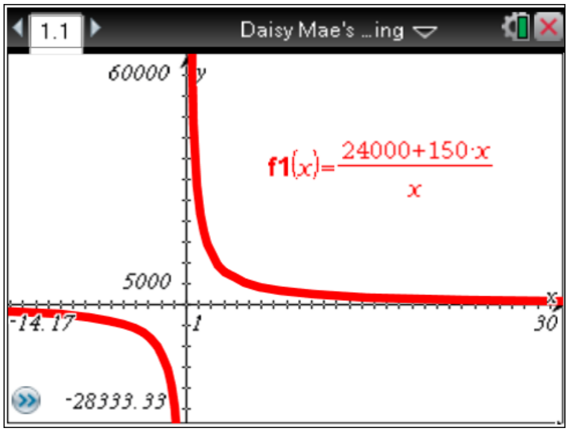
(a) What is the cost per half-hour of Daisy Mae's wedding reception if it lasts 3 hours?

$$\frac{24{,}000+150(6)}{6} \approx \$4150$$ ***per hour since there are 6 half-hour periods in 3 hours.***

(b) Develop a function that gives the cost per half-hour of the wedding reception as a function of the number of half-hours that the wedding reception lasts.

$$\mathbf{f(h) = \frac{24{,}000+150h}{h}}$$ **where h = number of half-hours of the wedding reception.**

(c) Sketch a graph of this function. Give any asymptotes from the graph.



Vertical Asymptotes: $\boldsymbol{h = 0}$

End-Behavior Asymptotes: $\boldsymbol{f(h) = 0}$

(d) Explain the meaning of the horizontal asymptote in terms of the wedding reception.

The horizontal asymptote exists because as the number of half-hours (h) continues to increase, the sum of the sunk cost and variable cost is being divided by a larger and larger value (while only adding a multiple of 150 to the numerator), eventually (if the wedding reception were to last for weeks or months) approaching an average cost per half-hour of $0. Of course, this isn't practical for this situation, but it is important for students to see that the average cost per half-hour does decline rapidly, and of course, it is also important for them to understand the practical need for a domain restriction for this rational function.

Say Yes to the Dress! ...or, A Model Marriage

1. Dress Sales

Mary Knupp-Shull runs a very posh (and therefore, very expensive) wedding dress boutique in Atlanta's Buckhead neighborhood. A lot of people think that Mary's life is all fabulous dresses and glamorous customers, but what makes Mary very successful is that she keeps a close watch on her sales and inventory.

Mary does not keep dresses in the store long, and she usually doesn't do repeat orders on dresses because she likes to be on the cutting edge. The average time it takes for Mary to sell out of a dress is 13 months. The sales of one wedding dress model designed by Fabio Fabulisi is modeled by the function

$$f(x) = -0.000795x^4 + 0.0256x^3 - 0.2834x^2 + 1.161x,$$

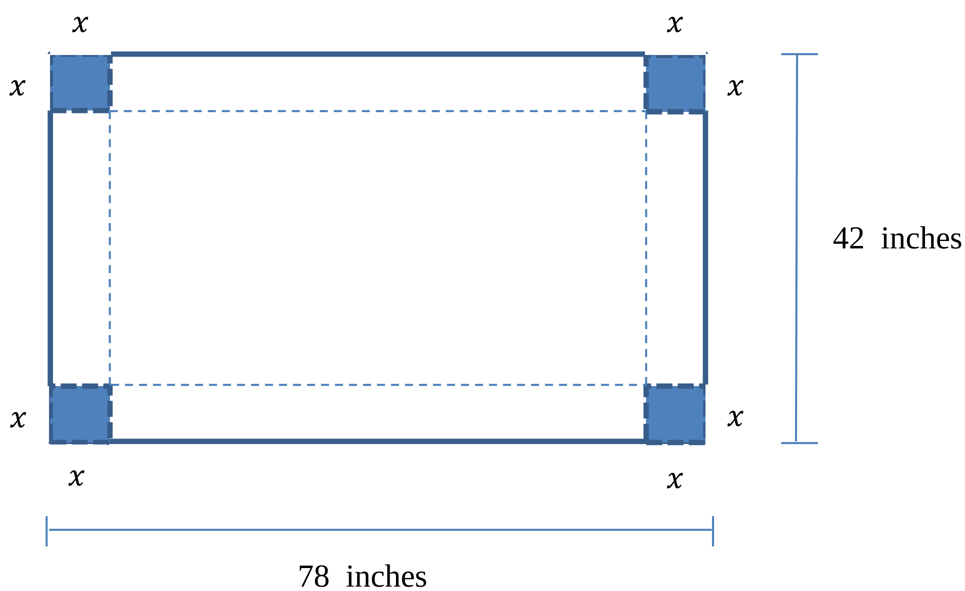
where x is the number of months since the release of the dress and $f(x)$ is the number of dresses sold in multiples of ten.

(a) About how long did the dress stay on the shelves? How did the sales of this dress perform compared to the average dress model in Mary's boutique?

(b) What was the most dresses sold in any one month?

2. Making Boxes

Mary needs open-topped boxes to store her excess inventory at year's end. Mary purchases large rectangles of thick cardboard with a length of 78 inches and width of 42 inches to make the boxes. Mary is interested in maximizing the volume of the boxes and wants to know what size squares to cut out at each corner of the cardboard (which will allow the corners to be folded up to form the box) in order to do this.



(a) Volume is a three-dimensional measure. What is the third dimension that the value x represents?

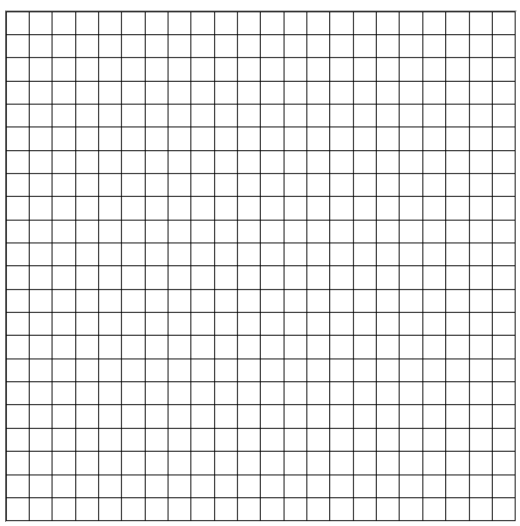
(b) Using the table below, choose five values of x and find the corresponding volumes.

x	**Length**	**Width**	**Volume**

You tested several different values of x above, and calculated five different volumes. There is a way to guarantee that you use dimensions that will maximize volume, and now we're going to work through that process.

(c) Write an equation for volume in terms of the three dimensions of the box.

(d) Graph the equation from part (c).



(e) From your graph, what are the values of the three dimensions that maximize the volume of the box? What is the maximum volume of the box?

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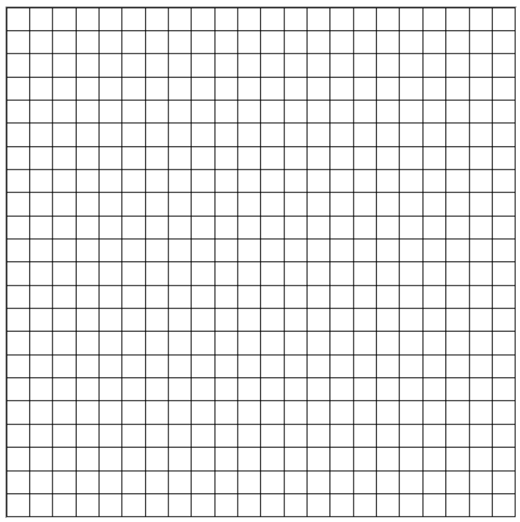
(a) Graph the function that represents the fee structure for Mary's bridal consultants.



(b) How much would a customer be charged if she stayed

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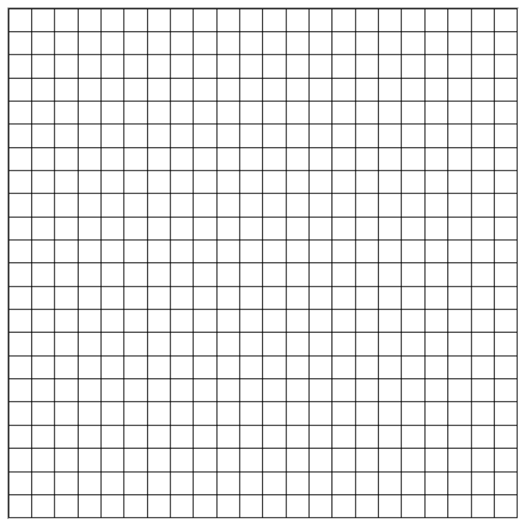

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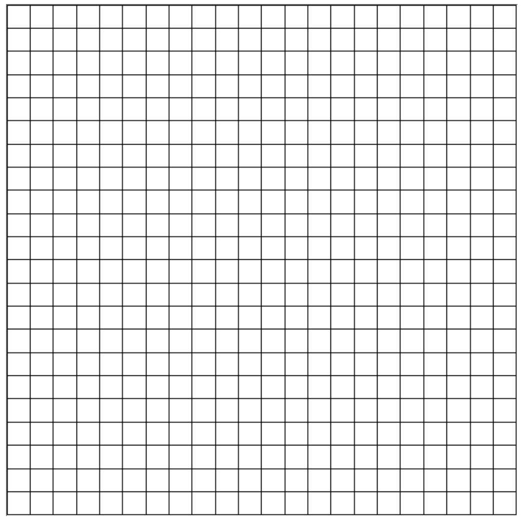

(g) Is this function a polynomial? Why or why not?

(h) Write a single equation for this function.

(i) Write a piecewise equation for this function.

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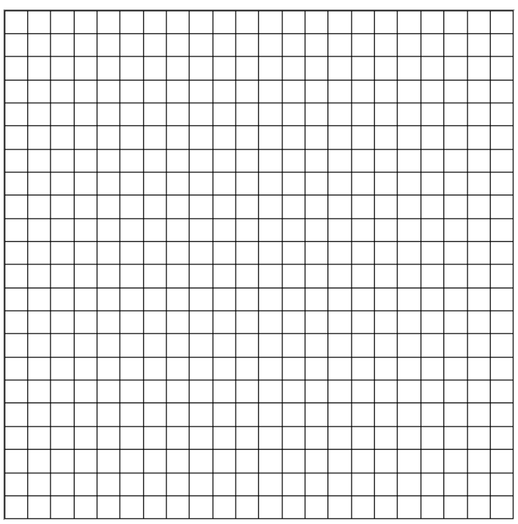

Vertical Asymptotes:

End-Behavior Asymptotes:

(d) Explain the meaning of the horizontal asymptote in terms of the wedding reception.