Georgia
Standards of Excellence
Curriculum Frameworks

Mathematics

Accelerated GSE Geometry B/Algebra II

Unit 2: Geometric and Algebraic Connections

Georgia Department of Education

Richard Woods, Georgia's School Superintendent

"Educating Georgia's Future"

Unit 2

Geometric and Algebraic Connections

Table of Contents

OVERVIEW

In this unit students will:

- prove the slope relationship that exists between parallel lines and between perpendicular lines and then use those relationships to write the equations of lines
- extend the Pythagorean Theorem to the coordinate plane
- develop and use the formulas for the distance between two points and for finding the point that partitions a line segment in a given ratio
- revisit definitions of polygons while using slope and distance on the coordinate plane
- use coordinate algebra to determine perimeter and area of defined figures
- use Algebra to model Geometric ideas
- spend time developing equations from geometric definition of circles
- address equations in standard and general forms
- graph by hand and by using graphing technology
- develop the idea of algebraic proof in conjunction with writing formal geometric proofs

Although the units in this instructional framework emphasize key standards and big ideas at specific times of the year, routine topics such as estimation, mental computation, and basic computation facts should be addressed on an ongoing basis. Ideas related to the eight practice standards should be addressed constantly as well. This unit provides much needed content information and excellent learning activities. However, the intent of the framework is not to provide a comprehensive resource for the implementation of all standards in the unit. A variety of resources should be utilized to supplement this unit. The tasks in this unit framework illustrate the types of learning activities that should be utilized from a variety of sources. To assure that this unit is taught with the appropriate emphasis, depth, and rigor, it is important that the **"Strategies for Teaching and Learning"** and the tasks listed under **"Evidence of Learning"** be reviewed early in the planning process.

STANDARDS ADDRESSED IN THIS UNIT

Mathematical standards are interwoven and should be addressed throughout the year in as many different units and activities as possible in order to emphasize the natural connections that exist among mathematical topics.

KEY STANDARDS

Translate between the geometric description and the equation for a conic section

> **MGSE9-12.G.GPE.1** Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.

Use coordinates to prove simple geometric theorems algebraically

MGSE9-12.G.GPE.4 Use coordinates to prove simple geometric theorems algebraically. *For example, prove or disprove that a figure defined by four given points in the coordinate plane is a rectangle; prove or disprove that the point (1, $\sqrt{3}$) lies on the circle centered at the origin and containing the point (0,2).*
(Focus on quadrilaterals, right triangles, and circles.)

MGSE9-12.G.GPE.5 Prove the slope criteria for parallel and perpendicular lines and use them to solve geometric problems (e.g., find the equation of a line parallel or perpendicular to a given line that passes through a given point).

MGSE9-12.G.GPE.6 Find the point on a directed line segment between two given points that partitions the segment in a given ratio.

MGSE9-12.G.GPE.7 Use coordinates to compute perimeters of polygons and areas of triangles and rectangles, e.g., using the distance formula.

<u>Apply geometric concepts in modeling situations</u>

MGSE9-12.G.MG.1 Use geometric shapes, their measures, and their properties to describe objects
(e.g., modeling a tree trunk or a human torso as a cylinder)

MGSE9-12.G.MG.2 Apply concepts of density based on area and volume in modeling situations
(e.g., persons per square mile, BTUs per cubic foot)

MGSE9-12.G.MG.3 Apply geometric methods to solve design problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios).

STANDARDS FOR MATHEMATICAL PRACTICE

Refer to the Comprehensive Course Overview for more detailed information about the Standards for Mathematical Practice.

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.

7. Look for and make use of structure.

8. Look for and express regularity in repeated reasoning.

ENDURING UNDERSTANDINGS

- Algebraic formulas can be used to find measures of distance on the coordinate plane.
- The coordinate plane allows precise communication about graphical representations.
- The coordinate plane permits use of algebraic methods to obtain geometric results.
- Derive the formula for a circle using the Pythagorean Theorem
- Apply algebraic formulas and ideas to geometric figures and definitions
- Model everyday objects using three dimensional shapes and describe the object using characteristics of the shape.
- Solve real world problems that can be modeled using density, area, and volume concepts.

ESSENTIAL QUESTIONS

- How can a line be partitioned?
- How can the distance between two points be determined?
- How are the slopes of lines used to determine if the lines are parallel, perpendicular, or neither?
- How do we write the equation of a line that goes through a given point and is parallel or perpendicular to another line?
- How can slope and the distance formula be used to determine properties of polygons and circles?
- How can slope and the distance formula be used to classify polygons?
- How do I apply what I have learned about coordinate geometry to a real–world situation?
- How can I use the Pythagorean Theorem to derive the equation of a circle?
- How are the graph of a circle and its equation related?
- How are the equation of a circle and its graph related?
- How can I prove properties of geometric figures algebraically?
- How can I minimize cost and maximize the volume of a topless box?

CONCEPTS AND SKILLS TO MAINTAIN

It is expected that students will have prior knowledge/experience related to the concepts and skills identified below. It may be necessary to pre–assess in order to determine if time needs to be spent on conceptual activities that help students develop a deeper understanding of these ideas.

- approximating radicals
- calculating slopes of lines
- graphing lines
- writing equations for lines

- number sense
- computation with whole numbers and decimals, including application of order of operations
- addition and subtraction of common fractions with like denominators
- applications of the Pythagorean Theorem
- graphing on a coordinate plane
- completing the square
- operations with radicals
- methods of proof

SELECTED TERMS AND SYMBOLS

The following terms and symbols are often misunderstood. These concepts are not an inclusive list and should not be taught in isolation. However, due to evidence of frequent difficulty and misunderstanding associated with these concepts, instructors should pay particular attention to them and how their students are able to explain and apply them.

The definitions below are for teacher reference only and are not to be memorized by the students. Students should explore these concepts using models and real life examples. Students should understand the concepts involved and be able to recognize and/or demonstrate them with words, models, pictures, or numbers.

The websites below are interactive and include a math glossary suitable for high school. **Note – At the elementary level, different sources use different definitions. Please preview any website for alignment to the definitions given in the frameworks.**

http://www.teachers.ash.org.au/jeather/maths/dictionary.html

This web site has activities to help students more fully understand and retain new vocabulary (i.e. the definition page for *dice* actually generates rolls of the dice and gives students an opportunity to add them).

http://intermath.coe.uga.edu/dictnary/homepg.asp

Definitions and activities for these and other terms can be found on the Intermath website because Intermath is geared towards middle and high school. Links to external sites are particularly useful.

- **Distance Formula:** $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

- **Formula for finding the point that partitions a directed segment AB at the ratio of *a* : *b* from A(x_1, y_1) to B(x_2, y_2):**

$$\left(x_1 + \frac{a}{a+b}(x_2 - x_1),\ \ y_1 + \frac{a}{a+b}(y_2 - y_1)\right)$$

or $$\left(\frac{a}{a+b}(x_2 - x_1) + x_1,\ \ \frac{a}{a+b}(y_2 - y_1) + y_1\right)$$

or $$\left(\frac{bx_1 + ax_2}{b+a},\ \ \frac{by_1 + ay_2}{b+a}\right)$$ **← weighted average approach**

- **Center of a Circle**: The point inside the circle that is the same distance from all of the points on the circle.
- **Circle:** The set of all points in a plane that are the same distance, called the radius, from a given point, called the center. Standard form: $(\mathrm{x}-\mathrm{h})^2+(\mathrm{y}-\mathrm{k})^2=\mathrm{r}^2$
- **Diameter**: The distance across a circle through its center. The line segment that includes the center and whose endpoints lie on the circle. **Directrix of a Parabola:** every point on a parabola is equidistant from a fixed point (focus) and a fixed line (**directrix**)
- **Pythagorean Theorem**: A theorem that states that in a right triangle, the square of the length of the hypotenuse equals the sum of the squares of the lengths of the legs.
- **Radius:** The distance from the center of a circle to any point on the circle. Also, the line segment that has the center of the circle as one endpoint and a point on the circle as the other endpoint.
- **Standard Form of a Circle:** $(x-h)^2+(y-k)^2=r^2$, where (h,k) is the center and r is the radius.

EVIDENCE OF LEARNING

At the conclusion of the unit, students should be able to:

- find the point that partitions a directed segment into a given ratio.
- determine if a given pair of lines are parallel, perpendicular, or neither.
- determine the equation of the line parallel or perpendicular to a given line and passing through a given point.
- use distance and slope concepts to prove geometric theorems algebraically.
- find perimeter of polygons and area of triangles and quadrilaterals.
- Write the equation for a circle given information such as a center, radius, point on the circle, etc.
- Prove simple geometric properties using coordinates.

TEACHER RESOURCES

The following pages include teacher resources that teachers may wish to use to supplement instruction.

- Web Resources
- Graphic Organizer: Partitioning a Directed Line Segment
- Compare / Contrast: Two Methods for Finding Distance

Web Resources

The following list is provided as a sample of available resources and is for informational purposes only. It is your responsibility to investigate them to determine their value and appropriateness for your district. GaDOE does not endorse or recommend the purchase of or use of any particular resource.

- Distance Formula Applet
  http://www.mathwarehouse.com/algebra/distance_formula/interactive–distance–formula.php
  This applet shows the distance formula in action based on different points on grid. This resource is helpful for an introduction on the distance formula.

- Quadrilaterals Overview
  http://www.cut–the–knot.org/Curriculum/Geometry/Quadrilaterals.shtml
  This page has a helpful overview of quadrilaterals and an applet that names a quadrilateral as you move its vertices. The page includes a flow chart of quadrilaterals with inclusive definition of trapezoid.

Graphic Organizer: Partitioning a Directed Line Segment

MGSE9–12.G.GPE.6

EQ: How do you partition a directed line segment?

Steps to Use:

1. Subtract the *x*–coordinates (point D – point C).
2. Change the ratio a:b to $\frac{a}{a+b}$.
3. Multiply the answers from step 1 and step 2.
4. Add the beginning *x* coordinate (of C) to step 3's result.

Problem: Given the points C (3, 4) and D(6, 10), find the coordinates of point P on a directed line segment $\overline{CD}$ that partitions $\overline{CD}$ in the ratio 1:2 (a:b).

Steps to Use:

1. Subtract the *y*–coordinates (point D – point C).
2. Change the ratio a:b to $\frac{a}{a+b}$.
3. Multiply the answers from step 1 and step 2.
4. Add the beginning *y* coordinate (of C) to step 3's result.

Solution: Point P (,)

How does the step by step process above relate to the portioning formula below?

$$\left((x_2 - x_1)\frac{a}{a+b} + x_1, (y_2 - y_1)\frac{a}{a+b} + y_1\right)$$

Compare / Contrast: Two Methods for Finding Distance

Focus Question: How does the Pythagorean Theorem relate to the distance formula?	
Use the Pythagorean Theorem to find the distance between (2, 7) and (–1, –4)	**Use the distance formula to find the distance between (2, 7) and (–1, –4)**

FORMATIVE ASSESSMENT LESSONS (FAL)

Formative Assessment Lessons are intended to support teachers in formative assessment. They reveal and develop students' understanding of key mathematical ideas and applications. These lessons enable teachers and students to monitor in more detail their progress towards the targets of the standards. They assess students' understanding of important concepts and problem solving performance, and help teachers and their students to work effectively together to move each student's mathematical reasoning forward.

More information on Formative Assessment Lessons may be found in the Comprehensive Course Overview.

SPOTLIGHT TASKS

A Spotlight Task has been added to each GSE mathematics unit in the Georgia resources for middle and high school. The Spotlight Tasks serve as exemplars for the use of the Standards for Mathematical Practice, appropriate unit–level Georgia Standards of Excellence, and research–based pedagogical strategies for instruction and engagement. Each task includes teacher commentary and support for classroom implementation. Some of the Spotlight Tasks are revisions of existing Georgia tasks and some are newly created. Additionally, some of the Spotlight Tasks are 3–Act Tasks based on 3–Act Problems from Dan Meyer and Problem–Based Learning from Robert Kaplinsky.

3–ACT TASKS

A Three–Act Task is a whole group mathematics task consisting of 3 distinct parts: an engaging and perplexing Act One, an information and solution seeking Act Two, and a solution discussion and solution revealing Act Three.

More information along with guidelines for 3–Act Tasks may be found in the Comprehensive Course Overview.

TASKS

The following tasks represent the level of depth, rigor, and complexity expected of all Geometry students. These tasks, or tasks of similar depth and rigor, should be used to demonstrate evidence of learning. It is important that all elements of a task be addressed throughout the learning process so that students understand what is expected of them. While some tasks are identified as a performance task, they may also be used for teaching and learning (learning/scaffolding task).

Task Name	Task Type Grouping Strategy	Content Addressed	Standards	SMP's Addressed
Analyzing the Pentagon (Spotlight Task) *30–45 minutes*	Discovery Task *Partner/ Individual*	• Investigating area, perimeter and other properties of polygons	G.GPE.4, 7	1,5
New York City *90–120 minutes*	Learning Task *Partner / Small Group*	• Partition a line segment into a given ratio.	G.GPE.6	1,2,4
Slopes of Special Pairs of Lines *90–120 minutes*	Discovery Task *Partner / Individual*	• Show that the slopes of parallel lines are the same. • Show that the slopes of perpendicular lines are opposite reciprocals. • Given the equation of a line and a point not on the line, find the equation of the line that passes through the point and is parallel/perpendicular to the given line.	G.GPE.5	2,5
Geometric Properties in the Plane *90–120 minutes*	Performance Task *Partner Task*	• Use coordinates, slope relationships, and distance formula to prove simple geometric theorems algebraically. • Compute the perimeters of polygons using the coordinates of the vertices and the distance formula. • Find the areas of rectangles and triangles using the coordinates of the vertices and the distance formula.	G.GPE.4, 7	6,7

Equations of Parallel & Perpendicular Lines (FAL) *90–120 minutes* PDF	Formative Assessment Lesson *Individual / Small Group*	• Use equations of parallel and perpendicular lines to form geometric figures.	G.GPE.4, 5, 6, 7	1,3,7
Square *20–30 minutes* PDF	Short Cycle Task *Individual / Small Group*	• Use slope and length to determine whether a figure with given vertices is a square.	G.GPE.4, 5, 6, 7	3,7
Proving Quadrilaterals in a Coordinate Plane	Performance Task *Individual / Partner*	• Prove theorems pertaining to lines and angles. • Prove theorems pertaining to triangles. • Prove theorems pertaining to parallelograms.	G.CO.11 G.GPE.4	1,2,3,4,5,6, 7,8
Euler's Village *2 – 3 hours*	Culminating Task *Small Group / Partner*	• Use coordinates, slope relationships, and distance formula to prove simple geometric theorems algebraically. • Compute the perimeters of polygons using the coordinates of the vertices and the distance formula. • Find the areas of rectangles and triangles using the coordinates of the vertices and the distance formula. • Given the equation of a line and a point not on the line, find the equation of the line that passes through the point and is parallel/perpendicular to the given line.	G.GPE.4, 5, 6, 7	4,6
Euler's Village is a culminating task of the algebraic and geometric connections on the coordinate				

plane. This task ties the concepts of distance, midpoint, and slope together to verify relationships with geometric figures such as triangles and quadrilaterals.

Rolling Cups (Spotlight Task)	Performance Task *Large Group/Partner Task*	• Introducing the second part of the unit along with choosing appropriate mathematics to solve a non-routine problem	G.MG.1 G.MG.3	1, 2, 3, 4
Deriving the General Equation of Circles	Performance Task *Large Group/Partner Task*	• Generalizing the formula for the equation of a circle.	G.GPE.1	2, 3, 7, 8
Equations of Circles – 1 (FAL)	Formative Assessment Lesson	• Translate between geometric features of a circle and its equation.	G.GPE.1	1, 5, 7
Equations of Circles – 2 (FAL)	Formative Assessment Lesson	• Translate between a circle's equation and its geometric features.	G.GPE.1	1, 7
Converting Standard to General Form	Learning Task *Individual/Partner Task*	• Algebraic manipulations necessary to change an equation from standard form to general form.	G.GPE.1	2, 3, 7, 8
Completing the Square in a Circle?	Learning Task *Individual/Partner Task*	• Completing the square to find the center and radius of a given circle.	G.GPE.1	2, 3, 7, 8
Graphing Circles on a Graphing Calculator	Extension Task *Partner/Small Group Task*	• Using technology to graph a circle.	G.GPE.1	2, 3, 7, 8
Radio Station Listening Areas	Performance Task *Individual/Partner Task*	• Real-world applications of writing the equation of a circle.	G.GPE.1	2, 3, 7, 8
Algebraic Proof	Learning Task *Individual/Partner Task*	• Using coordinates to prove simple geometric theorems algebraically.	G.GPE.4	2, 3, 7
A Day at the Beach	Performance Task *Individual/Partner Task*	• To visualize and identify the dimensions of geometric shapes • To determine the volume relationships of cylinders, pyramids, cones, and spheres • To justify geometric arguments	G.MG.1 G.MG.3	1, 4, 6

How many cells are in the human body?	Performance Task *Partner Task*	• To apply the concepts of mass, volume, and density in a real-world context.	G.MG.1 G.MG.2	2,3,5,6
Maximize Volume	Learning Task *Individual/Partner Task*	• Maximize Volume	G.MG.3	2-3, 7-8
Culminating Task: Dr. Cone's New House	Performance Task *Individual/Partner Task*	• Write the equations of circles and parabolas and use coordinates to prove simple geometric theorems algebraically.	G.GPE.1 G.GPE.4	2, 3, 7, 8
Dr. Cone's New House is a culminating task requiring understanding and manipulations of circles and parabolas. This task incorporates all understanding of circles and writing equations of circles to design a new home.				

Analyzing the Pentagon (Spotlight Task)

Back to Task Table

This spotlight task follows the 3 Act–Math task format originally developed by Dan Meyer. More information on these type tasks may be found at http://blog.mrmeyer.com/category/3acts/

Georgia Standards of Excellence

MGSE9–12.G.GPE.4 Use coordinates to prove simple geometric theorems algebraically. *For example, prove or disprove that a figure defined by four given points in the coordinate plane is a rectangle; prove or disprove that the point (1, $\sqrt{3}$) lies on the circle centered at the origin and containing the point (0,2).*
(Focus on quadrilaterals, right triangles, and circles.)

This standard is addressed through students assigning coordinates to key points on the picture and then using those coordinates to answer questions that they develop themselves. One example might be proving that all regular polygons with the same number of sides are similar. Students will use coordinates to explore properties and characteristics of polygons.

MGSE9–12.G.GPE.7 Use coordinates to compute perimeters of polygons and areas of triangles and rectangles, e.g., using the distance formula.
This standard is addressed through use of the distance formula to calculate the area and perimeter of the pentagon. Several other applications are possible, depending on the interests and questions of the students.

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them. In this task, students will formulate their own problem to solve. They will need to decide on a reasonable question to answer using the mathematics they have available to them. Several possible investigations could require extend effort to solve. Students will need to persevere through the problem solving process in order to arrive at a solution.

5. Use appropriate tools strategically. Students will need to select appropriate tools (graph paper, calculator, formulas) in order to be successful at this task.

ESSENTIAL QUESTIONS

- How do I construct a mathematical question that can be answered?
- How do I calculate the distance between two points?
- How can I calculate the area and perimeter of a figure given only coordinates of vertices?

MATERIALS REQUIRED

- Copy of student handout (picture of the Pentagon)
- Graph paper

TIME NEEDED

- 30–45 minutes based on the depth of investigation

More information along with guidelines for 3–Act Tasks may be found in the Comprehensive Course Overview.

Act One Description (Dan Meyer http://blog.mrmeyer.com/2011/the–three–acts–of–a–mathematical–story/)
"Introduce the central conflict of your story/task clearly, visually, viscerally, using as few words as possible."

Act One:
Present the students with the aerial photograph of the Pentagon. Pose the question: "What do you wonder?"
It should be noted here that students will likely come up with many questions that are non–mathematical in nature. The teacher's task is to help them to refine their questions so that they can be answered using the mathematics that they know or can discover.
Here are some questions that students developed based on the picture:

- What is the perimeter of the building?
- What is the area of the building?
- What is the area of the courtyard?
- Are the outer pentagon and inner pentagon similar?

In order to introduce a more real–world feel to the questions, they could be modified to the following:

- How long would it take an average person to walk around the exterior of the Pentagon?
- How many acres does the Pentagon cover?
- How many football fields could fit in the courtyard?
- How many times bigger is the outer pentagon than the inner pentagon?

This is also a great opportunity to revisit unit conversions and other standards from Unit 1.
The important part is to honor the students' curiosity. They will engage with the activity more if it is their own questions that they are answering. If a student develops a question that is outside the scope of the course, be sure to honor that student's curiosity by pointing them in the right direction and encouraging them to continue on their own. Hopefully, students will begin the process of formulating mathematical questions and then using mathematical models to answer the questions.
Here are some questions to help guide the discussion, but be sure not to give too much away. The goal is to have students formulate the questions and the methods to answer them.

- Can your question be answered using mathematics?
- How could you model an answer to your questions with an equation?
- Do you have all the information you need to answer your question?

Act Two Description (Dan Meyer http://blog.mrmeyer.com/2011/the–three–acts–of–a–mathematical–story/)
The protagonist/student overcomes obstacles, looks for resources, and develops new tools.

During Act Two, students will discuss the question in Act One and decide on the facts that are needed to answer the question. Students will also look for formulas and conversions that are needed to solve the problem. When students decide what they need to solve the problem, they should ask for the facts or use technology to find them.

Note: It is pivotal to the problem solving process that students decide what is needed without being given the information up front. Some groups might need scaffolds to guide them. The teacher should question groups who seem to be moving in the wrong direction or might not know where to begin.

The main content intended for use in this task is calculating areas and perimeters of polygons using the distance formula. Students may choose to try and measure the sides using a ruler. Instead, suggest using graph paper. Students were introduced to the distance formula in middle school so it should not be too much of a jump to apply the formula to the situation.

Students can be supplied with the following information when they ask for it:
- Typical graph paper is scaled 4 squares per inch.
- The scale of the photo is 1 inch = 263.14 feet
- 1 acre contains 43,560 square feet

Use your discretion on what other information students may look up on the internet. Calculating the area of a regular pentagon could be done using a formula, but resist the temptation to reduce it to that. Encourage the students to use other methods for calculating the area, such as decomposing the figure into triangles or trapezoids. This could even extend into students developing their own formula for the area of a polygons.

ACT 3
Students will compare and share solution strategies.
- Reveal the answer.
  - Each side of the Pentagon is 921 feet.
  - It covers 28.7 acres, and the interior courtyard is 5 acres
- Discuss the theoretical math versus the practical outcome.
- How close was your answer to the actual answer?
- What could account for the difference?
- Share student solution paths. Start with most common strategy.
- Revisit any initial student questions that weren't answered.

The answers that students come up with will vary based on their estimates for the coordinates of the vertices on the graph paper. Discuss the role of estimation and variation in their answers. The teacher also needs to be flexible and adapt the lesson to the curiosity of the class. Use this activity as a guide, but do not be afraid to deviate from it if the mathematics dictates that you do so.

The Sequel: "The goals of the sequel task are to a) challenge students who finished quickly so b) I can help students who need my help. It can't feel like punishment for good work. It can't seem like drudgery. It has to entice and activate the imagination." Dan Meyer
http://blog.mrmeyer.com/2013/teaching–with–three–act–tasks–act–three–sequel/
For a sequel, allow students to look up other aerial photographs of other famous buildings or landmarks. They could then proceed through a similar process. Google Earth or Google maps could be a good resource for this sequel.

Analyzing the Pentagon (Spotlight Task)
Image taken from www.googlemapsmania.blogsplot.com

What do you wonder?

New York City (Learning Task)

[Back to Task Table]

Introduction

This task provides a guided discovery of the procedure for partitioning a segment into a given ratio.

Mathematical Goals

- Find the point on a line segment that separates the segments into a given ratio.

Essential Questions

- How can a line be partitioned?

Georgia Standards of Excellence

MGSE9–12.G.GPE.6 Find the point on a directed line segment between two given points that partitions the segment in a given ratio.

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
Students must make sense of the layout of the town and represent it in a way that they can answer questions about the town.

2. Reason abstractly and quantitatively.
When partitioning pathways, students must apply their reasoning in both specific and general situations.

4. Model with mathematics.
Students must represent the problem situation mathematically and use their model to answer questions about the situation.

Background Knowledge

- Students understand fractions as representing part of a whole.
- Students recognize the relationship between ratios and fractions. e.g., breaking a segment into two pieces with a ratio of 3:5 means the two pieces are $^3/_8$ and $^5/_8$ of the whole.

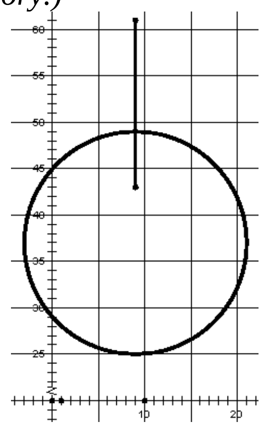
Common Misconceptions

- Students may have difficulty representing the situation graphically. Focus on the difference between Avenues and Streets in the problem, and relate this to (x, y) coordinates.
- Students may use the fraction $^3/_5$ instead of $^3/_8$ when partitioning into a ratio of 3:5. Remind students that fractions represent a part of the *whole*. 3 and 5 are both parts.
- Conversely, if the fraction "part / whole" (rather than the ratio "part : part") is given, then students don't need to change the denominator.

Materials
- Graph paper

Grouping
- Partner / small group

Differentiation
Extension:
- Emily (at work) and Gregory (at his hotel) want to walk to a location so that each person walks the same distance. The corner restaurant is halfway *between* them, but there are other locations that are also equidistant from each of them. Describe <u>all</u> points that are equidistant from Emily and Gregory. *(Solution: Points on the perpendicular bisector of the segment connecting Emily and Gregory.)*
- Describe all locations that are twice [half, three times, etc.] as far from Emily as they are from Gregory.
  Solution will be a circle centered at (9, b) for some value of b. The solution here shows all points that are <u>twice</u> as far from Emily as they are from Gregory.



Intervention:
- Students can highlight the right triangles they form to help them focus on them only. This helps motivate the idea of using the Pythagorean Theorem for distance, and the idea of partitioning horizontal and vertical components separately for partitioning.
- The partitioning formula may be challenging to remember and/or apply for many students. Encourage students to think of partitioning as a multi–step process rather than a complicated formula.

Formative Assessment Questions
- Describe how you would find the point Z that partitions the directed line segment XY in the ratio of 4:3 using the points $X(–5, 7)$ and $Y(6, 13)$.

New York City – Teacher Notes

The streets of New York City are laid out in a rectangular pattern, with all blocks approximately square and approximately the same size. **Avenues** run in a north–south direction, and the numbers increase as you move west. **Streets** run in an east–west direction, and the numbers increase as you move north.

Emily works at a building located on the corner of 9th Avenue and 61st Street in New York City. Her brother, Gregory, is in town on business. He is staying at a hotel at the corner of 9th Avenue and 43rd Street.

1. Gregory called Emily at work, and they agree to meet for lunch. They agree to meet at a corner half way between Emily's work and Gregory's hotel. Then Gregory's business meeting ends early so he decides to walk to the building where Emily works.

   a. How many blocks does he have to walk? Justify your answer using a diagram on grid paper.

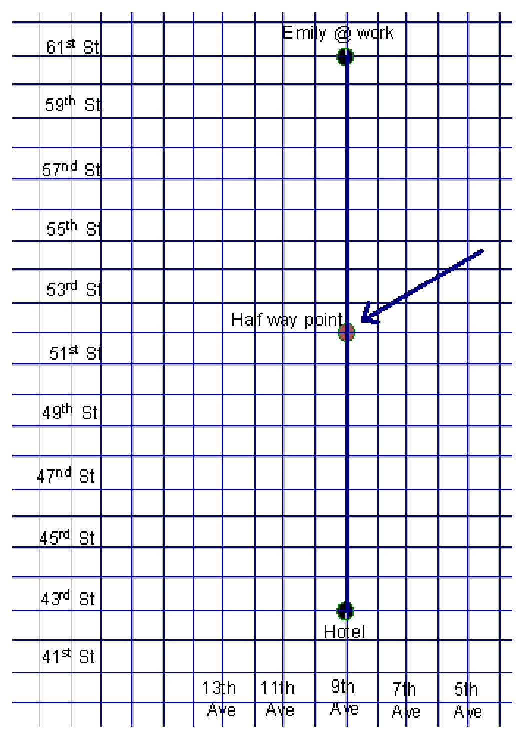
   b. After meeting Emily's coworkers, they walk back toward the corner restaurant. How many blocks must they walk? Justify your answer using your diagram.\

Comments:

Watch students carefully as they begin drawing a picture for this. Make sure they understand what is meant by North–South and East–West Avenues and Streets. It may be necessary to address this as a class so students understand the lay–out of the city streets.

Solutions:

The locations are 18 blocks apart. If each person walks 9 blocks they can meet at 9th Avenue and 52nd Street.



2. After lunch, Emily has the afternoon off so she walks back to the hotel with Gregory before turning to go to her apartment. Her apartment is three blocks north and four blocks west of the hotel.

   a. At what intersection is her apartment building located?

   Solution:
   Her apartment is located at 13^{th} Avenue and 46^{th} Street

   b. How many blocks south of the restaurant will they walk before Emily turns to go to her apartment?

   Solution:
   They will walk 6 blocks south of the restaurant.

   c. When Emily turns, what fraction of the distance from the restaurant to the hotel have the two of them walked? Express this fraction as a ratio of distance walked to distance remaining for Gregory.

   Solution:
   They will walk 6 blocks south of the restaurant which is $\frac{6}{9}$ or $\frac{2}{3}$ of the total distance Gregory will walk. This is a 6:3 or 2:1 ratio.

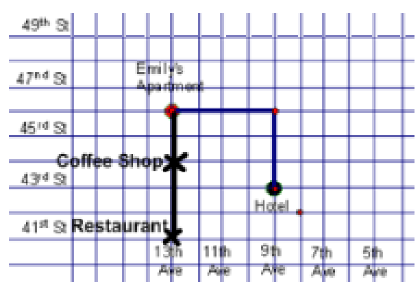
3. Gregory and Emily are going to meet for dinner at a restaurant 5 blocks south of her apartment.

   a. At which intersection is the restaurant located?

   Solution:
   The restaurant is at the corner of 13^{th} avenue and 41^{st} street.

   b. After dinner, they walk back towards her apartment, but stop at a coffee shop that is located three–fifths of the distance to the apartment. What is the location of the coffee shop?

   Solution:
   Since the restaurant is 5 blocks south of the apartment, $\frac{3}{5}$ of the distance back to the apartment means they will walk from 41^{st} up to 44^{th}. The coffee shop is located at 13^{th} and 44^{th}.



By investigating the situations that follow, you will determine a procedure for finding a point that partitions a segment into a given ratio.

4. Here, you will find a point that partitions a directed line segment from *C*(4, 3) to *D*(10, 3) in a given ratio.

   Comments:
   The task begins with fractions of horizontal and vertical segments so that students can reason through the step, clarifying distinctions between fractions of the whole and ratios of parts as well as direction of the partition.

   a. Plot the points on a grid. (Notice that the points lie on the same horizontal line.) What is the distance between the points?
   Solution:
   Distance from C to D is |10 – 4| = 6

   b. Use the fraction of the total length of CD to determine the location of Point A which partitions the segment from C to D in a ratio of 5:1. What are the coordinates of A?
   Solution:
   $$\boldsymbol{A\left(\left(\frac{5}{6}\right)6+4, 3\right)=A(9,3)}$$

   c. Find point B that partitions a segment from C to D in a ratio of 1:2 by using the fraction of the total length of CD to determine the location of Point B. What are the coordinates of B?
   Solution:
   $$\boldsymbol{B\left(\left(\frac{1}{3}\right)6+4, 3\right)=B(6,3)}$$

5. Find the coordinates of Point X along the directed line segment YZ.

   a. If Y(4, 5) and Z(4, 10), find X so the ratio is of YX to XZ is 4:1.
   Solution:
   $$\boldsymbol{X\left(4, \left(\frac{4}{5}\right)5+5\right)=X(4,9)}$$

   b. If Y(4, 5) and Z(4, 10), find X so the ratio is of YX to XZ is 3:2.
   Solution:
   $$\boldsymbol{X\left(4, \left(\frac{3}{5}\right)5+5\right)=X(4,8)}$$

So far, the situations we have explored have been with directed line segments that were either horizontal or vertical. Use the situations below to determine how the procedure used for Questions 4 and 5 changes when the directed line segment has a defined, nonzero slope.

Comments:
Students must treat the x and y values separately. Encourage them to plot the points on a grid and construct the vertical and horizontal components.

6. Find the coordinates of Point A along a directed line segment from C(1, 1) to D(9, 5) so that A partitions CD in a ratio of 3:1. Since CD is neither horizontal nor vertical, the *x* and *y* coordinates have to be considered distinctly.

   a. Find the *x*–coordinate of A using the fraction of the horizontal component of the directed line segment (i.e., the horizontal distance between C and D).
   Solution:
   Horizontal distance |9 – 1| = 8

   b. Find the *y*–coordinate of A using the fraction of the vertical component of the directed line segment (i.e., the vertical distance between C and D).
   Solution:
   Vertical distance |5 – 1| = 4

   c. What are the coordinates of A?
   Solution:
   $A\left(\left(\frac{3}{4}\right)8+1,\left(\frac{3}{4}\right)4+1\right)=A(7,4)$

7. Find the coordinates of Point A along a directed line segment from C(3, 2) to D(5, 8) so that A partitions CD in a ratio of 1:1. Since CD is neither horizontal nor vertical, the *x* and *y* coordinates have to be considered distinctly.

   a. Find the *x*–coordinate of A using the fraction of the horizontal component of the directed line segment (i.e., the horizontal distance between C and D).
   Solution:
   Horizontal distance |5 – 3| = 2

   b. Find the *y*–coordinate of A using the fraction of the vertical component of the directed line segment (i.e., the vertical distance between C and D).
   Solution:
   Vertical distance |8 – 2| = 6

   c. What are the coordinates of A?
   Solution:
   $A\left(\left(\frac{1}{2}\right)2+3,\left(\frac{1}{2}\right)6+2\right)=A(4,5)$

8. Now try a few more …

a. Find Point Z that partitions the directed line segment XY in a ratio of 5:3.
X(–2, 6) and Y(–10, –2)
Solution:
Horizontal distance – 10 – –2 = –8 Vertical distance –2 – 6= –8
$$Z\left(\left(\frac{5}{8}\right)(-8)+-2,\left(\frac{5}{8}\right)(-8)+6\right)=Z(-7,1)$$

b. Find Point Z that partitions the directed line segment XY in a ratio of 2:3.
X(2,–4) and Y(7,2)
Solution:
Horizontal distance 7– 2=5 Vertical distance 2 – –4 = 6
$$Z\left(\left(\frac{2}{5}\right)5+2,\left(\frac{2}{5}\right)6+\ -4\right)=Z(4,-1\tfrac{3}{5})$$

c. Find Point Z that partitions the directed line segment YX in a ratio of 1:3.
X(–2, –4) and Y(–7, 5) (Note the direction change in this segment.)
Solution:
Horizontal distance –2—7=5 Vertical distance –4– 5=–9
$$Z\left(\left(\frac{1}{4}\right)5+\ -7,\left(\frac{1}{4}\right)(-9)+\ 5\right)=Z(-5\tfrac{3}{4},2\tfrac{3}{4})$$

Comments:

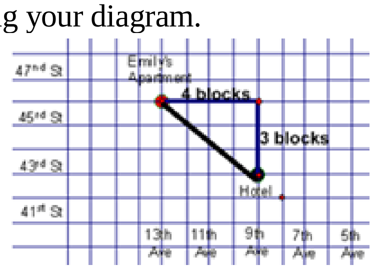
The next section of this task addresses using the Pythagorean Theorem to find the distance between two points in a coordinate system.

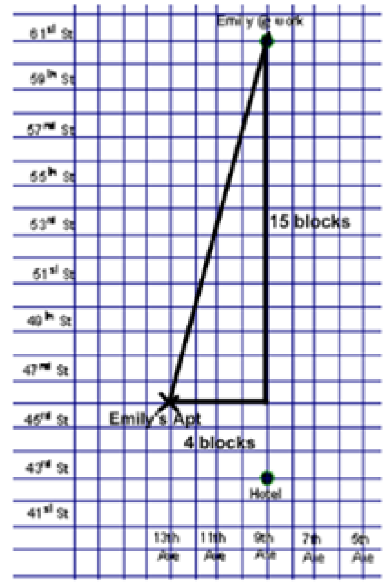
Back to Gregory and Emily….

9. When they finished their coffee, Gregory walked Emily back to her apartment, and then walked from there back to his hotel.

   a. How many blocks did he walk?
   Solution:
   Gregory walked 3 + 4 = 7 blocks.

   b. If Gregory had been able to walk the direct path to the hotel from Emily's apartment, how far would he have walked? Justify your answer using your diagram.
   Solution:
   Gregory walked 7 blocks from Emily's apartment back to his hotel. If he had been able to walk the most direct route, he would walk 5 blocks.
   $a^2 + b^2 = c^2$
   $4^2 + 3^2 = c^2$
   $5 = c$





b. What is the distance Emily walks to work from her apartment?
Solution:
Emily walks 19 blocks to work from her apartment.

c. What is the length of the direct path between Emily's apartment and the building where she works? Justify your answer using your diagram.
Solution:
If she had been able to walk the most direct route, he would walk approximately15.5 blocks.
$a^2 + b^2 = c^2$
$15^2 + 4^2 = c^2$
$c = \sqrt{241} \approx 15.5$

Determine a procedure for determining the distance between points on a coordinate grid by investigating the following situations.

10. What is the distance between 5 and 7? 7 and 5? –1 and 6? 5 and –3?
Comments:
This question is intended to get students thinking about using a formula to find the distance between two points. Students can easily draw a number line and count to find the distance between the given points. Help them recall that in 8th grade they learned how to find the distance between two points on a number line using $d = |a-b|$

Solutions:
Distance between 5 and 7 is 2. This can be found be simply subtracting 5 from 7. It can also be found by subtracting 7 from 5. The difference is whether the answer is positive or negative. Since distance should always be positive, taking the absolute value of the difference between the numbers will give you the distance between the two points.

$|7-5| = 2$ *or* $|5-7| = |-2| = 2$
$|6 - -1| = |6+1| = 7$ *or* $|-1-6| = |-7| = 7$
$|5 - -3| = |5+3| = 8$ *or* $|-3-5| = |-8| = 8$

11. Can you find a formula for the distance between two points, a and b, on a number line?

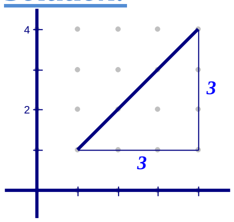
Comments:
At this point, students need to formalize their findings from above.

Solutions:
Distance between a and b is $|a-b|$ ***or*** $|b-a|$

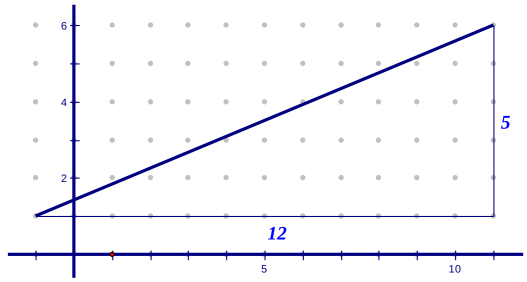
12. Using the same graph paper, find the distance between:

(1, 1) and (4, 4) (–1, 1) and (11, 6) (–1, 2) and (2, –6)

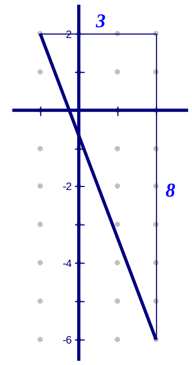
Solution:



$h^2 = 3^2 + 3^2$
$h^2 = 9 + 9$
$h^2 = 18$
$h = \sqrt{18} = 3\sqrt{2} \approx 4.2$



$h^2 = 12^2 + 5^2$
$h^2 = 144 + 25$
$h^2 = 169$
$h = \sqrt{169} = 13$



$h^2 = 3^2 + 8^2$
$h^2 = 9 + 64$
$h^2 = 73$
$h = \sqrt{73} \approx 8.5$

13. Find the distance between points (a, b) and (c, d) shown below.

Solution:

$$\sqrt{(c-a)^2 + (d-b)^2}$$

Comments:

Students need to look at the three problems from #13 to determine how they can find the distance between these points. Labeling the points and lengths on the earlier problems can help students see the pattern that is developing.

In the examples above, one leg of the right triangle is always parallel to the x–axis while the other leg is always parallel to the y–axis. Using the coordinates of the given points, the vertical length is always the difference of the x–coordinates of the points while the horizontal length is always the difference of the y–coordinates of the points. Help students relate this to #11.

14. Using your solutions from 13, find the distance between the point (x_1, y_1) and the point (x_2, y_2). Solutions written in this generic form are often called formulas.

Comments:

Encourage students to write one simple formula that will work all the time. To help students understand why the absolute value signs are not needed, discuss what happens to a number when you square it. Since the value, when squared, is always positive, it's not necessary to keep the absolute value signs.

<u>Solution:</u>

Groups may come up with slightly different solutions to this problem. All of the answers below are correct. Students should discuss the similarities and differences and why they are all valid formulas. Make sure to include a discussion of the role of mathematical properties.

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \qquad d = \sqrt{(y_1 - y_2)^2 + (x_1 - x_2)^2}$$
$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \qquad d = \sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}$$
$$d = \sqrt{(x_1 - x_2)^2 + (y_2 - y_1)^2} \qquad d = \sqrt{(y_1 - y_2)^2 + (x_2 - x_1)^2}$$
$$d = \sqrt{(x_2 - x_1)^2 + (y_1 - y_2)^2} \qquad d = \sqrt{(y_2 - y_1)^2 + (x_1 - x_2)^2}$$

15. Do you think your formula would work for any pair of points? Why or why not?

<u>Solution:</u>

Answers will vary. The formula from #14 should work for any pair of points.

Learning Task: New York City

Name________________________________ **Date**_________________

The streets of New York City are laid out in a rectangular pattern, with all blocks approximately square and approximately the same size. **Avenues** run in a north–south direction, and the numbers increase as you move west. **Streets** run in an east–west direction, and the numbers increase as you move north.

Emily works at a building located on the corner of 9th Avenue and 61st Street in New York City. Her brother, Gregory, is in town on business. He is staying at a hotel at the corner of 9th Avenue and 43rd Street.

1. Gregory calls Emily at work, and they agree to meet for lunch. They agree to meet at a corner half way between Emily's work and Gregory's hotel. Then Gregory's business meeting ends early so he decides to walk to the building where Emily works.

   a. How many blocks does he have to walk? Justify your answer using a diagram on grid paper.

   b. After meeting Emily's coworkers, they walk back toward the corner restaurant halfway between Emily's work and Gregory's hotel. How many blocks must they walk? Justify your answer using your diagram.

2. After lunch, Emily has the afternoon off, so she walks back to the hotel with Gregory before turning to go to her apartment. Her apartment is three blocks north and four blocks west of the hotel.

   a. At what intersection is her apartment building located?

   b. How many blocks south of the restaurant will they walk before Emily turns to go to her apartment?

   c. When Emily turns, what fraction of the distance from the restaurant to the hotel have the two of them walked? Express this fraction as a ratio of distance walked to distance remaining for Gregory.

3. Gregory and Emily are going to meet for dinner at a restaurant 5 blocks south of her apartment.

a. At which intersection is the restaurant located?

b. After dinner, they walk back towards her apartment, but stop at a coffee shop that is located three–fifths of the distance to the apartment. What is the location of the coffee shop?

By investigating the situations that follow, you will determine a procedure for finding a point that partitions a segment into a given ratio.

4. Here, you will find a point that partitions a directed line segment from $C(4, 3)$ to $D(10, 3)$ in a given ratio.

a. Plot the points on a grid. What is the distance between the points?

b. Use the fraction of the total length of CD to determine the location of Point A which partitions the segment from C to D in a ratio of 5:1. What are the coordinates of A?

c. Find point B that partitions a segment from C to D in a ratio of 1:2 by using the fraction of the total length of CD to determine the location of Point B. What are the coordinates of B?

5. Find the coordinates of Point X along the directed line segment YZ.

a. If $Y(4, 5)$ and $Z(4, 10)$, find X so the ratio is of YX to XZ is 4:1.

b. If $Y(4, 5)$ and $Z(4, 10)$, find X so the ratio is of YX to XZ is 3:2.

So far, the situations we have explored have been with directed line segments that were either horizontal or vertical. Use the situations below to determine how the procedure used for Questions 4 and 5 changes when the directed line segment has a defined, nonzero slope.

6. Find the coordinates of Point *A* along a directed line segment from *C*(1, 1) to *D*(9, 5) so that *A* partitions *CD* in a ratio of 3:1. *NOTE:* Since *CD* is neither horizontal nor vertical, the *x* and *y* coordinates have to be considered distinctly.

a. Find the *x*–coordinate of *A* using the fraction of the horizontal component of the directed line segment (i.e., the **horizontal** distance between *C* and *D*).

b. Find the *y*–coordinate of *A* using the fraction of the vertical component of the directed line segment (i.e., the **vertical** distance between *C* and *D*).

c. What are the coordinates of *A*?

7. Find the coordinates of Point *A* along a directed line segment from *C*(3, 2) to *D*(5, 8) so that *A* partitions *CD* in a ratio of 1:1. *NOTE:* Since *CD* is neither horizontal nor vertical, the *x* and *y* coordinates have to be considered distinctly.

a. Find the *x*–coordinate of *A* using the fraction of the horizontal component of the directed line segment (i.e., the **horizontal** distance between *C* and *D*).

b. Find the *y*–coordinate of *A* using the fraction of the vertical component of the directed line segment (i.e., the **vertical** distance between *C* and *D*).

c. What are the coordinates of *A*?

8. Now try a few more …

a. Find Point Z that partitions the directed line segment XY in a ratio of 5:3.
$X(–2, 6)$ and $Y(–10, –2)$

b. Find Point Z that partitions the directed line segment XY in a ratio of 2:3.
$X(2, –4)$ and $Y(7, 2)$

c. Find Point Z that partitions the directed line segment YX in a ratio of 1:3.
$X(–2, –4)$ and $Y(–7, 5)$ (Note the direction change in this segment.)

Back to Gregory and Emily….

9. When they finished their coffee, Gregory walked Emily back to her apartment, and then walked from there back to his hotel.

a. How many blocks did he walk?

b. If Gregory had been able to walk the direct path (“as the crow flies”) to the hotel from Emily’s apartment, how far would he have walked? Justify your answer using your diagram.

c. What is the distance Emily walks to work from her apartment?

d. What is the length of the direct path between Emily’s apartment and the building where she works? Justify your answer using your diagram.

Determine a procedure for determining the distance between points on a coordinate grid by investigating the following situations.

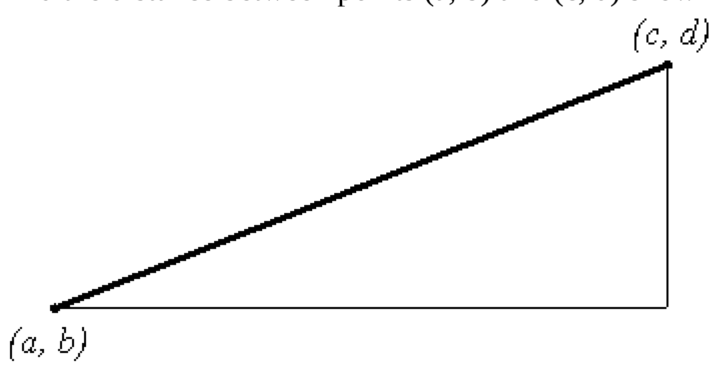
10. What is the distance between 5 and 7?   7 and 5? –1 and 6? 5 and –3?

11. Find a formula for the distance between two points, *a* and *b,* on a number line.

12. Using the same graph paper, find the distance between:

(1, 1) and (4, 4)   (–1, 1) and (11, 6)   (–1, 2) and (2, –6)

13. Find the distance between points (a, b) and (c, d) shown below.



14. Using your solutions from #13, find the distance between the point (x_1, y_1) and the point (x_2, y_2). Solutions written in this generic form are often called formulas.

15. Do you think your formula would work for any pair of points? Why or why not?

Slopes of Special Pairs of Lines (Discovery Task)

[Back to Task Table]

Introduction

This task provides a guided discovery of the relationship between the slopes of parallel lines and the slopes of perpendicular lines.

Mathematical Goals

- Show that the slopes of parallel lines are the same.
- Show that the slopes of perpendicular lines are opposite reciprocals.
- Given the equation of a line and a point not on the line, find the equation of the line that passes through the point and is parallel/perpendicular to the given line.

Essential Questions

- How do we write the equation of a line that goes through a given point and is parallel or perpendicular to another line?

Georgia Standards of Excellence

MGSE9–12.G.GPE.5 Prove the slope criteria for parallel and perpendicular lines and use them to solve geometric problems (e.g., find the equation of a line parallel or perpendicular to a given line that passes through a given point).

Standards for Mathematical Practice

2. Reason abstractly and quantitatively.
Students interpret the meaning of parallel and perpendicular lines graphically and numerically, and they generalize their findings.

5. Use appropriate tools strategically.
Students use straightedges, protractors

Background Knowledge

- Students know the graphical definition of parallel and perpendicular lines.
- For the proof about slopes of parallel lines, scaffolded in #4, students need background knowledge of similar triangles. (This proof can be changed to eliminate the need for similar triangles. See “Intervention,” second bullet, below.)
- Students need to know the meaning of slope and how to calculate it.

Common Misconceptions

- The phrase “negative reciprocal” can be confusing for students if the slope is already negative. Using the phrase “opposite reciprocal” instead can mitigate this issue.
- Students sometimes think “perpendicular” means only that the lines intersect. Emphasize that they must form right angles. Additionally, *segments* can be perpendicular, forming an “X,” a “T”, or an “L” shape; they do not have to cross through each other.

Materials

- Graph paper
- Protractor
- Ruler

Grouping

- Partner / individual

Differentiation

Extension:

- Create another proof for the relationship of slopes of parallel and perpendicular lines.
  Solution: See intervention notes about using transformations of right triangles.
- If a line is written in standard form $Ax + By = C$, what would be similar / different for a line that was parallel / perpendicular to it?
  Solution: $Ax + By = C_2$ *for parallel;* $-Bx + Ay = C_2$ *or* $Bx - Ay = C_2$ *for perpendicular*

Intervention:

- Students may need remediation in writing equations of lines given a point and the slope of the line.
- Students can cut out triangles to perform transformations to serve as an entry point to proving the properties. Cut out a right triangle and label its legs appropriately as "rise" and "run." Translating the triangle and extending the hypotenuse creates a parallel line. (The fact that it's a translation means the "rise" and "run" sides haven't changed orientation.) Rotating the triangle 90º and extending the hypotenuse creates a perpendicular line. (The rotation causes the side labeled "rise" to now become horizontal, and the side labeled "run" to now become vertical. Visually, students can also see that the slope has changed from positive to negative or vice–versa.)

Formative Assessment Questions

- Describe how to write the equation of a line <u>parallel</u> to $2x + 3y = 15$ that passes through (8, –3).
- Describe how to write the equation of a line <u>perpendicular</u> to $2x + 3y = 15$ that passes through (8, –3).

Slopes of Special Pairs of Lines – Teacher Notes

Parallel Lines

1. On an *xy*–plane, graph lines ℓ_1, ℓ_2, and ℓ_3, containing the given points. ℓ_1 contains points *A* (0, 7) and *B* (8, 9); ℓ_2 contains points *C* (0, 4) and *D* (8, 6); ℓ_3 contains points *E* (0, 0) and *F* (8, 2). Make sure to carefully extend the lines past the given points.
Solutions are below.

a. Find the distance between points *A* and *C* and between points *B* and *D*. What do you notice?

What word describes lines ℓ_1 and ℓ_2?

b. Find the distance between points *C* and *E* and between points *D* and *F*. What do you notice?

What word describes lines ℓ_2 and ℓ_3?

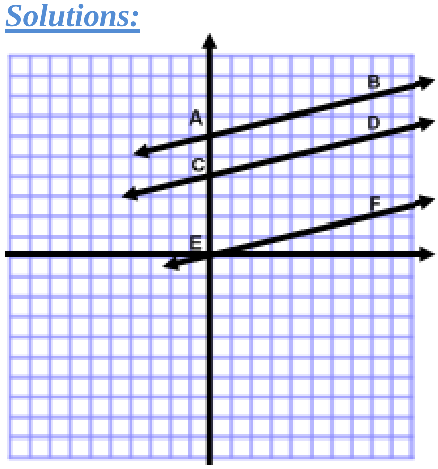
c. Find the distance between points *A* and *E* and between points *B* and *F*. What do you notice?

What word describes lines ℓ_1 and ℓ_3?

d. Now find the slopes of ℓ_1, ℓ_2, and ℓ_3.

What do you notice?

Solutions:



a. A and C are 2 units apart, as are B and D. l_1 and l_2 are parallel.

b. C and E are 4 units apart, as are D and F. l_2 and l_3 are parallel.

c. A and E are 6 units apart, as are B and F. l_1 and l_3 are parallel.

d. Slope of $l_1 = \frac{2}{8} = \frac{1}{4}$
Slope of $l_2 = \frac{2}{8} = \frac{1}{4}$
Slope of $l_3 = \frac{2}{8} = \frac{1}{4}$

All slopes are the same.

2. Now plot line ℓ_4 through points W (–1, 3) and X(–3, 6) and line ℓ_5 through points Y (–2, 1) and Z (–4, 4) carefully extending the lines across the y–axis.

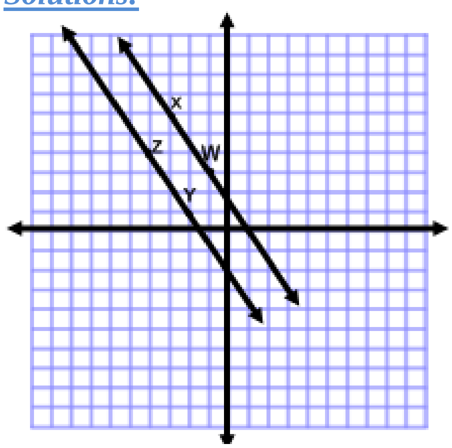
Solutions are below.

a. Use a ruler to measure the distance from W vertically to ℓ_5. Then measure the distance from X vertically to ℓ_5. What do you notice?

b. What word describes these lines?

c. Find the slope of each line. What do you notice?

Solutions:



a. The distances are the same

b. parallel

c. $-\frac{3}{2}$ & $-\frac{3}{2}$; *they are the same.*

3. What appears to be true about the slopes of parallel lines?
Solution:
Parallel lines have the same slope.

4. Follow the steps below to prove this true for all pairs of parallel lines.

a. Let the straight lines ℓ and m be parallel. Sketch these on grid paper.

b. Plot any points U and V on line ℓ and the point W so that WV is the rise and UW is the run of the slope of line ℓ. (A straight line can have only one slope.)

That is, slope of line ℓ is $\dfrac{WV}{UW}$.

c. Draw the straight line UW so that it intersects line m at point X and extends to include Point Z such that segment YZ is perpendicular to UW.

d. What is the slope of line m?

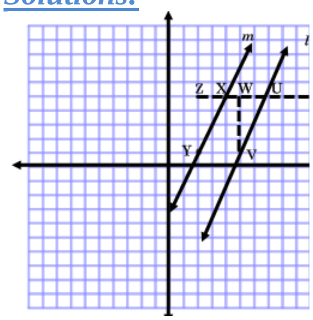
e. Line UZ is the ______________________ of the lines ℓ and m, so $\angle VUW$ and $\angle YXZ$ are ____ angles, so $\angle VUW$ ____ $\angle YXZ$.

f. Why is it true that $\angle UWV \cong \angle YXZ$?

g. Now, ΔUWV and ΔYXZ are similar, so the ratio of their sides is proportional. Write the proportion that relates the vertical leg to the horizontal leg of the triangles.

h. Note that this proportion shows the slope of line ℓ is the same as the slope of line m. Therefore, parallel lines have the same slope.

Solutions:



d. The slope of line m is $\frac{YZ}{XZ}$.

e. transversal; corresponding; $\cong$

f. The angles are right angles.

g. $\frac{WV}{UW} = \frac{YZ}{XZ}$.

5. Write equations of two lines that are parallel to the line. $y = \frac{2}{3}x + 4$

Solution:
Answers will vary, but all should have a slope of $^2/_3$

6. Determine which of the following lines is / are parallel to $2x - 3y = 21$. Explain why.

a. $y = -\frac{2}{3}x + 2$ **b.** $-6x + 9y = 12$ **c.** $\frac{1}{3}x + y = 6$ **d.** $2x + 3y = 7$ **e.** $3y = 2x + 1$

Comments:
A review of writing equations in slope–intercept form may be necessary prior to this problem.

Solution:
The given equation shows a slope of $\frac{2}{3}$. ***When each of the others are written in slope–intercept form, their equations are:***
a. $y = -\frac{2}{3}x + 2$ *b.* $y = \frac{2}{3}x + \frac{4}{3}$ *c.* $y = -\frac{1}{3}x + 6$ *d.* $y = -\frac{2}{3}x + \frac{7}{3}$ *e.* $y = \frac{2}{3}x + \frac{1}{3}$
So, only choices b and e are parallel to the given line.

7. Line *m* is parallel to the line $y = -\frac{1}{2}x + 2$ and contains the point (–6, 1). What is the equation of line *m* in slope–intercept form?

Solution:

y = mx + b
1 = (–½)(–6) + b
1 = 3 + b
b = 2

The slope of the given line is $-\frac{1}{2}$ Since line m is parallel, it has the same slope but a different y–intercept. By substituting a point known to lie on line m and the slope of line m into the slope–intercept form for the equation of the line, b can be found. Then the equation can be written using the slope and the newly found y–intercept: $y = -\frac{1}{2}x + 2$

8. What is the equation of the line that passes through (5, 2) and is parallel to the line that passes through (0, 5) and (–4, 8)?

Solution:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = -\frac{3}{4}$$

y = mx + b
2 = (–¾)(5) + b
2 = –15/4 + b
b = 23/4

The slope of the given line is found using the slope formula.. Since line m is parallel, it has the same slope but a different y–intercept. By substituting a point known to lie on line m and the slope of line m into the slope–intercept form for the equation of the line, b can be found. Then the equation can be written using the slope and the newly found y–intercept, $y = -\frac{3}{4}x + \frac{23}{4}$.

Perpendicular Lines

1. On a coordinate grid, graph the following pairs of lines. For each pair, answer: Do these lines intersect? If so, describe the angles formed at their intersection. Use a protractor if necessary. If not, describe the lines.

a. $y = -\frac{3}{4}x + 5$ and $y = \frac{4}{3}x + 1$

b. $y = 3x - 1$ and $y = -\frac{1}{3}x - 1$

c. $y = -7x + 2$ and $y = \frac{1}{7}x - 3$

d. $y = x$ and $y = -x - 8$

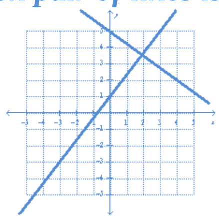
Comments:

Expect students to see the relationship between the slopes of perpendicular lines as negative reciprocals, but not necessarily see that the product of the slopes is –1. In addition, the proof of the relationship will be challenging for many students. One proof based on transformations is given, but there are other ways to prove the statement.

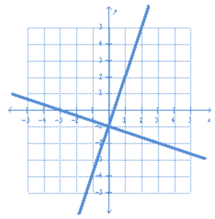
Solutions:

Each pair of lines is perpendicular. The slopes are negative reciprocals.

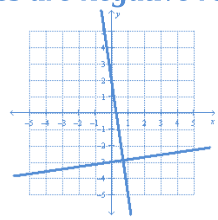
a.



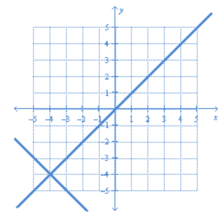
b.



c.



d.



2. Create two equations that have the same type relationship as the lines in Question 1. Draw the lines on a grid to show this relationship. What characteristics do the equations of these lines possess?

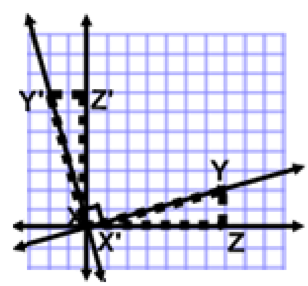
Solution:

Student answers will vary, but slopes of the lines should be opposite reciprocals.

3. Will all lines with these characteristics have the same graphical relationship? If so, prove it. If not, give a counterexample.

Solution:

Yes, all perpendicular lines will have slopes that are negative reciprocals.
Proofs will vary. A sample proof is below.



Sample Proof:
On a coordinate grid, use a protractor to draw two lines l and m perpendicular to each other at the origin. Lines l and m should be neither horizontal nor vertical.
Locate Points Y and Z such that the slope of Line l is $\frac{YZ}{XZ}$.
Rotate ΔXYZ around Point X 90°.
Name the new triangle ΔX'Y'Z'.
X' and Y' lie on Line m so that the slope of Line m is $-\frac{X'Z'}{Y'Z'}$.
Since the lengths of the sides of the figure do not change in a rotation, we have: $-\frac{X'Z'}{Y'Z'} \times \frac{YZ}{XZ} = -1$.

4. Use the relationship between slopes of perpendicular lines to answer the following questions.

a. Line *m* has the equation $y = \frac{5}{4}x + 1$. What is the slope of a line perpendicular to line *m*?

Solution:
Since line m has the slope of $\frac{5}{4}$, the slope of the new line is $-\frac{4}{5}$, the opposite reciprocal.

b. Write the equation of the line perpendicular to $y = -2x + 5$ whose *y*–intercept is 12.

Solution:
Since the given line has the slope of –2, the slope of the new line is $\frac{1}{2}$, the opposite reciprocal. Substituting the slope and y–intercept of the new line into the slope–intercept form of a line gives $= \frac{1}{2}x + 12$.

c. Write the equation of the line perpendicular to $y = \frac{1}{5}x - 6$ which passes through the point (1, –3).

Solution:
Since the given line has the slope of $\frac{1}{5}$, the slope of the new line is–5, the opposite reciprocal. Substituting the slope and given point that lies on the new line into the slope–intercept form of a line gives:
$-3 = -5(1) + b \rightarrow -3 = -5 + b \rightarrow b = 2$
Substituting the slope and y–intercept into slope–intercept form yields, y = –5x+2.

d. What is the equation of the line that passes through (5, 2) and is perpendicular to the line that passes through (0, 5) and (–4, 8)?

Solution:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = -\frac{3}{4}$$

$2 = (^4/_3)(5) + b$

$b = -\,^{14}/_3$

$$y = \frac{4}{3}x - \frac{14}{3}$$

The slope of the given line is found using the slope formula. Then the opposite reciprocal of the result is found. The new slope and the given point on the line is substituted into slope–intercept form, so that the y–intercept can be determined. The slope and y–intercept are then put into slope–intercept form.

Discovery Task: Slopes of Special Pairs of Lines

Name________________________________ **Date**_________________

Mathematical Goals

- Show that the slopes of parallel lines are the same.
- Show that the slopes of perpendicular lines are opposite reciprocals.
- Given the equation of a line and a point not on the line, find the equation of the line that passes through the point and is parallel/perpendicular to the given line.

Essential Questions

- How do we write the equation of a line that goes through a given point and is parallel or perpendicular to another line?

Georgia Standards of Excellence

MGSE9–12.G.GPE.5 Prove the slope criteria for parallel and perpendicular lines and use them to solve geometric problems (e.g., find the equation of a line parallel or perpendicular to a given line that passes through a given point).

Standards for Mathematical Practice

2. Reason abstractly and quantitatively.
5. Use appropriate tools strategically.

Discovery Task: Slopes of Special Pairs of Lines

Name_________________________________ **Date**__________________

Parallel Lines

1. On an *xy*–plane, graph lines ℓ_1, ℓ_2, and ℓ_3, containing the given points. ℓ_1 contains points *A* (0, 7) and *B* (8, 9); ℓ_2 contains points *C* (0, 4) and *D* (8, 6); ℓ_3 contains points *E* (0, 0) and *F* (8, 2). Make sure to carefully extend the lines past the given points.

a. Find the distance between *A* and *C* and between *B* and *D*. What do you notice?

What word describes lines ℓ_1 and ℓ_2?

b. Find the distance between *C* and *E* and between *D* and *F*. What do you notice?

What word describes lines ℓ_2 and ℓ_3?

c. Find the distance between *A* and *E* and between *B* and *F*. What do you notice?

What word describes lines ℓ_1 and ℓ_3?

d. Now find the slopes of ℓ_1, ℓ_2, and ℓ_3.

What do you notice?

2. Now plot line ℓ_4 through points W (–1, 3) and X(–3, 6) and line ℓ_5 through points Y (–2, 1) and Z (–4, 4) carefully extending the lines across the y–axis.

a. Use a ruler to measure the distance from W vertically to ℓ_5. Then measure the distance from X vertically to ℓ_5. What do you notice?

b. What word describes these lines?

c. Find the slope of each line. What do you notice?

3. What appears to be true about the slopes of parallel lines?

4. Follow the steps below to prove this true for all pairs of parallel lines.

a. Let the straight lines ℓ and m be parallel. Sketch these on grid paper.

b. Plot any points U and V on line ℓ and the point W so that WV is the rise and UW is the run of the slope of line ℓ. (A straight line can have only one slope.)
That is, the slope of line ℓ is $\dfrac{WV}{UW}$.

c. Draw the straight line UW so that it intersects line m at point X and extends to include Point Z such that segment YZ is perpendicular to UW.

d. What is the slope of line m?

e. Line UZ is the ______________________ of the lines ℓ and m, so $\angle VUW$ and $\angle YXZ$

are ____ angles, so $\angle VUW$ ____ $\angle YXZ$.

f. Why is it true that $\angle UWV \cong \angle YXZ$?

g. Now, ΔUWV and ΔYXZ are similar, so the ratio of their sides is proportional. Write the proportion that relates the vertical leg to the horizontal leg of the triangles.

h. Note that this proportion shows the slope of line ℓ is the same as the slope of line m. **<u>Therefore, parallel lines have the same slope.</u>**

5. Write equations of two lines that are parallel to the line. $y = \frac{2}{3}x + 4$

6. Determine which of the following lines is / are parallel to $2x - 3y = 21$. Explain why.

a. $y = -\frac{2}{3}x + 2$ **b.** $-6x + 9y = 12$ **c.** $\frac{1}{3}x + y = 6$

d. $2x + 3y = 7$ **e.** $3y = 2x + 1$

7. Line m is parallel to the line $y = -\frac{1}{2}x + 2$ and contains the point (–6, 1). What is the equation of line m in slope–intercept form?

8. What is the equation of the line that passes through (5, 2) and is parallel to the line that passes through (0, 5) and (–4, 8)?

Perpendicular Lines

1. On a coordinate grid, graph the following pairs of lines. For each pair, answer: Do these lines intersect? If so, describe the angles formed at their intersection. Use a protractor if necessary. If not, describe the lines.

a. $y = -\frac{3}{4}x + 5$ and $y = \frac{4}{3}x + 1$

b. $y = 3x - 1$ and $y = -\frac{1}{3}x - 1$

c. $y = -7x + 2$ and $y = \frac{1}{7}x - 3$

d. $y = x$ and $y = -x - 8$

2. Create two equations that have the same type relationship as the lines in Question 1. Draw the lines on a grid to show this relationship. What characteristics do the equations of these lines possess?

3. Will all lines with these characteristics have the same graphical relationship? If so, prove it. If not, give a counterexample.

4. Use the relationship between slopes of perpendicular lines to answer the following questions.

a. Line *m* has the equation $y = \frac{5}{4}x + 1$. What is the slope of a line perpendicular to line *m*?

b. Write the equation of the line perpendicular to $y = -2x + 5$ whose y–intercept is 12.

c. Write the equation of the line perpendicular to $y = \frac{1}{5}x - 6$ which passes through the point (1, –3).

d. What is the equation of the line that passes through (5, 2) and is perpendicular to the line that passes through (0, 5) and (–4, 8)?

Geometric Properties in the Plane (Performance Task)

Back to Task Table

Introduction

This task provides students an opportunity to apply the algebraic concepts of slope, line segment partitioning, distance formula and the Pythagorean Theorem to geometric figures constructed on the coordinate plane. Students will also use their knowledge of the properties of various polygons to justify their solution of the problems.

Mathematical Goals

- Use coordinates, slope relationships, and distance formula to prove simple geometric theorems.
- Compute the perimeters of polygons using the coordinates of the vertices and the distance formula.
- Find the areas of rectangles and triangles using the coordinates of the vertices and the distance formula.

Essential Questions

- How can slope and distance be used to determine properties of polygons?

Georgia Standards of Excellence

MGSE9–12.G.GPE.4 Use coordinates to prove simple geometric theorems algebraically. *For example, prove or disprove that a figure defined by four given points in the coordinate plane is a rectangle; prove or disprove that the point (1, $\sqrt{3}$) lies on the circle centered at the origin and containing the point (0,2).*
(Focus on quadrilaterals, right triangles, and circles.)

MGSE9–12.G.GPE.7 Use coordinates to compute perimeters of polygons and areas of triangles and rectangles, e.g., using the distance formula.

Standards for Mathematical Practice

6. Attend to precision.
7. Look for and make use of structure.
Students use distance and slope to prove properties of various geometric figures.

Background Knowledge

- Students know some basic geometric definitions (scalene, isosceles, equilateral, trapezoid, kite, parallelogram, rhombus, rectangle, square, midpoint, diagonal).
- Students can find perimeter of polygons and area of triangles and rectangles.
- Students can work with radicals: $(\sqrt{3})^2 = 3$ and $\sqrt{9} = 3$ (not $\sqrt{9} = \sqrt{3}$, a common misconception)

Common Misconceptions

- Students may stop after finding one description of a quadrilateral without checking to see if a more specific quadrilateral better describes it. For example, in #4, Set 2, the quadrilateral is a parallelogram, but further calculations show it is better described as a rhombus (but not a square).

Materials

- Graph paper

Grouping

- Partners

Differentiation

Extension:

- The segment from (1, 1) to (1, 3) is a leg of an isosceles trapezoid. Give possible coordinates for the other vertices of the trapezoid.
  (Possible solution: (5, 5); (3, 5); (1, 3))

Intervention:

- Encourage students to *predict* answers visually, but emphasize the importance of justifying mathematically.
- Putting slopes and lengths in a table (as in solutions, below) can help students keep work organized.

Formative Assessment Questions

- Describe the similarities between a parallelogram and a rectangle referencing slope and distance of the sides.

Geometric Properties in the Plane – Teacher Notes

1. Determine the coordinates of a scalene triangle. Support your answer mathematically and justify with a drawing on a coordinate grid.

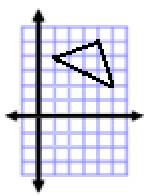
   Answers will vary

2. Classify the triangle as scalene, isosceles, or equilateral. Determine if it is also a right triangle. Then find the perimeter and area.

   a. (1, 4), (4, 5), (5, 2)

   Comments:
   For perimeter and area, students do not have the skills (yet) to add, subtract, and multiply radicals. Decimal approximations are appropriate for these problems.

Solution:



Points	*Slope*	*Distance*
(1, 4) *(4, 5)*	$\frac{1}{3}$	$\sqrt{10}$
(4, 5) *(5, 2)*	-3	$\sqrt{10}$
(5, 2) *(1, 4)*	$-\frac{1}{2}$	$\sqrt{20}$

Two sides have the same length, so the triangle is isosceles.

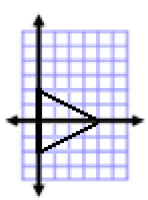
Two sides have slopes that are opposite reciprocals, so the triangle is a right triangle.

Perimeter: $\sqrt{10}+\sqrt{10}+\sqrt{20}\approx$ *10.8 units*

Area: $\frac{\sqrt{10}\cdot\sqrt{10}}{2}=5$ *square units*

b. (0, –2), (0, 2), (4, 0)

Solution:



Points	*Slope*	*Distance*
(0, –2) *(0, 2)*	*undef.*	*4*
(0, 2) *(4, 0)*	$-\frac{1}{2}$	$\sqrt{20}$
(4, 0) *(0, –2)*	$\frac{1}{2}$	$\sqrt{20}$

Two sides have the same length, so the triangle is isosceles.

No two sides have slopes that are opposite reciprocals, so the triangle is not a right triangle.

Perimeter: $4+\sqrt{20}+\sqrt{20}\approx$ *12.9 units*

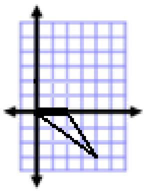
Area: $\frac{4\cdot 4}{2}=8$ *square units*

c. (0, 0), (2, 0), (4, –3)

Solution:

Points	*Slope*	*Distance*

No two sides have the same length, so the triangle is not isosceles.



(0, 0) *(2, 0)*	*0*	*2*
(2, 0) *(4, –3)*	$-\frac{3}{2}$	$\sqrt{13}$
(4, –3) *(0, 0)*	$-\frac{3}{4}$	*5*

No two sides have slopes that are opposite reciprocals, so the triangle <u>is not a right triangle.</u>

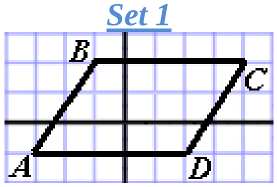
Perimeter: $7+\sqrt{13}\approx$ ***10.6 units***
Area: $\frac{2\cdot 3}{2}=3$ ***square units***

3. Find the following information for each set of points below.
   a. Plot points and connect to form a quadrilateral.
   b. Determine whether the quadrilateral is a trapezoid, kite, parallelogram, rhombus, rectangle, or square. Justify with math.
   c. Find the midpoints of the diagonals. What do you notice?
   d. Find the slope of the diagonals. Are the diagonals perpendicular?
   e. Find the perimeter of each.

Set 1: *A* (–3, –1) *B* (–1, 2) *C* (4, 2) *D* (2, –1)
Set 2: *E* (1, 2) *F* (2, 5) *G* (4, 3) *H* (5, 6)

<u>Solutions:</u>

<u>Set 1</u>



b.

Side	*Slope*	*Length*
$\overline{AB}$	*0*	*5*
$\overline{BC}$	$\frac{3}{2}$	$\sqrt{13}$
$\overline{CD}$	*0*	*5*
$\overline{DA}$	$\frac{3}{2}$	$\sqrt{13}$

c. Midpoint (ratio 1:1)

$\overline{AC}$: $\left(\frac{-3+4}{2},\frac{-1+2}{2}\right)=\left(\frac{1}{2},\frac{1}{2}\right)$

$\overline{BD}$: $\left(\frac{-1+2}{2},\frac{2+-1}{2}\right)=\left(\frac{1}{2},\frac{1}{2}\right)$

The midpoints are the same.

Since opposite sides are parallel (same slope), ABCD is a <u>parallelogram</u>.

Not all sides are the same length, so the shape cannot be a rhombus.

Adjacent sides do not have slopes that are opposite reciprocals, so there are no right angles and the shape cannot be a rectangle.

d. Diagonals' slopes are –1 and $\frac{3}{7}$
They are not perpendicular, since the slopes are not opposite reciprocals.

e. Perimeter is $10+2\sqrt{13}\approx$ ***17.2 units.***

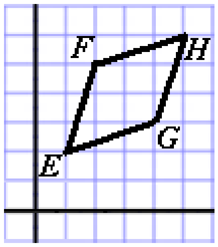
<u>Set 2</u>

b.

Side	*Slope*	*Length*
$\overline{EF}$	*3*	$\sqrt{10}$
$\overline{FH}$	$\frac{1}{3}$	$\sqrt{10}$

c. Midpoint (ratio 1:1)

$\overline{EH}$: $\left(\frac{1+5}{2},\frac{2+6}{2}\right)=\left(\frac{6}{2},\frac{8}{2}\right)=(3,4)$



$\overline{HG}$	3	$\sqrt{10}$
$\overline{GE}$	$\frac{1}{3}$	$\sqrt{10}$

Since all sides are the same length, EFHG is a rhombus.

Adjacent sides do not have slopes that are opposite reciprocals, so there are no right angles and the shape cannot be a square.

$\overline{FG}$: $\left(\frac{2+4}{2}, \frac{3+5}{2}\right) = \left(\frac{6}{2}, \frac{8}{2}\right) = (3, 4)$

The midpoints are the same.

d. Diagonals' slopes are –1 and 1. They are perpendicular, since the slopes are opposite reciprocals.

e. Perimeter is $4\sqrt{10} \approx 12.6$ units.

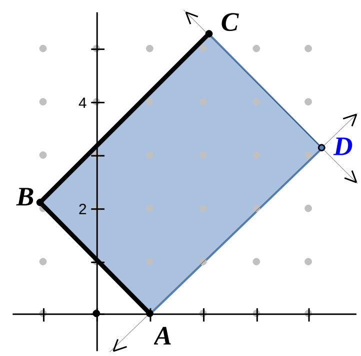
4. Plot points $A = (1, 0)$, $B = (–1, 2)$, and $C = (2, 5)$.

   a. Find the coordinates of a fourth point D that would make $ABCD$ a rectangle. Justify that $ABCD$ is a rectangle.

   b. Find the area of the rectangle.

<u>Comments:</u>

One way to solve this problem is to create lines that are perpendicular to segments AB and BC. The point of intersection of the two lines is the 4th point.



<u>Solution:</u>

a. D is located at (4, 3).
This makes ABCD a rectangle because:
The slope of $\overline{AB}$ = the slope of $\overline{CD}$ = –1.
The slope of $\overline{BC}$ = the slope of $\overline{AD}$ = 1.
So all pairs of adjacent sides are perpendicular.

b. $AB = CD = \sqrt{8}$, $CB = AD = \sqrt{18}$
$AB \perp AD$, so area is $\sqrt{8} \cdot \sqrt{18} = 12$ square units.

Performance Task: Geometric Properties in the Plane

Name_________________________________ **Date**_________________

Mathematical Goals

- Use coordinates, slope relationships, and distance formula to prove simple geometric theorems.
- Compute the perimeters of polygons using the coordinates of the vertices and the distance formula.
- Find the areas of rectangles and triangles using the coordinates of the vertices and the distance formula.

Essential Questions

- How can slope and distance be used to determine properties of polygons and circles?

Georgia Standards of Excellence

MGSE9–12.G.GPE.4 Use coordinates to prove simple geometric theorems algebraically. *For example, prove or disprove that a figure defined by four given points in the coordinate plane is a rectangle; prove or disprove that the point (1, $\sqrt{3}$) lies on the circle centered at the origin and containing the point (0,2).*
(Focus on quadrilaterals, right triangles, and circles.)

MGSE9–12.G.GPE.7 Use coordinates to compute perimeters of polygons and areas of triangles and rectangles, e.g., using the distance formula.

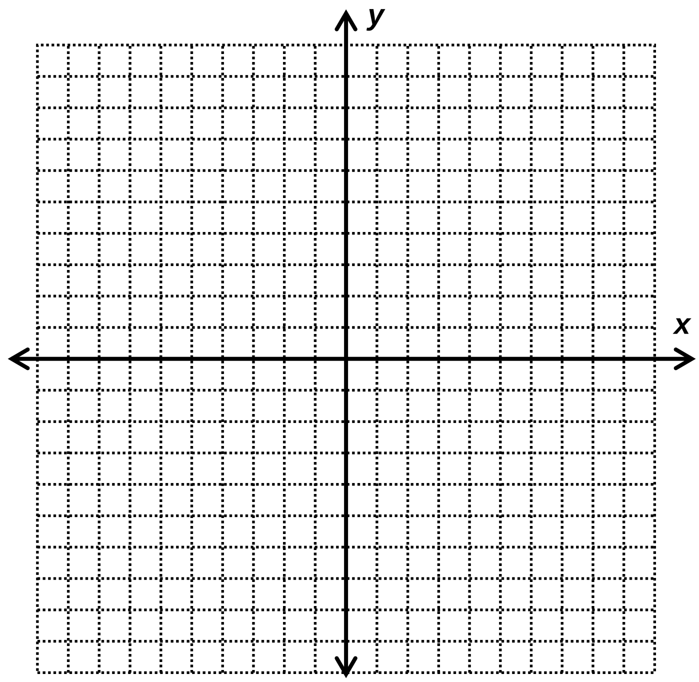
Standards for Mathematical Practice

6. Attend to precision.
7. Look for and make use of structure.

Performance Task: Geometric Properties in the Plane

Name_________________________________ **Date**__________________

1. Determine the coordinates of a scalene triangle. Support your answer mathematically and justify with a drawing on a coordinate grid.



2. Classify the triangle with the given vertices as scalene, isosceles, or equilateral. Determine if it is also a right triangle. Then find the perimeter and area.

a. (1, 4) (4, 5) (5, 2)

b. (0, –2) (0, 2) (4, 0)

c. (0, 0) (2, 0) (4, –3)

3. Find the following information for each set of points below.

a. Plot points and connect to form a quadrilateral.
b. Determine whether the quadrilateral is a trapezoid, kite, parallelogram, rhombus, rectangle, or square. Justify mathematically.
c. Find the midpoints of the diagonals. What do you notice?
d. Find the slope of the diagonals. Are the diagonals perpendicular?
e. Find the perimeter of each quadrilateral.

Set 1: $A(-3, -1)$ $B(-1, 2)$ $C(4, 2)$ $D(2, -1)$

Set 2: $E(1, 2)$ $F(2, 5)$ $G(4, 3)$ $H(5, 6)$

4. Plot points $A(1, 0)$ $B(-1, 2)$ $C(2, 5)$.

a. Find the coordinates of a fourth point D that would make $ABCD$ a rectangle. Justify that $ABCD$ is a rectangle.

b. Find the area of the rectangle.

Equations of Parallel & Perpendicular Lines (Formative Assessment Lesson)

Back to Task Table

Source: Formative Assessment Lesson Materials from Mathematics Assessment Project
http://map.mathshell.org/materials/download.php?fileid=703

Task Comments and Introduction

Tasks and lessons from the Mathematics Assessment Project are specifically designed to help teachers effectively formatively assess their students. The way the tasks and lessons are designed gives the teacher a clear understanding of what the students are able to do and not do. Within the lesson, teachers will find suggestions and question prompts that will help guide students towards understanding. For more information access the MAP website:
http://www.map.mathshell.org/materials/background.php?subpage=formative

The task, *Finding Equations of Parallel and Perpendicular Lines*, is a Formative Assessment Lesson (FAL) that can be found at the website:
http://map.mathshell.org/materials/lessons.php?taskid=226&subpage=concept

The FAL document provides a clear lesson design, from the opening of the lesson to the closing of the lesson.

The PDF version of the task can be found at the link below:
http://map.mathshell.org/materials/download.php?fileid=703

Mathematical Goals

- Find, from their equations, lines that are parallel and perpendicular.
- Identify and use intercepts.

Essential Questions

- How do we use equations of parallel and perpendicular lines to form geometric figures?

Georgia Standards of Excellence

MGSE9–12.G.GPE.4 Use coordinates to prove simple geometric theorems algebraically. *For example, prove or disprove that a figure defined by four given points in the coordinate plane is a rectangle; prove or disprove that the point (1, $\sqrt{3}$) lies on the circle centered at the origin and containing the point (0,2).*
(Focus on quadrilaterals, right triangles, and circles.)

MGSE9–12.G.GPE.5 Prove the slope criteria for parallel and perpendicular lines and use them to solve geometric problems (e.g., find the equation of a line parallel or perpendicular to a given line that passes through a given point).

MGSE9–12.G.GPE.6 Find the point on a directed line segment between two given points that partitions the segment in a given ratio.

MGSE9–12.G.GPE.7 Use coordinates to compute perimeters of polygons and areas of triangles.

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
Students must determine what each question is asking and how to approach it.

3. Construct viable arguments and critique the reasoning of others.
Students must justify why two lines are parallel, perpendicular, or neither.

7. Look for and make use of structure.
Students use patterns relating the slopes of parallel and perpendicular lines to generalize to form rules about these pairs of lines.

Background Knowledge

- Students know how to find the slope, *x*–intercept, and *y*–intercept of a line.
- Students know how to write the equation of a line.
- Students know the definition of a rectangle and connect this to the coordinate plane.

Common Misconceptions

- Students may believe a rectangle needs only to have two pairs of congruent sides, but this is only sufficient to show the figure is a parallelogram. Students must show that adjacent sides are perpendicular.
- The phrase “negative reciprocal” can be confusing for students if the slope is already negative. Using the phrase “opposite reciprocal” instead can mitigate this issue.

Materials

- See FAL website.

Grouping

- Individual / small group

Square (Short Cycle Task)

Back to Task Table

Source: Balanced Assessment Materials from Mathematics Assessment Project
http://www.map.mathshell.org/materials/download.php?fileid=792

Task Comments and Introduction

Tasks and lessons from the Mathematics Assessment Project are specifically designed to help teachers effectively formatively assess their students. The way the tasks and lessons are designed gives the teacher a clear understanding of what the students are able to do and not do. Within the lesson, teachers will find suggestions and question prompts that will help guide students towards understanding. For more information access the MAP website:
http://www.map.mathshell.org/materials/background.php?subpage=summative

The task, *Square*, is a Mathematics Assessment Project Assessment Task that can be found at the website:
http://www.map.mathshell.org/materials/tasks.php?taskid=270&subpage=apprentice

The PDF version of the task can be found at the link below:
http://www.map.mathshell.org/materials/download.php?fileid=792

The scoring rubric can be found at the following link:
http://www.map.mathshell.org/materials/download.php?fileid=793

Mathematical Goals

- Use slope and length to determine whether a figure with given vertices is a square.

Essential Questions

- How do you use slope and distance to classify a geometric figure?

Georgia Standards of Excellence

MGSE9–12.G.GPE.4 Use coordinates to prove simple geometric theorems algebraically. *For example, prove or disprove that a figure defined by four given points in the coordinate plane is a rectangle; prove or disprove that the point (1, $\sqrt{3}$) lies on the circle centered at the origin and containing the point (0,2).*
(Focus on quadrilaterals, right triangles, and circles.)

MGSE9–12.G.GPE.5 Prove the slope criteria for parallel and perpendicular lines and use them to solve geometric problems (e.g., find the equation of a line parallel or perpendicular to a given line that passes through a given point).

MGSE9–12.G.GPE.6 Find the point on a directed line segment between two given points that partitions the segment in a given ratio.

MGSE9–12.G.GPE.7 Use coordinates to compute perimeters of polygons and areas of triangles.

Standards for Mathematical Practice

3. Construct viable arguments and critique the reasoning of others.
Students must mathematically justify whether the shape is a square.

7. Look for and make use of structure.
Students use the properties of squares to determine what to look for in their calculations.

Background Knowledge

- Students know the definition of a square.
- Students know how to find and interpret slope and length of segments.

Common Misconceptions

- Students may stop after finding one description of a quadrilateral without checking to see if a more specific quadrilateral better describes it.

Materials

- See FAL website.

Grouping

- Partner / small group

Proving Quadrilaterals in the Coordinate Plane

Back to Task Table

Mathematical Goals

- Prove theorems pertaining to lines and angles.
- Prove theorems pertaining to triangles.
- Prove theorems pertaining to parallelograms.

STANDARDS ADDRESSED IN THIS TASK

MGSE9-12.G.CO.11 Prove theorems about parallelograms. Theorems include: opposite sides are congruent, opposite angles are congruent, the diagonals of a parallelogram bisect each other, and conversely, rectangles are parallelograms with congruent diagonals. (Note: This standard is a review from Unit 2)

MGSE9-12.G.GPE.4 Use coordinates to prove simple geometric theorems algebraically. *For example, prove or disprove that a figure defined by four given points in the coordinate plane is a rectangle; prove or disprove that the point (1, $\sqrt{3}$) lies on the circle centered at the origin and containing the point (0,2).*
(Focus on quadrilaterals, right triangles, and circles.)

Standards for Mathematical Practice

1. **Make sense of problems and persevere in solving them** by requiring students to interpret and make meaning of a problem and find a logical starting point, and to monitor their progress and change their approach to solving the problem, if necessary.
2. **Reason abstractly and quantitatively** by requiring students to make sense of quantities and their relationships to one another in problem situations.
3. **Construct viable arguments and critique the reasoning of others** by engaging students on discussion of why they agree or disagree with responses, decide whether they make sense, and ask useful questions to clarify or improve the arguments.
4. **Model with mathematics** by expecting students to apply the mathematics concepts they know in order to solve problems arising in everyday situations, and reflect on whether the results are sensible for the given scenario.
5. **Use appropriate tools strategically** by expecting students to consider available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a compass, a calculator, software, etc.
6. **Attend to precision** by requiring students to calculate efficiently and accurately; and to communicate precisely with others by using clear mathematical language to discuss their reasoning.
7. **Look for and make use of structure** by expecting students to apply rules, look for patterns and analyze structure.
8. **Look for and express regularity in repeated reasoning** by expecting students to understand broader applications and look for structure and general methods in similar situations.

Introduction

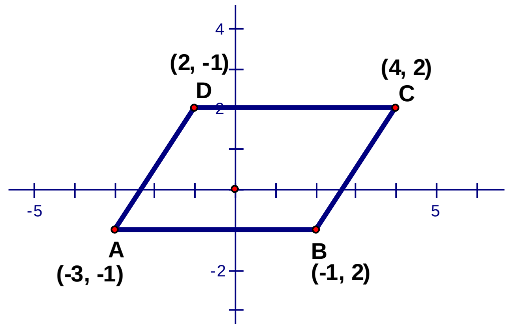
This task provides students an opportunity to apply the algebraic concepts of slope, midpoint, distance formula and the Pythagorean Theorem to geometric figures constructed on the coordinate plane. Students will also use their knowledge of the properties of quadrilaterals to justify their solution of the problems.

Materials

- graph paper

Plot points A = (-3, -1), B = (-1, 2), C = (4, 2), and D = (2, -1).

1. What specialized geometric figure is quadrilateral ABCD? Support your answer mathematically.

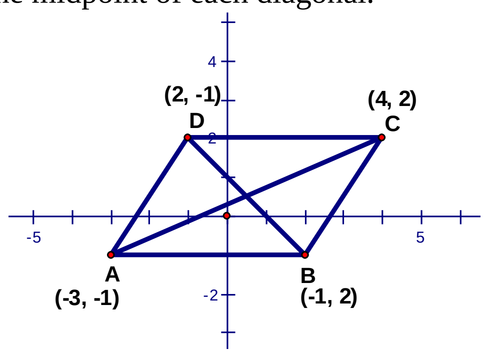


Solution

The slope of $\overline{AB}$ = 0 and the slope of $\overline{DC}$ = 0, so $\overline{AB} \parallel \overline{DC}$.

The slope of $\overline{AD}$ = 3/2 and the slope of $\overline{BC}$ = 3/2, so $\overline{AD} \parallel \overline{BC}$.
Since opposite sides are parallel, ABCD is a parallelogram.

2. Draw the diagonals of ABCD. Find the coordinates of the midpoint of each diagonal. What do you notice?



Solution

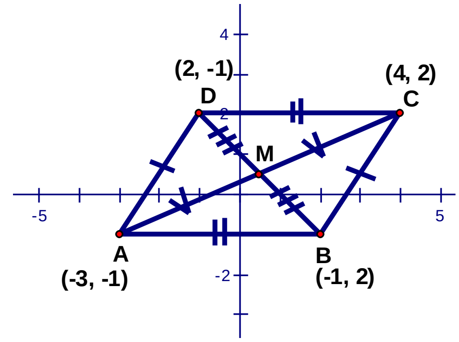
Midpoint of $\overline{AC}$ = $\left(\frac{-3+4}{2}, \frac{2+(-1)}{2}\right) = \left(\frac{1}{2}, \frac{1}{2}\right)$
Midpoint of $\overline{BD}$ = $\left(\frac{-1+2}{2}, \frac{2+(-1)}{2}\right) = \left(\frac{1}{2}, \frac{1}{2}\right)$
The midpoint of $\overline{AC}$is the same as the midpoint of $\overline{BD}$.
The diagonals bisect each other.

3. Find the slopes of the diagonals of ABCD. What do you notice?

Solution

The slope of $\overline{AC}$ = $\frac{-1-2}{-3-4} = \frac{-3}{-7} = \frac{3}{7}$
The slope of $\overline{BD}$ = $\frac{2-(-1)}{-1-2} = \frac{3}{-3} = -1$
The diagonals are not perpendicular.

4. The diagonals of ABCD create four small triangles. Are any of these triangles congruent to any of the others? Why or why not?

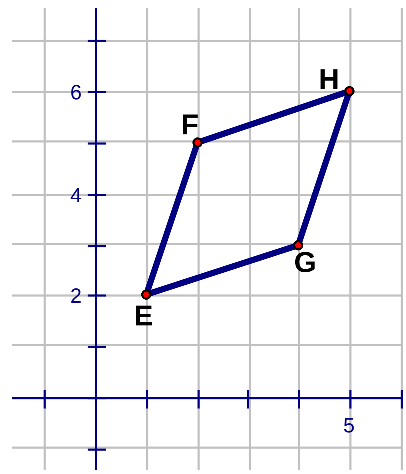


Solution

Yes.
$\overline{AD} \cong \overline{BC}$ and $\overline{AB} \cong \overline{DC}$ because opposite sides of a parallelogram are congruent.
$\overline{AM} \cong \overline{MC}$ by definition of midpoint.
$\overline{DM} \cong \overline{MB}$ by definition of midpoint.
$\Delta AMD \cong \Delta CMB$ by SSS.
$\Delta ABM \cong \Delta CDM$ by SSS.

Plot points E = (1, 2), F = (2, 5), G = (4, 3) and H = (5, 6).

5. What specialized geometric figure is quadrilateral EFHG? Support your answer mathematically using two different methods.



Comments

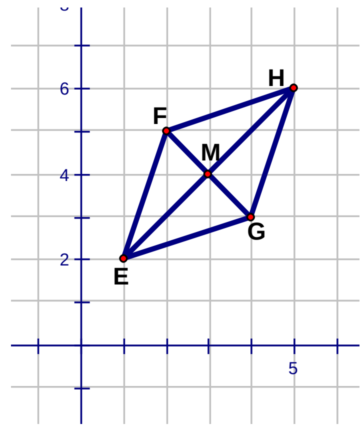
Students can use different methods to show EFHG is a rhombus. A couple of solutions are given but students may have other ways to justify their solutions.

Solution

Option #1:
$\overline{EF} \parallel \overline{GH}$ because the slopes are both 3.
$\overline{FH} \parallel \overline{EG}$ because the slopes are both 1/3.
So, EFHG is a parallelogram.
Using Pythagorean Theorem:
$EF = FH = HG = GE = \sqrt{1^2 + 3^2} = \sqrt{1 + 9} = \sqrt{10}$
Since EFHG is a parallelogram with congruent sides it is a rhombus.

Option #2:
$\overline{EF} \parallel \overline{GH}$ because the slopes are both 3.
$\overline{FH} \parallel \overline{EG}$ because the slopes are both 1/3.
So, EFHG is a parallelogram.
$\overline{FG} \perp \overline{EH}$ because the slope of $\overline{FG}$ is -1 and the slope of $\overline{EH}$ is 1 which are opposite reciprocals.
Therefore, EFHG is a rhombus.

6. Draw the diagonals of EFHG. Find the coordinates of the midpoint of each diagonal. What do you notice?



Solution

Midpoint of $\overline{EH} = \left(\frac{1+5}{2}, \frac{2+6}{2}\right) = \left(\frac{6}{2}, \frac{8}{2}\right) = (3, 4)$
Midpoint of $\overline{BD} = \left(\frac{2+4}{2}, \frac{3+5}{2}\right) = \left(\frac{6}{2}, \frac{8}{2}\right) = (3, 4)$
The midpoint of $\overline{EH}$ ***is the same as the midpoint of*** $\overline{FG}$.
The diagonals bisect each other.

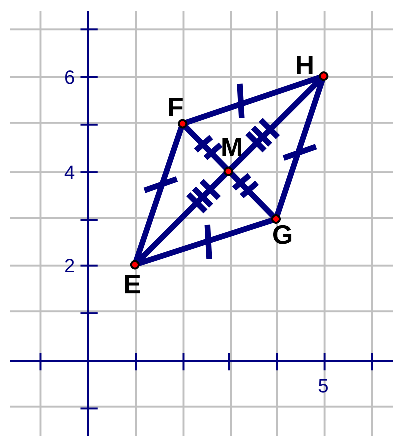
7. Find the slopes of the diagonals of EFHG. What do you notice?

Solution

The slope of $\overline{EH} = \frac{2-6}{1-5} = \frac{-4}{-4} = 1$
The slope of $\overline{FG} = \frac{5-3}{2-4} = \frac{2}{-2} = -1$
The diagonals are perpendicular because the slopes are opposite reciprocals.

8. The diagonals of EFHG create four small triangles. Are any of these triangles congruent to any of the others? Why or why not?

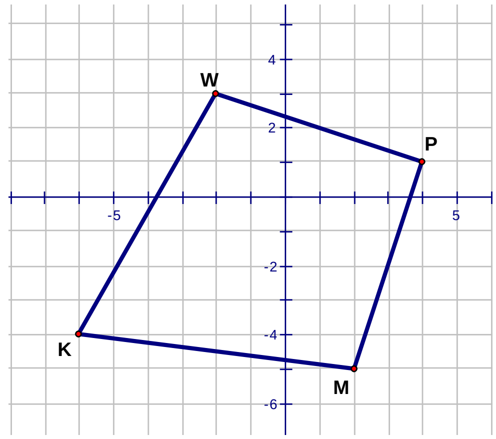
Solution



All four triangles are congruent to each other. There are several ways to prove this. Using the fact that the four sides are congruent to each other and the diagonals are bisected, SSS proves they are all congruent. (See diagram to the right.)
Or
Since the diagonals are perpendicular, the four triangles are right triangles. $\overline{FM} \cong \overline{MG}$ ***and*** $\overline{EM} \cong \overline{MH}$ ***because the diagonals bisect each other. The four triangles are congruent using LL or SAS.***

Plot points P = (4, 1), W = (-2, 3), M = (2,-5), and K = (-6, -4).

9. What specialized geometric figure is quadrilateral PWKM? Support your answer mathematically.



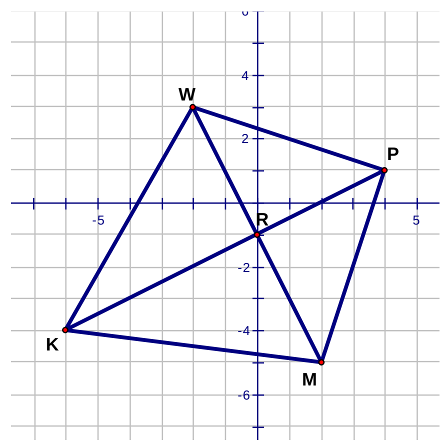
Comments

Students may need to answer questions 10 – 13 before they answer #9.

Solution

The figure is a kite.
$\overline{WP} \cong \overline{PM}$ and $\overline{MK} \cong \overline{KW}$. This can be shown using the Pythagorean Theorem or the distance formula. By definition, this means the quadrilateral is a kite (2 pairs of congruent adjacent sides).

10. Draw the diagonals of PWKM. Find the coordinates of the midpoint of each diagonal. What do you notice?



Solution

Midpoint of $\overline{KP} = \left(\frac{-6+4}{2}, \frac{-4+1}{2}\right) = \left(\frac{-2}{2}, \frac{-3}{2}\right) = \left(-1, -1\frac{1}{2}\right)$
Midpoint of $\overline{WM} = \left(\frac{-2+2}{2}, \frac{3+(-5)}{2}\right) = \left(\frac{0}{2}, \frac{-2}{2}\right) = (0, -1)$
$\overline{KP}$ **bisects** $\overline{WM}$ **but** $\overline{WM}$ **is not bisected.**

11. Find the lengths of the diagonals of PWKM. What do you notice?

Solution

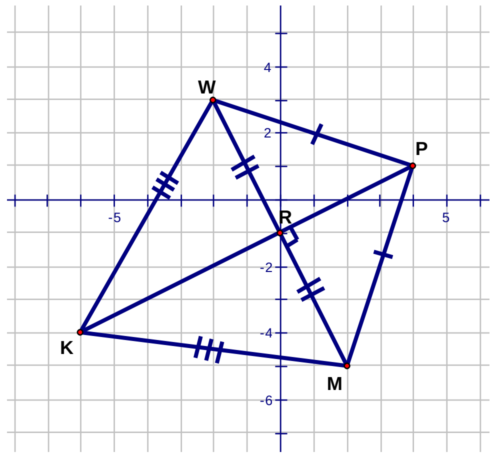
Using Pythagorean Theorem:
$KP = \sqrt{10^2 + 5^2} = \sqrt{100 + 25} = \sqrt{125}$
$WM = \sqrt{8^2 + 4^2} = \sqrt{64 + 16} = \sqrt{80}$
The diagonals are not the same length.

12. Find the slopes of the diagonals of PWKM. What do you notice?

Solution

The slope of $\overline{KP} = \frac{-4-1}{-6-4} = \frac{-5}{-10} = 2$
The slope of $\overline{WM} = \frac{-5-3}{2-(-2)} = \frac{-8}{4} = -2$
The diagonals are perpendicular because the slopes are opposite reciprocals.

13. The diagonals of ABCD create four small triangles. Are any of these triangles congruent to any of the others? Why or why not?

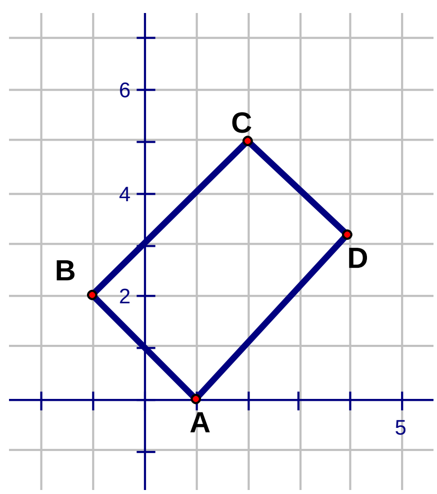


Solution

Yes.
All 4 triangles are right triangles (the diagonals are perpendicular)
ΔWRP ≅ ΔMRP by HL.
ΔMRK ≅ ΔWRK by HL.

Plot points A = (1, 0), B = (-1, 2), and C = (2, 5).

14. Find the coordinates of a fourth point D that would make ABCD a rectangle. Justify that ABCD is a rectangle.



Comments

One way to solve this problem is to create lines perpendicular to segments AB and BC. The point of intersection of the two lines is the 4th point.

Solution

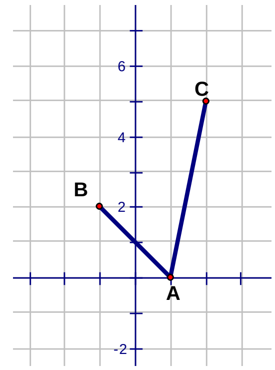
D is located at (4, 3).
The slope of $\overline{AB}$ ***= the slope of*** $\overline{CD}$***= -1.***
The slope of $\overline{BC}$ ***= the slope of*** $\overline{AD}$***= 1.***
ABCD is a rectangle because the opposite sides are parallel and the adjacent sides are all perpendicular.

15. Find the coordinates of a fourth point D that would make ABDC a parallelogram that

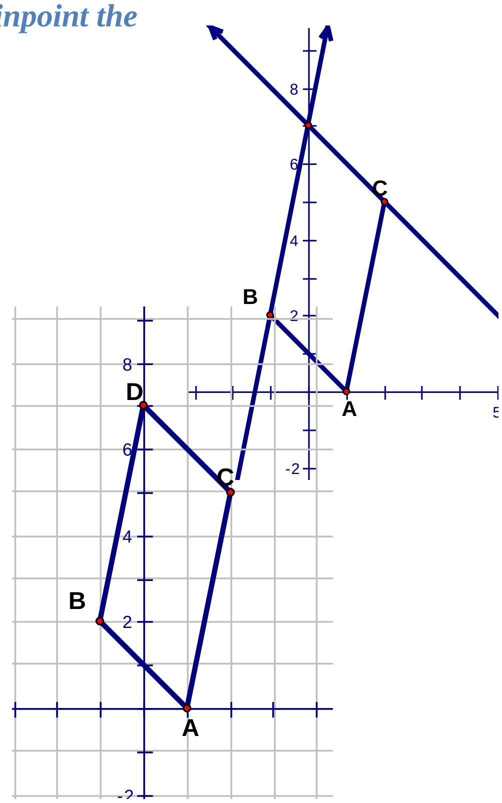
is not also a rectangle. Justify that ABDC is a parallelogram but is not a rectangle.

Comments

This may be more difficult for students to solve. It is easier to solve this if the students will redraw the problem. See the diagram to the right.



Once they have redrawn the diagram they can use a similar strategy to create a parallelogram. Since the opposite sides of a parallelogram have to be parallel, constructing lines parallel to the given sides, that go through points B and C, helps pinpoint the 4th point.



Solution

D is located at (0, 7).
The slope of $\overline{AC}$ = the slope of $\overline{BD}$= 5.
The slope of $\overline{AB}$ = the slope of $\overline{CD}$= 1.
Opposite sides are parallel but not perpendicular.

Proving Quadrilaterals in the Coordinate Plane

Mathematical Goals

- Prove theorems pertaining to lines and angles.
- Prove theorems pertaining to triangles.
- Prove theorems pertaining to parallelograms.

STANDARDS ADDRESSED IN THIS TASK

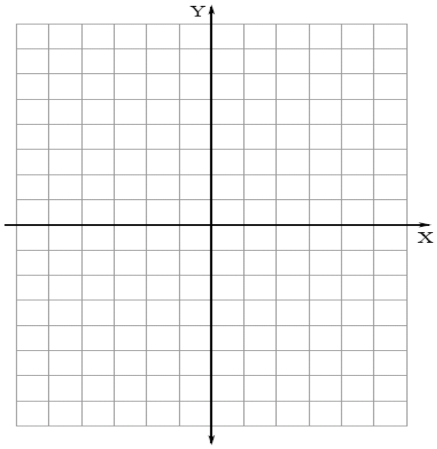
MGSE9-12.G.CO.11 Prove theorems about parallelograms. Theorems include: opposite sides are congruent, opposite angles are congruent, the diagonals of a parallelogram bisect each other, and conversely, rectangles are parallelograms with congruent diagonals.

MGSE9-12.G.GPE.4 Use coordinates to prove simple geometric theorems algebraically. *For example, prove or disprove that a figure defined by four given points in the coordinate plane is a rectangle; prove or disprove that the point (1, $\sqrt{3}$) lies on the circle centered at the origin and containing the point (0,2).*
(Focus on quadrilaterals, right triangles, and circles.)

Standards for Mathematical Practice

1. **Make sense of problems and persevere in solving them** by requiring students to interpret and make meaning of a problem and find a logical starting point, and to monitor their progress and change their approach to solving the problem, if necessary.
2. **Reason abstractly and quantitatively** by requiring students to make sense of quantities and their relationships to one another in problem situations.
3. **Construct viable arguments and critique the reasoning of others** by engaging students on discussion of why they agree or disagree with responses, decide whether they make sense, and ask useful questions to clarify or improve the arguments.
4. **Model with mathematics** by expecting students to apply the mathematics concepts they know in order to solve problems arising in everyday situations, and reflect on whether the results are sensible for the given scenario.
5. **Use appropriate tools strategically** by expecting students to consider available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a compass, a calculator, software, etc.
6. **Attend to precision** by requiring students to calculate efficiently and accurately; and to communicate precisely with others by using clear mathematical language to discuss their reasoning.
7. **Look for and make use of structure** by expecting students to apply rules, look for patterns and analyze structure.
8. **Look for and express regularity in repeated reasoning** by expecting students to understand broader applications and look for structure and general methods in similar situations.

Plot points A = (-3, -1), B = (-1, 2), C = (4, 2), and D = (2, -1).



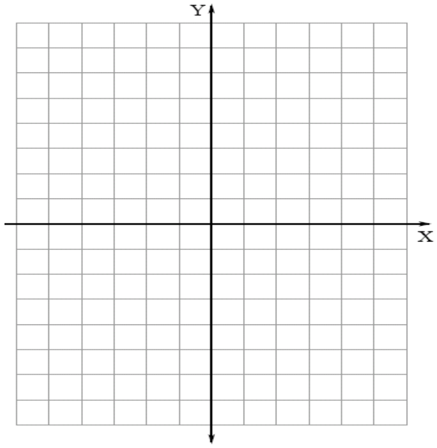
1. What specialized geometric figure is quadrilateral ABCD? Support your answer mathematically.

2. Draw the diagonals of ABCD. Find the coordinates of the midpoint of each diagonal. What do you notice?

3. Find the slopes of the diagonals of ABCD. What do you notice?

4. The diagonals of ABCD create four small triangles. Are any of these triangles congruent to any of the others? Why or why not?

Plot points E = (1, 2), F = (2, 5), G = (4, 3) and H = (5, 6).



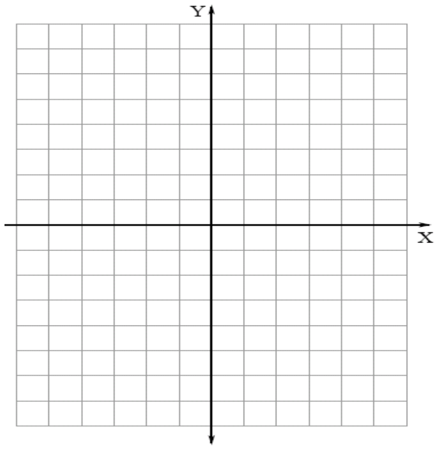
5. What specialized geometric figure is quadrilateral EFHG? Support your answer mathematically using two different methods.

6. Draw the diagonals of EFHG. Find the coordinates of the midpoint of each diagonal. What do you notice?

7. Find the slopes of the diagonals of EFHG. What do you notice?

8. The diagonals of EFHG create four small triangles. Are any of these triangles congruent to any of the others? Why or why not?

Plot points P = (4, 1), W = (-2, 3), M = (2,-5), and K = (-6, -4).



9. What specialized geometric figure is quadrilateral PWKM? Support your answer mathematically.

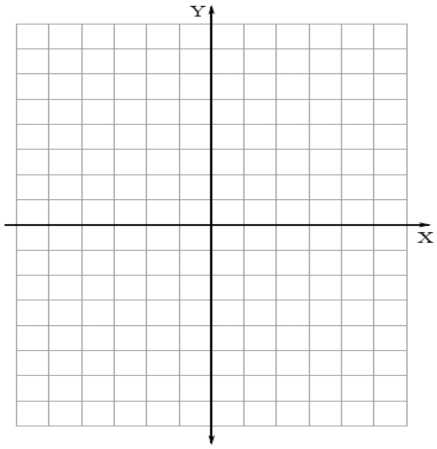
10. Draw the diagonals of PWKM. Find the coordinates of the midpoint of each diagonal. What do you notice?

11. Find the lengths of the diagonals of PWKM. What do you notice?

12. Find the slopes of the diagonals of PWKM. What do you notice?

13. The diagonals of ABCD create four small triangles. Are any of these triangles congruent to any of the others? Why or why not?

Plot points A = (1, 0), B = (-1, 2), and C = (2, 5).



14. Find the coordinates of a fourth point D that would make ABCD a rectangle. Justify that ABCD is a rectangle.

15. Find the coordinates of a fourth point D that would make ABCD a parallelogram that is not also a rectangle. Justify that ABCD is a parallelogram but is not a rectangle.

Euler's Village (Culminating Task)

Back to Task Table

Introduction

This task provides students an opportunity to apply the algebraic concepts of slope, intersection of two lines, distance from a point to line, and the distance formula.

Mathematical Goals

- Find the point on a line segment that separates the segments into a given ratio.
- Given the equation of a line and a point not on the line, find the equation of the line that passes through the point and is parallel/perpendicular to the given line.

Essential Questions

- How can a line be partitioned?
- How do we write the equation of a line that goes through a given point and is parallel or perpendicular to another line?

Georgia Standards of Excellence

MGSE9–12.G.GPE.4 Use coordinates to prove simple geometric theorems algebraically. *For example, prove or disprove that a figure defined by four given points in the coordinate plane is a rectangle; prove or disprove that the point (1, $\sqrt{3}$) lies on the circle centered at the origin and containing the point (0,2).*
(Focus on quadrilaterals, right triangles, and circles.)

MGSE9–12.G.GPE.5 Prove the slope criteria for parallel and perpendicular lines and use them to solve geometric problems (e.g., find the equation of a line parallel or perpendicular to a given line that passes through a given point).

MGSE9–12.G.GPE.6 Find the point on a directed line segment between two given points that partitions the segment in a given ratio.

MGSE9–12.G.GPE.7 Use coordinates to compute perimeters of polygons and areas of triangles and rectangles, e.g., using the distance formula.

Standards for Mathematical Practice

4. Model with mathematics.
   Students use mathematical properties to represent and answer questions about the problem situation.
6. Attend to precision.
   Students must decide on an appropriate answer form (radical, decimal, etc.) and level of accuracy.

Background Knowledge

- The distance between a point and a line is the length of the *shortest* segment connecting the point and the line. This segment is always perpendicular to the line.
- Students can write equations of lines and solve systems of linear equations.
- Students know the definition of a parallelogram.
- Students know how to find and interpret slope and length of segments.

Common Misconceptions

- Students might measure the distance from a point to a line only vertically or horizontally, rather than perpendicular to the line.

Grouping

- Partners / small group

Differentiation

Extension:

- Find another way (without using systems of equations) to find the point on the road that is closest to the well.
  (Possible solution: Draw three segments from the well to the road: one that is perpendicular to the road, one that is horizontal, and another that is vertical. This forms similar right triangles—two small and one large. The lengths of the large triangle's sides can be found, and proportional reasoning can be used to find other lengths to locate the house.)
  (Alternate solution: Draw three segments as described above. This happens to be an isosceles triangle, so the best location for the house is the midpoint of the large triangle's hypotenuse.)
- What if the road were curved? While we can't use calculations to determine the best location of the house, we can use tools like a compass and straightedge to estimate its location. Describe your method.
  (Possible solution: Draw a series of concentric circles centered at the well, increasing the radius until a circle intersects the road in one point. This intersection is the best location of the house.)

Intervention:

- Scaffolding may be necessary to find the minimum distance from the well to the road.

Formative Assessment Questions

- Describe how you can find the distance between the line $y = \frac{1}{3}x + 7$ and the point (–4, 5)?
- How do you use slope and distance to classify a figure?
- How do you interpret perimeter and area in real-world situations?

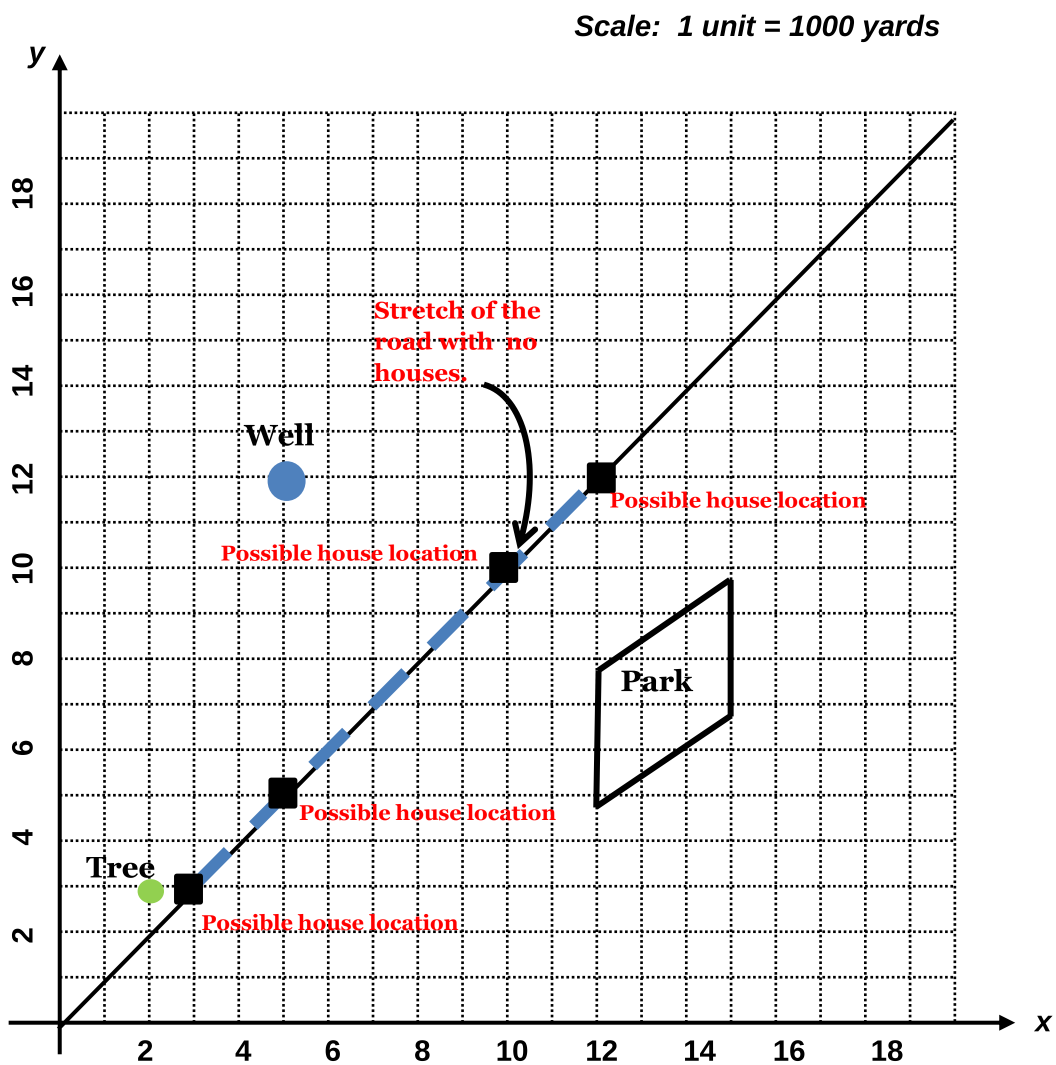
Euler's Village – Teacher Notes

You would like to build a house close to the village of Euler. There is a beautiful park just outside the village, and the road you would like to build your house on begins right at the town square and goes by this park.

The road follows an approximately north east direction as you leave town and continues for 3,000 feet. It passes right by a large shade tree located approximately 200 yards east and 300 yards north of the town square. There is a stretch of the road, between 300 and 1200 yards to the east of town, which currently has no houses. This stretch of road is where you would like to locate your house. All water supplies are linked to town wells, and the closest well to this part of the road is 500 yards east and 1200 yards north of the town square.

Comments:

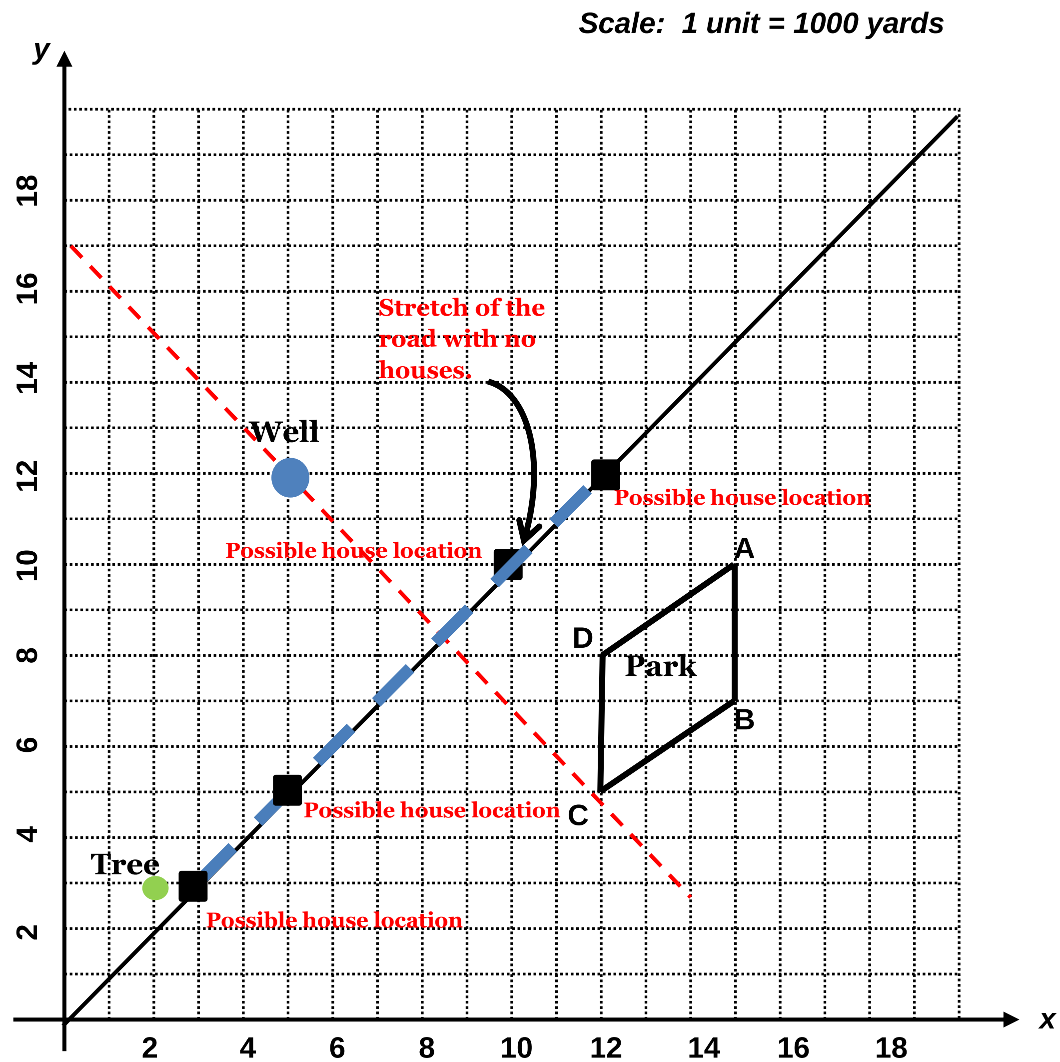
The first thing students will need to do is make a sketch of the town, the road, the tree and the well. The idea of a road traveling 'approximately north east' may need to be discussed. The easiest way to model this is to use the line f(x) = x.



1. How far from the well would it be if the house was located on the road 300 yards east of town? 500 yards east of town? 1,000 yards east of town? 1,200 yards east of town? (For the sake of calculations, do not consider how far the house is from the road, just use the road to make calculations)

Comments:
Students can use the distance formula to calculate these distances. The key to the problem is making sure they can find the coordinates for the well, houses, etc.

The calculation for the distance to the points at (500,500) and (1200, 1200) is shown below. It is not necessary for students to use the distance formula, but if they do, watch to see how they complete the calculations. This could provide an opportunity to focus on how the rule $\sqrt{x^2} = |x|$ makes the calculations simpler here.



Solutions:

The well is located at (500, 1200).

300 yards east of town would place the house at (300, 300)
$\sqrt{(\mathbf{500}-\mathbf{300})^{2}+(\mathbf{1200}-\mathbf{300})^{2}}=\sqrt{\mathbf{850000}}\approx \mathbf{921.95}$ → ***922 yards***

500 yards east of town would place the house at (500, 500)
The distance to the well is 700 yards. This can be found directly from the graph because it is directly below the well.

1000 yards east of town would place the house at (1000, 1000)
$\sqrt{(\mathbf{500}-\mathbf{1000})^{2}+(\mathbf{1200}-\mathbf{1000})^{2}}=\sqrt{\mathbf{290000}}\approx \mathbf{538.52}$ → ***539 yards***

1200 yards east of town would place the house at (1200, 1200)
The distance to the well is 700 yards. This can be found directly from the graph because this location is directly to the right of the well.

2. The cost of the piping leading from the well to the house is a major concern. Where should you locate your house in order to have the shortest distance to the well? (*Remember: the shortest distance between a line and a point is the length of the segment perpendicular to the line that passes through the point*). Justify your answer mathematically.

Comment:
The students need to determine the point on the line closest to the well. To do this they can find a line that is perpendicular to the road and goes through the point where the well is located. The point where the perpendicular line intersects the road would be the best place to locate the house. There are several ways the students can find the equation of the line: slope–intercept (shown), point–slope, etc.

<table>
<tr><td colspan="2">Solution:
The slope of the road is 1. A line perpendicular to the road would have a slope of –1. But it also needs to go through the point (500, 1200).</td></tr>
<tr><td>$m = -1$
$y = -1x + b$
$1200 = -1(500) + b$
$1700 = b$
$y = -x + 1700$</td><td>Graphing this shows the point we are interested in for the location of the house. Now we need to calculate the point of intersection of the two lines.</td></tr>
<tr><td>Using elimination:
$y = x$
$\underline{y = -x + 1700}$
$2y = 1700$
$y = 850$
$x = 850$</td><td>Using substitution:
$y = x$
$y = -x + 1700$
$x = -x + 1700$
$2x = 1700$
$x = 850$
$y = 850$</td></tr>
</table>

The house should be located at the point (850, 850) to be the shortest distance to the well.

3. If the cost of laying pipes is $22.50 per linear yard, how much will it cost to connect your house to the well?

Comments:
The students should use the location they found in #2. To find the length of the piping they need to use the distance formula.

Solution:
First, we need to determine the distance from (500, 1200), the location of the well, to the point (850, 850) the location of the house. Using the distance formula we find:
$\sqrt{(\mathbf{500-850})^2+(\mathbf{1200-850})^2}=\sqrt{\mathbf{245000}}\approx\mathbf{494.97}$

To calculate cost we need to multiply the cost of the pipes by the distance:
494.97 yds · $22.50 / yd = $11,136.83

4. You also want to install a swimming pool on the line with the pipes. You want the front edge of the pool to be $\frac{3}{5}$ the distance from the road to the well. What are the coordinates of the front corner of the swimming pool?

Solution: $\left(\frac{3}{5}(\mathbf{500-850})+\mathbf{850},\frac{3}{5}(\mathbf{1200-850})+\mathbf{850}\right)=(\mathbf{640},\mathbf{1060})$

5. The builder of your house is impressed by your calculations and wants to use the same method for placing other houses. Describe the method you used. Would you want him to place the other houses in the same manner?

Comment:
The students need to remember they were trying to find the point that would be closest to the well. If the builder wants to place other houses along the road, he will not need to calculate a perpendicular line. He will simply need to calculate the distance between the two points.

Solution:

Answers will vary. Sample answer:

The perpendicular distance from any point to a line is the shortest. So, we needed to find a line perpendicular to the road that would go through the point where the well was located. To do this we used the point slope formula. We knew the slope of the perpendicular line would be –1 and the point it had to go through was (500, 1200). Once we had the equation of the line, we had to find the point of intersection of the two lines, which was (850, 850). We then had to find the distance between that point and the well. Once we had that distance we could multiply it by the cost of the piping.

The same method (with different numbers, of course) would always yield the correct answer.

6. Write a formula that the builder could use to find the cost of laying pipes to any house along this road. How would you have to change your formula for another road?

Solution:

Let (a, b) represent the location of the house. The distance from the well to the house would be:

distance from (500, 1200) to (a, b): $\sqrt{(500-a)^2+(1200-b)^2}$

The builder would then need to multiply this distance by the cost of the piping, $22.50 / yard.

This equation would give him the cost of the piping to any point along the line.
Cost = 22.50 · $\sqrt{(500-a)^2+(1200-b)^2}$

7. One day you were wondering what geometric shape the park formed, so you looked at the map and took some measurements. Using those measurements, prove what specific geometric shape is formed by the park boundaries.

Solution:
Students will need to calculate the lengths of the sides using the distance formula. They will also need to find the slope of each side to test for parallel/perpendicular sides. To aid in communicating, the students will need to label each vertex.

SIDE	SLOPE	LENGTH
$\overline{AB}$	*Undefined*	*3000*
$\overline{BC}$	$\frac{2}{3}$	$\sqrt{2000^2+3000^2}\approx 3605$
$\overline{CD}$	*Undefined*	*3000*
$\overline{DA}$	$\frac{2}{3}$	$\sqrt{2000^2+3000^2}\approx 3605$

Since opposite sides of the figure have the same slope, they are parallel. Since opposite sides of the figure have the same length, they are congruent. Therefore, the figure is a PARALLELOGRAM.

8. Your service club noticed that the fence around the park needed to be replaced. How much fencing would be needed?

Solution:
Students should recognize that they are finding the perimeter of the park. So, adding the lengths found in #7, the amount of fencing needed is 13,210 yards.

9. While the fence was being replaced, some damage occurred to a portion the grass in the park. The entire park is covered with grass, but only ¼ of the total area needs to be replaced. Grass seed is sold in bags that cover 5000 square yards each. How many bags of grass seed would your club need to purchase to reseed the entire park?

Solution:
Using the formula A = bh, students will find the entire area of the park:
$A = (3000)(3000)$
$A = 9{,}000{,}000\ yd^2$

Since only ¼ of the grass needs to be replaced, the area to be reseeded is found by:
$\frac{1}{4}(9{,}000{,}000) = 2{,}250{,}000\ yd^2$

Each bag will cover 5000 yd², so the club will need to purchase:
$\frac{2{,}250{,}000}{5000} = 450$ ***bags of grass seed***

Culminating Task: Euler's Village

Name__________________________________ **Date**___________________

Mathematical Goals
- Find the point on a line segment that separates the segments into a given ratio.
- Given the equation of a line and a point not on the line, find the equation of the line that passes through the point and is parallel/perpendicular to the given line.

Essential Questions
- How can a line be partitioned?
- How do we write the equation of a line that goes through a given point and is parallel or perpendicular to another line?

Georgia Standards of Excellence
MGSE9–12.G.GPE.4 Use coordinates to prove simple geometric theorems algebraically. *For example, prove or disprove that a figure defined by four given points in the coordinate plane is a rectangle; prove or disprove that the point (1, √3) lies on the circle centered at the origin and containing the point (0,2).*
(Focus on quadrilaterals, right triangles, and circles.)

MGSE9–12.G.GPE.5 Prove the slope criteria for parallel and perpendicular lines and use them to solve geometric problems (e.g., find the equation of a line parallel or perpendicular to a given line that passes through a given point).

MGSE9–12.G.GPE.6 Find the point on a directed line segment between two given points that partitions the segment in a given ratio.

MGSE9–12.G.GPE.7 Use coordinates to compute perimeters of polygons and areas of triangles and rectangles, e.g., using the distance formula.

Standards for Mathematical Practice

4. Model with mathematics.

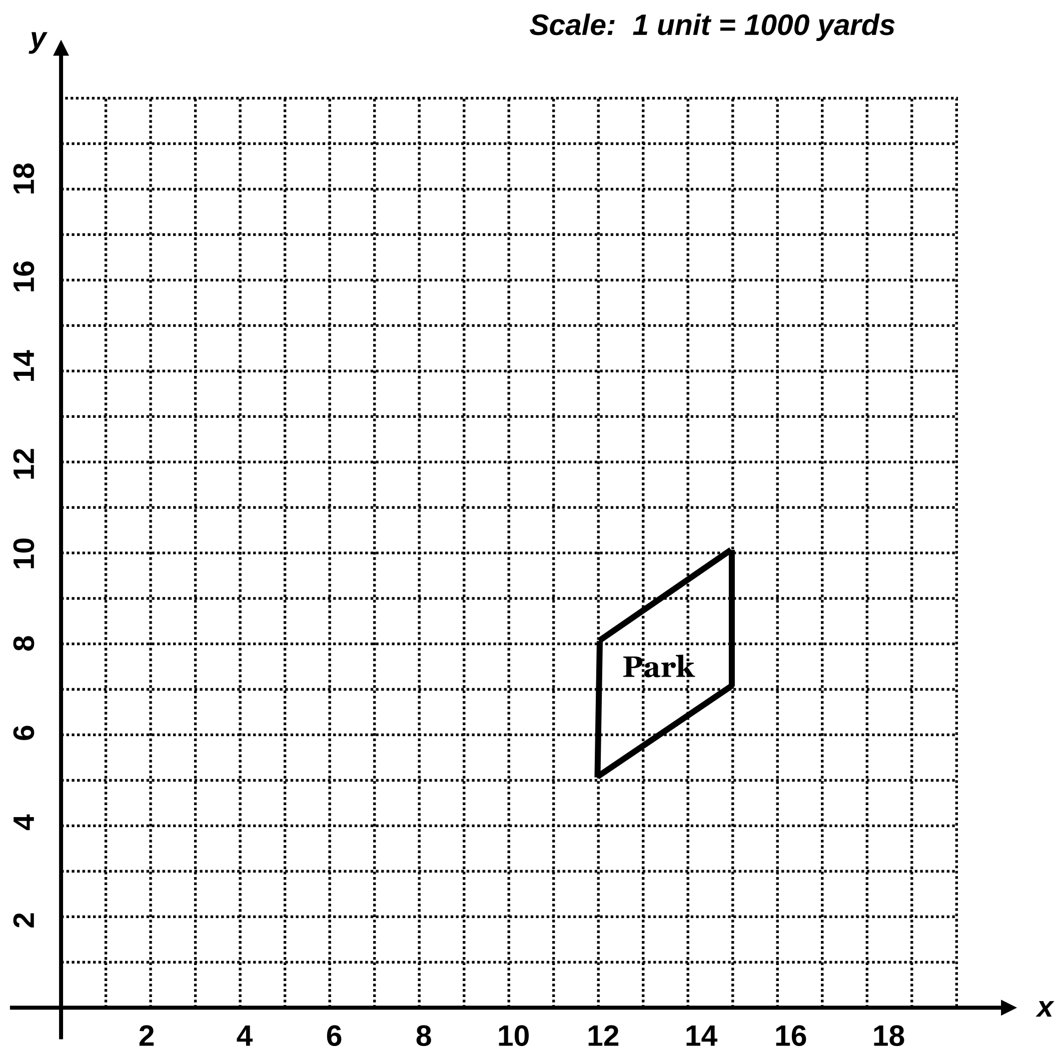
6. Attend to precision.

Culminating Task: Euler's Village

Name__________________________________ **Date**___________________

You would like to build a house close to the village of Euler. There is a beautiful park just outside the village, and the road you would like to build your house on begins right at the town square and goes by this park.

The road follows an approximately north east direction as you leave town and continues for 3,000 feet. It passes right by a large shade tree located approximately 200 yards east and 300 yards north of the town square. There is a stretch of the road, between 300 and 1200 yards to the east of town, which currently has no houses. This stretch of road is where you would like to locate your house. All water supplies are linked to town wells and the closest well to this part of the road is 500 yards east and 1200 yards north of the town square.



1. How far from the well would it be if the house was located on the road 300 yards east of town? 500 yards east of town? 1,000 yards east of town? 1,200 yards east of town? (For the sake of calculations, assume the house is exactly *on* the road.)

2. The cost of the piping leading from the well to the house is a major concern. Where should you locate your house in order to have the shortest distance to the well? (*Remember: the shortest distance between a line and a point is the length of the segment perpendicular to the line that passes through the point*). Justify your answer mathematically.

3. If the cost of laying pipes is $22.50 per linear yard, how much will it cost to connect your house to the well?

4. You also want to install a swimming pool on the line with the pipes. You want the front edge of the pool to be $\frac{3}{5}$ the distance from the road to the well. What are the coordinates of the front corner of the swimming pool?

5. The builder of your house is impressed by your calculations and wants to use the same method for placing other houses. Describe the method you used. Would you want him to place the other houses in the same manner?

6. Write a formula that the builder could use to find the cost of laying pipes to any house along this road. How would you have to change your formula for another road?

7. One day you were wondering what geometric shape the park formed, so you looked at the map and took some measurements. Using those measurements, prove what specific geometric shape is formed by the park boundaries.

8. Your service club noticed that the fence around the park needed to be replaced. How much fencing would be needed?

9. While the fence was being replaced, some damage occurred to a portion the grass in the park. The entire park is covered with grass, but only ¼ of the total area needs to be replaced. Grass seed is sold in bags that cover 5000 square yards each. How many bags of grass seed would your club need to purchase to reseed the entire park?

Rolling Cups

[Back to Task Table]

Modification of Source: Lesson Materials from Mathematics Assessment Project
http://map.mathshell.org/materials/download.php?fileid=1254

ESSENTIAL QUESTIONS:

- How do you choose appropriate mathematics to solve a non-routine problem?
- How do you generate useful data by systematically controlling variables?
- How do you develop experimental and analytical models of a physical situation?

TASK COMMENTS:

Tasks and lessons from the Mathematics Assessment Project are specifically designed to help teachers effectively formatively assess their students. The way the tasks and lessons are designed gives the teacher a clear understanding of what the students are able to do and not do. Within the lesson, teachers will find suggestions and question prompts that will help guide students towards understanding. For more information access the MAP website:
http://www.map.mathshell.org/materials/background.php?subpage=formative

The task that this task is based on, *Modeling: Rolling Cups*, is a Formative Assessment Lesson (FAL) that can be found at the website:
http://map.mathshell.org/materials/lessons.php?taskid=428&subpage=problem

The FAL document provides a clear lesson design, from the opening of the lesson to the closing of the lesson.

The PDF version of the task can be found at the link below:
http://map.mathshell.org/materials/download.php?fileid=1254

GEORGIA STANDARDS OF EXCELLENCE

MGSE9-12.G.MG.1 Use geometric shapes, their measures, and their properties to describe objects (e.g., modeling a tree trunk or a human torso as a cylinder).

MGSE9-12.G.MG.3 Apply geometric methods to solve design problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios).

Standards for Mathematical Practice

This lesson uses all of the practices with emphasis on:

1. Make sense of problems and persevere in solving them by requiring students to interpret and make meaning of a problem and find a logical starting point, and to monitor their progress and change their approach to solving the problem, if necessary.

2. Reason abstractly and quantitatively by requiring students to make sense of quantities and their relationships to one another in problem situations.

3. Construct viable arguments and critique the reasoning of others by engaging students on discussion of why they agree or disagree with responses, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

4. Model with mathematics by expecting students to apply the mathematics concepts they know in order to solve problems arising in everyday situations, and reflect on whether the results are sensible for the given scenario.

More information along with guidelines for 3-Act Tasks may be found in the Comprehensive Course Guide.

Act I

Have the students watch a video of each cup being rolled. If you have cups, you can also bring them in and have students roll their own cups along with watching the video. If the MAP website must be used, as opposed to the youtube link or source video, only show the parts of each cup video that shows the cup rolling. A youtube link is provided below for the video or Act I – Rolling Cups.mov can be used.

After students have watched the video, leave the video on loop and ask the students to discuss questions that they might ask about the situation (e.g., the dimensions of the glass, how long it takes to complete a roll, how big the roll circle is etc.). Keep a running tab on the whiteboard of the questions that students generate. If not posed, introduce, "How big is the roll circle?" as the focal question. As the students work on the problem, leave the video playing on loop.

Watch the presented video, which can be found at:
http://youtu.be/psMYzAqRJVQ
or at:
http://real.doe.k12.ga.us/vod/gso/math/Act-I-Rolling-Cups.mp4

Act II

With the focal question introduced, have students work in groups pursuing this question, as well as the other questions that they have raised. Write, "What information do you need?" and remind them they should be asking this as they go along. The most relevant information is:
Green Cup:
Diameter: 3 inches
Slant Length: 3.5 inches
Bottom Diameter: 2 inches
Clear Short Cup:
Diameter 3.5 inches
Slant Length: 3.75 inches
Botton Diamter: 3 inches

Soup Can:
Diameter 3 inches
Slant Length: 4.25 inches

Tall Cup:
Diameter: 2.5 inches
Slant Length: 5.75 inches
Bottom Diameter: 2 inches

This information is on the video at: http://real.doe.k12.ga.us/vod/gso/math/videos/Act-II-Rolling-Cups-with-Cup-Measurements.mp4

As you work on your problems, think about and determine what information you need.

Act III

Video reveal is done, using Act III – Rolling Cups.mp4, the MAP website, or the following Youtube link.

Watch the presented video, which can be found at:
http://youtu.be/d9yCNzA4fDg
or at
http://real.doe.k12.ga.us/vod/gso/math/Act-III-Rolling-Cups.mp4

The following questions can be used as extension questions. Later in the unit, you can also return to this task and have them define equations for the roll circles using the center of the circle as the origin.

Design your own cup and then determine what its roll radius will be.

Choose a roll radius and design at least 2 cups with this roll radius.

Rolling Cups

Act I

Watch the presented video, which can be found at:
http://youtu.be/psMYzAqRJVQ
or at:
http://real.doe.k12.ga.us/vod/gso/math/Act-I-Rolling-Cups.mp4

Act II

As you work on your problems, think about and determine what information you need.

Act III

Watch the presented video, which can be found at:
http://youtu.be/d9yCNzA4fDg
or at:
http://real.doe.k12.ga.us/vod/gso/math/Act-III-Rolling-Cups.mp4

Design your own cup and then determine what its roll radius will be.

Choose a roll radius and design at least 2 cups with this roll radius.

Deriving the General Equation of a Circle

Back to Task Table

Standard Addressed in this Task

MGSE9-12.G.GPE.1 Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.

Standards for Mathematical Practice

2. Reason abstractly and quantitatively by requiring students to make sense of quantities and their relationships to one another in problem situations.

3. Construct viable arguments and critique the reasoning of others by engaging students on discussion of why they agree or disagree with responses, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

7. Look for and make use of structure by expecting students to apply rules, look for patterns and analyze structure.

8. Look for and express regularity in repeated reasoning by expecting students to understand broader applications and look for structure and general methods in similar situations.

Common Student Misconceptions

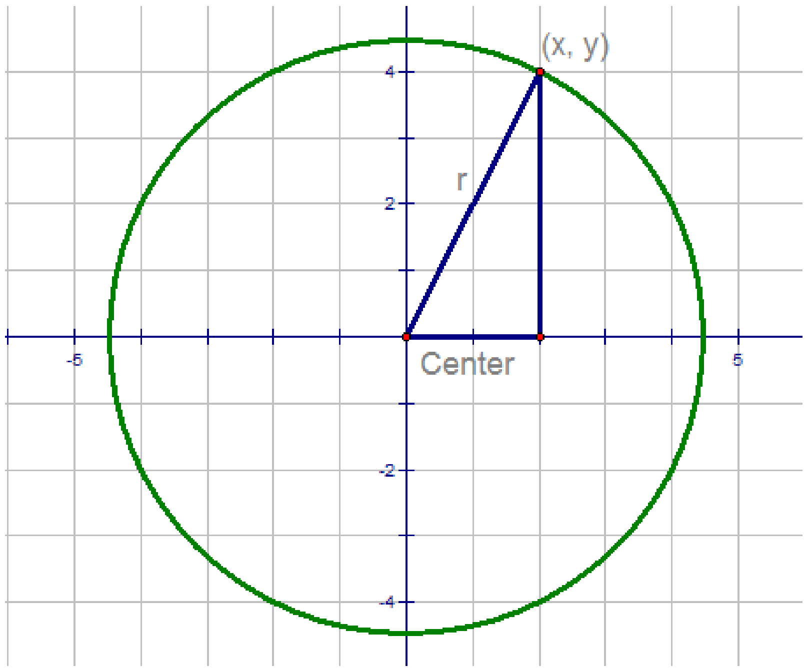
- Because new vocabulary is being introduced in this cluster, remembering the names of the conic sections can be problematic for some student
- The Euclidean distance formula involves squared, subscripted variables whose differences are added. The notation and multiplicity of steps can be a serious stumbling block for some students.
- The method of completing the square is a multi-step process that takes time to assimilate.
- A geometric demonstration of completing the square can be helpful in promoting conceptual understanding

Teacher Notes

This task is designed to walk a student through the process of generalizing the formula for the equation of a circle. Hopefully the teacher will be needed less and less as the students become more familiar with the process. As a teacher, it is not your job to provide the students with the answer, but to encourage them to persevere through the problem solving process. Students will benefit greatly from practice outside of class through thoughtful homework assignments that develop fluency with the equations.

Part 1: Finding the Radius

Consider the circle below. Notice the center is at the origin and a point is on the circle (x, y).



Answer the following questions or perform the requested constructions.

1. Construct a line segment from the center to the point (x, y) on the circle and label it "r". What is this line segment called?

   Solution
   The line segment is the radius of the circle

2. Construct a right triangle with r as the hypotenuse. What are the coordinates of the point (x, y)?

   Solution
   The point is (2, 4)

3. What is the measure of r? Explain your method for calculating it.

   Solution
   The measure can be found several ways. One way is the Pythagorean Theorem:

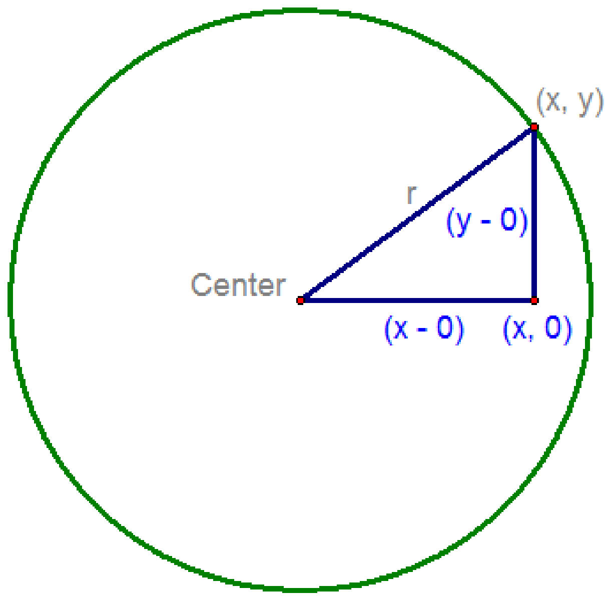
$$2^2 + 4^2 = c^2$$
$$20 = c^2$$
$$c = \sqrt{20} = 2\sqrt{5}$$

Another possibility is by using the distance formula:

$$d = \sqrt{(2-0)^2 + (4-0)^2}$$
$$d = \sqrt{20} = 2\sqrt{5}$$

Part 2: Circles Centered at the Origin.

Consider the circle below. The center is located at the origin.



Answer the following questions or perform the requested constructions.

1. Construct a radius from the center to the point (x, y). Label it "r".

2. Construct a right triangle with r as the hypotenuse. What are the coordinates of the point where the legs meet?

   Comments
   It is important here that students begin the process of generalizing the point. This is at the heart of deriving a formula. Don't be afraid to spend a little extra time on developing the idea that this is not a specific point, but could be any point on the circle.

   Solution
   The point is (x, 0)

3. Write an expression for the distance from the center to the point from #2. Label the triangle accordingly.

   Solution
   (x – 0)

4. Write an expression for the distance from (x, y) to the point from #2. Label the triangle accordingly.

Solution
(y – 0)

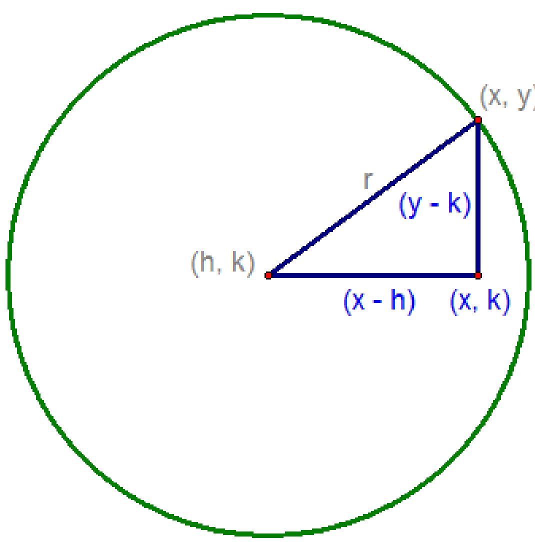
5. Now use your method from part one to write an expression for r^2

Solution
From the Pythagorean Theorem: $(x-0)^2+(y-0)^2=r^2$

Part 3: Circles centered anywhere!

In the previous section, you found that $x^2+y^2=r^2$. This is the general equation for a circle centered at the origin. However, circles are not always centered at the origin. Use the following circle and directions to find the general equation for a circle centered anywhere.

Solution
This is the main point of the activity. By now, the students should have examples for reference and may be able to complete part 3 on their own.



Answer the following questions and perform the requested constructions.

1. Construct a radius between (h, k) and (x, y). Then create a right triangle with the radius as the hypotenuse. Find the coordinates for the point where the legs meet.

Solution
(x, k)

2. Write an expression for the distance between (x, y) and the point from #1. Label the triangle.

Solution
(y – k)

3. Write an expression for the distance between (h, k) and the point from #1. Label the triangle.

Solution
(x – h)

4. Now write an expression for r^2.

Solution
From the Pythagorean Theorem: $(x-h)^2+(y-k)^2=r^2$

Deriving the General Equation of a Circle

Standard Addressed in this Task

MGSE9-12.G.GPE.1 Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.

Standards for Mathematical Practice

2. Reason abstractly and quantitatively by requiring students to make sense of quantities and their relationships to one another in problem situations.

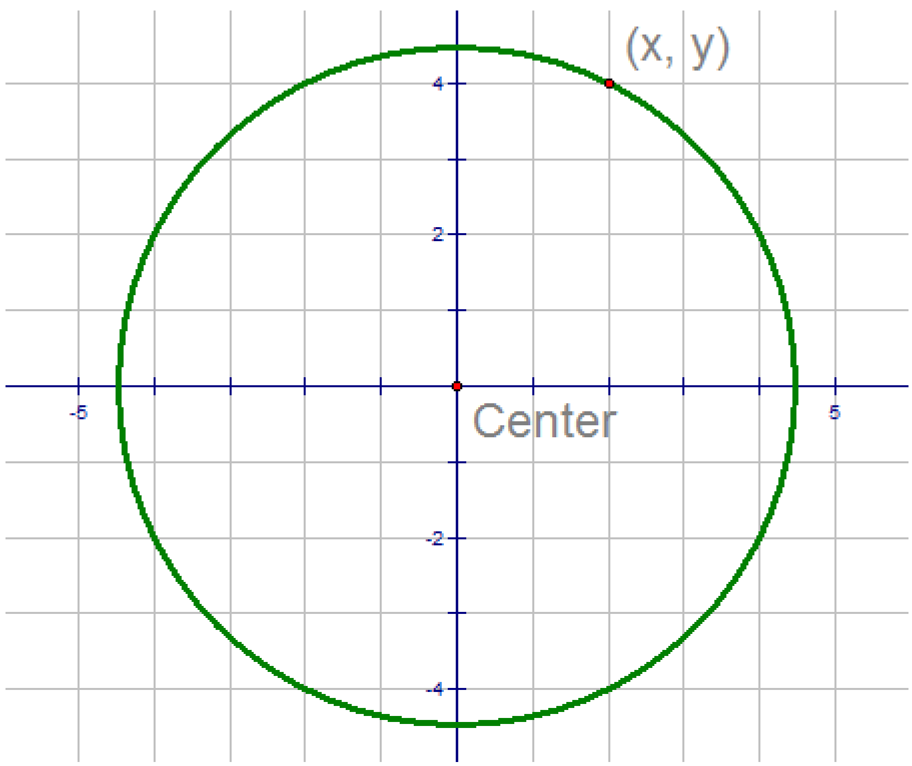
3. Construct viable arguments and critique the reasoning of others by engaging students on discussion of why they agree or disagree with responses, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

7. Look for and make use of structure by expecting students to apply rules, look for patterns and analyze structure.

8. Look for and express regularity in repeated reasoning by expecting students to understand broader applications and look for structure and general methods in similar situations.

Part 1: Finding the Radius

Consider the circle below. Notice the center is at the origin and a point is on the circle (x, y).



Answer the following questions

or perform the requested constructions.

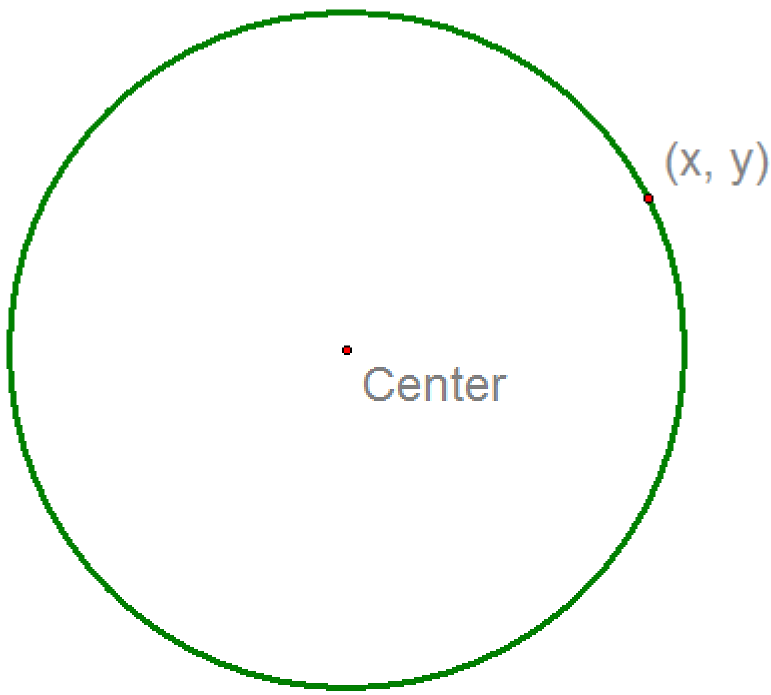
1. Construct a line segment from the center to the point (x, y) on the circle and label it “r”. What is this line segment called?

2. Construct a right triangle with r as the hypotenuse. What are the coordinates of the point (x, y)?

3. What is the measure of r? Explain your method for calculating it.

Part 2: Circles Centered at the Origin.

Consider the circle below. The center is located at the origin.



Answer the following questions or perform the requested constructions.

1. Construct a radius from the center to the point (x, y). Label it “r”.

2. Construct a right triangle with r as the hypotenuse. What are the coordinates of the point where the legs meet?

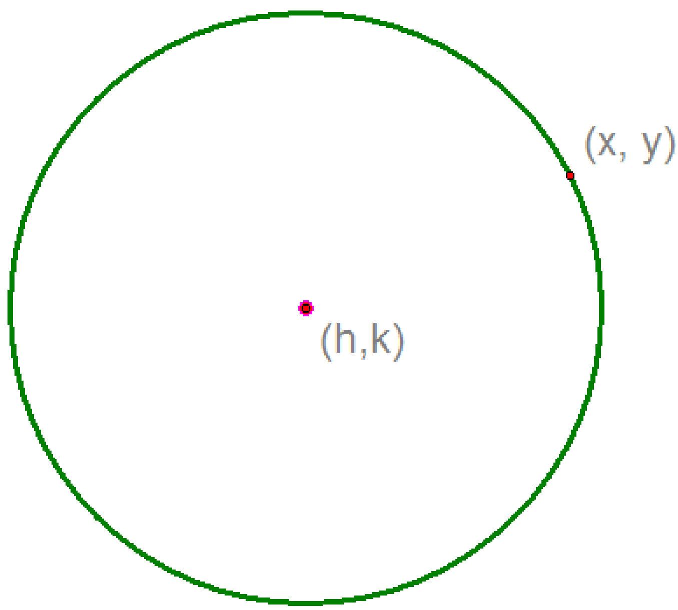
3. Write an expression for the distance from the center to the point from #2. Label the triangle accordingly.

4. Write an expression for the distance from (x, y) to the point from #2. Label the triangle accordingly.

5. Now use your method from part one to write an expression for r^2

Part 3: Circles centered anywhere!

In the previous section, you found that $x^2+y^2=r^2$. This is the general equation for a circle centered at the origin. However, circles are not always centered at the origin. Use the following circle and directions to find the general equation for a circle centered anywhere.



Answer the following questions and perform the requested constructions.

1. Construct a radius between (h, k) and (x, y). Then create a right triangle with the radius as the hypotenuse. Find the coordinates for the point where the legs meet.

2. Write an expression for the distance between (x, y) and the point from #1. Label the triangle.

3. Write an expression for the distance between (h, k) and the point from #1. Label the triangle.

4. Now write an expression for r^2.

Formative Assessment Lesson: Equations of Circles – 1

Back to Task Table

Source: Formative Assessment Lesson Materials from Mathematics Assessment Project
http://map.mathshell.org/materials/download.php?fileid=1202

ESSENTIAL QUESTIONS:

- How do you use the Pythagorean theorem to derive the equation of a circle?
- How do you translate between the geometric features of circles and their equations?

TASK COMMENTS:

Tasks and lessons from the Mathematics Assessment Project are specifically designed to help teachers effectively formatively assess their students. The way the tasks and lessons are designed gives the teacher a clear understanding of what the students are able to do and not do. Within the lesson, teachers will find suggestions and question prompts that will help guide students towards understanding. For more information access the MAP website:
http://www.map.mathshell.org/materials/background.php?subpage=formative

The task, *Equations of Circles - 1*, is a Formative Assessment Lesson (FAL) that can be found at the website: http://map.mathshell.org/materials/lessons.php?taskid=406&subpage=concept

The FAL document provides a clear lesson design, from the opening of the lesson to the closing of the lesson.

The PDF version of the task can be found at the link below:
http://map.mathshell.org/materials/download.php?fileid=1202

STANDARDS ADDRESSED IN THIS TASK:

Translate between the geometric description and the equation for a conic section

MGSE9-12.G.GPE.1 Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.

Standards for Mathematical Practice
This lesson uses all of the practices with emphasis on:

1. Make sense of problems and persevere in solving them by requiring students to interpret and make meaning of a problem and find a logical starting point, and to monitor their progress and change their approach to solving the problem, if necessary.

5. Use appropriate tools strategically by expecting students to consider available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a compass, a calculator, software, etc.

7. Look for and make use of structure by expecting students to apply rules, look for patterns and analyze structure.

Formative Assessment Lesson: Equations of Circles – 2 [Back to Task Table]

Source: Formative Assessment Lesson Materials from Mathematics Assessment Project
http://map.mathshell.org/materials/download.php?fileid=1247

ESSENTIAL QUESTIONS:

- How do you translate between the equations of circles and their geometric features?
- How do you sketch a circle from its equation?

TASK COMMENTS:

Tasks and lessons from the Mathematics Assessment Project are specifically designed to help teachers effectively formatively assess their students. The way the tasks and lessons are designed gives the teacher a clear understanding of what the students are able to do and not do. Within the lesson, teachers will find suggestions and question prompts that will help guide students towards understanding. For more information access the MAP website:
http://www.map.mathshell.org/materials/background.php?subpage=formative

The task, *Equations of Circles - 2*, is a Formative Assessment Lesson (FAL) that can be found at the website http://map.mathshell.org/materials/lessons.php?taskid=425&subpage=concept

The FAL document provides a clear lesson design, from the opening of the lesson to the closing of the lesson.

The PDF version of the task can be found at the link below:
http://map.mathshell.org/materials/download.php?fileid=1247

STANDARDS ADDRESSED IN THIS TASK:

Translate between the geometric description and the equation for a conic section

MGSE9-12.G.GPE.1 Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.

Standards for Mathematical Practice
This lesson uses all of the practices with emphasis on:

1. Make sense of problems and persevere in solving them by requiring students to interpret and make meaning of a problem and find a logical starting point, and to monitor their progress and change their approach to solving the problem, if necessary.

7. Look for and make use of structure by expecting students to apply rules, look for patterns and analyze structure.

Converting Standard Form to General Form

Standard Addressed in this Task

MGSE9-12.G.GPE.1 Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.

Standards for Mathematical Practice

2. Reason abstractly and quantitatively by requiring students to make sense of quantities and their relationships to one another in problem situations.

3. Construct viable arguments and critique the reasoning of others by engaging students on discussion of why they agree or disagree with responses, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

7. Look for and make use of structure by expecting students to apply rules, look for patterns and analyze structure.

8. Look for and express regularity in repeated reasoning by expecting students to understand broader applications and look for structure and general methods in similar situations.

Common Student Misconceptions

- Because new vocabulary is being introduced in this cluster, remembering the names of the conic sections can be problematic for some students.
- The Euclidean distance formula involves squared, subscripted variables whose differences are added. The notation and multiplicity of steps can be a serious stumbling block for some students.
- The method of completing the square is a multi-step process that takes time to assimilate.
- A geometric demonstration of completing the square can be helpful in promoting conceptual understanding.

Teacher Notes

This task is a brief look at the algebraic manipulations necessary to change an equation from standard form to general form. This task should in no way take the place of regular classroom instruction when it comes to writing the equation of a circle. Students should feel comfortable writing the equation of a circle given a center and a radius.

In Task 1, you used the Pythagorean Theorem to derive: $(x-h)^2+(y-k)^2=r^2$, which is known as the **Standard Form of a Circle**.

By expanding the binomial terms this equation can be written as

$x^2-2hx+h^2+y^2-2ky+k^2=r^2$ or

$x^2+y^2-2hx-2ky+h^2+k^2-r^2=0$

Then, by using variables for coefficients, and realizing that h^2, k^2 and r^2 are all real numbers and can be added, we derive the **General Form equation of a Circle**:

$$Ax^2+By^2+Cx+Dy+E=0$$

***Note:** In order to be a circle, A and B must be equal.*

Occasionally, it becomes necessary to convert the equation of a circle from Standard to General Form. Take the circle with a center at (3, 4) and a radius of 6, for example.

The Standard Equation would be: $(x-3)^2+(y-4)^2=6^2$

By expanding the binomial terms, we would then have: $x^2-6x+9+y^2-8y+16=36$.

Grouping the monomials according to degree would yield:
$x^2+y^2-6x-8y+9+16-36=0$

Through arithmetic, the General Form equation would be: $x^2+y^2-6x-8y-11=0$

Write the General form equations for the following circles:

1. A circle with center *(1, -6)* and radius *4* ***Solution:*** $x^2+y^2-2x+12y+21=0$
2. A circle with center *(6, 8)* and radius *10* ***Solution:*** $x^2+y^2-12x-16y=0$
3. A circle with center *(0, 3)* and radius $2\sqrt{3}$ ***Solution:*** $x^2+y^2-6y-3=0$
4. A circle with center *(-0.5, 5.5)* and radius *8.4* ***Solution:*** $x^2+y^2+x-11y-40.06=0$
5. A circle with center *(a, b)* and radius *c* ***Solution:***
   $x^2+y^2-2ax-2by+a^2+b^2-c^2=0$

Converting Standard Form to General Form

[Back to Task Table]

Standard Addressed in this Task
MGSE9-12.G.GPE.1 Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.

Standards for Mathematical Practice
2. Reason abstractly and quantitatively by requiring students to make sense of quantities and their relationships to one another in problem situations.

3. Construct viable arguments and critique the reasoning of others by engaging students on discussion of why they agree or disagree with responses, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

7. Look for and make use of structure by expecting students to apply rules, look for patterns and analyze structure.

8. Look for and express regularity in repeated reasoning by expecting students to understand broader applications and look for structure and general methods in similar situations.

In Task 1, you used the Pythagorean Theorem to derive: $(x-h)^2+(y-k)^2=r^2$, which is known as the **Standard Form of a Circle**.

By expanding the binomial terms this equation can be written as

$x^2-2hx+h^2+y^2-2ky+k^2=r^2$ or

$x^2+y^2-2hx-2ky+h^2+k^2-r^2=0$

Then, by using variables for coefficients, and realizing that h^2, k^2 and r^2 are all real numbers and can be added, we derive the **General Form equation of a Circle**:

$$Ax^2+By^2+Cx+Dy+E=0$$

*****Note: In order to be a circle, A and B must be equal.*

Occasionally, it becomes necessary to convert the equation of a circle from Standard to General Form. Take the circle with a center at (3, 4) and a radius of 6, for example.

The Standard Equation would be: $(x-3)^2+(y-4)^2=6^2$

By expanding the binomial terms, we would then have: $x^2-6x+9+y^2-8y+16=36$.
Grouping the monomials according to degree would yield:
$x^2+y^2-6x-8y+9+16-36=0$

Through arithmetic, the General Form equation would be: $x^2+y^2-6x-8y-11=0$

Write the General form equations for the following circles:

1. A circle with center *(1, -6)* and radius *4*

2. A circle with center *(6, 8)* and radius *10*

3. A circle with center *(0, 3)* and radius $2\sqrt{3}$

4. A circle with center *(-0.5, 5.5)* and radius *8.4*

5. A circle with center *(a, b)* and radius *c*

Completing the Square in a Circle?

Back to Task Table

Standard Addressed in this Task
MGSE9-12.G.GPE.1 Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.

Standards for Mathematical Practice
2. Reason abstractly and quantitatively by requiring students to make sense of quantities and their relationships to one another in problem situations.

3. Construct viable arguments and critique the reasoning of others by engaging students on discussion of why they agree or disagree with responses, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

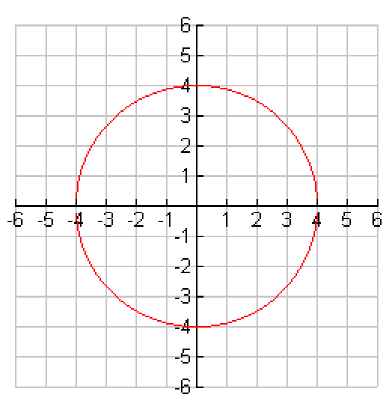
7. Look for and make use of structure by expecting students to apply rules, look for patterns and analyze structure.

8. Look for and express regularity in repeated reasoning by expecting students to understand broader applications and look for structure and general methods in similar situations.

Common Student Misconceptions
- Because new vocabulary is being introduced in this cluster, remembering the names of the conic sections can be problematic for some students.
- The Euclidean distance formula involves squared, subscripted variables whose differences are added. The notation and multiplicity of steps can be a serious stumbling block for some students.
- The method of completing the square is a multi-step process that takes time to assimilate.
- A geometric demonstration of completing the square can be helpful in promoting conceptual understanding.

1. Write equations for the following circle graphs in both standard form and general form.

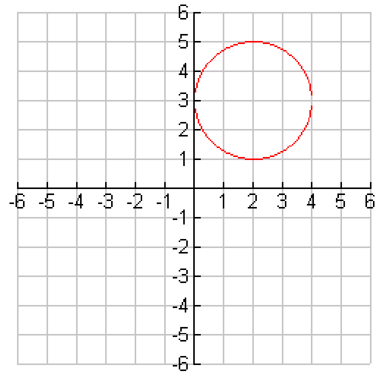
a.



Solution:
center at (0, 0) and radius r = 4

$x^2 + y^2 = 4$ **(standard form)**
$x^2 + y^2 - 4 = 0$ **(general form)**

b.



Solution:
center at (2, 3) and radius r = 2

$(x - 2)^2 + (y - 3)^2 = 2^2$ **(standard form)**

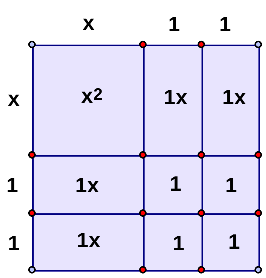
$x^2 - 4x + 4 + y^2 - 6y + 9 = 4$

$x^2 + y^2 - 4x - 6y + 9 = 0$ (general form)

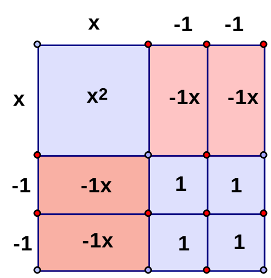
Teacher Notes:
Using a geometric area model approach will lend a visual to the idea of "completing the square". Students often get wrapped up in an algorithm for completing the square and never understand the idea behind it. Some students may find this confusing if they have not been exposed to algebra tiles earlier in their curriculum. This is only a suggested presentation, not the only way to teach it.

To change from general form to standard form, it is necessary to complete the square for x and y. **Completing the square** is an algebraic tool used to change equations of conic sections given in general form, $Ax^2 + Cy^2 + Dx + Ey + F = 0$, to standard form, $\mathbf{(x - h)^2 + (y - k)^2 = r^2}$. Standard form is the form used to graph conic sections.

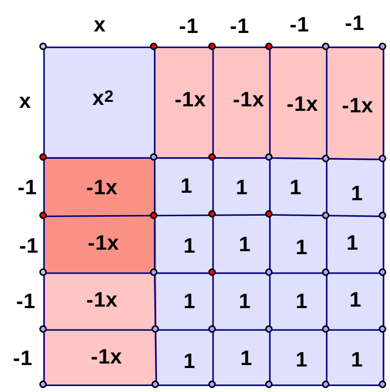
Perfect squares are numbers or expressions which have exactly two identical factors.
$(2)(2) = 4$ $(-5)(-5) = 25$ $(3x)(3x) = 9x^2$ $(-6y)(-6y) = 36y^2$ $(x + 2)(x + 2) = x^2 + 4x + 4$
Consider the following geometric area models of three perfect squares. The area is given as both factors and as a quadratic expression.



area = (x + 2)(x + 2)
= x2 + 4x + 4



area = (x - 2)(x - 2)
= x2 - 4x + 4



area = (x - 4)(x - 4)
= x2 - 8x + 16

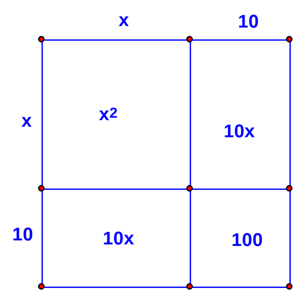
2. Find the products of the following expressions.

a. $(x + 1)^2 = (x + 1)(x + 1) =$ ***Solution:*** $x^2 + 1x + 1x + 1 = x^2 + 2x + 1$
b. $(x - 3)^2 = (x - 3)(x - 3) =$ ***Solution:*** $x^2 - 3x - 3x + 9 = x^2 - 6x + 9$
c. $(x - 5)^2 = (x - 5)(x - 5) =$ ***Solution:*** $x^2 - 5x - 5x + 25 = x^2 - 10x + 25$
d. $(x + 7)^2 = (x + 7)(x + 7) =$ ***Solution:*** $x^2 + 7x + 7x + 49 = x^2 + 14x + 49$
e. $(x + n)^2 = (x + n)(x + n) =$ ***Solution:*** $x^2 + nx + nx + n^2 = x^2 + 2nx + n^2$

3. Each of the products in #2 is a perfect square. Use the results of #2 to complete each of the squares and show their factored forms. Include Geometric diagrams to illustrate the perfect squares.

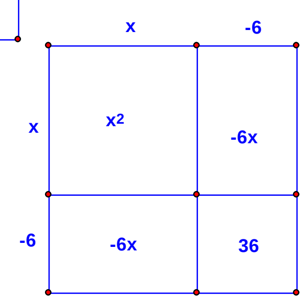
a. $x^2 + 20x +$ ____ $= (x +$ ____$)^2$

Solution: $x^2 + 20x + 100 = (x + 10)^2$



b. $x^2 - 12x +$ ____ $= (x -$ ____$)^2$

Solution: $x^2 - 12x + 36 = (x - 6)^2$



c. $x^2 + 18x +$ ____ $= (x +$ ____$)^2$ ***Solution:*** $x^2 + 18x + 81 = (x + 9)^2$

d. $x^2 - 7x +$ ____ $= (x -$ ____$)^2$ ***Solution:*** $x^2 - 7x + \frac{49}{4} = \left(x - \frac{7}{2}\right)^2$

e. $x^2 + 2nx +$ ____ $= (x +$ ____$)^2$ ***Solution:*** $x^2 + 2nx + n^2 = (x + n)^2$

In order to graph a circle given in general form, it is necessary to change to standard form. In order to rewrite $x^2 + y^2 + 2x - 4y - 11 = 0$ in standard form to facilitate graphing, it is necessary to complete the square for both x and y.

Teacher Notes:
The teacher should emphasize here the benefits of having the equation expressed in standard form. It is easier to see the center and the radius, therefore it is easier to graph.

$\mathbf{x^2 + y^2 + 2x - 4y - 11 = 0}$

$(x^2 + 2x \quad) + (y^2 - 4y \quad) = 11$	group the x and y terms
$(x^2 + 2x + 1) + (y^2 - 4y + 4) = 11 + 1 + 4$	complete the square on x and y
	balance the equation by adding 1 and 4 to both sides of the equation
$\mathbf{(x + 1)^2 + (y - 2)^2 = 16}$	factor

Circle with center at (-1, 2) and radius 4

To change $x^2 + y^2 + 2x - 4y - 11 = 0$ to standard form, it is necessary to remove a factor of 2 before completing the square for both x and y.

$\mathbf{2x^2 + 2y^2 - 4x + 6y - 4 = 0}$	remove a factor of 2
$(x^2 - 2x \quad) + (y^2 + 3y \quad) = 2$	group the x and y terms
$(x^2 - 2x + 1) + (y^2 + 3y + {}^9/_4) = 2 + 1 + {}^9/_4$	complete the square on x and y
	balance the equation by adding 1 and ${}^9/_4$ to both sides of the equation
$(x - 1)^2 + (y + {}^3/_2)^2 = {}^{21}/_4$	factor

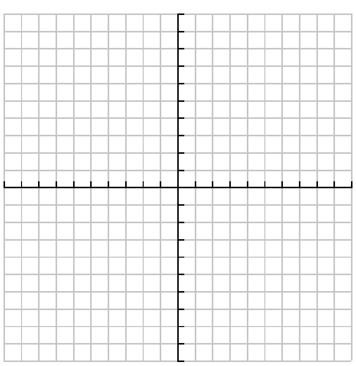
$\mathbf{(x - 1)^2 + (y + 1.5)^2 = 5.25}$

circle with center at (1, -1.5) and radius 5.25

4. Change the following equations to standard form. Graph the circles; identify the centers and the radii.

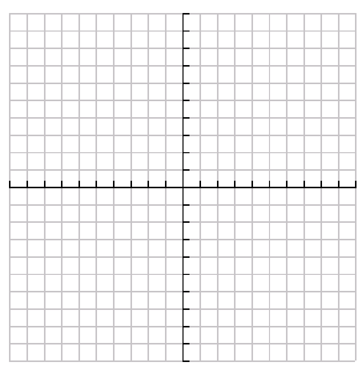
a. $x^2 + y^2 + 2x + 4y - 20 = 0$

$(x + 1)^2 + (y + 2)^2 = 25$
center (-1, -2) radius 5



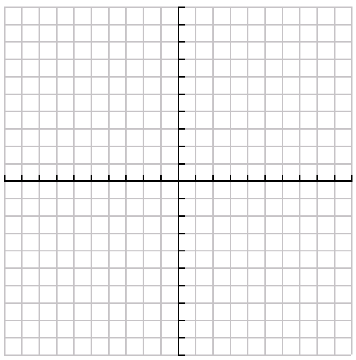
b. $x^2 + y^2 - 4y = 0$

$(x + 0)^2 + (y - 2)^2 = 4$
center (0, 2) radius 2



c. $x^2 + y^2 - 6x - 10y = 2$

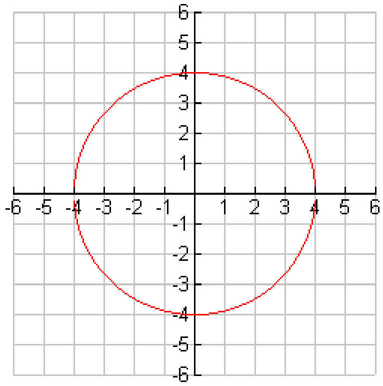
$(x - 3)^2 + (y - 5)^2 = 36$
center (3, 5) radius 6



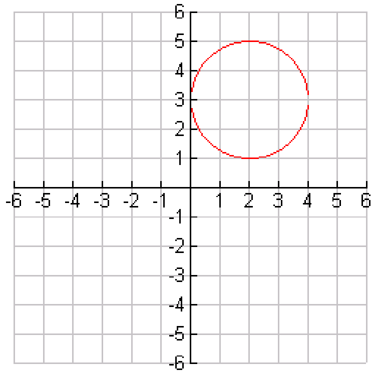
Completing the Square in a Circle?

1. Write equations for the following circle graphs in both standard form and general form.

a.



b.



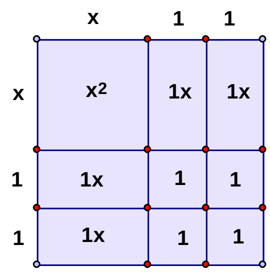
2. Take a moment to compare your General form and Standard form equations. Which form would be easier to graph? Why do you think so?

In Task 2, you converted the Standard form equation to a General form equation. Today you will convert from General form to Standard form.

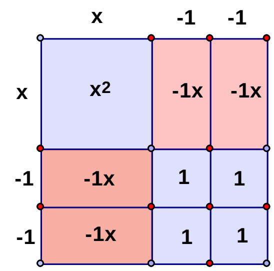
To change from general form to standard form, it is necessary to "complete the square" for x and y. **Completing the square** is an algebraic tool used to change equations of circles given in general form, $Ax^2 + By^2 + Cx + Dy + E = 0$, to standard form, $(x-h)^2 + (y-k)^2 = r^2$. Standard form is the form used to graph circles.
Perfect squares are numbers or expressions which have exactly two identical factors.

(2)(2) = 4 (-5)(-5) = 25 (3x)(3x) = $9x^2$ (x + 2)(x + 2) = $x^2 + 4x + 4$

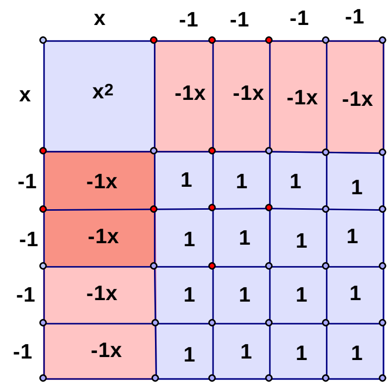
Consider the following geometric area models of three perfect squares. The area is given as both factors and as a quadratic expression.



area = (x + 2)(x + 2)
= x^2 + 4x + 4



area = (x - 2)(x - 2)
= x^2 - 4x + 4



area = (x - 4)(x - 4)
= x^2 - 8x + 16

3. Find the products of the following expressions.

a. $(x + 1)^2 = (x + 1)(x + 1) =$
b. $(x - 3)^2 = (x - 3)(x - 3) =$
c. $(x - 5)^2 = (x - 5)(x - 5) =$
d. $(x + 7)^2 = (x + 7)(x + 7) =$
e. $(x + n)^2 = (x + n)(x + n) =$

4. Each of the products in #3 is a perfect square. Use the results of #3 to complete each of the squares and show their factored forms. Include Geometric diagrams to illustrate the perfect squares.

a. $x^2 + 20x +$ ____ = (x + ____$)^2$

b. $x^2 - 12x +$ ____ = (x – ____$)^2$

c. $x^2 + 18x +$ ____ = (x + ____$)^2$

d. $x^2 - 7x +$ ____ = (x – ____$)^2$

e. $x^2 + 2nx +$ ____ = (x + ____$)^2$

In order to graph a circle given in general form, it is necessary to change to standard form. In order to rewrite $x^2 + y^2 + 2x - 4y - 11 = 0$ in standard form to facilitate graphing, it is necessary to complete the square for both x and y.

$\mathbf{x^2 + y^2 + 2x - 4y - 11 = 0}$	
$(x^2 + 2x \quad) + (y^2 - 4y \quad) = 11$	group the x and y terms
$(x^2 + 2x + 1) + (y^2 - 4y + 4) = 11 + 1 + 4$	complete the square on x and y balance the equation by adding 1 and 4 to both sides of the equation
$\mathbf{(x + 1)^2 + (y - 2)^2 = 16}$	factor

circle with center at (-1, 2) and radius 4

To change $x^2 + y^2 + 2x - 4y - 11 = 0$ to standard form, it is necessary to remove a factor of 2 before completing the square for both x and y.

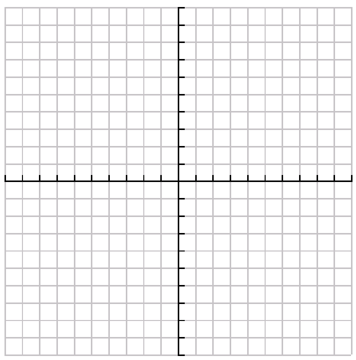
$\mathbf{2x^2 + 2y^2 - 4x + 6y - 4 = 0}$	remove a factor of 2
$(x^2 - 2x \quad) + (y^2 + 3y \quad) = 2$	group the x and y terms
$(x^2 - 2x + 1) + (y^2 + 3y + {}^9/_4) = 2 + 1 + {}^9/_4$	complete the square on x and y balance the equation by adding 1 and ${}^9/_4$ to both sides of the equation
$(x - 1)^2 + (y + {}^3/_2)^2 = {}^{21}/_4$	factor

$\mathbf{(x - 1)^2 + (y + 1.5)^2 = 5.25}$

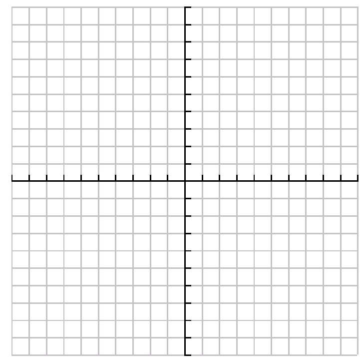
circle with center at (1, -1.5) and radius 5.25

5. Change the following equations to standard form. Graph the circles; identify the centers and the radii.

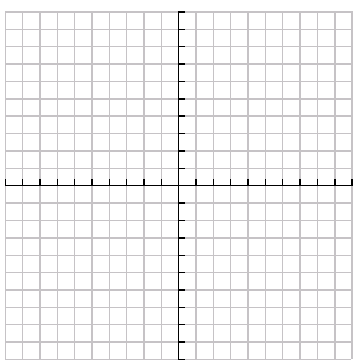
a. $x^2 + y^2 + 2x + 4y - 20 = 0$



b. $x^2 + y^2 - 4y = 0$



c. $x^2 + y^2 - 6x - 10y = 2$



Graphing Circles on a Graphing Calculator

Back to Task Table

Standard Addressed in this Task

MGSE9-12.G.GPE.1 Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.

Standards for Mathematical Practice
2. Reason abstractly and quantitatively by requiring students to make sense of quantities and their relationships to one another in problem situations.

3. Construct viable arguments and critique the reasoning of others by engaging students on discussion of why they agree or disagree with responses, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

7. Look for and make use of structure by expecting students to apply rules, look for patterns and analyze structure.

8. Look for and express regularity in repeated reasoning by expecting students to understand broader applications and look for structure and general methods in similar situations.

Common Student Misconceptions

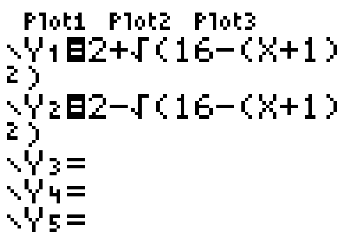
- Because new vocabulary is being introduced in this cluster, remembering the names of the conic sections can be problematic for some students.
- The Euclidean distance formula involves squared, subscripted variables whose differences are added. The notation and multiplicity of steps can be a serious stumbling block for some students.
- The method of completing the square is a multi-step process that takes time to assimilate.
- A geometric demonstration of completing the square can be helpful in promoting conceptual understanding.

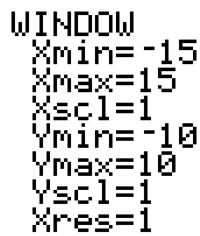
Teacher Notes:
This is an extension activity. Ultimately it is an exercise in symbolic manipulation, but you may find it useful as a way to get students to use technology.

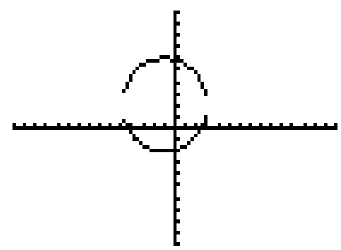
To graph the circle $x^2 + y^2 + 2x - 4y - 11 = 0$ using a TI83/TI84 it is necessary to solve for y after changing the equation to standard form.

$$(x + 1)^2 + (y - 2)^2 = 16$$
$$(y - 2)^2 = 16 - (x + 1)^2$$
$$\sqrt{(y-2)^2} = \pm\sqrt{16-(x+1)^2}$$
$$y - 2 = \pm\sqrt{16-(x+1)^2}$$
$$y = 2 \pm \sqrt{16-(x+1)^2}$$

Enter this result as two functions $y1 = 2 + \sqrt{(16-(x+1)^2)}$ and $y2 = 2 - \sqrt{(16-(x+1)^2)}$. In order to minimize the distortion caused by the rectangular screen of the graphing calculator, use a window with a x to y ratio of 3 to 2. Otherwise circles appear as ellipses.







1. Write the equations as you would enter them in a graphing calculator and list an appropriate graphing window to show the entire circle graph.

a. $x^2 + y^2 + 2x + 4y - 20 = 0$ ***Solution:*** $y_1 = -2 + \sqrt{\left(25-(x+1)^2\right)}$, $y_2 = -2 - \sqrt{\left(25-(x+1)^2\right)}$ [-15, 15] by [-10, 10]

b. $x^2 + y^2 - 4y = 0$ ***Solution:*** $y_1 = 2 + \sqrt{\left(4-x^2\right)}$, $y_2 = 2 - \sqrt{\left(4-x^2\right)}$ [-6, 6] by [-4, 4]

c. $x^2 + y^2 - 6x - 10y = 2$ ***Solution:*** $y_1 = 5 + \sqrt{\left(36-(x-3)^2\right)}$, $y_2 = 5 - \sqrt{\left(36-(x-3)^2\right)}$ [-18, 18] by [-12, 12]

Graphing Circles on a Graphing Calculator

Standard Addressed in this Task
MGSE9-12.G.GPE.1 Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.

Standards for Mathematical Practice
2. Reason abstractly and quantitatively by requiring students to make sense of quantities and their relationships to one another in problem situations.

3. Construct viable arguments and critique the reasoning of others by engaging students on discussion of why they agree or disagree with responses, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

7. Look for and make use of structure by expecting students to apply rules, look for patterns and analyze structure.

8. Look for and express regularity in repeated reasoning by expecting students to understand broader applications and look for structure and general methods in similar situations.

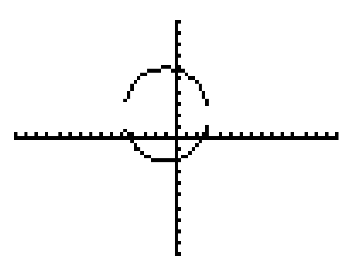
To graph the circle $x^2 + y^2 + 2x - 4y - 11 = 0$ using a TI83/TI84 it is necessary to solve for y after changing the equation to standard form.

$$(x + 1)^2 + (y - 2)^2 = 16$$
$$(y - 2)^2 = 16 - (x + 1)^2$$
$$\sqrt{(y-2)^2} = \pm\sqrt{16-(x+1)^2}$$
$$y - 2 = \pm\sqrt{16-(x+1)^2}$$
$$y = 2 \pm \sqrt{16-(x+1)^2}$$

Enter this result as two functions $y1 = 2 + \sqrt{(16-(x+1)^2)}$ and $y2 = 2 - \sqrt{(16-(x+1)^2)}$. In order to minimize the distortion caused by the rectangular screen of the graphing calculator, use a window with a x to y ratio of 3 to 2. Otherwise circles appear as ellipses.

```
Plot1 Plot2 Plot3
\Y1=2+√(16-(X+1)
2)
\Y2=2-√(16-(X+1)
2)
\Y3=
\Y4=
\Y5=
```

```
WINDOW
 Xmin=-15
 Xmax=15
 Xscl=1
 Ymin=-10
 Ymax=10
 Yscl=1
 Xres=1
```



1. Write the equations as you would enter them in a graphing calculator and list an appropriate graphing window to show the entire circle graph.

   a. $x^2 + y^2 + 2x + 4y - 20 = 0$

   b. $x^2 + y^2 - 4y = 0$

   c. $x^2 + y^2 - 6x - 10y = 2$

Radio Station Listening Areas

[Back to Task Table]

Standard Addressed in this Task
MGSE9-12.G.GPE.1 Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.

Standards for Mathematical Practice
2. Reason abstractly and quantitatively by requiring students to make sense of quantities and their relationships to one another in problem situations.

3. Construct viable arguments and critique the reasoning of others by engaging students on discussion of why they agree or disagree with responses, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

7. Look for and make use of structure by expecting students to apply rules, look for patterns and analyze structure.

8. Look for and express regularity in repeated reasoning by expecting students to understand broader applications and look for structure and general methods in similar situations.

Common Student Misconceptions
- Because new vocabulary is being introduced in this cluster, remembering the names of the conic sections can be problematic for some students.
- The Euclidean distance formula involves squared, subscripted variables whose differences are added. The notation and multiplicity of steps can be a serious stumbling block for some students.
- The method of completing the square is a multi-step process that takes time to assimilate.
- A geometric demonstration of completing the square can be helpful in promoting conceptual understanding.

Teacher Notes:
This activity is an application of writing equations of circles. It gives students a taste of a real life application of the formula that they wrote in the first task.

1. Radio signals emitted from a transmitter form a pattern of concentric circles. Write equations for three concentric circles.

Solution:
Answers vary. The student should realize that concentric circles have the same center, but different radii.

2. Randy listens to radio station WYAY from Atlanta. Randy's home is located 24 miles east and 32 miles south of the radio station's transmitter. His house is located on the edge of WYAY's maximum broadcast range.

a. When a radio signal reaches Randy's house, how far has it traveled? Sketch WYAY's listening area of the partial map of Georgia given. On the map let Atlanta's WYAY have coordinates (0, 0) and use the scale as 100 miles = 60 mm.

Solution:
See map. Randy's house at (24mi, -32mi) or (14.4mm, -19.2mm)

a. Find an equation which represents the station's maximum listening area.

Solution:
miles: $x^2 + y^2 = 40^2$ or mm: $x^2 + y^2 = 24^2$

c. Determine four additional locations on the edge of WYAY's listening area, give coordinates correct to tenths.

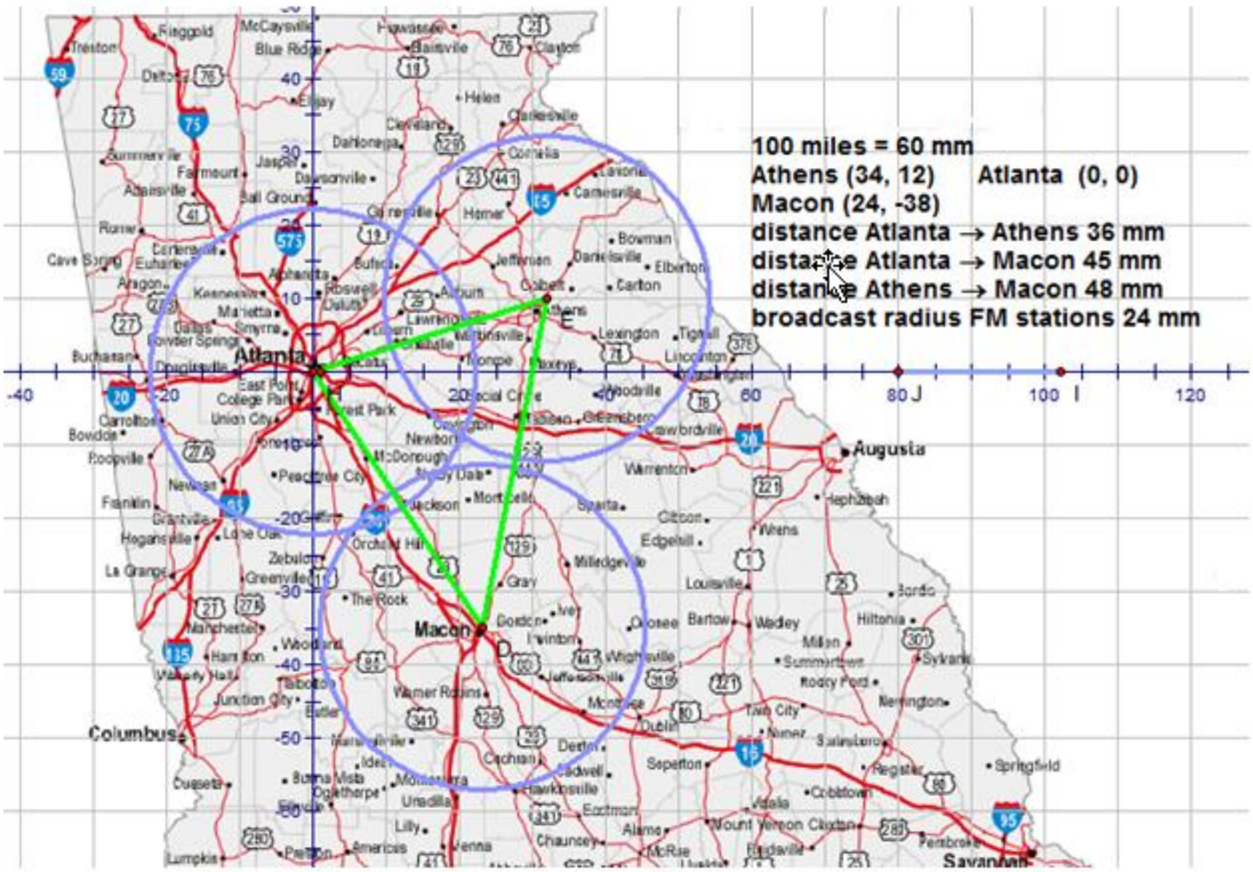
Solution:
some possible solutions in miles: (±26, ±30.4) (±22, ±33.4) (±19, ±35.2)

Teacher Notes:
Before starting this exercise, check the scale on the map. Some distortion occurred when the map was pasted into Geometer's Sketchpad to set the scale. However values should be close enough to allow students to draw graphs on the map and locate points.

3. Randy likes to listen to country music. Several of his friends have suggested that in addition to WYAY, he try station WXAG in Athens and WDEN in Macon. WYAY, WXAG, and WDEN are FM stations which normally have an average broadcast range of 40 miles. Use the map included with the indicated measures to answer the following questions.

a. Given the location of Randy's home, can he expect to pick up radio signals from WXAG and WDEN? Explain how you know.

Solution:
Randy's home should be close to Jackson. He should get signals from Macon, but not from Athens.



Broadcast area of WXAG in Athens is given by the equation $(x - 56.7)^2 + (y - 20)^2 = 40^2$ or $(x - 34)^2 + (y - 12)^2 = 24^2$

Broadcast area of WDEN in Macon is given by the equation $(x - 40)^2 + (y + 63.3)^2 = 40^2$ or $(x - 24)^2 + (y + 38)^2 = 24^2$

Broadcast areas drawn to scale in mms.

Radio Station Listening Areas

Standard Addressed in this Task

MGSE9-12.G.GPE.1 Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.

Standards for Mathematical Practice
2. Reason abstractly and quantitatively by requiring students to make sense of quantities and their relationships to one another in problem situations.

3. Construct viable arguments and critique the reasoning of others by engaging students on discussion of why they agree or disagree with responses, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

7. Look for and make use of structure by expecting students to apply rules, look for patterns and analyze structure.

8. Look for and express regularity in repeated reasoning by expecting students to understand broader applications and look for structure and general methods in similar situations.

1. Radio signals emitted from a transmitter form a pattern of concentric circles. Write equations for three concentric circles.

2. Randy listens to radio station WYAY from Atlanta. Randy's home is located 24 miles east and 32 miles south of the radio station's transmitter. His house is located on the edge of WYAY's maximum broadcast range.

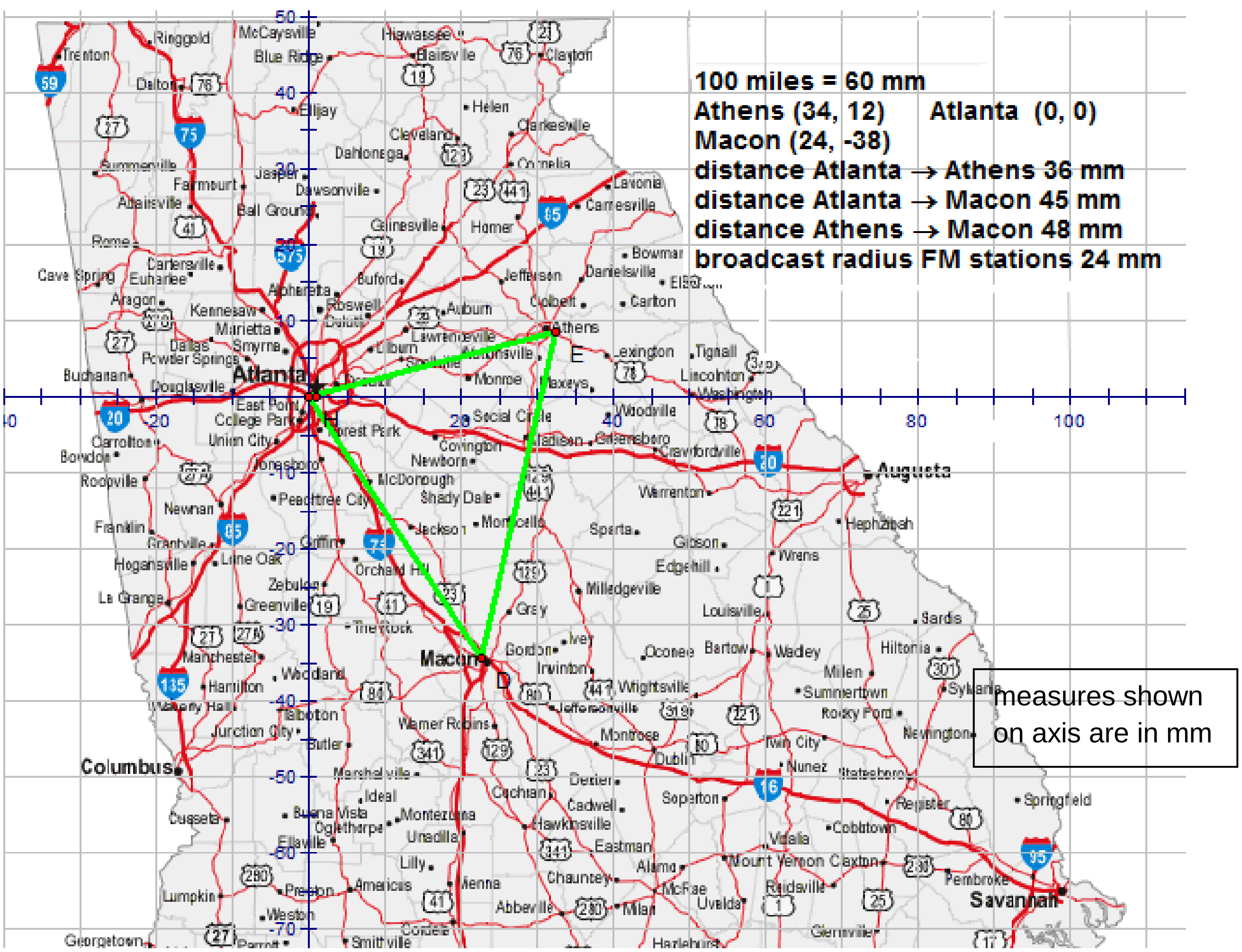
   a. When a radio signal reaches Randy's house, how far has it traveled? Sketch WYAY's listening area of the partial map of Georgia given. On the map let Atlanta's WYAY have coordinates (0, 0) and use the scale as 100 miles = 60 mm.

   b. Find an equation which represents the station's maximum listening range.

   c. Determine four additional locations on the edge of WYAY's listening area, give coordinates correct to tenths.

3. Randy likes to listen to country music. Several of his friends have suggested that in addition to WYAY, he try station WXAG in Athens and WDEN in Macon. WYAY, WXAG, and WDEN are FM stations which normally have an average broadcast range of 40 miles. Use the map included with the indicated measures to answer the following questions.

a. Given the location of Randy's home, can he expect to pick up radio signals from WXAG and WDEN? Explain how you know.



Algebraic Proof

[Back to Task Table](#)

Standard Addressed in this Task
MGSE9-12.G.GPE.4 Use coordinates to prove simple geometric theorems algebraically. *For example, prove or disprove that a figure defined by four given points in the coordinate plane is a rectangle; prove or disprove that the point (1, √3) lies on the circle centered at the origin and containing the point (0,2).*
(Focus on quadrilaterals, right triangles, and circles.)

Standards for Mathematical Practice
2. Reason abstractly and quantitatively by requiring students to make sense of quantities and their relationships to one another in problem situations.

3. Construct viable arguments and critique the reasoning of others by engaging students on discussion of why they agree or disagree with responses, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

7. Look for and make use of structure by expecting students to apply rules, look for patterns and analyze structure.

Common Student Misconceptions

- Students may claim that a vertical line has infinite slopes. This suggests that infinity is a number. Since applying the slope formula to a vertical line leads to division by zero, we say that the slope of a vertical line is undefined. Also, the slope of a horizontal line is 0. Students often say that the slope of a vertical and/or horizontal lines is "no slope," which is incorrect.

Teacher Notes:
The importance of this standard cannot be overstated. Using Algebra to prove ideas and confirm hypotheses is a critical mathematical thinking skill. Do not focus on the specific proofs presented in this task, but rather the overarching idea of Algebraic proof. The examples presented here are meant as a starting point. More examples should be presented in class.

Already in Geometry you have been exposed to and written proofs about geometric theorems and properties. It is now time to mix in some algebraic proofs. For this unit, we will restrict our algebraic proofs to problems involving the coordinate plane.

First, we should examine our "toolbox" to see what math concepts we have at our disposal for these types of proofs:

Distance Formula: Useful for determining distances between two points.

Slope Formula: Useful for determining if lines are parallel or perpendicular.

Substitution: Useful for determining if points satisfy given equations.

1. Proof #1: Prove or disprove that the point $\left(1,\sqrt{3}\right)$ lies on the circle centered at the origin and passing through the point (0, 2).

Teacher Notes:
The teacher must avoid the temptation to just substitute and move on. The idea behind these types of proofs is to understand that the points given are equidistant from the center. Always take the algebra back to the geometry and the definition.

a. What do we need to show in order to prove or disprove this statement?

Solution:
Students need to show that the two points are equidistant from the center. This could be shown in multiple ways, including using the equation of a circle.

b. Write an equation for the circle described in the problem.

Comments: This is only one approach to solving the problem. It also refers back to skills introduced at the beginning of the unit. While this proof is step-by-step you should challenge students to complete similar proofs on their own.

Solution:

$$\left(0-0\right)^2+(2-0)^2=r^2$$
$$4=r^2$$
$$2=r$$
$$x^2+y^2=2^2$$

c. Substitute the point in for the equation and comment on the results. Did you prove the statement or disprove it?

Solution:
$$(1)^2+\left(\sqrt{3}\right)^2=4$$

Because the equation is true, the given point is the same distance from the center as the given point on the circle.

Now you are ready to try some on your own. Use the questions above as a guide and write algebraic proofs for the following.

2. Prove or disprove that the point A(10, 3) lies on a circle centered at C(5, -2) and passing through the point B(6, 5).

   Solution:
   The point does lie on the circle. The equation of the circle is: $(x-5)^2+(y+2)^2=50$
   Substituting in the point gives: $(10-5)^2+(3+2)^2=50$

Algebraic Proof

Standard Addressed in this Task

MGSE9-12.G.GPE.4 Use coordinates to prove simple geometric theorems algebraically. *For example, prove or disprove that a figure defined by four given points in the coordinate plane is a rectangle; prove or disprove that the point (1, √3) lies on the circle centered at the origin and containing the point (0,2).*
(Focus on quadrilaterals, right triangles, and circles.)

Standards for Mathematical Practice

2. Reason abstractly and quantitatively by requiring students to make sense of quantities and their relationships to one another in problem situations.

3. Construct viable arguments and critique the reasoning of others by engaging students on discussion of why they agree or disagree with responses, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

7. Look for and make use of structure by expecting students to apply rules, look for patterns and analyze structure.

Already in Geometry you have been exposed to and written proofs about geometric theorems and properties. It is now time to mix in some algebraic proofs. For this unit, we will restrict our algebraic proofs to problems involving the coordinate plane.

First, we should examine our “toolbox” to see what math concepts we have at our disposal for these types of proofs:

Distance Formula: Useful for determining distances between two points.

Slope Formula: Useful for determining if lines are parallel or perpendicular.

Substitution: Useful for determining if points satisfy given equations.

1. Proof #1: Prove or disprove that the point $\left(1,\sqrt{3}\right)$ lies on the circle centered at the origin and passing through the point (0, 2).

a. What do we need to show in order to prove or disprove this statement?

b. Write an equation for the circle described in the problem.

c. Substitute the point in for the equation and comment on the results. Did you prove the statement or disprove it?

d. Now use the facts in a-c to write a paragraph proof.

Now you are ready to try some on your own. Use the questions above as a guide and write algebraic proofs for the following.

2. Prove or disprove that the point A(10, 3) lies on a circle centered at C(5, -2) and passing through the point B(6, 5).

A Day at the Beach

Back to Task Table

Source: NYC Department of Education
https://www.weteachnyc.org/media2016/filer_public/f3/12/f312f45e-a203-4400-8648-c40a2b090cb9/g10_math_a_day_at_the_beach.pdf

Mathematical Goals

- To visualize and identify the dimensions of geometric shapes
- To determine the volume relationships of cylinders, pyramids, cones, and spheres
- To justify geometric arguments

Essential Questions

- Which geometric shape will fill a pail full of sand faster?

TASK COMMENTS

The students will identify and label geometric shapes and apply the appropriate formulas and measurements to calculate the volume of the figures. After finding this information, students will determine which geometric figure will fill a pail full of sand faster. In this task, the student routinely interprets the mathematical results, applies geometric concepts in the context of the situation, reflects on whether the results make sense and uses all appropriate tools strategically.
http://schools.nyc.gov/NR/rdonlyres/C03D80B2-9213-43A9-AAA3-BB0032C62F4F/139657/NYCDOE_G10_ADayattheBeach_FINAL1.pdf

The task, *A Day at the Beach*, is a Performance task that can be found at the website:
https://www.weteachnyc.org/media2016/filer_public/f3/12/f312f45e-a203-4400-8648-c40a2b090cb9/g10_math_a_day_at_the_beach.pdf

The scoring rubric can be found at the following link:
https://www.weteachnyc.org/media2016/filer_public/f3/12/f312f45e-a203-4400-8648-c40a2b090cb9/g10_math_a_day_at_the_beach.pdf

GEORGIA STANDARDS OF EXCELLENCE

MGSE9-12.G.MG.1 Use geometric shapes, their measures, and their properties to describe objects (e.g., modeling a tree trunk or a human torso as a cylinder).

MGSE9-12.G.MG.3 Apply geometric methods to solve design problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios).

STANDARDS FOR MATHEMATICAL PRACTICE

This task uses all of the practices with emphasis on:

1. **Make sense of problems and persevere in solving them** by requiring students to interpret and make meaning of a problem and find a logical starting point, and to monitor their progress and change their approach to solving the problem, if necessary.

4. **Model with mathematics** by expecting students to apply the mathematics concepts they know in order to solve problems arising in everyday situations, and reflect on whether the results are sensible for the given scenario.

6. **Attend to precision** by requiring students to calculate efficiently and accurately; and to communicate precisely with others by using clear mathematical language to discuss their reasoning.

Grouping

- Individual/Partner

Time Needed

- 90 minutes

How Many Cells are in the Human Body?

Back to Task Table

Source: Illustrative Mathematics
https://www.illustrativemathematics.org/content-standards/HSG/MG/A/1/tasks/1146

Mathematical Goals

- To apply the concepts of mass, volume, and density in a real-world context.

Essential Questions

- About how many cells are in the human body?

TASK COMMENTS

The purpose of this task is to help students apply the concepts of mass, volume, and density in a real-world context. The task allows someone to estimate the volume of a person to make calculations on the number of cells. However, in order to better adapt this task to the Georgia Standards of Excellence, students can make the comparison of the shape of a human to that of a geometric figure, for example a human torso could compare to a cylinder, make the appropriate calculations to determine the number of cells are in the human body.
https://www.illustrativemathematics.org/content-standards/HSG/MG/A/1/tasks/1146

The task, *How many cells are in the human body?*, is a Performance task that can be found at the website: https://www.illustrativemathematics.org/content-standards/HSG/MG/A/1/tasks/1146

GEORGIA STANDARDS OF EXCELLENCE

MGSE9-12.G.MG.1 Use geometric shapes, their measures, and their properties to describe objects (e.g., modeling a tree trunk or a human torso as a cylinder).

MGSE9-12.G.MG.2 Apply concepts of density based on area and volume in modeling situations (e.g, persons per square mile, BTUs per cubic foot).

STANDARDS FOR MATHEMATICAL PRACTICE

This task uses all of the practices with emphasis on:

2. **Make sense of problems and persevere in solving them** by requiring students to interpret and make meaning of a problem and find a logical starting point, and to monitor their progress and change their approach to solving the problem, if necessary.

3. **Reason abstractly and quantitatively** by requiring students to make sense of quantities and their relationships to one another in problem situations.

5. **Model with mathematics** by expecting students to apply the mathematics concepts they know in order to solve problems arising in everyday situations, and reflect on whether the results are sensible for the given scenario.

6. **Attend to precision** by requiring students to calculate efficiently and accurately; and to communicate precisely with others by using clear mathematical language to discuss their reasoning.

Grouping

- Partner

Time Needed

- 60-90 minutes

Maximize Volume

Back to Task Table

Standard Addressed in this Task
MGSE9-12.G.MG.3 Apply geometric methods to solve design problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios).

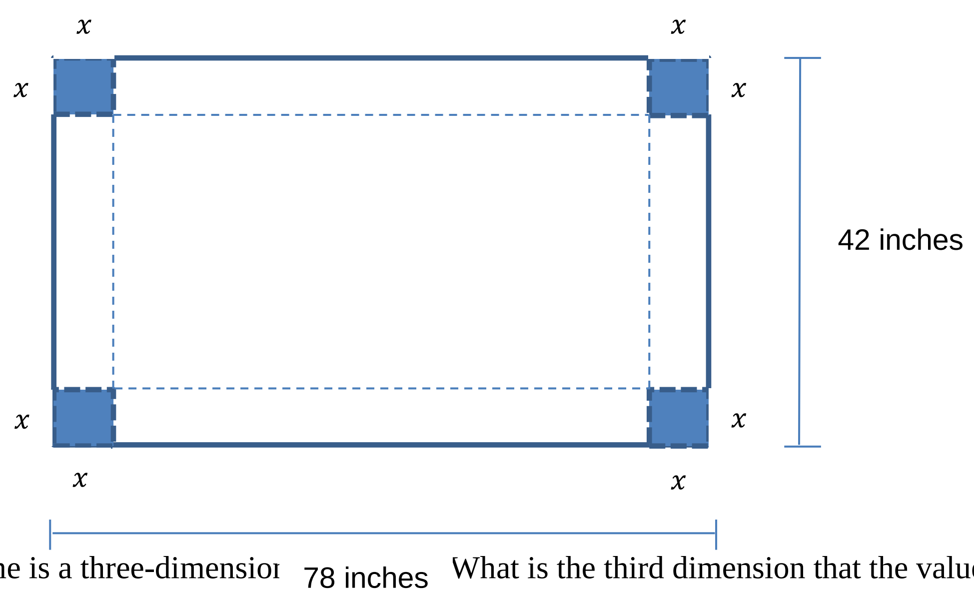
Standards for Mathematical Practice
2. Reason abstractly and quantitatively by requiring students to make sense of quantities and their relationships to one another in problem situations.

3. Construct viable arguments and critique the reasoning of others by engaging students on discussion of why they agree or disagree with responses, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

7. Look for and make use of structure by expecting students to apply rules, look for patterns and analyze structure.

8. Look for and express regularity in repeated reasoning by expecting students to understand broader applications and look for structure and general methods in similar situations.

Mary needs open-topped boxes to store her excess inventory at year's end. Mary purchases large rectangles of thick cardboard with a length of 78 inches and width of 42 inches to make the boxes. Mary is interested in maximizing the volume of the boxes and wants to know what size squares to cut out at each corner of the cardboard (which will allow the corners to be folded up to form the box) in order to do this.



(a) Volume is a three-dimensio What is the third dimension that the value x represents?

x represents the height of the box.

(b) Using the table below, choose five values of x and find the corresponding volumes.

Answers vary.

x	**Length**	**Width**	**Volume**

You tested several different values of x above, and calculated five different volumes. There is a way to guarantee that you use dimensions that will maximize volume, and now we're going to work through that process.

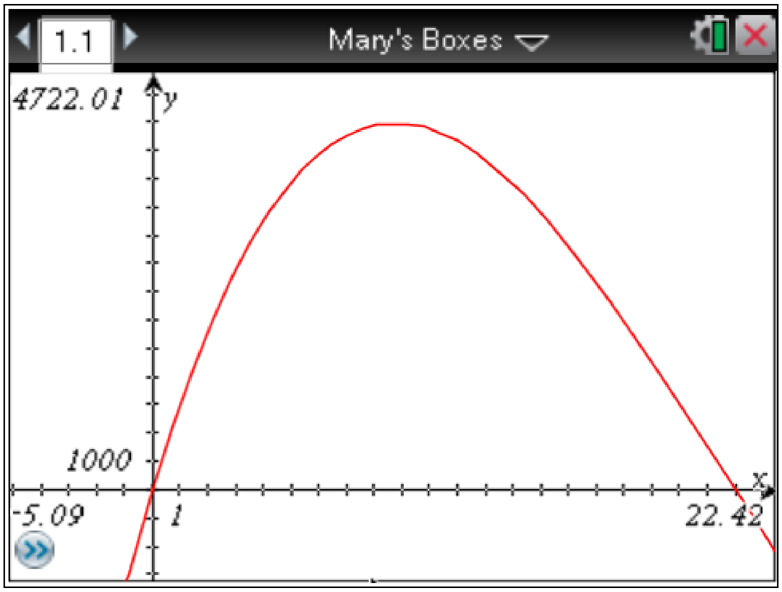
(c) Write an equation for volume in terms of the three dimensions of the box.

$$\boldsymbol{V = x(78 - 2x)(42 - 2x)}$$

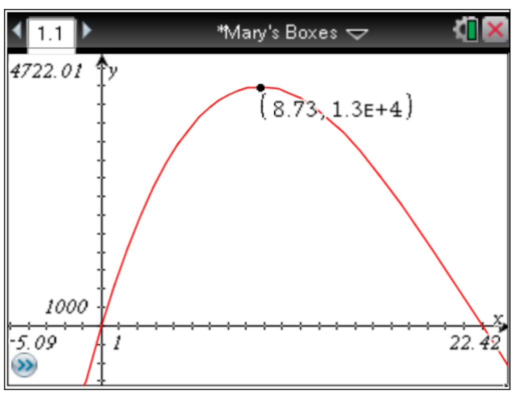
$$\boldsymbol{V = x(3276 - 240x + 4x^2)}$$

$$\boldsymbol{V = 3276x - 240x^2 + 4x^3}$$

(d) Graph the equation from part (c).



(e) From your graph, what are the values of the three dimensions that maximize the volume of the box? What is the maximum volume of the box?



From the graph, it appears that the maximum occurs at approximately $(8.73, 13000)$, so the maximum volume would be 13000 cubic inches with a height of 8.73 inches, a length of 60.54 inches, and a width of 24.54 inches.

Maximize Volume

Standard Addressed in this Task
MGSE9-12.G.MG.3 Apply geometric methods to solve design problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios).

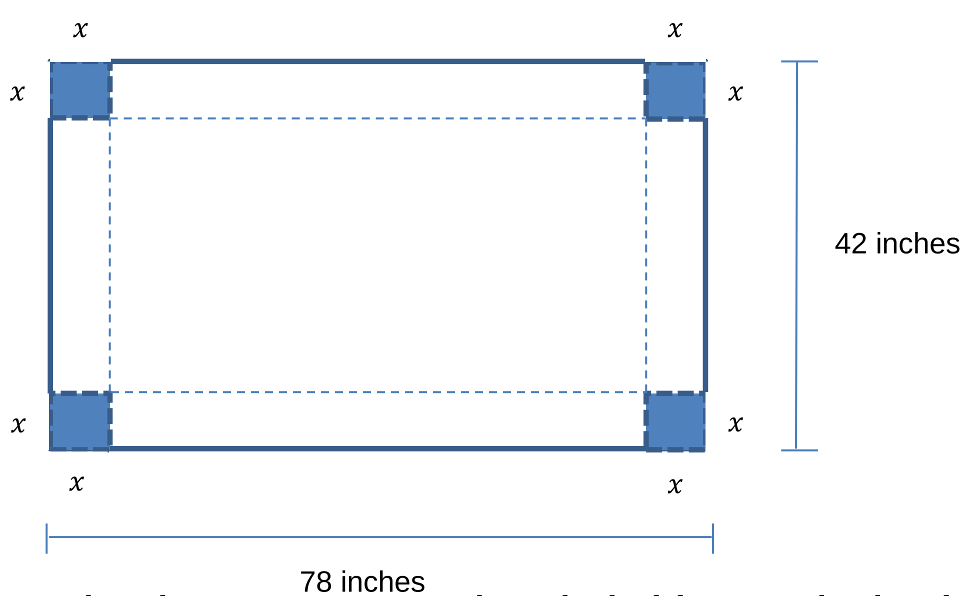
Standards for Mathematical Practice
2. Reason abstractly and quantitatively by requiring students to make sense of quantities and their relationships to one another in problem situations.

3. Construct viable arguments and critique the reasoning of others by engaging students on discussion of why they agree or disagree with responses, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

7. Look for and make use of structure by expecting students to apply rules, look for patterns and analyze structure.

8. Look for and express regularity in repeated reasoning by expecting students to understand broader applications and look for structure and general methods in similar situations.

Mary needs open-topped boxes to store her excess inventory at year's end. Mary purchases large rectangles of thick cardboard with a length of 78 inches and width of 42 inches to make the boxes. Mary is interested in maximizing the volume of the boxes and wants to know what size squares to cut out at each corner of the cardboard (which will allow the corners to be folded up to form the box) in order to do this.



(a) Volume is a three-dimensio[illegible] What is the third dimension that the value x represents?

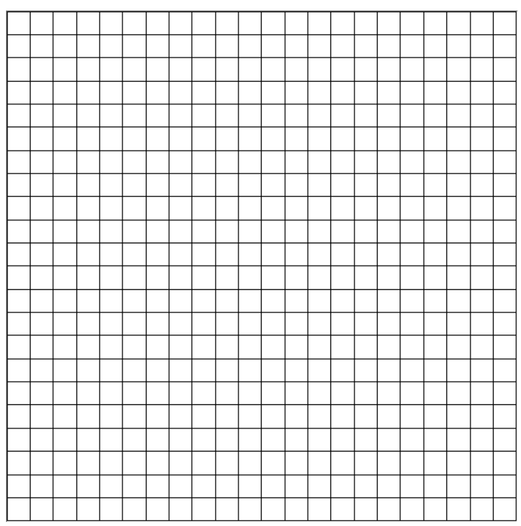
(b) Using the table below, choose five values of x and find the corresponding volumes.

x	**Length**	**Width**	**Volume**

You tested several different values of x above, and calculated five different volumes. There is a way to guarantee that you use dimensions that will maximize volume, and now we're going to work through that process.

(c) Write an equation for volume in terms of the three dimensions of the box.

(d) Graph the equation from part (c).



(e) From your graph, what are the values of the three dimensions that maximize the volume of the box? What is the maximum volume of the box?

Culminating Task: Dr. Cone's New House

[Back to Task Table]

Standard Addressed in this Task

MGSE9-12.G.GPE.1 Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.

MGSE9-12.G.GPE.4 Use coordinates to prove simple geometric theorems algebraically. *For example, prove or disprove that a figure defined by four given points in the coordinate plane is a rectangle; prove or disprove that the point (1, √3) lies on the circle centered at the origin and containing the point (0,2).*
(Focus on quadrilaterals, right triangles, and circles.)

Standards for Mathematical Practice

2. Reason abstractly and quantitatively by requiring students to make sense of quantities and their relationships to one another in problem situations.

3. Construct viable arguments and critique the reasoning of others by engaging students on discussion of why they agree or disagree with responses, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

7. Look for and make use of structure by expecting students to apply rules, look for patterns and analyze structure.

8. Look for and express regularity in repeated reasoning by expecting students to understand broader applications and look for structure and general methods in similar situations.

Teacher Notes:

This performance task is designed to apply students' understanding of equations of parabolas and circles. While completing this task, students should take care to write the equations and fit the graphs in the required specifications. This is a great time to revisit parts of a parabola, as the x-intercepts are likely to be where the doorframes are. Scale will be an important aspect of this task. There are many possible designs for the entry way.

A local mathematician, Dr. Cone, has hired your architecture firm to design his new house. Because your boss knows you are in Geometry, he has put you in charge of the design for the entrance of the house. The mathematician has given some very unconventional requests for the design of the entrance:

- He wants two doors, both shaped like parabolas.
- He wants at least two windows, both shaped like circles.

You also know the following information:

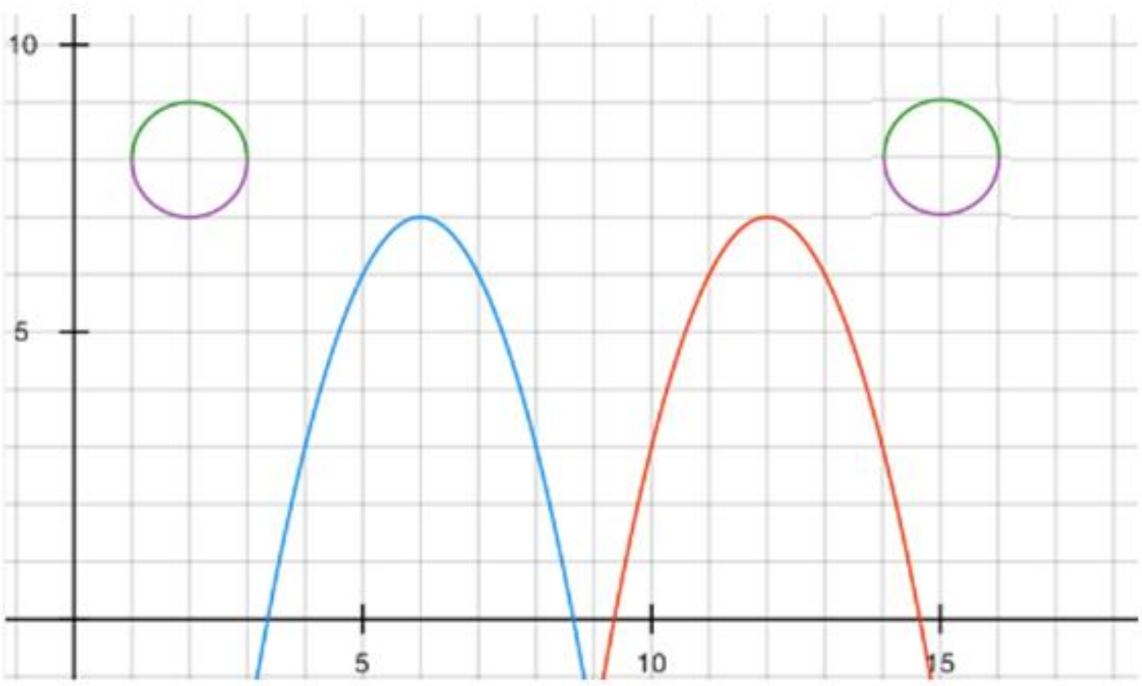
- The dimensions of the front entrance way are 18 feet long and 10 feet tall.
- A local window and door manufacturer can produce any shape window or door, given an equation for the shape.

State and Local guidelines also state:

- All entryways to residential property must be greater than or equal to 7 feet in height.

Using a piece of graph paper, draw a design for the entry way of the house. Be sure to label all important points for the builder. Include a "Specifications Sheet" that includes equations of the figures for the window and door manufacturer.

Teacher Notes:



This an example of a design for the front of the house given the specifications of the task. Students could design any layout as long as the door frames and windows follow the specifications of the task.

Once students have completed their plans, hand them the information below. This will give them a chance to prove algebraically that the box will or will not fit in the door.

Dr. Cone has come to you after seeing your design and expresses a concern. He has just ordered a new transmogrifer and he is worried that it will not fit in the front door. The transmogrifer ships in a box that is 3’x3’x9’. According to your design, will the box fit in one of your doors?

Culminating Task: Dr. Cone's New House

Standard Addressed in this Task

MGSE9-12.G.GPE.1 Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.

MGSE9-12.G.GPE.4 Use coordinates to prove simple geometric theorems algebraically. *For example, prove or disprove that a figure defined by four given points in the coordinate plane is a rectangle; prove or disprove that the point (1, √3) lies on the circle centered at the origin and containing the point (0,2).*
(Focus on quadrilaterals, right triangles, and circles.)

Standards for Mathematical Practice

2. Reason abstractly and quantitatively by requiring students to make sense of quantities and their relationships to one another in problem situations.

3. Construct viable arguments and critique the reasoning of others by engaging students on discussion of why they agree or disagree with responses, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

7. Look for and make use of structure by expecting students to apply rules, look for patterns and analyze structure.

8. Look for and express regularity in repeated reasoning by expecting students to understand broader applications and look for structure and general methods in similar situations.

A local mathematician, Dr. Cone, has hired your architecture firm to design his new house. Because your boss knows you are in Geometry, he has put you in charge of the design for the entrance of the house. The mathematician has given some very unconventional requests for the design of the entrance:

- He wants two doors, both shaped like parabolas.
- He wants at least two windows, both shaped like circles.

You also know the following information:

- The dimensions of the front entrance way are 18 feet long and 10 feet tall.
- A local window and door manufacturer can produce any shape window or door, given an equation for the shape.

State and Local guidelines also state:

- All entryways to residential property must be greater than or equal to 7 feet in height.

Using a piece of graph paper, draw a design for the entry way of the house. Be sure to label all important points for the builder. Include a "Specifications Sheet" that includes equations of the figures for the window and door manufacturer.

Dr. Cone has come to you after seeing your design and expresses a concern. He has just ordered a new transmogrifer and he is worried that it will not fit in the front door. The transmogrifer ships in a box that is 3'x3'x9'. According to your design, will the box fit in one of your doors?