

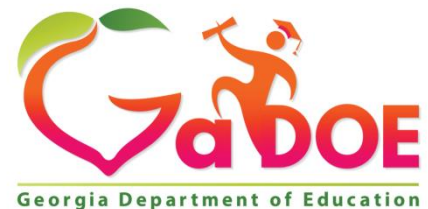


# Georgia Standards of Excellence Curriculum Frameworks

## Mathematics

Accelerated GSE Pre-Calculus

Unit 4: Trigonometric Identities



Richard Woods, Georgia's School Superintendent  
"Educating Georgia's Future"

## Unit 4

### Trigonometric Identities

#### Table of Contents

<u>OVERVIEW</u> .....	3
<u>STANDARDS ADDRESSED IN THIS UNIT</u> .....	3
<u>KEY STANDARDS</u> .....	3
<u>STANDARDS FOR MATHEMATICAL PRACTICE</u> .....	3
<u>ENDURING UNDERSTANDINGS</u> .....	4
<u>ESSENTIAL QUESTIONS</u> .....	4
<u>CONCEPTS/SKILLS TO MAINTAIN</u> .....	4
<u>SELECTED TERMS AND SYMBOLS</u> .....	5
<u>CLASSROOM ROUTINES</u> .....	6
<u>STRATEGIES FOR TEACHING AND LEARNING</u> .....	6
<u>EVIDENCE OF LEARNING</u> .....	7
<u>TASKS</u> .....	7
<u>PROVING THE SINE ADDITION AND SUBTRACTION IDENTITIES</u> .....	8
<u>PROVING THE COSINE ADDITION AND SUBTRACTION IDENTITIES</u> .....	16
<u>A DISTANCE FORMULA PROOF FOR THE COSINE ADDITION IDENTITY</u> .....	25
<u>PROVING THE TANGENT ADDITION AND SUBTRACTION IDENTITIES</u> .....	33
<u>DOUBLE-ANGLE IDENTITIES FOR SINE, COSINE, AND TANGENT</u> .....	37
<u>THE COSINE DOUBLE-ANGLE: A MAN WITH MANY IDENTITIES</u> .....	42
<u>DERIVING HALF-ANGLE IDENTITIES</u> .....	45
<u>CULMINATING TASK: HOW MANY ANGLES CAN YOU FIND?</u> .....	48

## **OVERVIEW**

In this unit, students will:

- build upon their work with trigonometric identities with addition and subtraction formulas
- will look at addition and subtraction formulas geometrically
- prove addition and subtraction formulas
- use addition and subtraction formulas to solve problems

Although the units in this instructional framework emphasize key standards and big ideas at specific times of the year, routine topics such as estimation, mental computation, and basic computation facts should be addressed on an ongoing basis. Ideas related to the eight practice standards should be addressed constantly as well. To assure that this unit is taught with the appropriate emphasis, depth, and rigor, it is important that the tasks listed under “Evidence of Learning” be reviewed early in the planning process. A variety of resources should be utilized to supplement this unit. This unit provides much needed content information, but excellent learning activities as well. The tasks in this unit illustrate the types of learning activities that should be utilized from a variety of sources.

## **STANDARDS ADDRESSED IN THIS UNIT**

Mathematical standards are interwoven and should be addressed throughout the year in as many different units and activities as possible in order to emphasize the natural connections that exist among mathematical topics. The standards call for a student to demonstrate mastery of the content by proving these addition and subtraction formulas using various methods.

## **KEY STANDARDS**

### **Prove and apply trigonometric identities**

**MGSE9-12.F.TF.9** Prove addition, subtraction, double, and half-angle formulas for sine, cosine, and tangent and use them to solve problems.

## **STANDARDS FOR MATHEMATICAL PRACTICE**

Refer to the Comprehensive Course Overview for more detailed information about the Standards for Mathematical Practice.

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.

3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

### **ENDURING UNDERSTANDINGS**

- Understand the concept of identity
- Prove the addition formula for sine, cosine and tangent
- Prove the subtraction formula for sine, cosine and tangent
- Use trigonometric functions to prove formulas

### **ESSENTIAL QUESTIONS**

- How can I add trigonometric functions?
- How can I subtract trigonometric functions?
- How can I prove the addition formula for trigonometric functions?
- How can I prove the subtraction formula for trigonometric functions?
- What is an identity?

### **CONCEPTS/SKILLS TO MAINTAIN**

It is expected that students will have prior knowledge/experience related to the concepts and skills identified below. It may be necessary to pre-assess in order to determine if time needs to be spent on conceptual activities that help students develop a deeper understanding of these ideas.

- number sense
- computation with whole numbers and decimals, including application of order of operations
- applications of the Pythagorean Theorem
- operations with trigonometric ratios
- operations with radians and degrees
- even and odd functions
- geometric constructions
- algebraic proofs
- geometric proofs
- methods of proof

## SELECTED TERMS AND SYMBOLS

The following terms and symbols are often misunderstood. These concepts are not an inclusive list and should not be taught in isolation. However, due to evidence of frequent difficulty and misunderstanding associated with these concepts, instructors should pay particular attention to them and how their students are able to explain and apply them.

**The definitions below are for teacher reference only and are not to be memorized by the students.** Students should explore these concepts using models and real life examples. Students should understand the concepts involved and be able to recognize and/or demonstrate them with words, models, pictures, or numbers.

The website below is interactive and includes a math glossary.

<http://intermath.coe.uga.edu/dictionary/homepg.asp>

Definitions and activities for these and other terms can be found on the InterMath website. Links to external sites are particularly useful.

- **Addition Identity for Cosine:**  $\cos(x + y) = \cos x \cos y - \sin x \sin y$
- **Addition Identity for Sine:**  $\sin(x + y) = \sin x \cos y + \cos x \sin y$
- **Addition Identity for Tangent:**  $\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$
- **Double Angle Identity for Sine:**  $\sin(2x) = 2 \sin x \cos x$
- **Double Angle Identity for Cosine:**  
 $\cos(2x) = \cos^2 x - \sin^2 x = 2 \cos^2 x - 1 = 1 - 2 \sin^2 x$
- **Double Angle Identity for Tangent:**  $\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$
- **Half Angle Identity for Sine:**  $\sin\left(\frac{x}{2}\right) = \pm \sqrt{\frac{1 - \cos x}{2}}$
- **Half Angle Identity for Cosine:**  $\cos\left(\frac{x}{2}\right) = \pm \sqrt{\frac{1 + \cos x}{2}}$
- **Half Angle Identity for Tangent:**  $\tan\left(\frac{x}{2}\right) = \pm \sqrt{\frac{1 - \cos x}{1 + \cos x}} = \frac{\sin x}{1 + \cos x} = \frac{1 - \cos x}{\sin x}$
- **Even Function:** a function with symmetry about the y-axis that satisfies the relationship  $f(x) = f(-x)$
- **Identity:** an identity is a relation that is always true, no matter the value of the variable.
- **Odd Function:** a function with symmetry about the origin that satisfies the relationship  $-f(x) = f(-x)$
- **Subtraction Identity for Cosine:**  $\cos(x - y) = \cos x \cos y + \sin x \sin y$
- **Subtraction Identity for Sine:**  $\sin(x - y) = \sin x \cos y - \cos x \sin y$
- **Subtraction Identity for Tangent:**  $\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$

## **CLASSROOM ROUTINES**

The importance of continuing the established classroom routines cannot be overstated. Daily routines must include such obvious activities as estimating, analyzing data, describing patterns, and answering daily questions. They should also include less obvious routines, such as how to select materials, how to use materials in a productive manner, how to put materials away, how to access classroom technology such as computers and calculators. An additional routine is to allow plenty of time for children to explore new materials before attempting any directed activity with these new materials. The regular use of routines is important to the development of students' number sense, flexibility, fluency, collaborative skills and communication. These routines contribute to a rich, hands-on standards based classroom and will support students' performances on the tasks in this unit and throughout the school year.

## **STRATEGIES FOR TEACHING AND LEARNING**

- Students should be actively engaged by developing their own understanding.
- Mathematics should be represented in as many ways as possible by using graphs, tables, pictures, symbols and words.
- Interdisciplinary and cross curricular strategies should be used to reinforce and extend the learning activities.
- Appropriate manipulatives and technology should be used to enhance student learning.
- Students should be given opportunities to revise their work based on teacher feedback, peer feedback, and metacognition which includes self-assessment and reflection.
- Students should write about the mathematical ideas and concepts they are learning.
- Consideration of all students should be made during the planning and instruction of this unit. Teachers need to consider the following:
  - What level of support do my struggling students need in order to be successful with this unit?
  - In what way can I deepen the understanding of those students who are competent in this unit?
  - What real life connections can I make that will help my students utilize the skills practiced in this unit?

## **EVIDENCE OF LEARNING**

By the conclusion of this unit, students should be able to demonstrate the following competencies:

- Demonstrate a method to prove addition or subtraction identities for sine, cosine, and tangent.
- Apply addition or subtraction identities for sine, cosine, and tangent.
- Use addition or subtraction identities to find missing values for sine, cosine and tangent functions.

## **TASKS**

The following tasks represent the level of depth, rigor, and complexity expected of all Pre-Calculus students. These tasks or tasks of similar depth and rigor should be used to demonstrate evidence of learning. It is important that all elements of a task be addressed throughout the learning process so that students understand what is expected of them. While some tasks are identified as a performance task, they may also be used for teaching and learning (learning task).

<b>Task Name</b>	<b>Task Type <i>Grouping Strategy</i></b>	<b>Content Addressed</b>
<a href="#"><u>Proving the Sine Addition and Subtraction Identities</u></a>	Learning Task <i>Large Group/Partner Task</i>	Developing a proof for the Sine addition and subtraction identities.
<a href="#"><u>Proving the Cosine Addition and Subtraction Identities</u></a>	Learning Task <i>Large Group/Partner Task</i>	Developing a proof for the Cosine addition and subtraction identities.
<a href="#"><u>A Distance Formula Proof for the Cosine Addition Identity</u></a>	Learning Task <i>Individual/Partner Task</i>	Using the Distance Formula to develop a proof for the Cosine addition identity.
<a href="#"><u>Proving the Tangent Addition and Subtraction Identities</u></a>	Learning Task <i>Partner/Small Group Task</i>	Developing a proof for the Tangent addition and subtraction identities.
<a href="#"><u>Double Angle Identities for Sine, Cosine, and Tangent.</u></a>	Learning Task <i>Individual/Partner</i>	Using the Addition Identities to Derive the Double Angle Identities
<a href="#"><u>The Cosine Double-Angle</u></a>	Learning Task <i>Individual/Partner</i>	Exploring the different forms of and uses for the double angle identity for cosine.
<a href="#"><u>Half Angle Identities for Sine, Cosine, and Tangent</u></a>	Learning Task <i>Individual/Partner</i>	Using the Double Angle Identities to Derive the Half Angle Identities
<a href="#"><u>Culminating Task: How Many Angles Can You Find?</u></a>	Performance Task <i>Individual/Partner Task</i>	Using trigonometric addition and subtraction identities to determine trigonometric values.

## **PROVING THE SINE ADDITION AND SUBTRACTION IDENTITIES**

### **[Back to Task Table](#)**

#### **Georgia Standards of Excellence:**

**MGSE9-12.F.TF.9** Prove addition, subtraction, double, and half-angle formulas for sine, cosine, and tangent and use them to solve problems.

#### **Introduction:**

This task guides students through one possible proof of addition and subtraction identities for sine. It is designed to guide the student step by step through the process, with several steps along the way to make sure they are on the right track. The emphasis should not be on this particular method, but rather the act of proof itself. Ask students to look beyond the steps to see mathematics happening through proof.

## **PROVING THE SINE ADDITION AND SUBTRACTION IDENTITIES**

When it becomes necessary to add and subtract angles, finding the sine of the new angles is not as straight forward as adding or subtracting the sine of the individual angles. Convince yourself of this by performing the following operation:

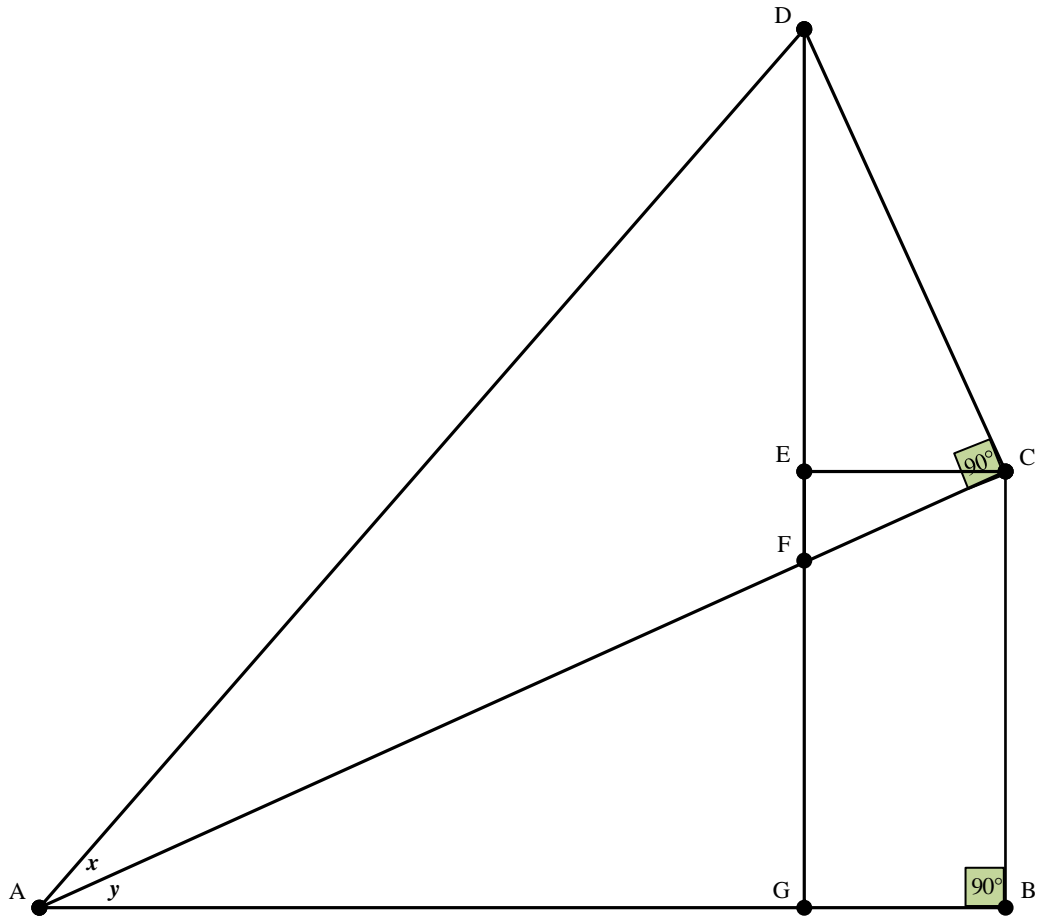
$$\sin(30^\circ) + \sin(60^\circ) = \sin(90^\circ)?$$

$$\frac{1}{2} + \frac{\sqrt{3}}{2} \neq 1$$

In this task, you will develop a formula for finding the sine of a sum or difference of two angles. Begin by looking at the diagram below. (note:  $\overline{DG} \perp \overline{AB}$ ):

Use the diagram to answer the following questions:





1. What type of triangles are  $\triangle ABC$ ,  $\triangle ACD$  and  $\triangle DAG$ ?

*Right Triangles*

2. Write an expression for the measurement of  $\angle DAB$ .

$$m\angle DAB = x + y$$

3. Write an expression for  $\sin \angle DAB$ .

$$\sin(x + y) = \frac{\overline{DG}}{\overline{AD}}$$

4.  $\overline{EG} \cong \overline{BC}$

Now we will use algebra to develop the formula for the sine of a sum of two angles.

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- a. Use your expression from #3.

$$\sin(x + y) = \frac{\overline{DG}}{\overline{AD}}$$

- b. Rewrite DG as the sum of two segments, EG and DE, then split the fraction.

$$\sin(x + y) = \frac{\overline{DG}}{\overline{AD}} = \frac{\overline{EG} + \overline{DE}}{\overline{AD}} = \frac{\overline{EG}}{\overline{AD}} + \frac{\overline{DE}}{\overline{AD}}$$

- c. Use the relationship from #4 to make a substitution.

$$\sin(x + y) = \frac{\overline{EG}}{\overline{AD}} + \frac{\overline{DE}}{\overline{AD}} = \frac{\overline{BC}}{\overline{AD}} + \frac{\overline{DE}}{\overline{AD}}$$

Did you arrive at this equation?  $\sin(x + y) = \frac{\overline{BC}}{\overline{AD}} + \frac{\overline{DE}}{\overline{AD}}$  If not, go back and try again.

- a. Now multiply the first term by  $\frac{\overline{AC}}{\overline{AC}}$ , and the second term by  $\frac{\overline{CD}}{\overline{CD}}$ . This will not change the value of our expressions, but will establish a link between some important pieces.

$$\sin(x + y) = \frac{\overline{BC}}{\overline{AD}} \cdot \frac{\overline{AC}}{\overline{AC}} + \frac{\overline{DE}}{\overline{AD}} \cdot \frac{\overline{CD}}{\overline{CD}}$$

*We are using AC and CD because they give us sine and cosine ratios when we multiply and divide them purposefully in each term.*

- b. Now change the order on the denominators of both terms.

$$\sin(x + y) = \frac{\overline{BC}}{\overline{AC}} \cdot \frac{\overline{AC}}{\overline{AD}} + \frac{\overline{DE}}{\overline{CD}} \cdot \frac{\overline{CD}}{\overline{AD}}$$

Did you arrive at this equation?  $\sin(x + y) = \frac{\overline{BC}}{\overline{AC}} \cdot \frac{\overline{AC}}{\overline{AD}} + \frac{\overline{DE}}{\overline{CD}} \cdot \frac{\overline{CD}}{\overline{AD}}$

Now is the most important step. Look at each factor. What relationship do each of those represent? For example,  $\overline{BC}$  is the opposite side from  $\angle X$ , and  $\overline{AC}$  is the hypotenuse in that triangle. This means that  $\frac{\overline{BC}}{\overline{AC}} = \sin X$ .

Substitute the other values and you have developed a formula for  $\sin(x+y)$ .

*Students may have trouble finding a substitution for  $\frac{\overline{DE}}{\overline{CD}}$ . Try to point them towards the fact that triangle CDE and triangle FCE are similar (CE is parallel to AB so angle ECA is congruent to angle BAC by alternate interior angles.)*

$$\sin(x+y) = \frac{\overline{BC}}{\overline{AC}} \cdot \frac{\overline{AC}}{\overline{AD}} + \frac{\overline{DE}}{\overline{CD}} \cdot \frac{\overline{CD}}{\overline{AD}} = \sin x \cos y + \cos x \sin y$$

Check your formula with the following exercises:

1.  $\sin(45^\circ + 45^\circ) = \sin(90^\circ)$ ?

$$\sin 45^\circ \cos 45^\circ + \cos 45^\circ \sin 45^\circ = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2} = 1 = \sin 90^\circ$$

2.  $\sin\left(\frac{\pi}{3} + \frac{\pi}{6}\right) = \sin\left(\frac{\pi}{2}\right)$ ?

$$\sin \frac{\pi}{3} \cos \frac{\pi}{6} + \cos \frac{\pi}{3} \sin \frac{\pi}{6} = \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{1}{2} \cdot \frac{1}{2} = 1 = \sin \frac{\pi}{2}$$

Write your formula here:

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$

Now that we have a formula for the sine of a sum of angles, we can use it to develop the formula for the difference of angles.

We can think of the difference of two angles like this:  $\sin(x-y) = \sin(x+(-y))$

Using the formula we developed above, we substitute in and get:

$$\sin(x-y) = \sin x \cos(-y) + \cos x \sin(-y)$$

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But, cosine is an even function meaning that:  $f(-x) = f(x)$  . Use this fact to simplify the equation.

$$\sin(x - y) = \sin x \cos y + \cos x \sin(-y)$$

Similarly, sine is an odd function meaning that:  $f(-x) = -f(x)$  . Use this fact to simplify the equation.

$$\sin(x - y) = \sin x \cos y - \cos x \sin y$$

Write your formula here:

$$\sin(x - y) = \sin x \cos y - \cos x \sin y$$

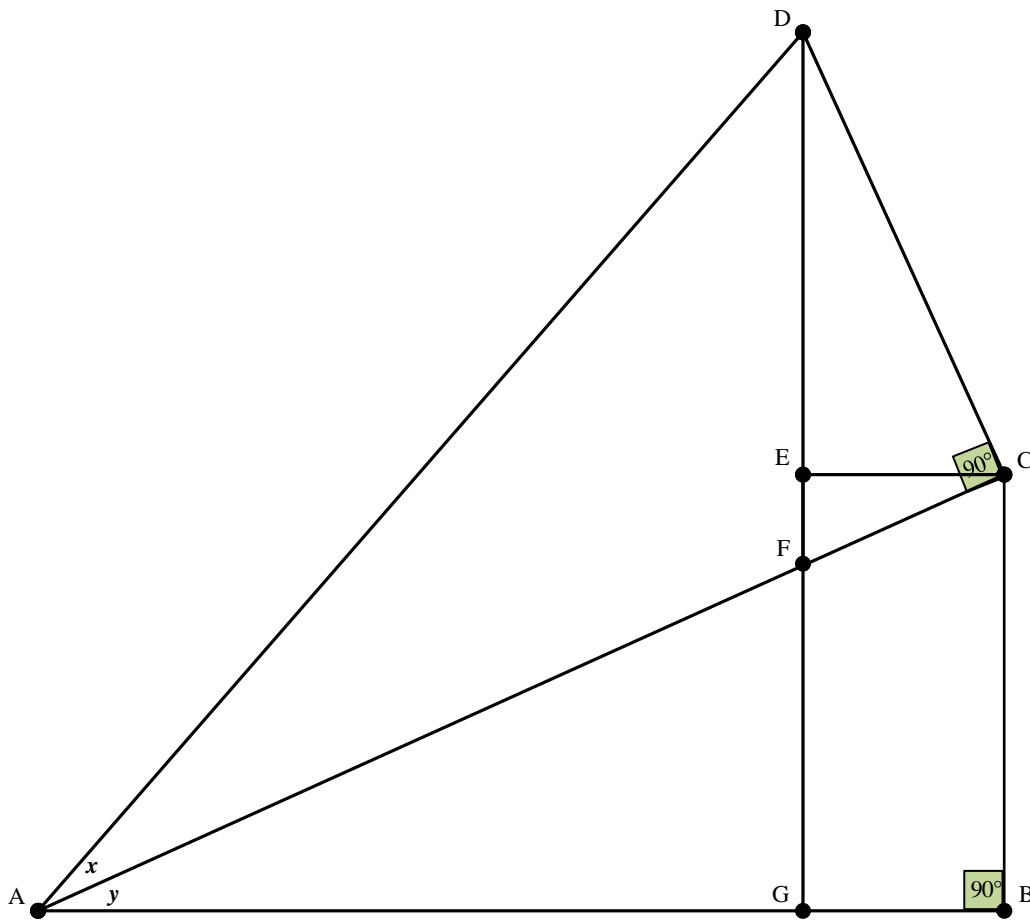
*A note to the teacher: This task only involves proving the addition and subtraction identities. The classroom teacher should build on this proof by providing problems that build competency and fluency in applying the identity. Such problems should include, but not be limited to: finding exact values, expressing as a trigonometric function of one angle, verifying identities and solving trigonometric equations.*

## PROVING THE SINE ADDITION AND SUBTRACTION IDENTITIES

When it becomes necessary to add and subtract angles, finding the sine of the new angles is not as straight forward as adding or subtracting the sine of the individual angles. Convince yourself of this by performing the following operation:

$$\sin(30^\circ) + \sin(60^\circ) = \sin(90^\circ)?$$

In this task, you will develop a formula for finding the sine of a sum or difference of two angles. Begin by looking at the diagram below. (note:  $\overline{DG} \perp \overline{AB}$ ):



Use the diagram to answer the following questions:

1. What type of triangles are  $\triangle ABC$ ,  $\triangle ACD$  and  $\triangle DAG$ ?
2. Write an expression for the measurement of  $\angle DAB$ .

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3. Write an expression for  $\sin \angle DAB$ .

4.  $\overline{EG} \cong$  \_\_\_\_\_

Now we will use algebra to develop the formula for the sine of a sum of two angles.

- a. Use your expression from #3.
- b. Rewrite DG as the sum of two segments, EG and DE, then split the fraction.
- c. Use the relationship from #4 to make a substitution.

Did you arrive at this equation?  $\sin(x + y) = \frac{\overline{BC}}{\overline{AD}} + \frac{\overline{DE}}{\overline{AD}}$  If not, go back and try again.

- a. Now multiply the first term by  $\frac{\overline{AC}}{\overline{AC}}$ , and the second term by  $\frac{\overline{CD}}{\overline{CD}}$ . This will not change the value of our expressions, but will establish a link between some important pieces.
- b. Now change the order on the denominators of both terms.

Did you arrive at this equation?  $\sin(x + y) = \frac{\overline{BC}}{\overline{AC}} \cdot \frac{\overline{AC}}{\overline{AD}} + \frac{\overline{DE}}{\overline{CD}} \cdot \frac{\overline{CD}}{\overline{AD}}$

Now is the most important step. Look at each factor. What relationship do each of those represent? For example,  $\frac{\overline{BC}}{\overline{AC}}$  is the opposite side from  $\angle X$ , and  $\overline{AC}$  is the hypotenuse in that triangle. This means that  $\frac{\overline{BC}}{\overline{AC}} = \sin X$ .

Substitute the other values and you have developed a formula for  $\sin(x+y)$ .

Check your formula with the following exercises:

1.  $\sin(45^\circ + 45^\circ) = \sin(90^\circ)$  ?

2.  $\sin\left(\frac{\pi}{3} + \frac{\pi}{6}\right) = \sin\left(\frac{\pi}{2}\right)$  ?

Write your formula here:

Now that we have a formula for the sine of a sum of angles, we can use it to develop the formula for the difference of angles.

We can think of the difference of two angles like this:  $\sin(x - y) = \sin(x + (-y))$

Using the formula we developed above, we substitute in and get:

$$\sin(x - y) = \sin x \cos(-y) + \cos x \sin(-y)$$

But, cosine is an even function meaning that:  $f(-x) = f(x)$  . Use this fact to simplify the equation.

Similarly, sine is an odd function meaning that:  $f(-x) = -f(x)$  . Use this fact to simplify the equation.

Write your formula here:

## **PROVING THE COSINE ADDITION AND SUBTRACTION IDENTITIES**

### **[Back to Task Table](#)**

### **Georgia Standards of Excellence:**

**MGSE9-12.F.TF.9** Prove addition, subtraction, double and half-angle formulas for sine, cosine, and tangent and use them to solve problems.

### **Introduction:**

This task follows the format of the first task. It is designed to guide the student step by step through the process, with several steps along the way to make sure they are on the right track. The emphasis should not be on this particular method, but rather the act of proof itself. Ask students to look beyond the steps to see mathematics happening through proof.

## **PROVING THE COSINE ADDITION AND SUBTRACTION IDENTITIES**

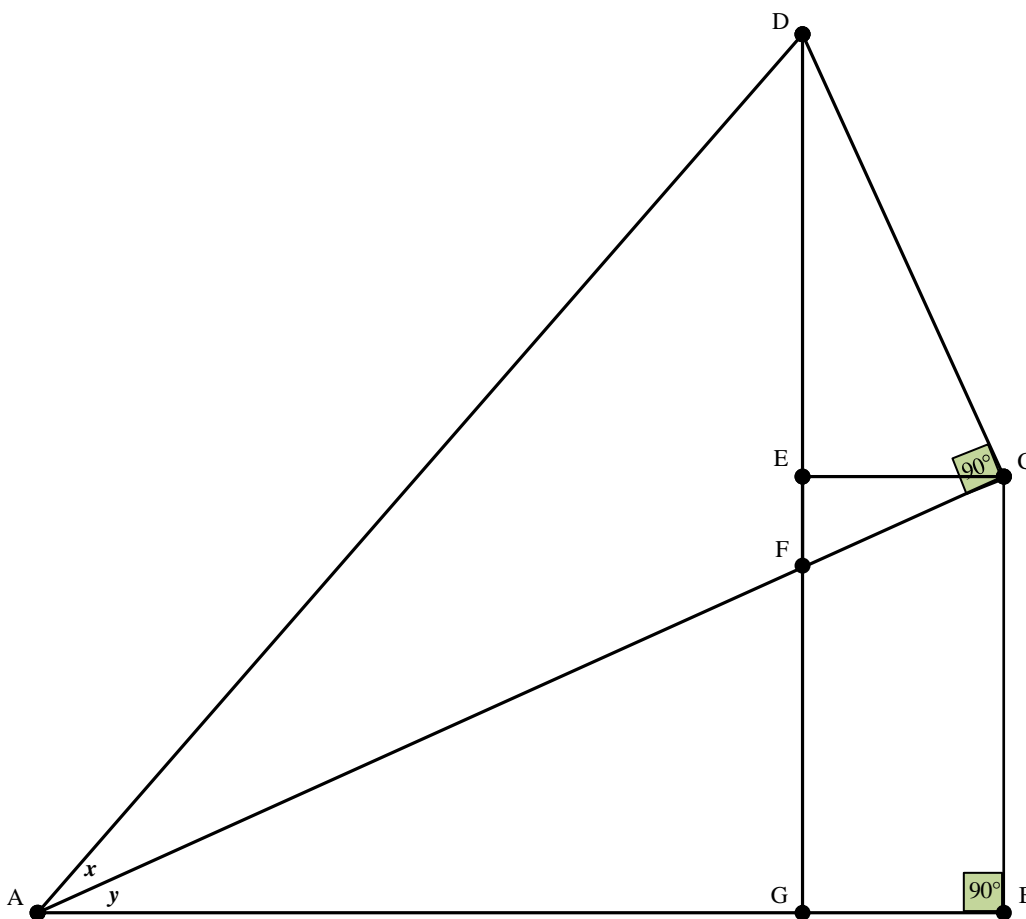
In the first task you saw that it is necessary to develop a formula for calculating the sine of sums or differences of angles. Can the same formula work for cosine? Experiment with your formula and see if you can use it to answer the exercise below:

$$\cos(30^\circ) + \cos(60^\circ) = \cos(90^\circ)?$$

$$\frac{\sqrt{3}}{2} + \frac{1}{2} \neq 0$$

In this task, you will develop a formula for finding the sine of a sum or difference of two angles. Begin by looking at the diagram below. (note:  $\overline{DG} \perp \overline{AB}$ ):





Use the diagram to answer the following questions:

1. What type of triangles are  $\triangle ABC$ ,  $\triangle ACD$  and  $\triangle DAG$ ?

*Right Triangles*

2. Write an expression for the measurement of  $\angle DAB$ .

$$m\angle DAB = x + y$$

3. Write an expression for  $\cos \angle DAB$ .

$$\cos(x + y) = \frac{\overline{AG}}{\overline{DA}}$$

4.  $\overline{BG} \cong \overline{EC}$

5.  $\angle ECA \cong \angle CAB$  by alternate interior angles.
6.  $\angle CDE \cong \angle ECA$  because they are corresponding angles in similar triangles.

Now we will use algebra to develop the formula for the cosine of a sum of two angles.

- a. Use your expression from #3.

$$\cos(x + y) = \frac{\overline{AG}}{\overline{DA}}$$

- b. Rewrite AG as the difference of two segments, AB and BG, then split the fraction.

$$\cos(x + y) = \frac{\overline{AG}}{\overline{DA}} = \frac{\overline{AB} - \overline{BG}}{\overline{AD}} = \frac{\overline{AB}}{\overline{AD}} - \frac{\overline{BG}}{\overline{AD}}$$

- c. Use the relationship from #4 to make a substitution.

$$\cos(x + y) = \frac{\overline{AB}}{\overline{AD}} - \frac{\overline{EC}}{\overline{AD}}$$

Did you arrive at this equation?  $\cos(x + y) = \frac{\overline{AB}}{\overline{AD}} - \frac{\overline{EC}}{\overline{AD}}$  If not, go back and try again.

- a. Now multiply the first term by  $\frac{\overline{AC}}{\overline{AC}}$ , and the second term by  $\frac{\overline{CD}}{\overline{CD}}$ . This will not change the value of our expressions, but will establish a link between some important pieces.

$$\cos(x + y) = \frac{\overline{AB}}{\overline{AD}} \cdot \frac{\overline{AC}}{\overline{AC}} - \frac{\overline{EC}}{\overline{AD}} \cdot \frac{\overline{CD}}{\overline{CD}}$$

*We are using AC and CD because they give us sine and cosine ratios when we multiply and divide them purposefully in each term.*

- b. Now change the order on the denominators of both terms.

$$\cos(x + y) = \frac{\overline{AB}}{\overline{AC}} \cdot \frac{\overline{AC}}{\overline{AD}} - \frac{\overline{EC}}{\overline{CD}} \cdot \frac{\overline{CD}}{\overline{AD}}$$

Did you arrive at this equation?  $\cos(x + y) = \frac{\overline{AB}}{\overline{AC}} \cdot \frac{\overline{AC}}{\overline{AD}} - \frac{\overline{EC}}{\overline{CD}} \cdot \frac{\overline{CD}}{\overline{AD}}$

Now is the most important step. Look at each factor. What relationship do each of those represent? For example,  $\overline{EC}$  is the opposite side from  $\angle X$  (from similarity), and  $\overline{CD}$  is a hypotenuse in that triangle. This means that  $\frac{\overline{EC}}{\overline{CD}} = \sin X$ .

Substitute the other values and you have developed a formula for  $\cos(x+y)$ .

$$\cos(x + y) = \cos x \cdot \cos y - \sin x \cdot \sin y$$

Check your formula with the following exercises:

1.  $\cos(30^\circ + 30^\circ) = \cos(60^\circ)$ ?

$$\cos(30^\circ + 30^\circ) = \cos 30^\circ \cdot \cos 30^\circ - \sin 30^\circ \cdot \sin 30^\circ = \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2} = \cos 60^\circ$$

2.  $\cos\left(\frac{\pi}{3} + \frac{\pi}{6}\right) = \cos\left(\frac{\pi}{2}\right)$ ?

$$\cos\left(\frac{\pi}{3} + \frac{\pi}{6}\right) = \cos \frac{\pi}{3} \cdot \cos \frac{\pi}{6} - \sin \frac{\pi}{3} \cdot \sin \frac{\pi}{6} = \frac{1}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} \cdot \frac{1}{2} = 0 = \cos \frac{\pi}{2}$$

Write your formula here:

$$\cos(x + y) = \cos x \cdot \cos y - \sin x \cdot \sin y$$

Now that we have a formula for the cosine of a sum of angles, we can use it to develop the formula for the difference of angles.

We can think of the difference of two angles like this:  $\cos(x - y) = \cos(x + (-y))$

Using the formula we developed above, we substitute in and get:

$$\cos(x - y) = \cos x \cos(-y) - \sin x \sin(-y)$$

But, cosine is an even function meaning that:  $f(-x) = f(x)$ . Use this fact to simplify the equation.

$$\cos(x - y) = \cos x \cdot \cos y - \sin x \cdot \sin(-y)$$

Similarly, sine is an odd function meaning that:  $f(-x) = -f(x)$ . Use this fact to simplify the equation.

$$\cos(x - y) = \cos x \cdot \cos y + \sin x \cdot \sin y$$

Write your formula here:

$$\cos(x - y) = \cos x \cdot \cos y + \sin x \cdot \sin y$$

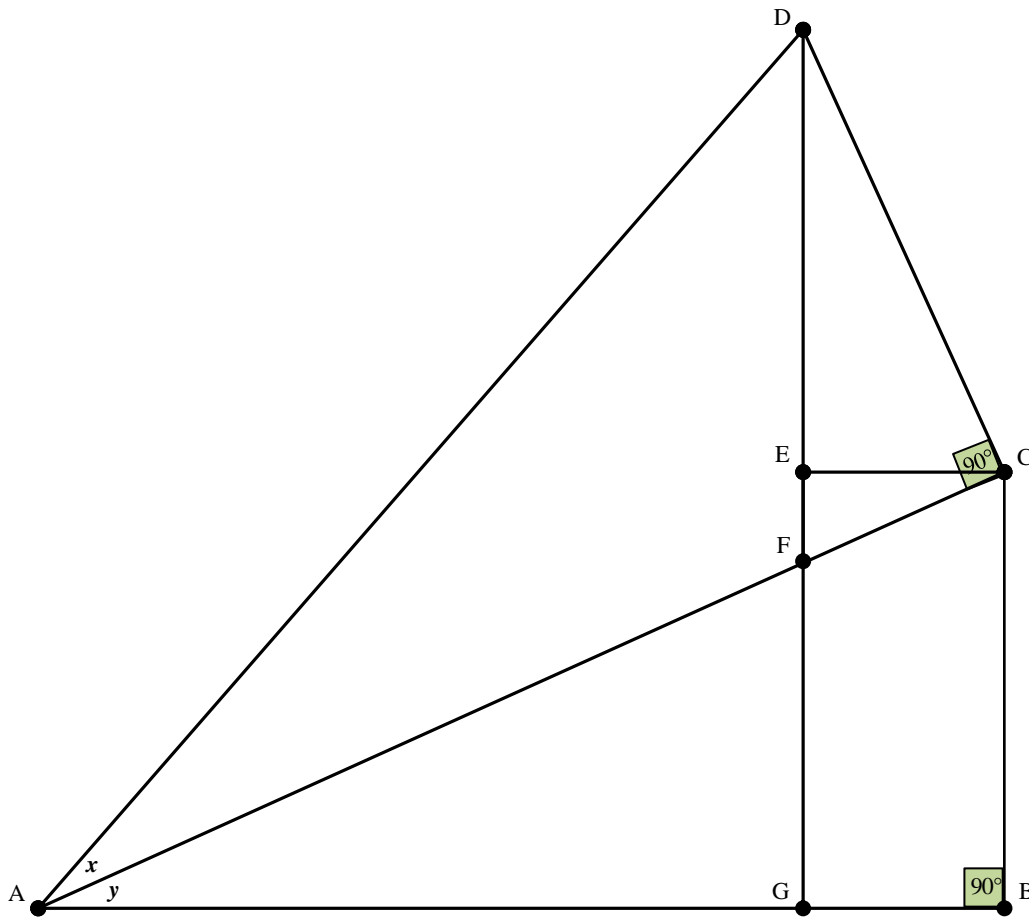
*A note to the teacher: This task only involves proving the addition and subtraction identities. The classroom teacher should build on this proof by providing problems that build competency and fluency in applying the identity. Such problems should include, but not be limited to: finding exact values, expressing as a trigonometric function of one angle, verifying identities and solving trigonometric equations.*

## PROVING THE COSINE ADDITION AND SUBTRACTION IDENTITIES

In the first task you saw that it is necessary to develop a formula for calculating the sine of sums or differences of angles. Can the same formula work for cosine? Experiment with your formula and see if you can use it to answer the exercise below:

$$\cos(30^\circ) + \cos(60^\circ) = \cos(90^\circ)?$$

In this task, you will develop a formula for finding the sine of a sum or difference of two angles. Begin by looking at the diagram below. (note:  $\overline{DG} \perp \overline{AB}$ ):



Use the diagram to answer the following questions:

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Georgia Standards of Excellence Framework  
*Accelerated GSE Pre-Calculus • Unit 4*

1. What type of triangles are  $\triangle ABC$ ,  $\triangle ACD$  and  $\triangle DAG$ ?
2. Write an expression for the measurement of  $\angle DAB$ .
3. Write an expression for  $\cos \angle DAB$ .
4.  $\overline{BG} \cong$  \_\_\_\_\_
5.  $\angle ECA \cong$  \_\_\_\_\_ by alternate interior angles.
6.  $\angle CDE \cong$  \_\_\_\_\_ because they are corresponding angles in similar triangles.

Now we will use algebra to develop the formula for the cosine of a sum of two angles.

- a. Use your expression from #3.
- b. Rewrite AG as the difference of two segments, AB and BG, then split the fraction.
- c. Use the relationship from #4 to make a substitution.

Did you arrive at this equation?  $\cos(x + y) = \frac{\overline{AB}}{\overline{AD}} - \frac{\overline{EC}}{\overline{AD}}$  If not, go back and try again.

- a. Now multiply the first term by  $\frac{\overline{AC}}{\overline{AC}}$ , and the second term by  $\frac{\overline{CD}}{\overline{CD}}$ . This will not change the value of our expressions, but will establish a link between some important pieces.
- b. Now change the order on the denominators of both terms.

Did you arrive at this equation?  $\cos(x + y) = \frac{\overline{AB}}{\overline{AC}} \cdot \frac{\overline{AC}}{\overline{AD}} - \frac{\overline{EC}}{\overline{CD}} \cdot \frac{\overline{CD}}{\overline{AD}}$

Now is the most important step. Look at each factor. What relationship do each of those represent? For example,  $\overline{EC}$  is the opposite side from  $\angle X$  (from similarity), and  $\overline{CD}$  is a hypotenuse in that triangle. This means that  $\frac{\overline{EC}}{\overline{CD}} = \sin X$ .

Substitute the other values and you have developed a formula for  $\cos(x+y)$ .

Check your formula with the following exercises:

1.  $\cos(30^\circ + 30^\circ) = \cos(60^\circ)$ ?

2.  $\cos\left(\frac{\pi}{3} + \frac{\pi}{6}\right) = \cos\left(\frac{\pi}{2}\right)$ ?

Write your formula here:

Now that we have a formula for the cosine of a sum of angles, we can use it to develop the formula for the difference of angles.

We can think of the difference of two angles like this:  $\cos(x - y) = \cos(x + (-y))$

Using the formula we developed above, we substitute in and get:

$$\cos(x - y) = \cos x \cos(-y) - \sin x \sin(-y)$$

But, cosine is an even function meaning that:  $f(-x) = f(x)$ . Use this fact to simplify the equation.

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Georgia Standards of Excellence Framework  
*Accelerated GSE Pre-Calculus • Unit 4*

Similarly, sine is an odd function meaning that:  $f(-x) = -f(x)$ . Use this fact to simplify the equation.

Write your formula here:



## **A DISTANCE FORMULA PROOF FOR THE COSINE ADDITION IDENTITY**

### **[Back to Task Table](#)**

#### **Georgia Standards of Excellence:**

**MGSE9-12.F.TF.9** Prove addition, subtraction, double and half-angle formulas for sine, cosine, and tangent and use them to solve problems.

#### **Introduction:**

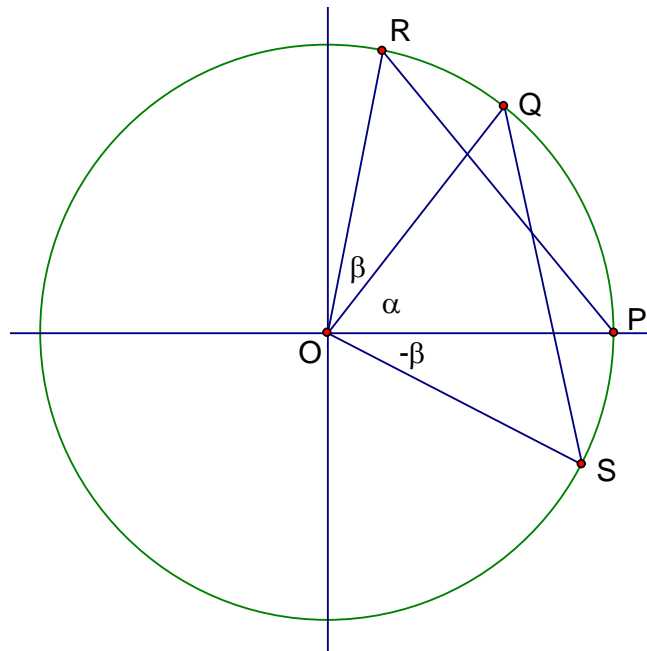
In this task, students derive the sum identity for the cosine function, in the process reviewing some of the geometric topics and ideas about proofs. This derivation also provides practice with algebraic manipulation of trigonometric functions that include examples of how applying the Pythagorean identities can often simplify a cumbersome trigonometric expression. Rewriting expressions in order to solve trigonometric equations is one of the more common applications of the sum and difference identities. This task is presented as an alternative proof to the ones the students performed earlier.

## **A DISTANCE FORMULA PROOF FOR THE COSINE ADDITION IDENTITY**

In this task, you will use the sum and difference identities to solve equations and find the exact values of angles that are not multiples of  $\frac{\pi}{6}$  and  $\frac{\pi}{4}$ . Before you apply these identities to problems, you will first derive them. The first identity you will prove involves taking the cosine of the sum of two angles.

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

We can derive this identity by making deductions from the relationships between the quantities on the unit circle below.



- Complete the following congruence statements:
  - $\angle ROP \cong \angle QOS$
  - $\overline{RO} \cong \overline{QO} \cong \overline{PO} \cong \overline{SO}$
  - By the **SAS** congruence theorem,  $\triangle ROP \cong \triangle QOS$
  - $\overline{RP} \cong \overline{QS}$
- Write the coordinates of each of the four points on the unit circle, remembering that the cosine and sine functions produce x- and y- values on the unit circle.
  - $R = (\cos(\alpha + \beta), \sin(\alpha + \beta))$
  - $Q = (\cos \alpha, \sin \alpha)$
  - $P = (1, 0)$
  - $S = (\cos(-\beta), \sin(-\beta))$

3. Use the coordinates found in problem 2 and the distance formula to find the length of chord  $\overline{RP}$ . *Note: Students may not simplify here, but will need to in part 5.*

**Solution:**

$$\begin{aligned} & \sqrt{(\cos(\alpha + \beta) - 1)^2 + (\sin(\alpha + \beta) - 0)^2} \\ &= \sqrt{\cos^2(\alpha + \beta) - 2\cos(\alpha + \beta) + 1 + \sin^2(\alpha + \beta)} \text{ *squaring each binomial*} \\ &= \sqrt{(\cos^2(\alpha + \beta) + \sin^2(\alpha + \beta)) + 1 - 2\cos(\alpha + \beta)} \text{ *rearranging terms*} \\ &= \sqrt{1 + 1 - 2\cos(\alpha + \beta)} \text{ *applying a Pythagorean identity*} \\ &= \sqrt{2 - 2\cos(\alpha + \beta)} \end{aligned}$$

4. a. Use the coordinates found in problem 2 and the distance formula to find the length of chord  $\overline{QS}$ . *Note: Students may not simplify here, but will need to in part 5.*

**Solution:**

$$\begin{aligned} & \sqrt{(\cos \alpha - \cos(-\beta))^2 + (\sin \alpha - \sin(-\beta))^2} \\ &= \sqrt{\cos^2 \alpha - 2\cos \alpha \cos(-\beta) + \cos^2(-\beta) + \sin^2 \alpha - 2\sin \alpha \sin(-\beta) + \sin^2(-\beta)} \text{ *squaring each binomial*} \\ &= \sqrt{(\cos^2 \alpha + \sin^2 \alpha) + (\cos^2(-\beta) + \sin^2(-\beta)) - 2\cos \alpha \cos(-\beta) - 2\sin \alpha \sin(-\beta)} \text{ *rearranging terms*} \\ &= \sqrt{1 + 1 - 2\cos \alpha \cos(-\beta) - 2\sin \alpha \sin(-\beta)} \text{ *applying a Pythagorean identity twice*} \\ &= \sqrt{2 - 2\cos \alpha \cos(-\beta) - 2\sin \alpha \sin(-\beta)} \end{aligned}$$

- b. Two useful identities that you may choose to explore later are  $\cos(-\theta) = \cos \theta$  and  $\sin(-\theta) = -\sin \theta$ . Use these two identities to simplify your solution to 4a so that your expression has no negative angles.

**Solution:**

$$\begin{aligned} & \sqrt{2 - 2\cos \alpha \cos(-\beta) - 2\sin \alpha \sin(-\beta)} \\ &= \sqrt{2 - 2\cos \alpha \cos \beta - 2\sin \alpha (-\sin \beta)} \\ &= \sqrt{2 - 2\cos \alpha \cos \beta + 2\sin \alpha \sin \beta} \end{aligned}$$

5. From 1d, you know that  $\overline{RP} \cong \overline{QS}$ . You can therefore write an equation by setting the expressions found in problems 3 and 4b equal to one another. Simplify this equation and

solve for  $\cos(\alpha + \beta)$ . Applying one of the Pythagorean Identities will be useful! When finished, you will have derived the angle sum identity for cosine.

**Solution:**

$$\begin{aligned}\sqrt{2-2\cos(\alpha+\beta)} &= \sqrt{2-2\cos\alpha\cos\beta+2\sin\alpha\sin\beta} \\ 2-2\cos(\alpha+\beta) &= 2-2\cos\alpha\cos\beta+2\sin\alpha\sin\beta && \text{squaring both sides} \\ 2\cos(\alpha+\beta) &= 2\cos\alpha\cos\beta+2\sin\alpha\sin\beta && \text{adding 2 to both sides} \\ \cos(\alpha\pm\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta && \text{dividing both sides by 2}\end{aligned}$$

The other three sum and difference identities can be derived from the identity found in problem 5. These four identities can be summarized with the following two statements.

$$\begin{aligned}\sin(\alpha\pm\beta) &= \sin\alpha\cos\beta\pm\cos\alpha\sin\beta \\ \cos(\alpha\pm\beta) &= \cos\alpha\cos\beta\mp\sin\alpha\sin\beta\end{aligned}$$

Recall that so far, you can only calculate the exact values of the sines and cosines of multiples of  $\frac{\pi}{6}$  and  $\frac{\pi}{4}$ . These identities will allow you to calculate the exact value of the sine and cosine of many more angles.

6. Evaluate  $\sin 75^\circ$  by applying the angle addition identity for sine and evaluating each trigonometric function:

$$\sin(30^\circ + 45^\circ) = \sin 30^\circ \cos 45^\circ + \cos 30^\circ \sin 45^\circ$$

**Solution:**

$$\frac{1}{2} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4} = \frac{\sqrt{2} + \sqrt{6}}{4}$$

7. Similarly, find the exact value of the following trigonometric expressions:
- a.  $\cos(15^\circ)$

**Solution:**  $\frac{\sqrt{2} + \sqrt{6}}{4}$

b.  $\sin\left(\frac{\pi}{12}\right)$

**Solution:**  $\frac{\sqrt{6} - \sqrt{2}}{4}$

c.  $\cos(345^\circ)$

**Solution:**  $\frac{\sqrt{2} + \sqrt{6}}{4}$

d.  $\sin\frac{19\pi}{12}$

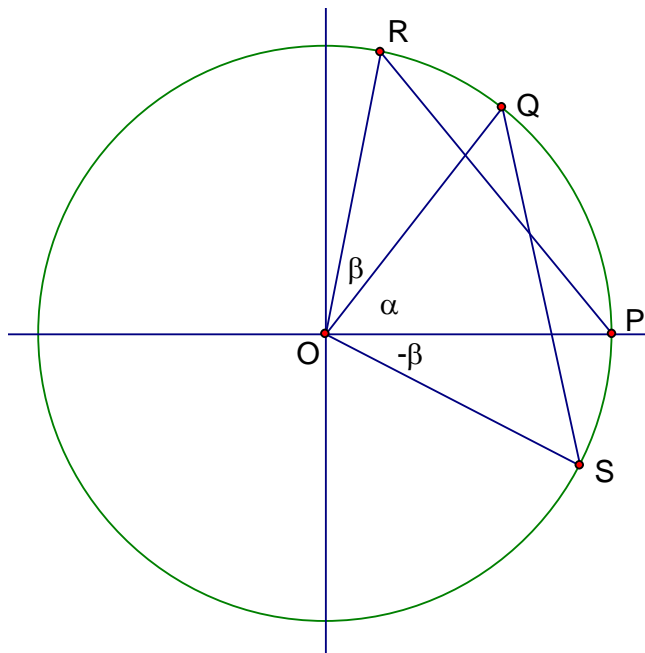
**Solution:**  $\frac{\sqrt{2} - \sqrt{6}}{4}$

## A DISTANCE FORMULA PROOF FOR THE COSINE ADDITION IDENTITY

In this task, you will use the sum and difference identities to solve equations and find the exact values of angles that are not multiples of  $\frac{\pi}{6}$  and  $\frac{\pi}{4}$ . Before you apply these identities to problems, you will first derive them. The first identity you will prove involves taking the cosine of the sum of two angles.

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

We can derive this identity by making deductions from the relationships between the quantities on the unit circle below.



1. Complete the following congruence statements:

- a.  $\angle ROP \cong$  \_\_\_\_\_
- b.  $\overline{RO} \cong$  \_\_\_\_\_  $\cong$  \_\_\_\_\_  $\cong$  \_\_\_\_\_
- c. By the \_\_\_\_\_ congruence theorem,  $\triangle ROP \cong \triangle QOS$
- d.  $\overline{RP} \cong$  \_\_\_\_\_

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Georgia Standards of Excellence Framework  
*Accelerated GSE Pre-Calculus • Unit 4*

2. Write the coordinates of each of the four points on the unit circle, remembering that the cosine and sine functions produce x- and y- values on the unit circle.
  - a. R =
  
  
  
  
  
  
  
  
  
  
  - b. Q =
  
  
  
  
  
  
  
  
  
  
  - c. P =
  
  
  
  
  
  
  
  
  
  
  - d. S =
  
3. Use the coordinates found in problem 2 and the distance formula to find the length of chord  $\overline{RP}$ .
  
  
  
  
  
  
  
  
  
  
4. Use the coordinates found in problem 2 and the distance formula to find the length of chord  $\overline{QS}$ .
  
  
  
  
  
  
  
  
  
  
5. From 1d, you know that  $\overline{RP} \cong \overline{QS}$ . You can therefore write an equation by setting the expressions found in problems 3 and 4b equal to one another. Simplify this equation and solve for  $\cos(\alpha + \beta)$ . Applying one of the Pythagorean Identities will be useful! When finished, you will have derived the angle sum identity for cosine.

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Georgia Standards of Excellence Framework  
*Accelerated GSE Pre-Calculus • Unit 4*

The other three sum and difference identities can be derived from the identity found in problem 5. These four identities can be summarized with the following two statements.

$$\sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta$$

$$\cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta$$

Recall that so far, you can only calculate the exact values of the sines and cosines of multiples of  $\frac{\pi}{6}$  and  $\frac{\pi}{4}$ . These identities will allow you to calculate the exact value of the sine and cosine of many more angles.

6. Evaluate  $\sin 75^\circ$  by applying the angle addition identity for sine and evaluating each trigonometric function:

$$\sin(30^\circ + 45^\circ) = \sin 30^\circ \cos 45^\circ + \cos 30^\circ \sin 45^\circ$$

7. Similarly, find the exact value of the following trigonometric expressions:

a.  $\cos(15^\circ)$

b.  $\sin\left(\frac{\pi}{12}\right)$

c.  $\cos(345^\circ)$

d.  $\sin \frac{19\pi}{12}$



## **PROVING THE TANGENT ADDITION AND SUBTRACTION IDENTITIES**

### **[Back to Task Table](#)**

### **Georgia Standards of Excellence:**

**MGSE9-12.F.TF.9** Prove addition, subtraction, double and half-angle formulas for sine, cosine, and tangent and use them to solve problems.

### **Introduction:**

This task uses algebraic manipulation and the previously developed identities to derive tangent addition and subtraction identities. Again, the emphasis should be on the mathematical processes and not the method itself. Students should feel confident in manipulating algebraic expressions and simplifying trigonometric expressions using identities.

## **PROVING THE TANGENT ADDITION AND SUBTRACTION IDENTITIES**

By this point, you should have developed formulas for sine and cosine of sums and differences of angles. If so, you are already most of the way to finding a formula for the tangent of a sum of two angles.

Let's begin with a relationship that we already know to be true about tangent.

1.  $\tan x = \frac{\sin x}{\cos x}$  so it stands to reason that  $\tan(x + y) = \frac{\sin(x + y)}{\cos(x + y)}$

Use what you already know about sum and difference formulas to expand the relationship above.

2.  $\tan(x + y) = \frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y}$

3. Now we want to simplify this. (Hint: Multiply numerator and denominator by  $\frac{1}{\cos x}$  )

*The reason to multiply by  $\frac{1}{\cos x}$  is to establish a tangent ratio and to divide out  $\cos x$ . It is important that students see why to choose that value as a part of the proof.*

$$\tan(x + y) = \frac{\frac{\sin x \cos y}{\cos x \cos y} + \frac{\cos x \sin y}{\cos x \cos y}}{\frac{\cos x \cos y}{\cos x} - \frac{\sin x \sin y}{\cos x}} = \frac{\tan x \cos y + \sin y}{\cos y - \tan x \sin y}$$

4. You can simplify it some more. Think about step 3 for a hint.

*Students should multiply by  $\frac{1}{\cos y}$  to establish a tangent ratio for the y variable and divide out  $\cos y$ . Lead them back to #3 if they need help.*

$$\tan(x + y) = \frac{\frac{\tan x \cos y}{\cos y} + \frac{\sin y}{\cos y}}{\frac{\cos y}{\cos y} - \frac{\tan x \sin y}{\cos y}} = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

5. Write your formula here for the tangent of a sum:

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

6. Now that you have seen the process, develop a formula for the tangent of a difference.

*Encourage students to attempt this on their own without referring back to the proof of the addition identity. If they need help, they may reference it. Encourage them to persevere through this process and not give up easily. Mathematics and especially proof is meant to be a productive struggle in which students work hard to construct their own knowledge.*

*When they have finished, they should have:*  $\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$

**Georgia Department of Education**  
Georgia Standards of Excellence Framework  
*Accelerated GSE Pre-Calculus • Unit 4*

Write your formula here:

$$\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

## **THE TANGENT ADDITION AND SUBTRACTION IDENTITIES**

By this point, you should have developed formulas for sine and cosine of sums and differences of angles. If so, you are already most of the way to finding a formula for the tangent of a sum of two angles.

Let's begin with a relationship that we already know to be true about tangent.

1.  $\tan x = \frac{\sin x}{\cos x}$  so it stands to reason that  $\tan(x + y) = \underline{\hspace{2cm}}$

Use what you already know about sum and difference formulas to expand the relationship above.

2.  $\tan(x + y) =$

3. Now we want to simplify this. (Hint: Multiply numerator and denominator by  $\frac{1}{\cos x}$  )

4. You can simplify it some more. Think about step 3 for a hint.

5. Write your formula here for the tangent of a sum:

6. Now that you have seen the process, develop a formula for the tangent of a difference.

Write your formula here:

## **DOUBLE-ANGLE IDENTITIES FOR SINE, COSINE, AND TANGENT**

### **[Back to Task Table](#)**

### **Georgia Standards of Excellence:**

**MGSE9-12.F.TF.9** Prove addition, subtraction, double, and half-angle formulas for sine, cosine, and tangent and use them to solve problems.

### **Introduction:**

This task guides students through a derivation of the double-angle identities for the primary trigonometric functions. It also aims to draw a distinction between doubling an angle and doubling the value of a trigonometric function associated with that angle. This distinction is a common misunderstanding that students have.

The task concentrates on the derivation of the sine double-angle formula by leading the students through a step-by-step process. The students are then left to derive the cosine and tangent identities through the use of the same process.

## **DOUBLE-ANGLE IDENTITIES FOR SINE, COSINE, AND TANGENT**

### **Before we begin...**

Evaluate the following expressions without a calculator.

- |   |  |
|---|--|
| 1a. $\cos 45^\circ = \sqrt{2}/2$        | 1b. $\cos 90^\circ = 0$                |
| 2a. $\sin 60^\circ = \sqrt{3}/2$        | 2b. $\sin 120^\circ = \sqrt{3}/2$      |
| 3a. $\sin \frac{\pi}{6} = 1/2$          | 3b. $\sin \frac{\pi}{3} = \sqrt{3}/2$  |
| 4a. $\cos \frac{\pi}{2} = 0$            | 4b. $\cos \pi = -1$                    |
| 5a. $\tan \frac{5\pi}{6} = -\sqrt{3}/3$ | 5b. $\tan \frac{5\pi}{3} = -\sqrt{3}$  |
| 6a. $\tan 45^\circ = 1$                 | 6b. $\tan 90^\circ = \text{undefined}$ |

Study the expressions in parts a and b of the problems above – they are related. Describe how the expressions in parts a and b of the problems *differ*.

*The angle measures in parts b are twice the measure of the angles in parts a.*

### **Consider the following...**

Based on the observations above decide which of these statements are ‘keepers’ & which are ‘trash’.

If an angle is doubled then the sine value of the angle is doubled too.

If an angle is halved then the cosine value of the angle is halved too.

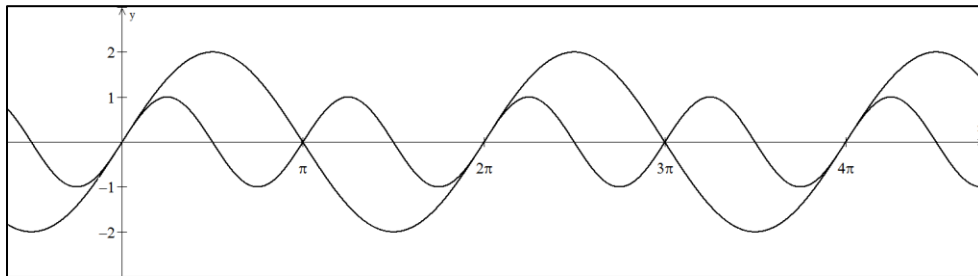
$$2 \sin x \equiv \sin 2x$$

“Double the angle” & “double the sine value” really mean the same thing.

The equation  $2 \cos x = \cos 2x$  has *no* solutions.

*Doubling angles does not double their trig values.*

**One more thing...**



Before we go on, let's look at the graph above – can you identify the two equations that have been graphed? Write them on opposite sides of the “equals” sign below.

$$\underline{\hspace{2cm}} \quad 2 \cos x \quad \underline{\hspace{2cm}} \quad = \quad \underline{\hspace{2cm}} \quad \cos 2x \quad \underline{\hspace{2cm}}$$

Discussion questions: Is the equation above an identity? Does the equation above have solutions? Does the graph above change your opinion of any of the “keepers” or “trash” on the previous page?

**Let's Dig In!**

Hopefully it is obvious to you by now that if an angle is doubled, we know very little about what happens to its trig values. One thing we are certain of, however, is that *doubling an angle does not double its trig values* (except in some special cases!).

Try this: You are familiar, by now, with the identity  $\sin(x + y) = \sin x \cos y + \cos x \sin y$ .  
A - In the space below, rewrite the identity replacing both  $x$  and  $y$  with  $x$ .

$$\sin(x + x) = \sin x \cos x + \cos x \sin x$$

**B** - Now, by combining like terms on the left side and like terms on the right side, simplify the identity you wrote above.

$$\sin(2x) = 2 \sin x \cos x$$

**C** - Use the new mathematical identity you found to complete the following statement. The words to the right may be helpful.

To find the sine value of an angle that I've doubled, I can...

*...double the product of the sine value and the cosine value*

sine
cosine
sum
product
double
half
identity
value
argument

Now, on a separate paper, repeat **A - B - C** - for cosine and tangent!

## **DOUBLE-ANGLE IDENTITIES FOR SINE, COSINE, AND TANGENT**

### **Before we begin...**

Evaluate the following expressions without a calculator.

1a.  $\cos 45^\circ =$

1b.  $\cos 90^\circ =$

2a.  $\sin 60^\circ =$

2b.  $\sin 120^\circ =$

3a.  $\sin \frac{\pi}{6} =$

3b.  $\sin \frac{\pi}{3} =$

4a.  $\cos \frac{\pi}{2} =$

4b.  $\cos \pi =$

5a.  $\tan \frac{5\pi}{6} =$

5b.  $\tan \frac{5\pi}{3} =$

6a.  $\tan 45^\circ =$

6b.  $\tan 90^\circ =$

Study the expressions in parts a and b of the problems above – they are related. Describe how the expressions in parts a and b of the problems *differ*.

### **Consider the following...**

Based on the observations above decide which of these statements are ‘keepers’ & which are ‘trash’.

If an angle is doubled then the sine value of the angle is doubled too.

If an angle is halved then the cosine value of the angle is halved too.

$$2 \sin x \equiv \sin 2x$$

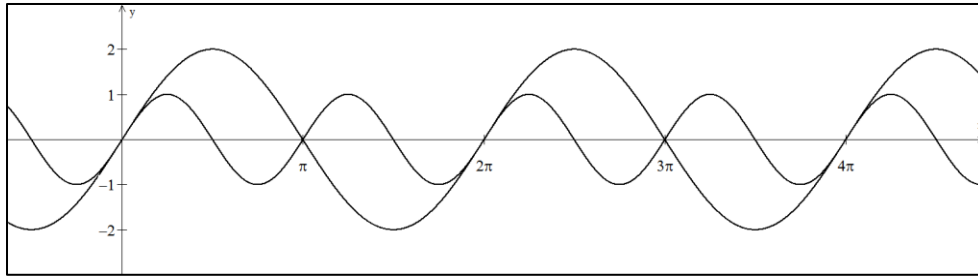
“Double the angle” & “double the sine value” really mean the same thing.

The equation  $2 \cos x = \cos 2x$  has *no* solutions.

Doubling angles does not double their trig values.



**One more thing...**



Before we go on, let's look at the graph above – can you identify the two equations that have been graphed? Write them on opposite sides of the “equals” sign below.

$$\underline{\hspace{10em}} = \underline{\hspace{10em}}$$

Discussion questions: Is the equation above an identity? Does the equation above have solutions? Does the graph above change your opinion of any of the “keepers” or “trash” on the previous page?

**Let's Dig In!**

Hopefully it is obvious to you by now that if an angle is doubled, we know very little about what happens to its trig values. One thing we are certain of, however, is that *doubling an angle does not double its trig values* (except in some special cases!).

Try this: You are familiar, by now, with the identity  $\sin(x + y) = \sin x \cos y + \cos x \sin y$ .

**A** - In the space below, rewrite the identity replacing both  $x$  and  $y$  with  $xs$ .

**B** - Now, by combining like terms on the left side and like terms on the right side, simplify the identity you wrote above.

**C** - Use the new mathematical identity you found to complete the following statement. The words to the right may be helpful.

*To find the sine value of an angle that I've doubled, I can...*

- sine
- cosine
- sum
- product
- double
- half
- identity
- value
- argument

Now, on a separate paper, repeat **A - B - C** - for cosine and tangent!

## **THE COSINE DOUBLE-ANGLE: A MAN WITH MANY IDENTITIES**

### **[Back to Task Table](#)**

### **Georgia Standards of Excellence:**

**MGSE9-12.F.TF.9** Prove addition, subtraction, double, and half-angle formulas for sine, cosine, and tangent and use them to solve problems.

### **Introduction:**

This task serves as a discovery of the alternate forms of the cosine double angle identity and also as a segue into the next task (deriving half-angle identities). A discussion about the advantages of the two alternate forms is encouraged. There is an extension included that challenges students to derive the double angle identity for tangent using the double angle identity for sine and cosine.

### **The Cosine Double-Angle: A man with many identities.**

Verify these two identities:

$$\begin{aligned}\cos^2 x - \sin^2 x &= 2 \cos^2 x - 1 \\ \cos^2 x - \sin^2 x &= 2 \cos^2 x - (\sin^2 x + \cos^2 x) \\ \cos^2 x - \sin^2 x &= 2 \cos^2 x - \sin^2 x - \cos^2 x \\ \cos^2 x - \sin^2 x &= 2 \cos^2 x - \cos^2 x - \sin^2 x \\ \cos^2 x - \sin^2 x &= \cos^2 x - \sin^2 x\end{aligned}$$

$$\begin{aligned}\cos^2 x - \sin^2 x &= 1 - 2 \sin^2 x \\ \cos^2 x - \sin^2 x &= (\sin^2 x + \cos^2 x) - 2 \sin^2 x \\ \cos^2 x - \sin^2 x &= \sin^2 x + \cos^2 x - 2 \sin^2 x \\ \cos^2 x - \sin^2 x &= \cos^2 x + \sin^2 x - 2 \sin^2 x \\ \cos^2 x - \sin^2 x &= \cos^2 x - \sin^2 x\end{aligned}$$

Earlier, you discovered that  $\cos(2x) = \cos^2 x - \sin^2 x$ . Use the transitive property of equality along with the identities above to rewrite two alternate forms of the double angle identity for cosine.

$$\cos(2x) = \cos^2 x - \sin^2 x$$

or

$$\cos(2x) = 2 \cos^2 x - 1$$

or

$$\cos(2x) = 1 - 2 \sin^2 x$$

Discussion Question: What advantages might one of the two alternate forms of the identity have over the original?

*Answers Vary: the two alternate forms express the double-angle identity for sine in terms of only one trig function instead of two.*

Try this!

You have written a double-angle identity for tangent already (based off the sum identity). Try simplifying  $\tan 2x = \frac{\sin 2x}{\cos 2x}$  to get the same thing. A helpful hint: You'll want to, at some point in the process, divide the top *and* bottom by... $\cos^2 x$

$$\tan 2x = \frac{\sin 2x}{\cos 2x}$$

$$\tan 2x = \frac{2 \sin x \cos x}{\cos^2 x - \sin^2 x}$$

$$\tan 2x = \frac{\frac{2 \sin x \cos x}{\cos^2 x}}{\frac{\cos^2 x - \sin^2 x}{\cos^2 x}}$$

$$\tan 2x = \frac{\frac{2 \sin x}{\cos x}}{\frac{\cos^2 x}{\cos^2 x} - \frac{\sin^2 x}{\cos^2 x}}$$

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$$

## The Cosine Double-Angle: A Man With Many Identities

Verify these two identities:

$$\cos^2 x - \sin^2 x = 2 \cos^2 x - 1$$

$$\cos^2 x - \sin^2 x = 1 - 2 \sin^2 x$$

Earlier, you discovered that  $\cos(2x) = \cos^2 x - \sin^2 x$ . Use the transitive property of equality along with the identities above to rewrite two alternate forms of the double angle identity for cosine.

$$\cos(2x) = \cos^2 x - \sin^2 x$$

or

$$\cos(2x) =$$

or

$$\cos(2x) =$$

Discussion Question: What advantages might one of the two alternate forms of the identity have over the original?

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You have written a double-angle identity for tangent already (based off the sum identity). Try simplifying  $\tan 2x = \frac{\sin 2x}{\cos 2x}$  to get the same thing. A helpful hint: You'll want to, at some point in the process, divide the top *and* bottom by...

## **DERIVING HALF-ANGLE IDENTITIES**

### **[Back to Task Table](#)**

### **Georgia Standards of Excellence:**

**MGSE9-12.F.TF.9** Prove addition, subtraction, double, and half-angle formulas for sine, cosine, and tangent and use them to solve problems.

### **Introduction:**

This task leads students through the derivation of the half-angle identities. It emphasizes algebraic manipulation and substitution as a means to transform a known identity into an identity that is more useful for a particular purpose. The opportunity to present math as a tool that has been created and can be manipulated by people is one that should not be missed by the teacher. It is recommended that students have completed the two tasks previous to this one.

## **DERIVING HALF-ANGLE IDENTITIES**

After deriving the double-angle identities for sine, cosine, and tangent, you're ready to try the same for *half*-angle identities. To do so, you'll use the double-angle identities you just derived.

Follow these steps to find a half-angle identity.

1. Begin with either of the alternate forms of the cosine double-angle identity.

Notice that, in the alternate identities, there are *two* instances of the variable  $x$  – one that is  $x$  alone and the other that is  $2x$ .

$$\cos 2x = 2 \cos^2 x - 1$$

$$\cos 2x = 1 - 2 \sin^2 x$$

2. Rearrange your chosen identity so that the term with  $x$  alone gets isolated on a side.

Think about this question: If you wanted to evaluate angle  $x$  for the trig function you isolated, what information would you need to know in order to use the identity you have?

$$\cos 2x = 2 \cos^2 x - 1$$

$$\cos 2x = 1 - 2 \sin^2 x$$

$$\cos x = \pm \sqrt{\frac{1 + \cos 2x}{2}}$$

$$\sin x = \pm \sqrt{\frac{1 - \cos 2x}{2}}$$

We'd like to have something more useful than what came out of step 2. *We'd like an identity that will give us the trig function value of half an angle if we know a trig function value of the full angle.* We can do this with a simple substitution.

3. Into the result you have from step 2 above, substitute  $\frac{u}{2}$  for  $x$ . Now simplify the result. Record your result below.

Half-Angle Identity for Sine

Half-Angle Identity for Cosine

Half-Angle Identity for Tangent

$$\sin \frac{u}{2} = \pm \sqrt{\frac{1 - \cos x}{2}}$$

$$\cos \frac{u}{2} = \pm \sqrt{\frac{1 + \cos x}{2}}$$

$$\tan \frac{u}{2} = \pm \sqrt{\frac{1 - \cos x}{1 + \cos x}}$$

4. After step 3, you have one of the three identities above. Now, return to step 1 and choose a different alternate form. Complete steps 2 and 3 again. Record the other identity in the correct place above.

5. Finally, there's tangent. It has been tricky in past identity sets (e.g. sums, differences, and double-angle) but is it surprisingly easy to come up with a working identity for  $\tan \frac{u}{2}$ .

Verify this identity (there are three here) and discuss your findings with a classmate:

$$\frac{\sin x}{1 + \cos x} = \pm \sqrt{\frac{1 - \cos x}{1 + \cos x}} = \frac{1 - \cos x}{\sin x}$$

This triple identity has three parts. The middle member of this identity is the half angle identity for tangent that was derived above. The two members on either side of the identity above are alternate forms of the half angle identity for tangent. Students could work through any one of these three identities above but should not miss the fact that *alternate forms of the half angle identity for tangent are being verified*.

## DERIVING HALF-ANGLE IDENTITIES

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Half-Angle Identity for Sine

Half-Angle Identity for Cosine

Half-Angle Identity for Tangent

$$\sin \frac{u}{2} = \pm$$

$$\cos \frac{u}{2} = \pm$$

$$\tan \frac{u}{2} = \pm$$

4. After step 3, you have one of the three identities above. Now, return to step 1 and choose a different alternate form. Complete steps 2 and 3 again. Record the other identity in the correct place above.

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Verify this identity (there are three here) and discuss your findings with a classmate:

$$\frac{\sin x}{1 + \cos x} = \pm \sqrt{\frac{1 - \cos x}{1 + \cos x}} = \frac{1 - \cos x}{\sin x}$$

## **CULMINATING TASK: HOW MANY ANGLES CAN YOU FIND?**

### **[Back to Task Table](#)**

### **Georgia Standards of Excellence:**

**MGSE9-12.F.TF.9** Prove addition, subtraction, double and half-angle formulas for sine, cosine, and tangent and use them to solve problems.

### **Introduction:**

The purpose of this culminating task is to give students an exercise in using the identities that they have developed. There are many possibilities for answers and students should demonstrate that they can apply each of the identities with other trigonometric relationships in order to find the answers.

You may consider suggesting different methods for students who need more guidance. For example, find 2 angles using the sine addition identity...etc.

## **CULMINATING TASK: HOW MANY ANGLES CAN YOU FIND?**

Using the following values, how many other trigonometric values of angles between 0 and 180 degrees can you find (without using a calculator)? Don't forget about complementary angles, co-terminal angles and other trig rules. Show work and justification for each value.

$$\sin 5^\circ = 0.0872 ; \cos 45^\circ = 0.7071 ; \sin 60^\circ = 0.8660 ; \cos 60^\circ = 0.5$$

*Example:  $\sin 50^\circ = \sin(5^\circ + 45^\circ) = \sin 5^\circ \cos 45^\circ + \cos 5^\circ \sin 45^\circ$  and remembering that  $\sin^2 5^\circ + \cos^2 5^\circ = 1$  allows us to find  $\cos 5^\circ = 0.9962$ , so  $\sin 50^\circ = 0.0872(0.7071) + 0.7071(0.9962) = 0.7661$  (which is quite close to the actual value of 0.7660).*

***EXTENSION: Have students develop an identity for  $\sin(x+y+z)$ ,  $\cos(x+y+z)$  and  $\tan(x+y+z)$ .***



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