Georgia
Standards of Excellence
Curriculum Frameworks

Mathematics

Accelerated GSE Pre-Calculus

Unit 7: Vectors

Georgia Department of Education

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"Educating Georgia's Future"

Unit 7
Vectors

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OVERVIEW

In this unit students will:

- recognize vectors as mathematical objects having both magnitude and direction
- add / subtract vectors using a variety of methods: end-to-end, parallelogram, and component-wise
- multiply vectors by a scalar
- interpret operations on vectors (+, –, ×) geometrically
- understand why the magnitude of the sum of two vectors is usually less than (sometimes equal to) the sum of their magnitudes
- use vectors to solve problems
- apply matrix transformations to vectors
- plot complex numbers in the complex plane
- write complex numbers in rectangular and polar form
- interpret operations on complex numbers (+, –, ×, ÷, conjugate) geometrically
- use the complex plane to find the distance between two complex numbers
- use the complex plane to find the average of two complex numbers

In Advanced Algebra/Algebra II, students were introduced to imaginary and complex numbers. In that year, students performed operations on complex numbers (+, –, ×, ÷, conjugate) purely algebraically. In this unit, students will extend their understanding of operations on complex numbers to include geometric interpretations in the complex plane. For the first time, students will learn that a point on the plane can be represented in both rectangular form and polar form. The polar form of complex numbers will allow for an elegant geometric interpretation of multiplication in the complex plane. Many operations on complex numbers that would be difficult algebraically—e.g., (3 + 4i)7—can be efficiently performed using a geometric interpretation of the operations.

In addition to developing a geometric understanding of complex numbers, students are introduced to vectors as geometric objects. Again, operations on vectors (+, –, scalar ×) will have geometric meaning to motivate the simple arithmetic processes involved. The fact that vector addition and subtraction can be quickly performed component-wise is obvious in a geometric representation, and relates to the component-wise nature of addition and subtraction of complex numbers. Scalar multiplication, geometrically, is simply a stretch or shrink (and perhaps 180° rotation). Using the simple properties of geometric interpretations of vector operations, students will be able to quickly perform a variety of operations on vectors. Matrices, which were the focus of the previous unit, allow students to perform more complex transformations on vectors, such as rotations.

The graphical interpretation of vectors lends itself easily to an exploration of real-world examples involving things that have both direction and magnitude—velocity, acceleration, force, etc. Students will apply their understanding of vector operations, especially addition and subtraction, to find resultant vectors and interpret these results in a problem context.

Although the units in this instructional framework emphasize key standards and big ideas at specific times of the year, routine topics such as estimation, mental computation, and basic computation facts should be addressed on an ongoing basis. Ideas related to the eight practice standards should be addressed constantly as well. To assure that this unit is taught with the appropriate emphasis, depth, and rigor, it is important that the tasks listed under "Evidence of Learning" be reviewed early in the planning process. A variety of resources should be utilized to supplement this unit. This unit provides much needed content information, but excellent learning activities as well. The tasks in this unit illustrate the types of learning activities that should be utilized from a variety of sources.

STANDARDS ADDRESSED IN THIS UNIT

Mathematical standards are interwoven and should be addressed throughout the year in as many different units and activities as possible in order to emphasize the natural connections that exist among mathematical topics.

KEY STANDARDS

Perform arithmetic operations with complex numbers.

MGSE9-12.N.CN.3 Find the conjugate of a complex number; use the conjugate to find the absolute value (modulus) and quotient of complex numbers.

Represent complex numbers and their operations on the complex plane.

MGSE9-12.N.CN.4 Represent complex numbers on the complex plane in rectangular and polar form (including real and imaginary numbers), and explain why the rectangular and polar forms of a given complex number represent the same number.

MGSE9-12.N.CN.5 Represent addition, subtraction, multiplication, and conjugation of complex numbers geometrically on the complex plane; use properties of this representation for computation.

MGSE9-12.N.CN.6 Calculate the distance between numbers in the complex plane as the modulus of the difference, and the midpoint of a segment as the average of the numbers at its endpoints.

<u>Represent and model with vector quantities.</u>

MGSE9-12.N.VM.1 Recognize vector quantities as having both magnitude and direction. Represent vector quantities by directed line segments, and use appropriate symbols for vectors and their magnitudes (e.g., **v, |v|, ||v||,** *v*).

MGSE9-12.N.VM.2 Find the components of a vector by subtracting the coordinates of an initial point from the coordinates of a terminal point.

MGSE9-12.N.VM.3 Solve problems involving velocity and other quantities that can be represented by vectors.

<u>Perform operations on vectors.</u>

MGSE9-12.N.VM.4 Add and subtract vectors.

MGSE9-12.N.VM.4a Add vectors end-to-end, component-wise, and by the parallelogram rule. Understand that the magnitude of a sum of two vectors is typically not the sum of the magnitudes.

MGSE9-12.N.VM.4b Given two vectors in magnitude and direction form, determine the magnitude and direction of their sum.

MGSE9-12.N.VM.4c Understand vector subtraction **v – w** as **v + (–w)**, where (–**w**) is the additive inverse of **w**, with the same magnitude as **w** and pointing in the opposite direction. Represent vector subtraction graphically by connecting the tips in the appropriate order, and perform vector subtraction component-wise.

MGSE9-12.N.VM.5 Multiply a vector by a scalar.

MGSE9-12.N.VM.5a Represent scalar multiplication graphically by scaling vectors and possibly reversing their direction; perform scalar multiplication component-wise, e.g., as $c(\mathbf{v}_x, \mathbf{v}_y) = (c \cdot \mathbf{v}_x, c \cdot \mathbf{v}_y)$.

MGSE9-12.N.VM.5b Compute the magnitude of a scalar multiple c**v** using $||c\mathbf{v}|| = |c|\mathbf{v}$. Compute the direction of c**v** knowing that when $|c|\mathbf{v} \neq 0$, the direction of c**v** is either along **v** (for $c > 0$) or against **v** (for $c < 0$).

<u>Perform operations on matrices and use matrices in applications.</u>

MGSE9-12.N.VM.11 Multiply a vector (regarded as a matrix with one column) by a matrix of suitable dimensions to produce another vector. Work with matrices as transformations of vectors.

STANDARDS FOR MATHEMATICAL PRACTICE

Refer to the Comprehensive Course Overview for more detailed information about the Standards for Mathematical Practice.

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

ENDURING UNDERSTANDINGS

- Complex numbers can be represented as points in the complex plane, with operations (+, –, ×, ÷, conjugate) and properties (modulus, distance, average) having geometric representations.
- Operations on vectors (addition, subtraction, scalar multiplication, and multiplying by a transformation matrix) have geometric representations.
- Vectors can be represented in component or direction-magnitude form.
- Complex numbers can be represented in rectangular or polar form.
- Geometric interpretations of operations on complex numbers and vectors can make computations easier.
- Problems involving quantities that have both magnitude and direction, such as velocity or displacement, can be modeled and solved using vectors.

ESSENTIAL QUESTIONS

- How can I represent complex numbers graphically?
- How does the complex plane show addition, subtraction, multiplication, and conjugation of complex numbers?
- What are two ways to represent a complex number, and what are the advantages of each form?

- How are operations on real numbers represented in the complex plane?
- When given two points on the complex plane, what does it mean to find the distance between them and the midpoint of the segment connecting them?
- How are vectors and scalars similar and different?
- How can I use vector operations to model, solve, and interpret real-world problems?
- How can I represent addition, subtraction, and scalar multiplication of vectors geometrically?
- How do geometric interpretations of addition, subtraction, and scalar multiplication of vectors help me perform computations efficiently?
- What are some different ways to add two vectors, and how are these representations related?
- In what ways can matrices transform vectors?

CONCEPTS/SKILLS TO MAINTAIN

The idea of "equivalent forms" has been emphasized for years: students found equivalent fractions in elementary school, used properties of real numbers to find equivalent expressions in middle school, rewrote systems of equations in equivalent forms, expressions and functions in various forms to identify important characteristics in previous studies of algebra. Here, writing vectors and complex numbers in equivalent forms can make operations much easier.

The components, magnitude, and direction of vectors can be calculated by drawing an appropriate right triangle and using the Pythagorean Theorem along with trigonometry and/or inverse trigonometry. By convention, angles are measured counterclockwise from the positive x-axis and are reported as angles between 0° and 360°, which students first learned with the basic introduction of the unit circle.

Operations on vectors are then applied to operations on complex numbers. Complex numbers were introduced in earlier courses, where students explored them algebraically. We now introduce the idea of complex numbers as points in the complex plane, and students explore how the operations they learned two years ago can be modeled geometrically. This geometric interpretation allows for easier calculations in such operations as z^n, where z is a complex number. These calculations were either difficult (for large n) or impossible (for non-whole n) in previous years. However, the operations students learned two years ago can be extended easily using geometric interpretations similar to those for vectors.

It is expected that students will have prior knowledge/experience related to the concepts and skills identified below. It may be necessary to pre-assess to determine whether instructional time should be spent on conceptual activities that help students develop a deeper understanding of these ideas.

- Use the Pythagorean Theorem.
- Solve right triangles using trigonometry and inverse trigonometry.
- Use reference angles to find the measure of an angle in standard position.
- Apply transformations (translations, rotations, reflections) to objects on a coordinate grid.

- Use the triangle inequality: The sum of the lengths of any two sides of a triangle is greater than the length of the third side.
- Use a ruler and protractor to measure lengths and angles.
- Multiply matrices.
- Perform operations on complex numbers—add, subtract, multiply, divide, and conjugate.
- Recognize multiplication as repeated addition and exponentiation as repeated multiplication.
- Use fractional exponents to write radicals.
- Find distance and midpoints in the plane.

SELECTED TERMS AND SYMBOLS

The following terms and symbols are often misunderstood. These concepts are not an inclusive list and should not be taught in isolation. However, due to evidence of frequent difficulty and misunderstanding associated with these concepts, instructors should pay particular attention to them and how their students are able to explain and apply them.

The definitions below are for teacher reference only and are not to be memorized by the students. Students should explore these concepts using models and real life examples. Students should understand the concepts involved and be able to recognize and/or demonstrate them with words, models, pictures, or numbers.

The websites below are interactive and include a math glossary suitable for middle school children. **Note – At the high school level, different sources use different definitions. Please preview any website for alignment to the definitions given in the frameworks.**

http://www.amathsdictionaryforkids.com/
This web site has activities to help students more fully understand and retain new vocabulary.

http://intermath.coe.uga.edu/dictnary/homepg.asp
Definitions and activities for these and other terms can be found on the Intermath website. Intermath is geared towards middle and high school students.

- **Vector:** A mathematical object that has both magnitude and direction. Vectors can be expressed as **v**, or $\langle a, b\rangle$, or as a directed line segment (arrow) in the plane.

- **Scalar:** A real number. A scalar has magnitude but not direction.

- **Initial Point:** The point at the "tail" of the arrow representing a vector.
  Often, the initial point is assumed to be (0, 0). This is the case in the notation $\langle a, b\rangle$**.**

- **Terminal Point:** The point at the "tip" of the arrow representing a vector.

- **Magnitude of a Vector:** The distance between a vector's initial and terminal points, denoted ||**v**|| or |**v**|. $\|\mathbf{v}\| = \|\langle a, b\rangle\| = \sqrt{a^2 + b^2}$
  Also called the length, norm, or absolute value of the vector.

- **Components of a Vector:** a and b in the vector $\langle a, b\rangle$.

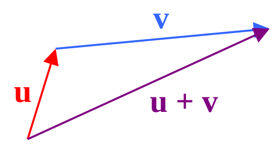
- **Parallel Vectors:** Two or more vectors whose directions are the same or opposite.

- **Equivalent Vectors:** Two or more vectors that have the same direction and magnitude—i.e., whose representations are the same in the form $\langle a, b\rangle$.
  Note that equivalent vectors may not have the same initial and terminal points.

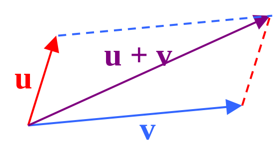
- **Zero Vector:** The vector $\langle 0, 0\rangle$.

- **Resultant Vector:** The vector that results from adding two or more vectors.

- **Tail-to-Head Representation:** A geometric representation of vector addition **u + v** wherein the initial point of **v** is placed at the terminal point of **u**. The vector beginning at the initial point of **u** and ending at the (translated) terminal point of **v** represents **u + v**.



- **Parallelogram Representation / Parallelogram Rule:** A geometric representation of vector addition **u + v** wherein a parallelogram is formed by placing the initial points of **u** and **v** at the same place and letting each vector represent the sides of a parallelogram. The diagonal of the resulting parallelogram, starting at this shared initial point, represents **u + v**.



- **Velocity:** A vector whose magnitude is an object's speed (a scalar) and whose direction is the direction of the object's motion.
  Note that speed is a scalar—magnitude, no direction—whereas velocity tells us how fast an object is moving and in what direction.

- **Complex Numbers:** A class of numbers including purely real numbers a, purely imaginary numbers bi, and numbers with both real and imaginary parts $a + bi$.

- **Complex Plane:** A 2-dimensional representation of complex numbers established by a horizontal real axis and a vertical imaginary axis.

- **Rectangular Form of a Complex Number:** $a + bi$

- **cis θ:** Shorthand for $\cos\theta + i\sin\theta$

- **Polar Form of a Complex Number:** $r(\cos\theta + i\sin\theta) = r\,\text{cis}\,\theta$

- **Complex Conjugate of $z = a + bi$:** $\bar{z} = a - bi$

- **Modulus of a Complex Number:** The distance between a number and 0 when plotted on the complex plane: $|z| = |a + bi| = \sqrt{z\bar{z}} = \sqrt{a^2 + b^2}$
  Also called absolute value or magnitude.

- **Argument of z, arg(z):** The angle—typically chosen in $(-\pi, \pi]$—formed by the positive-real axis and a segment connecting z to 0 in the complex plane.

- **Re(z):** a, the real part of the complex number $z = a + bi$

- **Im(z):** b, the coefficient of the imaginary part of the complex number $z = a + bi$

CLASSROOM ROUTINES

The importance of continuing the established classroom routines cannot be overstated. Daily routines must include such obvious activities as estimating, analyzing data, describing patterns, and answering daily questions. They should also include less obvious routines, such as how to select materials, how to use materials in a productive manner, how to put materials away, and how to access classroom technology such as computers and calculators. An additional routine is to allow plenty of time for students to explore new materials before attempting any directed activity with these new materials. The regular use of routines is important to the development of students' number sense, flexibility, fluency, collaborative skills and communication. These routines contribute to a rich, hands-on standards-based classroom and will support students' performances on the tasks in this unit and throughout the school year.

STRATEGIES FOR TEACHING AND LEARNING

- Students should be actively engaged by developing their own understanding.
- Mathematics should be represented in as many ways as possible by using graphs, tables, pictures, symbols and words.
- Interdisciplinary and cross-curricular strategies should be used to reinforce and extend the learning activities.
- Appropriate manipulatives and technology should be used to enhance student learning.
- Students should be given opportunities to revise their work based on teacher feedback, peer feedback, and metacognition**,** which includes self-assessment and reflection.
- Students should write about the mathematical ideas and concepts they are learning.
- Consideration of all students should be made during the planning and instruction of this unit. Teachers need to consider the following:
  - What level of support do my struggling students need in order to be successful with this unit?

- In what way can I deepen the understanding of those students who are competent in this unit?
- What real life connections can I make that will help my students utilize the skills practiced in this unit?

EVIDENCE OF LEARNING

By the conclusion of this unit, students should be able to demonstrate the following competencies:

- Convert vectors between component form and magnitude-direction form.
- Convert complex numbers between rectangular and polar coordinates.
- Use vectors to model and solve problems involving quantities that have both magnitude and direction.
- Convert between algebraic expressions and verbal descriptions / geometric representations of operations on vectors (add, subtract, multiply by a scalar, multiply by a transformation matrix) and algebraic equations or expressions.
- Convert between algebraic expressions and verbal descriptions / geometric representations of operations on complex numbers (add, subtract, multiply, divide, conjugate, average).
- Explain the effect of various operations on the magnitude / direction of vectors and on the modulus / argument of complex numbers.
- Use a variety of methods (tail-to-end, parallelogram, component-wise) to perform operations on vectors and explain the relationships among these methods.

TASKS

The following tasks represent the level of depth, rigor, and complexity expected of all Pre-calculus students. These tasks, or tasks of similar depth and rigor, should be used to demonstrate evidence of learning. It is important that all elements of a task be addressed throughout the learning process so that students understand what is expected of them. While some tasks are identified as a performance task, they may also be used for teaching and learning (learning/scaffolding task).

Task Name	Task Type *Grouping Strategy*	Content Addressed
Walking and Flying Around Hogsmeade	Scaffolding Task *Individual/Partner Task*	Definition of a vector Magnitude/direction of a vector
A Delicate Operation	Constructing Task *Partner/Small Group Task*	Sum and difference of vectors Scalar multiplication of vectors
Hedwig and Errol	Practice Task *Individual/Partner Task*	Sum and difference of vectors in magnitude-direction form Scalar multiplication of vectors Summary of operations in component form, mag.-dir. form, and geometrically
Putting Vectors to Use	Practice Task *Partner/Small Group Task*	Solve problems that can be modeled using vectors
He Who Must Not Be Named	Constructing Task *Individual/Partner Task*	Transformation matrices Matrices' effects on a vector's magnitude and/or direction
It's Not That Complex	Practice Task *Individual/Partner Task*	Review of complex number standards
A Plane You Can't Fly	Scaffolding Task *Individual/Partner Task*	Complex plane Modulus and argument Rectangular vs. polar form
Complex Operations	Constructing Task *Partner/Small Group Task*	Operations on complex numbers in rectangular and polar form Geometric interpretations of operations on complex #s
How Far and Halfway in Hogsmeade	Constructing Task *Partner/Small Group Task*	Distance between complex #s Average of complex numbers
Putting It All Together	Culminating Task *Partner/Small Group Task*	Application of all unit standards

Walking and Flying Around Hogsmeade

Back to Task Table

Mathematical Goals:
- Know that a vector has magnitude and direction.
- Know that vectors can be represented in component form or in magnitude-direction form.
- Convert vectors from one form to another.

GSE Standards:
- **MGSE9-12.N.VM.1** Recognize vector quantities as having both magnitude and direction. Represent vector quantities by directed line segments, ~~and use appropriate symbols for vectors and their magnitudes (e.g., **v**, |**v**|, ||**v**||, *v*).~~
- **MGSE9-12.N.VM.3** Solve problems involving ~~velocity and other~~ quantities that can be represented by vectors.

Standards for Mathematical Practice:
- Make sense of problems and persevere in solving them.
- Reason abstractly and quantitatively.
- Model with mathematics.
- Use appropriate tools strategically.

Introduction:

This task is intended to introduce students to the concept of a vector, and to see that vectors can be represented in component form ("Harry's form"—walking E/W or N/S) or magnitude-direction form ("Hedwig's form"—flying in any direction). Students will need a strong grasp of trigonometry and inverse trigonometry in order to convert from one form to the other.

Throughout this unit, there is an emphasis on converting between forms (magnitude-direction form vs. component form for vectors, introduced here, and rectangular vs. polar coordinates for complex numbers, introduced in A Plane You Can't Fly). Students should describe advantages and disadvantages of each form when performing various operations and should become skilled at converting from one form to another.

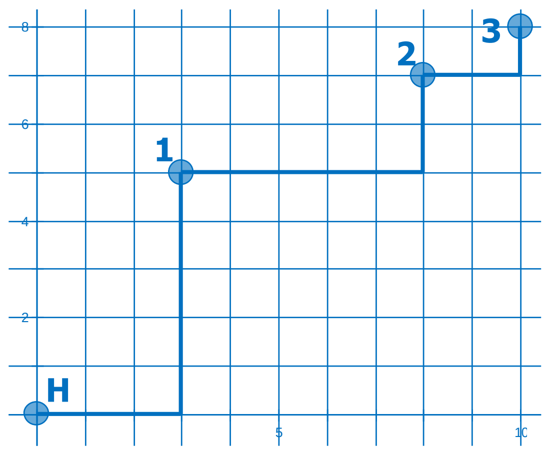
Walking and Flying Around Hogsmeade

Harry Potter needs to make a few stops around Hogsmeade. Harry's broom is broken, so he must walk between the buildings. The town is laid out in square blocks, which makes it easy to give directions. Here are the directions Harry must follow on Monday:

Monday – start at Hogwarts		
	E / W	**N / S**
Stop 1	3 blocks East	5 blocks North
Stop 2	5 blocks East	2 blocks North
Stop 3	2 blocks East	1 block North
back to Hogwarts	***10 blocks West***	***8 blocks South***
TOTAL	***10 blocks*** East ***10 blocks*** West	***8 blocks*** North ***8 blocks*** South

Note that

1. Use a piece of graph paper and draw Harry's trip. Put Hogwarts at the origin.



2. Fill in the blank parts of the table to give Harry directions to get back to Hogwarts. (Make your directions simple, so Harry must make only one turn.)

   10 blocks West, 8 blocks South

3. How many total blocks East did Harry walk? West? Record this in your table. What do you notice?

   10 blocks East, 10 blocks West. They are the same because he started and ended at the same place.

4. How many total blocks North did Harry walk? South? Record this in your table. What do you notice?

8 blocks North, 8 blocks South. Again, they are the same.

On Tuesday, Harry has more errands to run. Here are his directions:

Tuesday – start at Hogwarts		
	E / W	**N / S**
Stop 1	2 blocks East	3 blocks North
Stop 2	4 blocks West	2 blocks North
Stop 3	3 blocks East	1 block South
Stop 4	***4 blocks West***	***3 blocks South***
Hogwarts	3 blocks East	1 block South
TOTAL	***8 blocks*** East ***8 blocks*** West	***5 blocks*** North ***5 blocks*** South

5. Fill in the blank parts of the table to ensure that Harry's directions will get him back to Hogwarts.

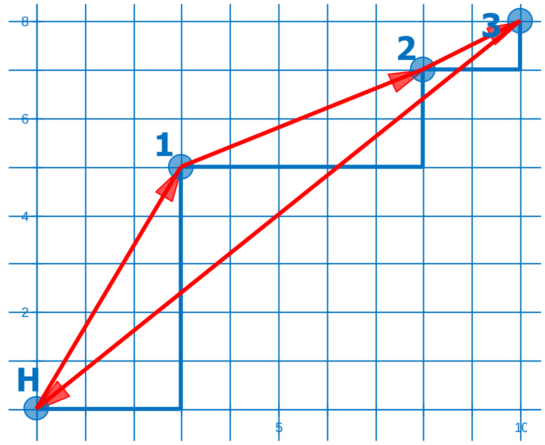
   4 blocks West, 3 blocks South. This problem requires students to think backward—essentially performing vector subtraction, although we do not introduce that formally yet.

6. Find the total distances East, West, North, and South Harry traveled, and record these in your table. What do you notice?

   8 blocks East, 8 blocks West, 5 blocks North, 5 blocks South. The E/W and N/S distances cancel each other out because Harry started and ended at the same place, Hogwart's.

Harry's trusted owl, Hedwig, can fly over buildings, so she travels in a straight line from each stop to the next and waits for Harry to arrive.

7. On your graph paper from #1, use a different color to draw arrows representing Hedwig's path on Monday.



By drawing both Harry's and Hedwig's paths on the same graph paper, the idea of right triangles and the Pythagorean Theorem, needed to calculate magnitude, will occur to students more naturally.

In mathematics, we use directed line segments, or **vectors**, to indicate a magnitude (length or distance) and a direction. Each part of Hedwig's trip has a distance and a direction, so the arrows you just drew are vectors.

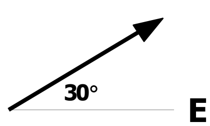
8. To get from Hogwarts to Stop 1 on Monday, how far did Hedwig fly?
(Hint: Use Harry's path on your graph paper as legs of a right triangle.)

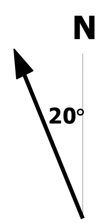
$3^2 + 5^2 = c^2 \rightarrow c = \sqrt{34}$

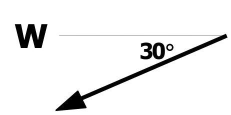
9. There are several ways to describe Hedwig's direction during this leg of the trip. We could simply say Hedwig traveled "northeast," but this would not be a very accurate description. Why not?

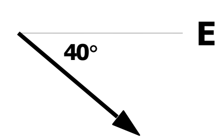
Just saying Hedwig traveled northeast tells us only that he traveled somewhere between due North and due East. We don't know if he was traveling more North than East or vice-versa. In some cases, people use Northeast to mean exactly 45° North of East.

10. For more accuracy, we can include an angle relating a direction to the nearest cardinal direction (N, S, E, W). Fill in the blanks below to describe each of these directions.





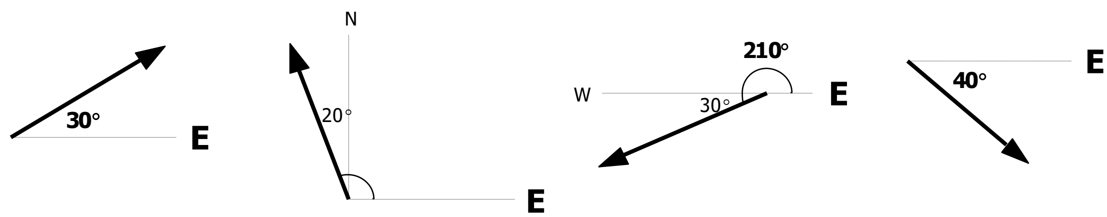




30° N of E   20° **W** of N   30° **S** of **W**   **40° S of E**

11. This notation can be cumbersome, so mathematicians measure directions as angles (possibly greater than 180°) measured counterclockwise from due East called "standard

position angles". Rewrite the angles above using this notation. Two of the four have been done for you.



30° **110°** 210° **320°**

12. Using inverse trigonometry, find Hedwig's direction going from Hogwarts to Stop 1 on Monday. Express your answer in the simple form introduced in #11.
(Hint: Use Harry's path on your graph paper as legs of a right triangle.)

Note:
In this problem, the inverse tangent function gives the correct angle because the vector makes an acute angle with the positive-x axis. Students will need guidance in future problems where the vectors point in directions other than NE.

Solution:
$\tan x = 5/3 \rightarrow x \approx 59.04°$

13. Find the magnitude (distance) and direction of Hedwig's path from Stop 1 to Stop 2 on Monday. Show your work neatly.

magnitude: $\sqrt{5^2 + 2^2} = \sqrt{29}$
direction: $\arctan(2/5) \approx 21.80°$

14. Find the magnitude and direction of Hedwig's path from Stop 2 to Stop 3 on Monday. Show your work neatly.

magnitude: $\sqrt{1^2 + 2^2} = \sqrt{5}$
direction: $\arctan(1/2) \approx 26.57°$

15. Find the magnitude and direction of Hedwig's path from Stop 3 to Hogwarts on Monday. Show your work neatly. BE CAREFUL! The angle of the triangle is *not* the same as the angle the path makes with due East. Look at #11 to see how to make the necessary adjustments.

Note:
Students are more likely to struggle with the necessary conversions from 38.66° into standard notation. Teachers should give students plenty of practice with such problems—those that have vectors not pointed NE—throughout the unit.

Solution:
magnitude: $\sqrt{(10^2 + 8^2)} = \sqrt{164} = 2\sqrt{41}$
direction: $\arctan(8/10) \approx 38.66°$ ***S of W*** → $38.66° + 180° = 218.66°$
OR $\arctan(-8/10) \approx -38.66°$ ***S of E*** → ***reflect across y-axis:*** $218.66°$

The way we have expressed Harry's path is known as **component form**, since it is split up into two parts, or components—a horizontal part and a vertical part. The way we have expressed Hedwig's path is known as **magnitude-direction form**, since it gives the magnitude and direction of the path.

It is important to be able to convert from one form to another. Practice this skill by filling in the table below using the Pythagorean Theorem, trigonometry, and inverse trigonometry. It will probably be helpful to draw a picture of Harry's path (with horizontal and vertical components) and Hedwig's path (a straight flight) to create a right triangle.

Note:
Students will need to use both inverse trigonometry (component→mag-dir. form) and regular trigonometry (mag-dir.→component form) in these problems. These problems also give students practice with vectors that are not pointed in the NE direction.

	Harry's description		Hedwig's description		
	horizontal	vertical	magnitude	direction	drawing
a.	3 blocks East	4 blocks North	$\sqrt{(3^2 + 4^2)}=$ ***5 blocks***	$\arctan(4/3)=$ ***53.13°***	***Vector points NE***
b.	$13\cos 113° =$ -5.08 ***5.08 West***	$13\sin 113° =$ 11.97 ***11.97 North***	13 blocks	113°	***Vector points NW***
c.	6 blocks West	2 blocks South	$\sqrt{(6^2 + 2^2)}=$ $\sqrt{40}=$ $2\sqrt{10}$	$\arctan(2/6)=$ $18.43°$ $180°+18.43°=$ ***198.43°***	***Vector points SW***
d.	$10\cos 315° =$ 7.07 ***7.07 East***	$10\sin 315° =$ -7.07 ***7.07 South***	10 blocks	315°	***Vector points SE***

Walking and Flying Around Hogsmeade

Mathematical Goals:

- Know that a vector has magnitude and direction.
- Know that vectors can be represented in component form or in magnitude-direction form.
- Convert vectors from one form to another.

GSE Standards: (strikethroughs indicate parts of the standard not addressed in this task)

- **MGSE9-12.N.VM.1** Recognize vector quantities as having both magnitude and direction. Represent vector quantities by directed line segments, ~~and use appropriate symbols for vectors and their magnitudes (e.g., **v, |v|, ||v||,** *v*).~~
- **MGSE9-12.N.VM.3** Solve problems involving ~~velocity and other~~ quantities that can be represented by vectors.

Standards for Mathematical Practice:

- Make sense of problems and persevere in solving them.
- Reason abstractly and quantitatively.
- Model with mathematics.
- Use appropriate tools strategically.

Task 1 – Walking and Flying Around Hogsmeade

Harry Potter needs to make a few stops around Hogsmeade. Harry's broom is broken, so he must walk between the buildings. The town is laid out in square blocks, which makes it easy to give directions. Here are the directions Harry must follow on Monday:

Monday – start at Hogwarts		
	E / W	**N / S**
Stop 1	3 blocks East	5 blocks North
Stop 2	5 blocks East	2 blocks North
Stop 3	2 blocks East	1 block North
back to Hogwarts		
TOTAL	East West	North South

1. Use a piece of graph paper and draw Harry's trip. Put Hogwarts at the origin.

2. Fill in the blank parts of the table to give Harry directions to get back to Hogwarts. (Make your directions simple so that Harry must make only one turn.)

3. How many total blocks East did Harry walk? West? Record this in your table. What do you notice?

4. How many total blocks North did Harry walk? South? Record this in your table. What do you notice?

On Tuesday, Harry has more errands to run. Here are his directions:

Tuesday – start at Hogwarts		
	E / W	**N / S**
Stop 1	2 blocks East	3 blocks North
Stop 2	4 blocks West	2 blocks North
Stop 3	3 blocks East	1 block South
Stop 4		
Hogwarts	3 blocks East	1 block South
TOTAL	East West	North South

5. Fill in the blank parts of the table to ensure that Harry's directions will get him back to Hogwarts.

6. Find the total distances East, West, North, and South Harry traveled, and record these in your table. What do you notice?

Harry's trusted owl, Hedwig, can fly over buildings, so she travels in a straight line from each stop to the next and waits for Harry to arrive.

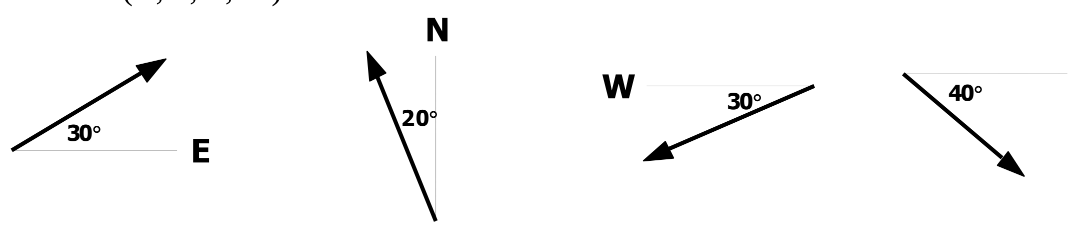
7. On your graph paper from #1, use a different color to draw arrows representing Hedwig's path.

In mathematics, we use directed line segments, or **vectors**, to indicate a magnitude (length or distance) and a direction. Each part of Hedwig's trip has a distance and a direction, so the arrows you just drew are vectors.

8. To get from Hogwarts to Stop 1 on Monday, how far did Hedwig fly?
(Hint: Use Harry's path on your graph paper as legs of a right triangle.)

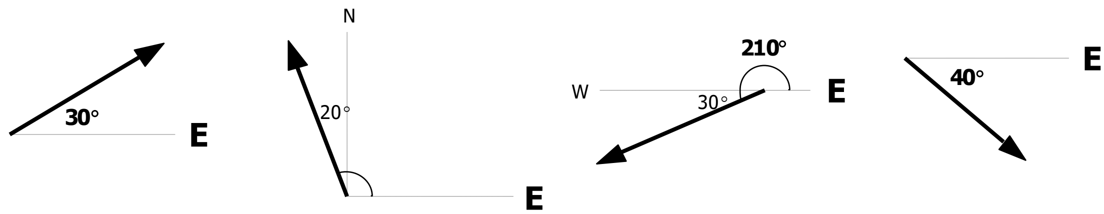
9. There are several ways to describe Hedwig's direction during this leg of the trip. We could simply say Hedwig traveled "northeast," but this would not be a very accurate description. Why not?

10. For more accuracy, we can include an angle relating a direction to the nearest cardinal direction (N, S, E, W). Fill in the blanks below to describe each of these directions.



30° N of E 20° ____ of N 30° ____ of ____ ________________

11. This notation can be cumbersome, so mathematicians measure directions as angles (possibly greater than 180°) measured counterclockwise from due East called "standard position angles". Rewrite the angles above using this notation. Two of the four have been done for you.



30° ______ 210° ______

12. Using inverse trigonometry, find Hedwig's direction going from Hogwarts to Stop 1 on Monday. Express your answer in the simple form introduced in #11.
(Hint: Use Harry's path on your graph paper as legs of a right triangle.)

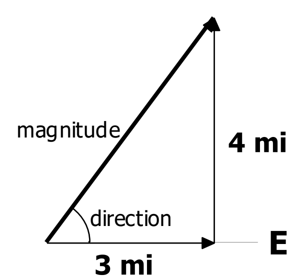
13. Find the magnitude (distance) and direction of Hedwig's path from Stop 1 to Stop 2 on Monday. Show your work neatly.

14. Find the magnitude and direction of Hedwig's path from Stop 2 to Stop 3 on Monday. Show your work neatly.

15. Find the magnitude and direction of Hedwig's path from Stop 3 to Hogwarts on Monday. Show your work neatly. BE CAREFUL! The angle of the triangle is *not* the same as the angle the path makes with due East. Look at #11 to see how to make the necessary adjustments.

The way we have expressed Harry's path is known as **component form**, since it is split up into two parts, or components—a horizontal part and a vertical part. The way we have expressed Hedwig's path is known as **magnitude-direction form**, since it gives the magnitude and direction of the path.

It is important to be able to convert from one form to another. Practice this skill by filling in the table below using the Pythagorean Theorem, trigonometry, and inverse trigonometry. It will probably be helpful to draw a picture of Harry's path (with horizontal and vertical components) and Hedwig's path (a straight flight) to create a right triangle.

	Harry's description		Hedwig's description		
	horizontal	vertical	magnitude	direction	drawing
a.	3 blocks East	4 blocks North			

b.			13 blocks	113°	
c.	6 blocks West	2 blocks South			
d.			10 blocks	315°	

A Delicate Operation

Back to Task Table

Mathematical Goals:
- Use common notation for component form and magnitude of a vector.
- Add and subtract vectors that are presented in component form.
- Multiply vectors, presented in component form, by a scalar.

GSE Standards:
- **MGSE9-12.N.VM.1** Recognize vector quantities as having both magnitude and direction. Represent vector quantities by directed line segments, and use appropriate symbols for vectors and their magnitudes (e.g., **v, |v|, ||v||,** *v*).
- **MGSE9-12.N.VM.2** Find the components of a vector by subtracting the coordinates of an initial point from the coordinates of a terminal point.
- **MGSE9-12.N.VM.3** Solve problems involving ~~velocity and other~~ quantities that can be represented by vectors.
- **MGSE9-12.N.VM.4** Add and subtract vectors.
- **MGSE9-12.N.VM.4a** Add vectors ~~end-to-end,~~ component-wise, ~~and by the parallelogram rule. Understand that the magnitude of a sum of two vectors is typically not the sum of the magnitudes.~~
- **MGSE9-12.N.VM.5** Multiply a vector by a scalar.
  - **MGSE9-12.N.VM.5a** Represent scalar multiplication graphically by scaling vectors and possibly reversing their direction; perform scalar multiplication component-wise, e.g., as $c(\mathbf{v}_x, \mathbf{v}_y) = (c\cdot\mathbf{v}_x, c\cdot\mathbf{v}_y)$.

Standards for Mathematical Practice:
- Make sense of problems and persevere in solving them.
- Reason abstractly and quantitatively.
- Construct viable arguments and critique the reasoning of others.
- Model with mathematics.
- Look for and make use of structure.

Introduction:

This task introduces three operations: the sum and difference of vectors and multiplying a vector by a scalar. All vectors in this task are presented in component form to make calculations easier; students will be given problems in magnitude-direction form in the next task. Students will explore these operations graphically somewhat, but will do so in more detail in the next task. Here, the idea of a sum of vectors as the total displacement when two or more "sub-paths" are followed is extremely important. Vector addition extends easily to subtraction (part + part = whole → whole – part = other part) and to scalar multiplication (a + a + a + a = 4a), and these connections should be emphasized and discussed.

A Delicate Operation

1. Writing the component form of a vector looks a lot like a coordinate pair, but uses the symbols "⟨" and "⟩" instead of "(" and ")". We can easily represent Harry's paths using this notation. We can also represent the magnitude of a vector ***u*** using double bars: ||***u***||. (Some people use single bars instead.) Complete the table below to practice this notation—remember positive and negative signs!

Tuesday – start at Hogwarts				
	E / W	**N / S**	**component form**	**magnitude**
Stop 1	2 blocks East	3 blocks North	$\boldsymbol{s} = \langle 2, 3\rangle$	$\lVert\boldsymbol{s}\rVert = \sqrt{13}$
Stop 2	4 blocks West	2 blocks North	$\boldsymbol{t} = \langle -4, 2\rangle$	$\lVert\boldsymbol{t}\rVert = \sqrt{20} = 2\sqrt{5}$
Stop 3	3 blocks East	1 block South	$\boldsymbol{u} = \langle 3, -1\rangle$	$\lVert\boldsymbol{u}\rVert = \sqrt{10}$
Stop 4	4 blocks West	3 blocks South	$\boldsymbol{v} = \langle -4, -3\rangle$	$\lVert\boldsymbol{v}\rVert = \sqrt{25} = 5$
Hogwarts	3 blocks East	1 block South	$\boldsymbol{w} = \langle 3, -1\rangle$	$\lVert\boldsymbol{w}\rVert = \sqrt{10}$

2. At Stop 2, how far East or West is Harry from Hogwarts?

2 blocks West

How far North or South?

5 blocks North

We can represent this using **vector addition:** $\langle 2, 3\rangle + \langle -4, 2\rangle = \langle -2, 5\rangle$. Adding two or more vectors results in what is called vector, called the **resultant vector.**

3. When given two or more vectors in component form, explain how to find their resultant vector and what it represents.

Add the first component (in this case, E/W directions)
Add the second component (in this case, N/W directions)

4. Find the sum of all five vectors in Harry's trip. What does this mean in the context?

$\langle 2,3\rangle + \langle -4,2\rangle + \langle 3,-1\rangle + \langle -4,-3\rangle + \langle 3,-1\rangle = \langle 2+-4+3+-4+3, 3+2+-1+-3+-1\rangle = \langle 0,0\rangle$
This means that Harry started and stopped at the same place, Hogwarts.

We can also easily perform **vector subtraction** in a similar way.

5. Ron and Hermione both begin at Hogwarts. Ron walks four blocks East and six blocks North to arrive at Zonko's Joke Shop. Hermione walks five blocks East and three blocks

South to arrive at the Shrieking Shack. Give Ron directions to get to the Shrieking Shack from Zonko's Joke Shop.

Comment:
In addition to being a vector subtraction problem, this problem is related to MGSE9-12.N.VM.2: Find the components of a vector by subtracting the coordinates of an initial point from the coordinates of a terminal point. This connection should be discussed, including the fact that a vector's initial point doesn't matter; the vector tells us the displacement from this point (wherever it is).

Solution:
1 block East and 9 blocks South

6. Write a vector subtraction problem to represent the steps you took to answer #5.

$\langle 5, -3\rangle - \langle 4, 6\rangle = \langle 1, -9\rangle$
total – part = other part

In addition to adding and subtracting, we can also perform multiplication on vectors. There are two ways to multiply a vector *by another vector*, called the dot product and the cross product, but they are beyond the scope of this class. Instead, we will focus only on multiplying a vector by a real number, which we can also call a **scalar.**

1. Use component-wise addition to find $\langle 4, -2\rangle + \langle 4, -2\rangle + \langle 4, -2\rangle$.

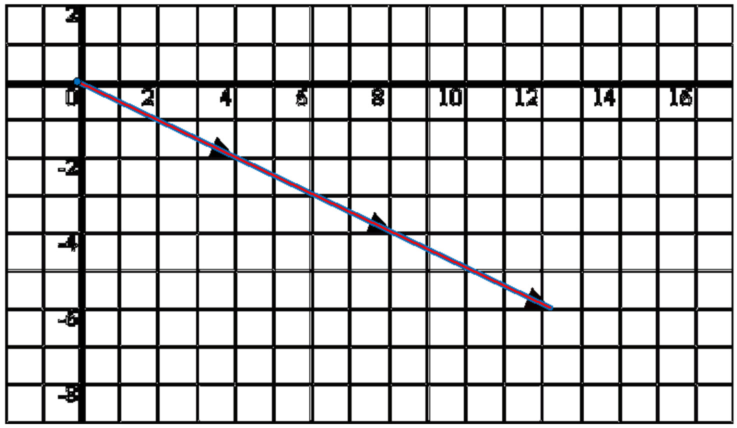
$\langle 4 + 4 + 4, -2 + -2 + -2\rangle = \langle 12, -6\rangle$

We can rewrite the sum $\langle 4, -2\rangle + \langle 4, -2\rangle + \langle 4, -2\rangle$ as the product $3\langle 4, -2\rangle$. This is similar to writing 7 + 7 + 7 as 3(7).

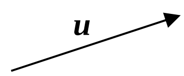
2. Describe a simple way to find $3\langle 4, -2\rangle$.

Multiply each component by the scalar: 3(4) and 3(-2)

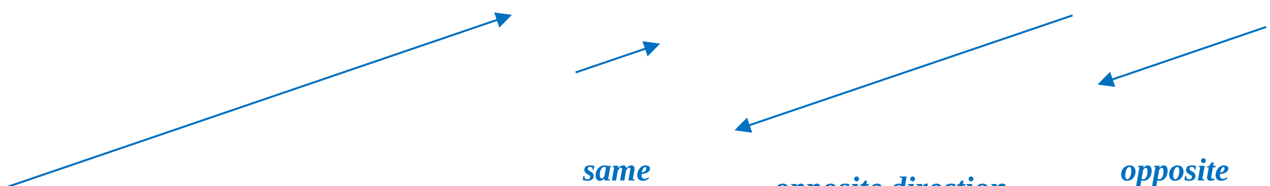
3. Let's see what this means graphically. Use the tail-to-end method to represent the sum $\langle 4, -2\rangle + \langle 4, -2\rangle + \langle 4, -2\rangle$ on the grid below.



4. The vector ***u*** is drawn below. For each problem, draw the new vector.



a. 3***u*** b. $\frac{1}{2}$***u*** c. -2***u*** d. -***u***



same direction, 3 times as long

same direction, half as long

opposite direction, twice as long

opposite direction, same length

5. What types of scalars result in a vector that…
…is longer than the original vector?

greater than 1 or less than -1: $|s| > 1$

…is shorter than the original vector?

between -1 and 1: $|s| < 1$

…points in the opposite direction of the original vector?

negative: $s < 0$

A Delicate Operation

Mathematical Goals:

- Use common notation for component form and magnitude of a vector.
- Add and subtract vectors that are presented in component form.
- Multiply vectors, presented in component form, by a scalar.

GSE Standards: (strikethroughs indicate parts of the standard not addressed in this task)

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A Delicate Operation

1. Writing the component form of a vector looks a lot like a coordinate pair, but uses the symbols "⟨" and "⟩" instead of "(" and ")". We can easily represent Harry's paths using this notation. We can also represent the magnitude of a vector $\boldsymbol{u}$ using double bars: $\|\boldsymbol{u}\|$. (Some people use single bars instead.) Complete the table below to practice this notation—remember positive and negative signs!

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Stop 2	4 blocks West	2 blocks North	$\boldsymbol{t} = \langle -4, 2\rangle$	$\|\boldsymbol{t}\| =$
Stop 3	3 blocks East	1 block South	$\boldsymbol{u} =$	$\|\boldsymbol{u}\| =$
Stop 4	4 blocks West	3 blocks South	$\boldsymbol{v} =$	$\|\boldsymbol{v}\| =$
Hogwarts	3 blocks East	1 block South	$\boldsymbol{w} =$	$\|\boldsymbol{w}\| =$

2. At Stop 2, how far East or West is Harry from Hogwarts?

   How far North or South?

We can represent this using **vector addition:** $\langle 2, 3\rangle + \langle -4, 2\rangle = \langle -2, 5\rangle$. Adding two or more vectors results in what is called vector, called the **resultant vector.**

3. When given two or more vectors in component form, explain how to find their resultant vector and what it represents.

4. Find the sum of all five vectors in Harry's trip. What does this mean in the context?

We can also easily perform **vector subtraction** in a similar way.

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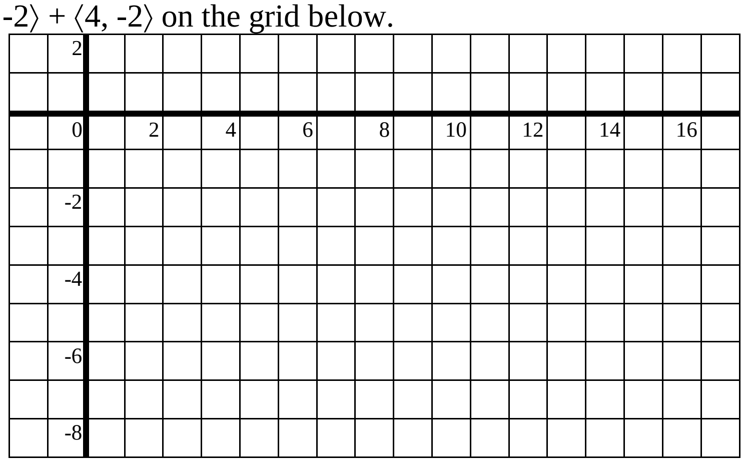
1. Use component-wise addition to find $\langle 4, -2\rangle + \langle 4, -2\rangle + \langle 4, -2\rangle$.

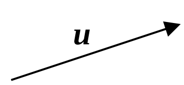
We can rewrite the sum $\langle 4, -2\rangle + \langle 4, -2\rangle + \langle 4, -2\rangle$ as the product $3\langle 4, -2\rangle$. This is similar to writing 7 + 7 + 7 as 3(7).

2. Describe a simple way to find $3\langle 4, -2\rangle$.

3. Let's see what this means graphically. Use the tail-to-end method to represent the sum $\langle 4, -2\rangle + \langle 4, -2\rangle + \langle 4, -2\rangle$ on the grid below.



4. The vector ***u*** is drawn below. For each problem, draw the new vector.



a. $3\boldsymbol{u}$ b. $\frac{1}{2}\boldsymbol{u}$ c. $-2\boldsymbol{u}$ d. $-\boldsymbol{u}$

5. What types of scalars result in a vector that…

…is longer than the original vector?

…is shorter than the original vector?

…points in the opposite direction of the original vector?

Hedwig and Errol

Back to Task Table

Mathematical Goals:

- Perform operations on vectors given in magnitude-direction form.
- Use the tail-to-end method to perform vector addition and subtraction.
- Understand advantages and disadvantages of component form and magnitude-direction form.
- Practice converting from one form to the other.

GSE Standards:

- **MGSE9-12.N.VM.3** Solve problems involving velocity and other quantities that can be represented by vectors.
- **MGSE9-12.N.VM.4** Add and subtract vectors.
- **MGSE9-12.N.VM.4a** Add vectors end-to-end, component-wise, and by the parallelogram rule. Understand that the magnitude of a sum of two vectors is typically not the sum of the magnitudes.
- **MGSE9-12.N.VM.4b** Given two vectors in magnitude and direction form, determine the magnitude and direction of their sum.
- **MGSE9-12.N.VM.4c** Understand vector subtraction **v** – **w** as **v** + **(–w)**, where (–**w**) is the additive inverse of **w**, with the same magnitude as **w** and pointing in the opposite direction. Represent vector subtraction graphically by connecting the tips in the appropriate order, and perform vector subtraction component-wise.
- **MGSE9-12.N.VM.5** Multiply a vector by a scalar.
  - **MGSE9-12.N.VM.5a** Represent scalar multiplication graphically by scaling vectors and possibly reversing their direction; perform scalar multiplication component-wise, e.g., as $c(\mathbf{v}_x, \mathbf{v}_y) = (c\cdot\mathbf{v}_x, c\cdot\mathbf{v}_y)$.
  - **MGSE9-12.N.VM.5b** Compute the magnitude of a scalar multiple c**v** using $||c\mathbf{v}|| = |c|\mathbf{v}$. Compute the direction of c**v** knowing that when $|c|\mathbf{v} \neq 0$, the direction of c**v** is either along **v** (for $c > 0$) or against **v** (for $c < 0$).

Standards for Mathematical Practice:

- Make sense of problems and persevere in solving them.
- Reason abstractly and quantitatively.
- Construct viable arguments and critique the reasoning of others.
- Model with mathematics.
- Use appropriate tools strategically.
- Attend to precision.
- Look for and make use of structure.

Introduction

This task gives students practice with vector operations, but presents the vectors in magnitude-direction form. Thus, to add or subtract, students must first convert each vector to component form (similar to Task 1) then perform operations component-wise (similar to Task 2).

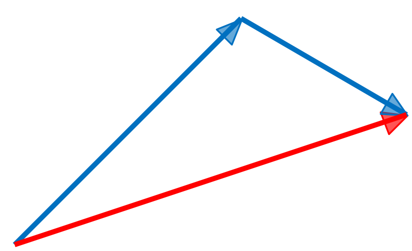
This task—and especially the summary at the end of the task—helps students solidify their understanding of vectors, operations, and geometric representations of those operations.

Hedwig and Errol

The vector addition and subtraction we have done so far has been quite easy, since we could do them **component-wise.** That is, we just added or subtracted each component separately. But if vectors are given in magnitude-direction form, this process becomes somewhat more difficult. We must first convert into component form so we can use the easy component-wise methods.

1. Starting from Hogwarts, Ron's clumsy owl Errol flies 5 miles at a 45° angle (remember, these are standard position angles, so they are measured counterclockwise from East). Then he realizes he's flying the wrong way, so he adjusts his path and flies 3 miles at a 330° angle (again, counterclockwise from East). On a separate piece of paper, carefully draw Errol's flight, letting 1 cm = 1 mile. Use a protractor to make your angles accurate.

   Errol's flight is shown in blue.



2. Hedwig has a better sense of direction, so she flies from Hogwarts directly to Errol's ending point. On the same paper, draw Hedwig's path. Use a ruler and protractor to estimate its magnitude and direction.

   Hedwig's flight is shown in red. Estimates will vary, but should be close to 6.5 miles at an angle of 18°.

Hedwig's flight vector is the **sum** of the vectors representing the two parts of Errol's flight. You have just used the **tail-to-end method** for adding two vectors, by placing the tail of Errol's second part at the end of his first part. This method obviously makes sense for a two-part flight, but we will see vectors in a variety of contexts.

3. Let's go back to Hedwig's flight. We could only *estimate* Hedwig's path using the tail-to-head method graphically. Let's use trigonometry to get more accurate results. Complete the table below to convert both parts of Errol's flight into component form:

	Magnitude-Direction Form (owl form)		Component Form (Harry's Form)	
	magnitude	direction	horizontal	vertical
Part 1	5 miles	45°	***5cos45°= <u>3.54 mi E</u>***	***5sin45°= <u>3.54 mi N</u>***
Part 2	3 miles	330°	***3cos330°= <u>2.60 mi E</u>***	***3sin330°= <u>1.50 mi S</u>***

Show your work here:
Students who recognize that the law of cosines works well here should be reminded to be careful about finding the angles they use. In fact, this is a great opportunity to remind students about the geometry in finding angles and to check their work using the law of cosines.

4. How far East or West must Hedwig fly to get to Errol's location from Hogwarts? How far North or South?

3.54 + 2.60 = 6.14 mi East
3.54 + (-1.50) = 2.04 mi North

5. Find the magnitude and direction of Hedwig's path. Compare to your estimates from #2.

<u>Comment:</u>
As before, since this vector points northeast, it is easier to determine the direction. Some problems in later tasks require an extra step.

<u>Solution:</u>
magnitude: $\sqrt{(6.14^2 + 2.04^2)} = 6.47$ ***miles***
direction: arctan(2.04/6.14) = 18.38°

6. How far did Errol fly altogether? Did Hedwig fly the same distance? Why does this make sense?

Errol: 5 + 3 = 8 miles
Hedwig flew only 6.47 miles (see #5), so he flew less distance than Errol. This makes sense because Errol did not take a direct path to the final destination, whereas Hedwig did.
Geometrically, this is the triangle inequality: Errol's two distances must add up to more than Hedwig's distance.

7. There is one particular situation where the sum of the magnitudes of two (or more) vectors *equals* the magnitude of their sum. Describe this situation by drawing a picture and explaining why the sum of the magnitudes equals the magnitude of the sum.

If the vectors point in the same direction, then there is no difference between the two paths—they are both direct paths; one just has a "break" in it.
Geometrically, this is the degenerate form of the triangle inequality. Students might have also see this as the segment addition property in geometry.

8. Is the direction (angle) of Hedwig's path equal to the sum of the angles of Errol's paths?

<u>Comment:</u>
The previous series of questions (#6-8) are intended to help students see the advantages of vectors given in component form, and that there are no similar shortcuts for problems presented in magnitude-direction form, except in very specific cases.

No. Errol's directions add up to 45° + 330° = 375° (or 15°), which is definitely not the same as Hedwig's direction of 18.38°.

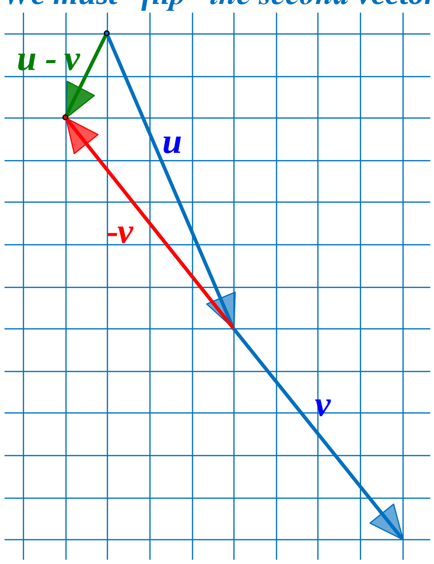
Let's see how we can modify the tail-to-end method of addition to apply to vector subtraction.

9. Use component-wise subtraction to find $\langle 3, -7\rangle - \langle 4, -5\rangle$.

$\langle 3 - 4, -7 - (-5)\rangle = \langle -1, -2\rangle$

10. Draw the vectors $\langle 3, -7\rangle$ and $\langle 4, -5\rangle$ tail-to-end. Describe a modified tail-to-end method that will make the resultant vector equal to your answer from #10.

We must "flip" the second vector (rotate it 180°) so we move in the opposite direction.



11. How is subtracting vectors similar to the common "add the opposite" method of subtracting integers?

    We keep the first vector, then we change the subtraction problem into an addition problem by changing the direction of the second vector.
    In other words, u – v = u + (-v)

12. What is the advantage of presenting this subtraction problem in component form rather than direction-magnitude form?

    We were able to do the subtraction component-wise without needing to convert to component form first.

Now let's look at scalar multiplication in each form.

13. From Hogwarts, Harry walks two blocks East and three blocks North then rests. He repeats this five times total. Describe how Ron can walk to Harry's ending position from Hogwarts without resting.

    10 blocks East and 15 blocks North.

14. From Hogwarts, Hedwig flies four miles at a direction of 70° then rests. He repeats this five times total. Describe how Errol can fly to Hedwig's ending position from Hogwarts.

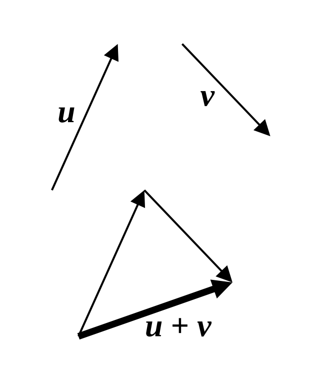
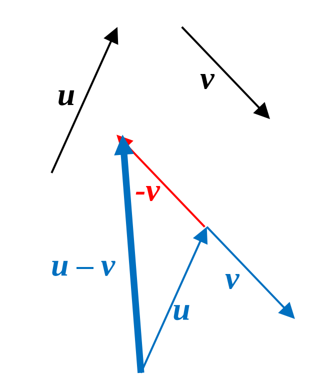
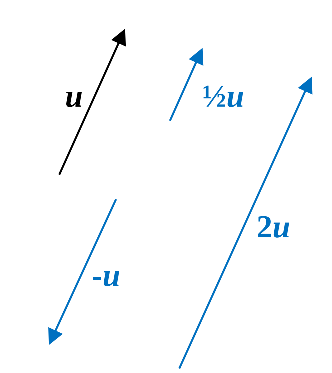
    fly 20 miles at a direction of 70°

15. We have seen that vector addition and subtraction is easier when the problem is presented in component form. Is the same true for scalar multiplication? Explain your answer.

    No, scalar multiplication is quite easy in both forms.

Summarize:

In each box, explain how to perform the indicated operation depending on the form in which the problem is presented. Two boxes have been completed for you.

	add $\mathbf{u} + \mathbf{v}$	**subtract** $\mathbf{u} - \mathbf{v}$	**multiply** $a\,\mathbf{u}$ (a is a real number)
given in component form	*add corresponding components*	*subtract corresponding components*	*multiply each component by the scalar*
given in direction-magnitude form	*convert to component form using trig* *add corresponding components* *(convert back if required)*	*convert to component form using trig* *subtract corresponding components* *(convert back if required)*	*multiply magnitude by the absolute value of the scalar* *keep the same direction, or add 180° if the scalar was negative (flips direction)*
geometric representation			
			stretch if $\lvert a \rvert > 1$ shrink if $\lvert a \rvert < 1$ reverse direction if $a < 0$

Practice:

Let $\boldsymbol{p} = \langle 3, 5\rangle$; $\boldsymbol{q} = \langle -1, 6\rangle$; $\boldsymbol{r} = \langle 4, -3\rangle$; $\boldsymbol{s} = \langle -2, -6\rangle$. Find each of the following:

1. $\boldsymbol{p} + \boldsymbol{q}$

   $\langle 3 + -1, 5 + 6\rangle = \langle 2, 11\rangle$

2. $\boldsymbol{p} + 5\boldsymbol{r} + 3\boldsymbol{s}$

   $\langle 3 + 5(4) + 3(-2), 5 + 5(-3) + 3(-6)\rangle = \langle 17, -28\rangle$

3. $\boldsymbol{s} - 2(\boldsymbol{p} + \boldsymbol{r})$

   $\langle -2 - 2(3 + 4), -6 - 2(5 + -3)\rangle = \langle -16, -10\rangle$

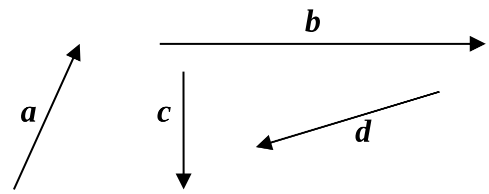
4. Find $\boldsymbol{t}$ so that $\boldsymbol{p} + \boldsymbol{t} = \boldsymbol{s}$

   $\langle 3 + x, 5 + y\rangle = \langle -2, -6\rangle \rightarrow \langle -5, -11\rangle$
   OR $\boldsymbol{s} - \boldsymbol{p} = \langle -2 - 3, -6 - 5\rangle = \langle -5, -11\rangle$

5. Find $\boldsymbol{u}$ so that $3\boldsymbol{u} - 4(\boldsymbol{p} + \boldsymbol{q}) = \boldsymbol{r}$

   $\langle 3x - 4(3 + -1), 3y - 4(5 + 6)\rangle = \langle 4, -3\rangle \rightarrow \langle 4, 41/3\rangle$

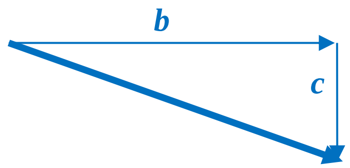
Let ***a***, ***b***, ***c***, and ***d*** be the vectors shown below. Sketch each of the following:



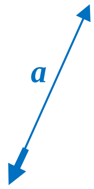
Comment:

These vectors are intentionally not presented on a grid. If they were, students could estimate their components and perform operations component-wise. Presenting the vectors without a grid forces students to interpret the operations geometrically.

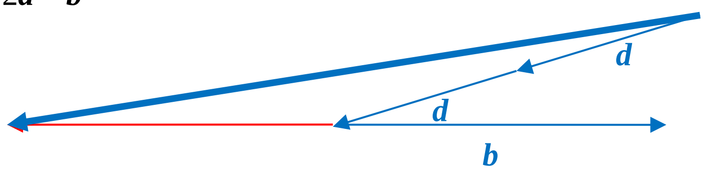
1. $\boldsymbol{b} + \boldsymbol{c}$

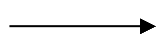


2. -¼ ***a***

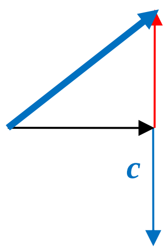


3. 2***d*** – ***b***

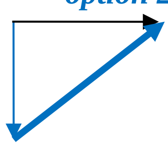


4. Find ***e*** so ***c*** + ***e*** = 

option 1: subtraction: e =  ***– c***

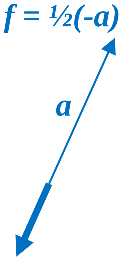


option 2: c and resultant start at same point



5. Find ***f*** so that 2***f*** + ***a*** = **0**.

f = ½(-a) = -½a



Hedwig and Errol

Mathematical Goals:
- Perform operations on vectors given in magnitude-direction form.
- Use the tail-to-end method to perform vector addition and subtraction.
- Understand advantages and disadvantages of component form and magnitude-direction form.
- Practice converting from one form to the other.

GSE Standards:
- **MGSE9-12.N.VM.3** Solve problems involving velocity and other quantities that can be represented by vectors.
- **MGSE9-12.N.VM.4** Add and subtract vectors.
- **MGSE9-12.N.VM.4a** Add vectors end-to-end, component-wise, and by the parallelogram rule. Understand that the magnitude of a sum of two vectors is typically not the sum of the magnitudes.
- **MGSE9-12.N.VM.4b** Given two vectors in magnitude and direction form, determine the magnitude and direction of their sum.
- **MGSE9-12.N.VM.4c** Understand vector subtraction **v – w** as **v + (–w)**, where (**–w**) is the additive inverse of **w**, with the same magnitude as **w** and pointing in the opposite direction. Represent vector subtraction graphically by connecting the tips in the appropriate order, and perform vector subtraction component-wise.
- **MGSE9-12.N.VM.5** Multiply a vector by a scalar.
  - **MGSE9-12.N.VM.5a** Represent scalar multiplication graphically by scaling vectors and possibly reversing their direction; perform scalar multiplication component-wise, e.g., as $c(\mathbf{v}_x, \mathbf{v}_y) = (c \cdot \mathbf{v}_x, c \cdot \mathbf{v}_y)$.
  - **MGSE9-12.N.VM.5b** Compute the magnitude of a scalar multiple c**v** using $\|c\mathbf{v}\| = |c|\mathbf{v}$. Compute the direction of c**v** knowing that when $|c|\mathbf{v} \neq 0$, the direction of c**v** is either along **v** (for c > 0) or against **v** (for c < 0).

Standards for Mathematical Practice:
- Make sense of problems and persevere in solving them.
- Reason abstractly and quantitatively.
- Construct viable arguments and critique the reasoning of others.
- Model with mathematics.
- Use appropriate tools strategically.
- Attend to precision.
- Look for and make use of structure.

Hedwig and Errol

The vector addition and subtraction we have done so far has been quite easy, since we could do them **component-wise.** That is, we just added or subtracted each component separately. But if vectors are given in magnitude-direction form, this process becomes somewhat more difficult. We must first convert into component form so we can use the easy component-wise methods.

1. Starting from Hogwarts, Ron's clumsy owl Errol flies 5 miles at a 45° angle (remember, these are standard position angles, so they are measured counterclockwise from East). Then he realizes he's flying the wrong way, so he adjusts his path and flies 3 miles at a 330° angle (again, counterclockwise from East). On a separate piece of paper, carefully draw Errol's flight, letting 1 cm = 1 mile. Use a protractor to make your angles accurate.

2. Hedwig has a better sense of direction, so she flies from Hogwarts directly to Errol's ending point. On the same paper, draw Hedwig's path. Use a ruler and protractor to estimate its magnitude and direction.

Hedwig's flight vector is the **sum** of the vectors representing the two parts of Errol's flight. You have just used the **tail-to-end method** for adding two vectors, by placing the tail of Errol's second part at the end of his first part. This method obviously makes sense for a two-part flight, but we will see vectors in a variety of contexts.

3. Let's go back to Hedwig's flight. We could only *estimate* Hedwig's path using the tail-to-head method graphically. Let's use trigonometry to get more accurate results. Complete the table below to convert both parts of Errol's flight into component form:

	Magnitude-Direction Form (owl form)		Component Form (Harry's Form)	
	magnitude	direction	horizontal	vertical
Part 1	5 miles	45°		
Part 2	3 miles	330°		

Show your work here:

4. How far East or West must Hedwig fly to get to Errol's location from Hogwarts? How far North or South?

5. Find the magnitude and direction of Hedwig's path. Compare to your estimates from #2.

6. How far did Errol fly altogether? Did Hedwig fly the same distance? Why does this make sense?

7. There is one particular situation where the sum of the magnitudes of two (or more) vectors *equals* the magnitude of their sum. Describe this situation by drawing a picture and explaining why the sum of the magnitudes equals the magnitude of the sum.

8. Is the direction (angle) of Hedwig's path equal to the sum of the angles of Errol's paths?

Let's see how we can modify the tail-to-end method of addition to apply to vector subtraction.

9. Use component-wise subtraction to find $\langle 3, -7\rangle - \langle 4, -5\rangle$.

10. Draw the vectors $\langle 3, -7\rangle$ and $\langle 4, -5\rangle$ tail-to-end. Describe a modified tail-to-end method that will make the resultant vector equal to your answer from #10.

11. How is subtracting vectors similar to the common "add the opposite" method of subtracting integers?

12. What is the advantage of presenting this subtraction problem in component form rather than direction-magnitude form?

Now let's look at scalar multiplication in each form.

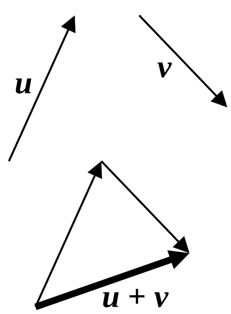
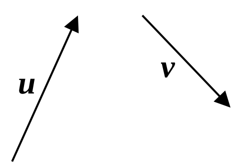
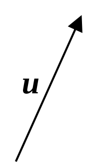
13. From Hogwarts, Harry walks two blocks East and three blocks North then rests. He repeats this five times total. Describe how Ron can walk to Harry's ending position from Hogwarts without stopping.

14. From Hogwarts, Hedwig flies four miles at a direction of 70° then rests. He repeats this five times total. Describe how Errol can fly to Hedwig's ending position from Hogwarts.

15. We have seen that vector addition and subtraction is easier when the problem is presented in component form. Is the same true for scalar multiplication? Explain your answer.

Summarize:

In each box, explain how to perform the indicated operation depending on the form in which the problem is presented. Two boxes have been completed for you.

	add $\boldsymbol{u} + \boldsymbol{v}$	**subtract** $\boldsymbol{u} - \boldsymbol{v}$	**multiply** $a\,\boldsymbol{u}$ (a is a real number)
given in component form		*subtract corresponding components*	
given in direction-magnitude form			
geometric representation			
			stretch if shrink if reverse direction if

Practice:

Let $\boldsymbol{p} = \langle 3, 5\rangle$; $\boldsymbol{q} = \langle -1, 6\rangle$; $\boldsymbol{r} = \langle 4, -3\rangle$; $\boldsymbol{s} = \langle -2, -6\rangle$. Find each of the following:

1. $\boldsymbol{p} + \boldsymbol{q}$

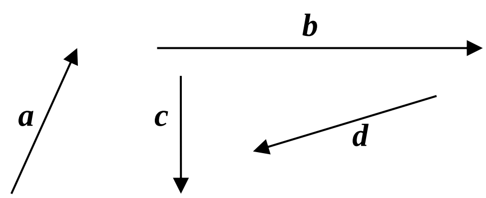
2. $\boldsymbol{p} + 5\boldsymbol{r} + 3\boldsymbol{s}$

3. $\boldsymbol{s} - 2(\boldsymbol{p} + \boldsymbol{r})$

4. Find $\boldsymbol{t}$ so that $\boldsymbol{p} + \boldsymbol{t} = \boldsymbol{s}$

5. Find $\boldsymbol{u}$ so that $3\boldsymbol{u} - 4(\boldsymbol{p} + \boldsymbol{q}) = \boldsymbol{r}$

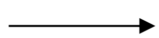
Let ***a***, ***b***, ***c***, and ***d*** be the vectors shown below. Sketch each of the following:



1. $\boldsymbol{b} + \boldsymbol{c}$

2. $-\frac{1}{4}\boldsymbol{a}$

3. $2\boldsymbol{d} - \boldsymbol{b}$

4. Find ***e*** so $\boldsymbol{c} + \boldsymbol{e} =$ 

5. Find ***f*** so that $2\boldsymbol{f} + \boldsymbol{a} = \boldsymbol{0}$.

Putting Vectors to Use

Back to Task Table

Mathematical Goals:

- Use vectors to represent and solve problems.

GSE Standards:

- **MGSE9-12.N.VM.1** Recognize vector quantities as having both magnitude and direction. Represent vector quantities by directed line segments, and use appropriate symbols for vectors and their magnitudes (e.g., **v, |v|, ||v||,** *v*).
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Standards for Mathematical Practice:

- Make sense of problems and persevere in solving them.
- Reason abstractly and quantitatively.
- Model with mathematics.
- Use appropriate tools strategically.
- Attend to precision.

Introduction:

This task gives students practice with problems that involve vectors other than distance / displacement vectors. You may wish to discuss with Physics teachers at your school how they set up similar problems in free-body diagrams, since many students may be taking this course concurrently with Physics. Using similar formats (here, an x- and y-component table) will help students see the connections between the two subjects.

Putting Vectors to Use

Many different problems can be solved using vectors. Any situation that involves quantities with both magnitude and direction can be represented using vectors. For each problem below, draw and label a diagram, then use what you know about vectors to answer the question.
Show all work on a separate piece of paper.

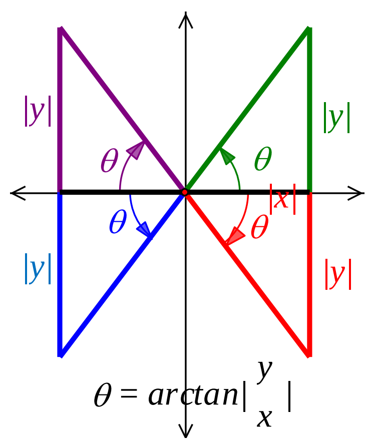
Comment:

When finding the direction of a vector (or, later, the argument of a complex number), we must make a decision about the possible answers. Some will allow these angles to be in [0°, 360°), whereas others use [-180°, 360°). To avoid some confusion with negative angles, throughout this unit we will use the interval [0°, 360°).

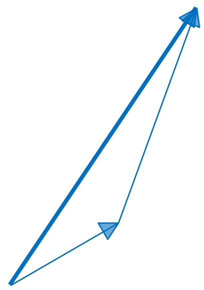
Regardless of the interval used, conversions must be made in certain situations. There are several ways to do this, some of which retain the sign of the components (e.g., using arctan(5/-4) for the vector ⟨4,-5⟩). Here, signs are ignored and only the quadrant of the vector's terminal point (when in standard position) is needed to tell us how to convert the angle to get it to be in the interval [0°, 360°):

Quadrant I:	***arctan(|y/x|)***
Quadrant II:	***180° – arctan(|y/x|)***
Quadrant III:	***180° + arctan(|y/x|)***
Quadrant IV:	***360° – arctan(|y/x|)***

The reasoning behind this can be shown easily with the following diagram, which does not require any negative angles. arctan(|y/x|) is used in each triangle, and students must simply decide whether they are moving "forward" (counterclockwise) or "backward" (clockwise) from the positive-x axis (0° and 360°) or the negative x-axis (180°).



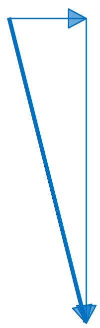
1. A ship leaves port and travels 49 miles at a standard position angle of 30°. The ship then travels for 89 miles in a standard position angle of 70°. At that point, the ship drops anchor. A helicopter, beginning from the same port, needs to join the ship as quickly as possible. Tell the helicopter's pilot how to get to the ship.



	horizontal	***vertical***
first portion	***$49\cos 30° \approx 42.44$ mi***	***$49\sin 30° \approx 24.50$ mi***
second portion	***$89\cos 70° \approx 30.44$ mi***	***$89\sin 70° \approx 83.63$ mi***
total	***+72.88 mi (East)***	***+108.13 mi (North)***

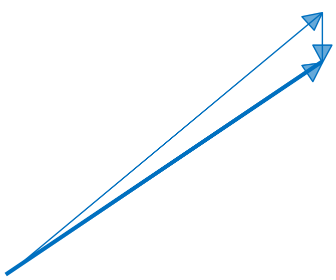
magnitude: $\sqrt{72.88^2 + 108.13^2} = 130.40$ mi
direction: arctan(108.13/72.88) = 56.02°

2. You jump into a river intending to swim straight across to the other side. But when you start swimming, you realize the current is traveling 4 miles per hour due south. You are trying to swim due East at 1 mile per hour, but the current is pulling on you. If you don't make any adjustment for the current, how far from your starting point will you be in 15 <u>minutes</u>?



resultant velocity = swimming + current
magnitude: $\sqrt{1^2 + 4^2} = \sqrt{17}$ mi / hr = $\sqrt{17}$ / 60 miles / minute
after 15 minutes: ($\sqrt{17}$ / 60)(15) = $\sqrt{17}$ / 4 ≈ 1.03 miles
Alternative solution method:
In 15 minutes, swimming without a current would get me (15/60)(1) = 0.25 miles East of my starting point.
In 15 minutes, the current pulls me (15/60)(4) = 1 mi S of my starting point.
The total distance from my starting point is given by the Pyth. Thm.:
$\sqrt{0.25^2 + 1^2} \approx 1.03$ miles

3. A plane is traveling at 400 mph along a path 40° North of East. A strong wind begins to blow at 50 mph from North to South. If no adjustment is made for the wind, what are the resulting bearing and groundspeed of the plane?



	horizontal	***vertical***
original velocity	***$400\cos 40°$= 306.42 mph***	***$400\sin 40°$= 257.12 mph***
wind	***0 mph***	***-50 mph***
resulting velocity	***306.42 mph***	***207.12 mph***

bearing: arctan(207.12/306.42) = 34.06°
groundspeed: $\sqrt{306.42^2 + 207.12^2} = 369.85$ mph

4. A motorboat traveling from one shore to the other at a rate of 5 m/s east encounters a current flowing at a rate of 3.5 m/s north.
   a. What is the resultant velocity?

   $\sqrt{5^2 + 3.5^2} = 6.10$ m/s

b. If the width of the river is 60 meters wide, then how much time does it take the boat to travel to the opposite shore?

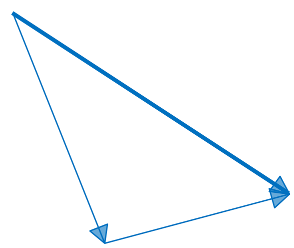
(60 m) / (5 m/s) = 12 s

c. What distance downstream does the boat reach the opposite shore?

(3.5 m/s) (12 s) = 42 m

5. A ship sails 12 hours at a speed of 8 knots (nautical miles per hour) at a heading of 68° south of east. It then turns to a heading of 15° north of east and travels for 5 hours at 15 knots.

a. Find the resultant displacement vector. Give your answer in component form.



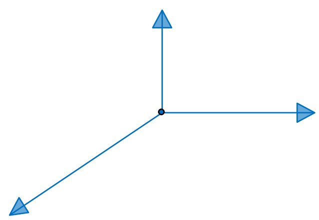
	horizontal	*vertical*
first portion: 96 miles, 68° S of E	*96cos(-68°) 35.96 mi*	*96sin(-68°) -89.01 mi*
second portion: 75 miles, 15° N of E	*75cos15° 72.44 mi*	*75sin15° 19.41 mi*
total	*+108.40 mi (East)*	*-69.60 mi (South)*

109.40 miles East, 69.60 miles South or <108.4, 69.6>

b. Convert your answer to magnitude-direction form.

magnitude: $\sqrt{(108.4^2 + 69.6^2)} = 128.8$ ***miles***
direction: $360° + \arctan(-69.6/108.4) = 327.3°$

6. In three-person tug-of-war, three ropes are tied at a point. Adam is pulling due East with a force of 600 Newtons, Barry is pulling due North with a force of 400 Newtons, and Cal is pulling the third rope. The knot in the middle is not moving. What are the direction and magnitude of Cal's effort?



	horizontal	***vertical***
Adam	***600 N***	***0 N***
Barry	***0 N***	***400 N***
Cal ***0 – (Adam + Barry)***	***0 – 600*** ***-600 N***	***0 – 400*** ***-400 N***
total = 0 ***(knot doesn't move)***	***0 N***	***0 N***

direction: $180° + \arctan(-400/-600) = 213.7°$

magnitude: $\sqrt{600^2 + 400^2} = 200\sqrt{13} \approx 721$ ***Newtons***

Problems #4 and #5 adapted from Arizona Department of Education: Standards and Assessment Division

Putting Vectors to Use

Mathematical Goals:
- Use vectors to represent and solve problems.

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- **MGSE9-12.N.VM.4c** Understand vector subtraction **v – w** as **v + (–w)**, where (–**w**) is the additive inverse of **w**, with the same magnitude as **w** and pointing in the opposite direction. Represent vector subtraction graphically by connecting the tips in the appropriate order, and perform vector subtraction component-wise.
- **MGSE9-12.N.VM.5** Multiply a vector by a scalar.
  - **MGSE9-12.N.VM.5a** Represent scalar multiplication graphically by scaling vectors and possibly reversing their direction; perform scalar multiplication component-wise, e.g., as $c(\mathbf{v}_x, \mathbf{v}_y) = (c\cdot\mathbf{v}_x, c\cdot\mathbf{v}_y)$.
  - **MGSE9-12.N.VM.5b** Compute the magnitude of a scalar multiple c**v** using $\|c\mathbf{v}\| = |c|\mathbf{v}$. Compute the direction of c**v** knowing that when $|c|\mathbf{v} \neq 0$, the direction of c**v** is either along **v** (for c > 0) or against **v** (for c < 0).

Standards for Mathematical Practice:
- Make sense of problems and persevere in solving them.
- Reason abstractly and quantitatively.
- Model with mathematics.
- Use appropriate tools strategically.
- Attend to precision.

Putting Vectors to Use

Many different problems can be solved using vectors. Any situation that involves quantities with both magnitude and direction can be represented using vectors. For each problem below, draw and label a diagram, then use what you know about vectors to answer the question.
Show all work on a separate piece of paper.

1. A ship leaves port and travels 49 miles at a standard position angle of 30°. The ship then travels for 89 miles in a standard position angle of 70°. At that point, the ship drops anchor. A helicopter, beginning from the same port, needs to join the ship as quickly as possible. Tell the helicopter's pilot how to get to the ship.

2. You jump into a river intending to swim straight across to the other side. But when you start swimming, you realize the current is traveling 4 miles per hour due south. You are trying to swim due East at 1 mile per hour, but the current is pulling on you. If you don't make any adjustment for the current, how far from your starting point will you be in 15 <u>minutes</u>?

3. A plane is traveling at 400 mph along a path 40° North of East. A strong wind begins to blow at 50 mph from North to South. If no adjustment is made for the wind, what are the resulting bearing and groundspeed of the plane?

4. A motorboat traveling from one shore to the other at a rate of 5 m/s east encounters a current flowing at a rate of 3.5 m/s north.
    a. What is the resultant velocity?
    b. If the width of the river is 60 meters wide, then how much time does it take the boat to travel to the opposite shore?
    c. What distance downstream does the boat reach the opposite shore?

5. A ship sails 12 hours at a speed of 8 knots (nautical miles per hour) at a heading of 68° <u>south</u> of east. It then turns to a heading of 75° <u>north</u> of east and travels for 5 hours at 15 knots.
    a. Find the resultant displacement vector. Give your answer in component form.
    b. Convert your answer to magnitude-direction form.

6. In three-person tug-of-war, three ropes are tied at a point. Adam is pulling due East with a force of 600 Newtons, Barry is pulling due North with a force of 400 Newtons, and Cal is pulling the third rope. The knot in the middle is not moving. What are the direction and magnitude of Cal's effort?

Problems #4 and #5 adapted from Arizona Department of Education: Standards and Assessment Division

He Who Must Not Be Named

Back to Task Table

Mathematical Goals:
- Apply transformations, represented as matrices, to vectors.

GSE Standards:
- **MGSE9-12.N.VM.11** Multiply a vector (regarded as a matrix with one column) by a matrix of suitable dimensions to produce another vector. Work with matrices as transformations of vectors.

Standards for Mathematical Practice:
- Make sense of problems and persevere in solving them.
- Reason abstractly and quantitatively.
- Model with mathematics.
- Look for and make use of structure.

Introduction

This task allows students to perform more operations on vectors through the use of transformation matrices. Students explore these transformations graphically and analytically, and practice finding magnitude and direction by comparing the original vector with the transformed vector. Some transformations are simple rotations (matrix T), others are dilations (matrix V in #7), and the rest are more complex transformations. Students will need to remember matrix multiplication from the previous unit.

Teachers wishing to review the Unit 6 standards in more depth might ask students to reverse a spell (inverse matrices). Or the teacher might ask for what would happen if two spells were cast by competing wizards (composition / matrix multiplication). The task as written here focuses only on the new standard of seeing matrices as transformations of vectors.

He Who Must Not Be Named

Voldemort wants to disrupt mail delivery to the wizards at Hogwarts. He has cast a spell that causes each owl to misinterpret directions to its destination. The spell can be represented as a **transformation matrix**:

$$T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

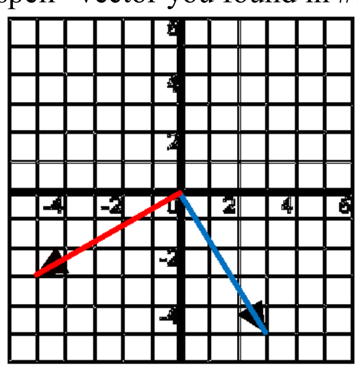
To apply this transformation, the owl's directions (in component form) is written as a 2×1 matrix *d*, then the owl follows the directions of the product *Td*.

To get a feel for how this spell works, let's see what Hedwig does when we give her certain directions. If we tell Hedwig to travel 3 miles East and 5 miles South, her direction vector would be written as the 2×1 matrix $d = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$

1. The spell will multiply the transformation matrix by this vector. Perform the calculation.

$$Td = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ -5 \end{bmatrix} = \begin{bmatrix} -5 \\ -3 \end{bmatrix}$$

2. On this grid, graph a vector representing the directions we gave Hedwig: ⟨3, -5⟩. In a different color, graph the "spell" vector you found in #1.



3. Repeat for the following. Graph the original and the spell vector on the same grid.

original directions	*d*	magnitude and direction of *d*	*Td*	spell vector	magnitude and direction of *Td*
4 miles West 3 miles North	$\begin{bmatrix} -4 \\ 3 \end{bmatrix}$	***mag: 5*** ***dir: 143.13°***	$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -4 \\ 3 \end{bmatrix}$	$\begin{bmatrix} 3 \\ 4 \end{bmatrix}$	***mag: 5*** ***dir: 53.13°***
2 miles West 5 miles South	$\begin{bmatrix} -2 \\ -5 \end{bmatrix}$	***mag: √29*** ***dir: 248.20°***	$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -2 \\ -5 \end{bmatrix}$	$\begin{bmatrix} -5 \\ 2 \end{bmatrix}$	***mag: √29*** ***dir: 158.20°***

4. Did the spell change the length (magnitude) of Hedwig's flight? ***No.***

   Did it change the direction of her flight? ***Yes. (90° rotation clockwise about origin)***

Dumbledore figures out what is going on, so Voldemort has to change his spell. The new transformation matrix is $U = \begin{bmatrix} -1 & -2 \\ 2 & -1 \end{bmatrix}$

5. Use the transformation matrix to find the "spell vector" when we tell Hedwig to fly 3 miles West and 1 mile South.

   $$\begin{bmatrix} -1 & -2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} -3 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ -5 \end{bmatrix}$$

6. Did the transformation change the vector's magnitude? Did it change its direction? Justify your answers by showing your calculations for magnitude and direction *and* by graphing the original vector and the spell vector below.

Calculations for magnitude and direction:

original:

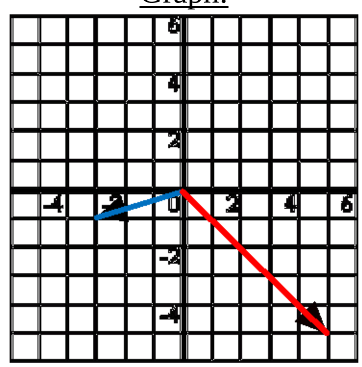
magnitude = $\sqrt{(3^2 + 1^2)} = \sqrt{10}$
direction: arctan(-1/-3) = 18.43° →
18.43° + 180° = 198.43°

transformed:

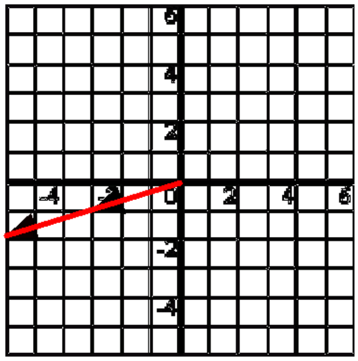
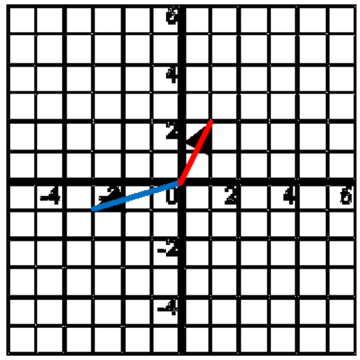
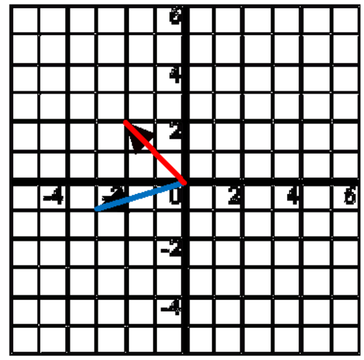
magnitude = $\sqrt{(5^2 + 5^2)} = \sqrt{50} = 5\sqrt{2}$
direction: arctan(-5/5) = -45° →
-45° + 360° = 315°

Both magnitude and direction changed.

Graph:



7. Perform transformations on the vector $\langle -3, -1\rangle$ using each of the following transformation matrices. Graph the original vector and the "spell vector" on the same graph, and state whether the vector's magnitude and/or direction has changed.

matrix	$V = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$	$W = \begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix}$	$Z = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$
Spell vector	$\begin{bmatrix} -6 \\ -2 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$	$\begin{bmatrix} -2 \\ 2 \end{bmatrix}$
graph			
changed…	$\boxed{\text{magnitude}}$ direction	$\boxed{\text{magnitude}}$ $\boxed{\text{direction}}$	$\boxed{\text{magnitude}}$ $\boxed{\text{direction}}$

He Who Must Not Be Named

Mathematical Goals:

- Apply transformations, represented as matrices, to vectors.

GSE Standards:

- **MGSE9-12.N.VM.11** Multiply a vector (regarded as a matrix with one column) by a matrix of suitable dimensions to produce another vector. Work with matrices as transformations of vectors.

Standards for Mathematical Practice:

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- Model with mathematics.
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<u>He Who Must Not Be Named</u>

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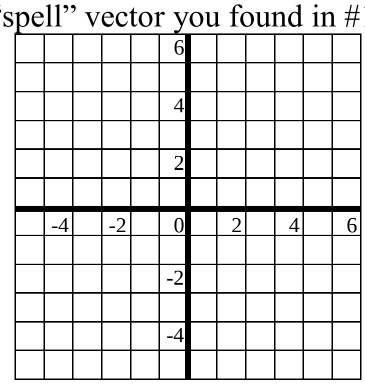
To get a feel for how this spell works, let's see what Hedwig does when we give her certain directions. If we tell Hedwig to travel 3 miles East and 5 miles South, her direction vector would be written as the 2×1 matrix $d = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$

1. The spell will multiply the transformation matrix by this vector. Perform the calculation.

$$Td = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ -5 \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$$

2. On this grid, graph a vector representing the directions we gave Hedwig: ⟨3, -5⟩. In a different color, graph the "spell" vector you found in #1.



3. Repeat for the following. Graph the original and the spell vector on the same grid.

original directions	*d*	magnitude and direction of *d*	*Td*	spell vector	magnitude and direction of *Td*
4 miles West 3 miles North	$\begin{bmatrix} -4 \\ 3 \end{bmatrix}$		$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -4 \\ 3 \end{bmatrix}$	$\begin{bmatrix} \quad \\ \quad \end{bmatrix}$	
2 miles West 5 miles South					

4. Did the spell change the length (magnitude) of Hedwig's flight?

Did it change the direction of her flight?

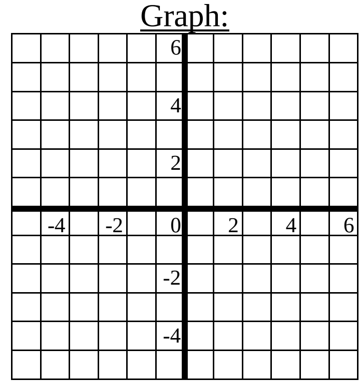
Dumbledore figures out what is going on, so Voldemort has to change his spell. The new transformation matrix is $U = \begin{bmatrix} -1 & -2 \\ 2 & -1 \end{bmatrix}$

5. Use the transformation matrix to find the "spell vector" when we tell Hedwig to fly 3 miles West and 1 mile South.

6. Did the transformation change the vector's magnitude? Did it change its direction? Justify your answers by showing your calculations for magnitude and direction *and* by graphing the original vector and the spell vector below.

Calculations for magnitude and direction:

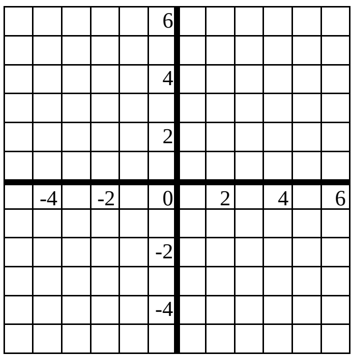
Graph:

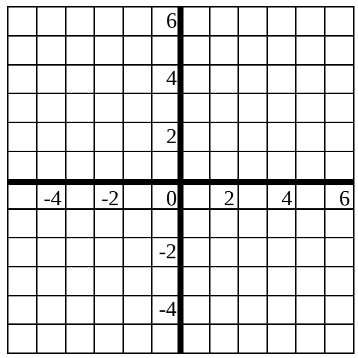


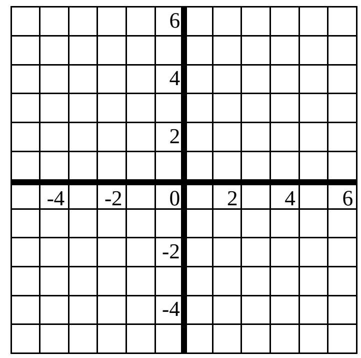
7. Perform transformations on the vector ⟨-3, -1⟩ using each of the following transformation matrices. Graph the original vector and the "spell vector" on the same graph, and state whether the vector's magnitude and/or direction has changed.

matrix	$V = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$	$W = \begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix}$	$Z = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$
Spell vector	$\begin{bmatrix} \quad \\ \quad \end{bmatrix}$	$\begin{bmatrix} \quad \\ \quad \end{bmatrix}$	$\begin{bmatrix} \quad \\ \quad \end{bmatrix}$

graph







changed…	magnitude	direction	magnitude	direction	magnitude	direction

It's Not That Complex!

Back to Task Table

Mathematical Goals:

- Recall the definition of imaginary and complex numbers.
- Recall how to add, subtract, multiply, and divide complex numbers.

GSE Standards (from previous courses):

- **MGSE9-12.N.CN.1** Know there is a complex number i such that $i^2 = -1$, and every complex number has the form $a + bi$ with a and b real.
- **MGSE9-12.N.CN.2** Use the relation $i^2 = -1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.
- **MGSE9-12.N.CN.3** Find the conjugate of a complex number; use the conjugate to find the absolute value (modulus) and quotient of complex numbers.

Standards for Mathematical Practice:

- Reason abstractly and quantitatively.
- Attend to precision.
- Look for and make use of structure.
- Look for and express regularity in repeated reasoning.

Introduction

This task is simply a review of complex numbers. Students will need to remember their foundation in operations on complex numbers in order to complete the remaining tasks in this unit.

<u>It's Not That Complex!</u>

Previously, you were introduced to *imaginary* and *complex* numbers. Let's review some of the operations you learned in that unit before we move forward.

For each of the following operations, give a brief explanation of how to perform the operation, then apply your process to the example. Write your answers in **standard form**: $a + bi$.

<table>
<tr><th></th><th>Explanation</th><th colspan="2">Example(s)</th></tr>
<tr><td>1. add</td><td>Add the real parts.
Add the imaginary parts.</td><td colspan="2">$(3 + 5i) + (4 - 7i)$
$(3 + 4) + (5 + -7)i$
$7 - 2i$</td></tr>
<tr><td>2. subtract</td><td>Subtract the real parts.
Subtract the imaginary parts.
--OR--
Change to an addition problem by adding the opposite of the second complex number.</td><td colspan="2">$(3 + 5i) - (4 - 7i)$
$(3 - 4) + (5 - -7)i$
$-1 + 12i$</td></tr>
<tr><td>3. multiply</td><td>Multiply as if the i's are variables.
Substitute: $i^2 = -1$; $i^3 = -1$; $i^4 = 1$
Combine like terms.</td><td>$3(4 - 7i)$
$3(4) + 3(-7)i$
$12 - 21i$</td><td>$(3 + 5i)(4 - 7i)$
$12-21i+20i-35i^2$
$12-21i+20i+35$
$47 - i$</td></tr>
<tr><td>4. conjugate
(The conjugate of a is written as $\bar{a}$.)</td><td>Change the sign of the imaginary part.

i.e., a + bi becomes a – bi</td><td colspan="2">conjugate of $(3 + 5i)$
$3 - 5i$</td></tr>
<tr><td>5. divide</td><td>If denominator is non-real, multiply numerator and denominator by the denominator's conjugate in order to make the denominator real.
If denominator is real, divide through by denominator.</td><td>$\frac{3+5i}{4}$
denominator is real
$(3/4) + (5/4)i$</td><td>$\frac{3+5i}{4-7i}$
$\frac{(3+5i)(4+7i)}{(4-7i)(4+7i)}$
$\frac{(12+21i+20i+35i^2)}{(16+28i-28i-49i^2)}$
$\frac{(-23+41i)}{65}$
$(-23/65) + (41/65)i$</td></tr>
</table>

It's Not That Complex!

Mathematical Goals:

- Recall the definition of imaginary and complex numbers.
- Recall how to add, subtract, multiply, and divide complex numbers.

GSE Standards (from previous courses):

- **MGSE9-12.N.CN.1** Know there is a complex number i such that $i^2 = -1$, and every complex number has the form $a + bi$ with a and b real.
- **MGSE9-12.N.CN.2** Use the relation $i^2 = -1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.
- **MGSE9-12.N.CN.3** Find the conjugate of a complex number; use the conjugate to find the absolute value (modulus) and quotient of complex numbers.

Standards for Mathematical Practice:

- Reason abstractly and quantitatively.
- Attend to precision.
- Look for and make use of structure.
- Look for and express regularity in repeated reasoning.

It's Not That Complex!

Previously, you were introduced to *imaginary* and *complex* numbers. Let's review some of the operations you learned in that unit before we move forward.

For each of the following operations, give a brief explanation of how to perform the operation, then apply your process to the example. Write your answers in **standard form**: $a + bi$.

	Explanation	Example(s)	
1. add		$(3 + 5i) + (4 - 7i)$	
2. subtract		$(3 + 5i) - (4 - 7i)$	
3. multiply		$3(4 - 7i)$	$(3 + 5i)(4 - 7i)$
4. conjugate		conjugate of $(3 + 5i)$	
5. divide		$\frac{3+5i}{4}$	$\frac{3+5i}{4-7i}$

A Plane You Can't Fly

Back to Task Table

Mathematical Goals:

- Plot complex numbers in the complex plane as points and as vectors.
- Find the modulus and argument of complex numbers.
- Convert between rectangular and polar form of complex numbers.
- Relate rectangular and polar form of complex numbers to component and magnitude-direction form of vectors.

GSE Standards:

- **MGSE9-12.N.CN.4** Represent complex numbers on the complex plane in rectangular and polar form (including real and imaginary numbers), and explain why the rectangular and polar forms of a given complex number represent the same number.

Standards for Mathematical Practice:

- Reason abstractly and quantitatively.
- Attend to precision.
- Look for and make use of structure.
- Look for and express regularity in repeated reasoning.

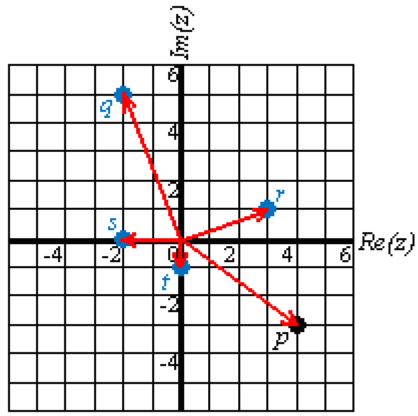
Introduction

This task introduces students to the complex plane. Following a discussion of vectors and vector operations, the complex plane allows us to use similar processes to perform operations on complex numbers, which we will begin in Complex Operations. The connection to vectors and their forms should be emphasized throughout the remaining tasks.

A Plane You Can't Fly

Since complex numbers have two parts—a real part and an imaginary part—it is helpful to view them as points on a two-dimensional plane, called the **complex plane**. Here, the horizontal axis represents the real part of the numbers, and the vertical axis represents the imaginary part. So we can represent the number -4 + 6*i* as the point (-4, 6).

1. Plot and label the following complex numbers on the complex plane.
The first one, *p*, has been done for you.
$p = 4 - 3i$ $q = -2 + 5i$ $r = 3 + i$ $s = -2$ (which is $-2 + 0i$) $t = -i$



The blue points represent the answers to this question.

2. Sometimes, we turn these points into **vectors** by drawing an arrow directed *from the origin* to the point. Add arrows to your graph above to turn the points into vectors.

The red arrows above represent the answers to this question.

Now that we have represented complex numbers as vectors, we can use what we know about vectors to learn more about complex numbers.

Like vectors, complex numbers have magnitude, usually called their **modulus**, **norm**, or **absolute value**. We use the notation $|z|$ to represent the modulus of a complex number *z*.

3. What is the definition of the absolute value of a real number?

The distance between a number and 0.

Many students use the "definition" that absolute value simply turns negative numbers positive. Since we are dealing with complex numbers now, it is important to emphasize that absolute value represents a <u>distance from 0</u>.

How is this related to the idea of the absolute value of a complex number?

The magnitude of a vector is its length. Since these vectors all have 0 + 0i as an initial point, the magnitude of that vector is equal to the terminal point's distance from 0.

4. Find the absolute value (magnitude) of each of the complex numbers you plotted in #1.

$p = 4 - 3i$ $|p| = \sqrt{4^2 + 3^2} = 5$

$q = -2 + 5i$ $|q| = \sqrt{2^2 + 5^2} = \sqrt{29}$

$r = 3 + i$ $|r| = \sqrt{3^2 + 1^2} = \sqrt{10}$

$s = -2$ $|s| = \sqrt{2^2 + 0^2} = 2$

$t = -i$ $|t| = \sqrt{0^2 + 1^2} = 1$

Also like vectors, complex numbers have a direction, usually called their **angle** or **argument**. We use the notation arg(z) to represent the argument of a complex number z.

There are different ways to measure this angle. Some mathematicians say complex numbers with negative imaginary parts (i.e., those below the Real axis) have a negative angle. Others say complex numbers below the Real axis have angles greater than 180°. To be consistent with how we described the direction of a vector, we will choose the latter. In other words, we will always measure counterclockwise from the positive-Real axis, which means our angles will be in the interval $0° \leq \arg(z) < 360°$.

5. Find the argument (direction) of each of the complex numbers you plotted in #1.

$p = 4 - 3i$ $\arg(p) =$ ***360° – arctan(3/4) = 323.13°***

$q = -2 + 5i$ $\arg(q) =$ ***180° – arctan(5/2) = 111.80°***

$r = 3 + i$ $\arg(r) =$ ***arctan(1/3) = 18.43°***

$s = -2$ $\arg(s) =$ ***directly left: 180°***

$t = -i$ $\arg(t) =$ ***directly down: 270°***

By finding the modulus and argument of complex numbers, we can convert between the form $a + bi$, called **rectangular form**, and the form $r(\cos\theta + i\sin\theta)$, called **polar form**. For simplicity, we will use the notation "cis θ" as shorthand for the expression $\cos\theta + i\sin\theta$.

Note:

Some textbooks use "cis θ" for polar form, whereas others keep the somewhat more daunting cos θ + isin θ notation. Still others write r cis θ as (r, θ). Teachers should expose students to all of these types of notation throughout the unit, emphasizing that the choice of one notation is simply a matter of preference.

Although polar form looks complicated, it isn't! In fact, we have essentially done this already during our time with Harry and Hedwig. Let's review:

6. If Harry walks 4 miles West and 6 miles North, what path would Hedwig take?

magnitude: $\sqrt{(4^2 + 6^2)} = \sqrt{52} = 2\sqrt{13}$
direction: $180° – \arctan(6/4) = 123.69°$

7. If Hedwig flies 2 miles at an angle of 210°, what path would Harry take?

E/W: $2\cos 210° = -1.73 = 1.73$ ***miles West***
N/S: $2\sin 210° = -1 = 1$ ***mile South***

In these examples, Harry's directions were written in **component form** of a vector, which is like **rectangular form** of a complex number: it has two parts (either E/W and N/S, or real and imaginary). Hedwig's directions were written in **magnitude-direction form** of a vector, which is like **polar form** of a complex number: it tells us the length and the direction of the path.

8. If Hegwig flew over the complex number plane, write Hedwig's directions from #6 and #7 in polar form of a complex number.

NOTE: We normally would not have a map on the complex plane since it would not represent the real world with Real and Imaginary directions. This is here to help students see the connection between imaginary numbers and vectors, though one would not represent traveled paths as complex numbers.

#6: $2\sqrt{13}(\cos 123.69° + i\sin 123.69°)$ ***or*** $2\sqrt{13}\text{ cis } 123.69°$
#7: $2(\cos 210° + i\sin 210°)$ ***or*** $2\text{ cis } 210°$

9. Write the complex numbers from #1 in polar form. The first one has been done for you. Use your answers to #4 and #5 to help.

$p = 4 - 3i = 5 \text{ cis } 323.13°$

$q = -2 + 5i = $ ***√29 cis 111.80°***

$r = 3 + i = $ ***√10 cis 18.43°***

$s = -2 = $ ***2 cis 180°***

$t = -i = $ ***1 cis 270°***

10. Write the following complex numbers in rectangular form, then plot them on the given complex plane. Remember to think of this as converting Hedwig's directions to Harry's directions. The first one has been done for you.

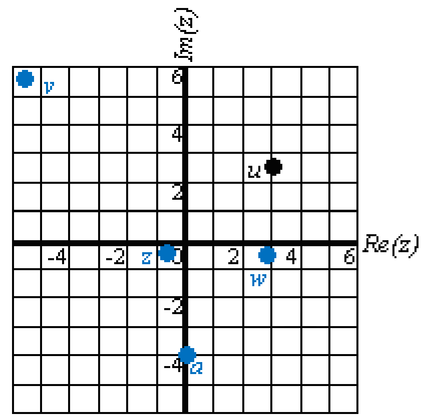
$u = 4 \text{ cis } 40° = 3.06 + 2.57i$

$v = 8 \text{ cis } 135° = $ ***-5.66 + 5.66i***

$w = 3 \text{ cis } 350° = $ ***2.95 – 0.52i***

$z = 1 \text{ cis } 200° = $ ***-0.94 – 0.34i***

$a = 4 \text{ cis } 270° = $ ***-4***



A Plane You Can't Fly

Mathematical Goals:

- Plot complex numbers in the complex plane as points and as vectors.
- Find the modulus and argument of complex numbers.
- Convert between rectangular and polar form of complex numbers.
- Relate rectangular and polar form of complex numbers to component and magnitude-direction form of vectors.

GSE Standards:

- **MGSE9-12.N.CN.4** Represent complex numbers on the complex plane in rectangular and polar form (including real and imaginary numbers), and explain why the rectangular and polar forms of a given complex number represent the same number.

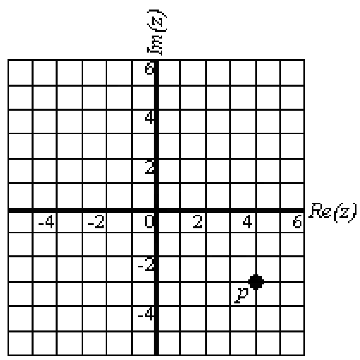
Standards for Mathematical Practice:

- Reason abstractly and quantitatively.
- Attend to precision.
- Look for and make use of structure.
- Look for and express regularity in repeated reasoning.

A Plane You Can't Fly

Since complex numbers have two parts—a real part and an imaginary part—it is helpful to view them as points on a two-dimensional plane, called the **complex plane**. Here, the horizontal axis represents the real part of the numbers, and the vertical axis represents the imaginary part. So we can represent the number -4 + 6*i* as the point (-4, 6).

1. Plot and label the following complex numbers on the complex plane.
The first one, *p*, has been done for you.
p = 4 – 3*i* *q* = -2 + 5*i* *r* = 3 + *i* *s* = -2 (which is -2 + 0*i*) *t* = -*i*



2. Sometimes, we turn these points into **vectors** by drawing an arrow directed *from the origin* to the point. Add arrows to your graph above to turn the points into vectors.

Now that we have represented complex numbers as vectors, we can use what we know about vectors to learn more about complex numbers.

Like vectors, complex numbers have magnitude, usually called their **modulus**, **norm**, or **absolute value**. We use the notation |*z*| to represent the modulus of a complex number *z*.

3. What is the definition of the absolute value of a real number?

How is this related to the idea of the absolute value of a complex number?

4. Find the absolute value (magnitude) of each of the complex numbers you plotted in #1.

$p = 4 - 3i$ $|p| =$

$q = -2 + 5i$ $|q| =$

$r = 3 + i$ $|r| =$

$s = -2$ $|s| =$

$t = -i$ $|t| =$

Also like vectors, complex numbers have a direction, usually called their **angle** or **argument**. We use the notation arg(z) to represent the argument of a complex number z.

There are different ways to measure this angle. Some mathematicians say complex numbers with negative imaginary parts (i.e., those below the Real axis) have a negative angle. Others say complex numbers below the Real axis have angles greater than 180°. To be consistent with how we described the direction of a vector, we will choose the latter. In other words, we will always measure counterclockwise from the positive-Real axis, which means our angles will be in the interval $0° \leq \arg(z) < 360°$.

5. Find the argument (direction) of each of the complex numbers you plotted in #1.

$p = 4 - 3i$ $\arg(p) =$

$q = -2 + 5i$ $\arg(q) =$

$r = 3 + i$ $\arg(r) =$

$s = -2$ $\arg(s) =$

$t = -i$ $\arg(t) =$

By finding the modulus and argument of complex numbers, we can convert between the form $a + bi$, called **rectangular form**, and the form $r(\cos\theta + i\sin\theta)$, called **polar form**. For simplicity, we will use the notation "cis θ" as shorthand for the expression $\cos\theta + i\sin\theta$.

Although polar form looks complicated, it isn't! In fact, we have essentially done this already during our time with Harry and Hedwig. Let's review:

6. If Harry walks 4 miles West and 6 miles North, what path would Hedwig take?

7. If Hedwig flies 2 miles at an angle of 210°, what path would Harry take?

In these examples, Harry's directions were written in **component form** of a vector, which is like **rectangular form** of a complex number: it has two parts (either E/W and N/S, or real and imaginary). Hedwig's directions were written in **magnitude-direction form** of a vector, which is like **polar form** of a complex number: it tells us the length and the direction of the path.

8. If Hegwig flew over the complex number plane, write Hedwig's directions from #6 and #7 in polar form of a complex number.

9. Write the complex numbers from #1 in polar form. The first one has been done for you. Use your answers to #4 and #5 to help.

$p = 4 - 3i = 5 \text{ cis } 323.13°$

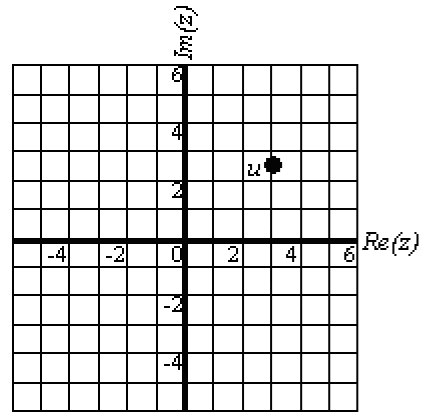
$q = -2 + 5i =$

$r = 3 + i =$

$s = -2 =$

$t = -i =$

10. Write the following complex numbers in rectangular form, then plot them on the given complex plane. Remember to think of this as converting Hedwig's directions to Harry's directions. The first one has been done for you.



u = 4 cis 40° = 3.06 + 2.57 i

v = 8 cis 135° =

w = 3 cis 350° =

z = 1 cis 200° =

a = 4 cis 270° =

Complex Operations

Back to Task Table

Mathematical Goals:

- Use rectangular and polar coordinates to represent complex numbers.
- Discuss advantages and disadvantages of each form for operations on complex numbers.
- Add, subtract, multiply, and divide complex numbers using geometric representations.
- Find powers and roots of complex numbers and represent this geometrically.

GSE Standards:

- **MGSE9-12.N.CN.5** Represent addition, subtraction, multiplication, and conjugation of complex numbers geometrically on the complex plane; use properties of this representation for computation.

Standards for Mathematical Practice:

- Reason abstractly and quantitatively.
- Construct viable arguments and critique the reasoning of others.
- Attend to precision.
- Look for and make use of structure.
- Look for and express regularity in repeated reasoning.

Introduction:

This task has students use what they already know about operations with complex numbers and interpret them in a geometric context. Students work with addition, subtraction, multiplication, division, and powers/roots of complex numbers. Students compare and contrast rectangular and polar form of a complex number to determine which form makes certain operations easier.

Complex Operations

With vectors, each form (component form and magnitude-direction form) had advantages and disadvantages for operations such as addition, subtraction, and scalar multiplication. The same is true for complex numbers. Let's see what the advantages and disadvantages of each form are.

1. First, find the modulus and argument (magnitude and direction) of each of the following complex numbers, then write each in polar form. We will be using these complex numbers throughout the task, so be careful! Round angles to the nearest whole number.

	$3 + 4i$	$2i$	$-5 - 12i$	$4 - 4i$
modulus	$\sqrt{(3^2 + 4^2)}$ $\sqrt{25}$ 5	$\sqrt{(2^2 + 0^2)}$ $\sqrt{4}$ 2	$\sqrt{(5^2 + 12^2)}$ $\sqrt{169}$ 13	$\sqrt{(4^2 + 4^2)}$ $\sqrt{32}$ $4\sqrt{2}$
argument	arctan(4/3)= 53°	directly down 270°	180° + arctan(12/5)= 247°	360° – arctan(4/4)= 315°
polar form	5 cis 53°	2 cis 270°	13 cis 247°	$4\sqrt{2}$ cis 315°

2. Use what you know about complex numbers to simplify each of these **addition and subtraction** expressions in standard form. Complete the table. The first one has been done for you.

expression in standard form	$(3 + 4i) + (2i)$	$(3 + 4i) + (-5 - 12i)$	$(-5 - 12i) - (4 - 4i)$
answer in standard form $a + bi$	$3 + 6i$	$(3 + -5) + (4 + -12)i=$ $-2 - 8i$	$(-5 - 4) + (-12 - -4)i=$ $-9 - 8i$
modulus of answer	$3\sqrt{5}$	$\sqrt{(2^2 + 8^2)}=$ $\sqrt{68}=$ $2\sqrt{17}$	$\sqrt{(9^2 + 8^2)}=$ $\sqrt{145}$
argument of answer	63°	180° + arctan(8/2)= 256°	180° + arctan(8/9)= 222°
answer in polar form r cis θ	$3\sqrt{5}$ cis 63°	$2\sqrt{17}$ cis 256°	$\sqrt{145}$ cis 222°

original in polar form	5 cis 53° + 2 cis 90°	*(5cis53°)+(13cis247°)*	*(13cis247°)–(4√2cis315°)*

3. Use what you know about complex numbers to simplify each of these **multiplication** expressions in standard form. Complete the table. The first one has been done for you.

expression in standard form	$(3 + 4i)(2i)$	$(3 + 4i)(-5 - 12i)$	$(-5 - 12i)(4 - 4i)$
answer in standard form $a + bi$	$-8 + 6i$	*-15 – 36i – 20i – 48i²* *-15 – 36i – 20i + 48* *33 – 56i*	*-20 + 20i – 48i + 48i²* *-20 + 20i – 48i – 48* *-68 – 28i*
modulus of answer	10	*√(33² + 56²)* *√4225* *65*	*√(68² + 28²)* *√5408* *52 √2*
argument of answer	143°	*360° – arctan(56/33)* *300°*	*180° + arctan(28/68)* *202°*
answer in polar form $r \text{ cis } \theta$	10 cis 143°	*65 cis 300°*	*52 √2 cis 202°*
original in polar form	(5 cis 53°)(2 cis 90°)	*(5cis53°)(13cis247°)*	*(13cis247°)(4 √2cis315°)*

4. Use what you know about complex numbers to simplify each of these **division** expressions in standard form. Complete the table. The first one has been done for you.

expression in standard form	$\dfrac{2i}{3 + 4i}$	$\dfrac{-5 - 12i}{3 + 4i}$
answer in standard form $a + bi$	$\frac{8}{25} + \frac{6}{25}i$	*[(-5–12i)(3–4i)]/[(3+4i)(3–4i)]* *(-63 – 16i) / 25* *(-63/25) – (16/25)i*
modulus of answer	$\frac{2}{5}$	*(1/25) √(63² + 16²)* *(1/25) √(4225* *13/5*
argument of answer	37°	*180° + arctan(16/63)* *194°*

answer in polar form $r \text{ cis } \theta$	$\frac{2}{5}$ cis 37°	***(13/5) cis 194°***
original in polar form	$\dfrac{2 \text{ cis } 90°}{5 \text{ cis } 53°}$	***$\dfrac{13 \text{ cis } 247°}{5 \text{ cis } 53°}$***

As we saw with vectors, there are advantages and disadvantages to each form—component and magnitude-direction form. The same is true with complex numbers: some operations are easy to perform in standard (rectangular) form, whereas others are easy in polar form.

5. Investigate your tables from #2, #3, and #4, and see if you can find quick methods in either (or both) forms to perform each operation. Write instructions in each box explaining how to perform each operation. The first column has been completed for you.

	addition (#2)	subtraction (#2)
standard form $a + bi$	Easy! :) Add real parts. Add imaginary parts.	***Easy!*** ***Subtract real parts.*** ***Subtract imaginary parts.***
polar form $r \text{ cis } \theta$	Convert to standard form first, then follow standard form instructions.	***Convert to standard form first, then follow standard form instructions.***

	multiplication (#3)	division (#4)
standard form $a + bi$	***Convert to polar form first, then follow polar form instructions...OR use the distributive property.***	***Convert to polar form first, then follow polar form instructions…OR multiply both numerator and denominator by the denominator's conjugate.***
polar form $r \text{ cis } \theta$	***Multiply moduli.*** ***Add arguments.*** ****subtract multiples of 360° if the sum of the arguments is greater than 360°.***	***Divide moduli.*** ***Subtract arguments.*** ****add multiples of 360° if the difference of the arguments is less than 0°.***

**** portions in red are added after #18.***

	exponentiation (#6-10)
standard form $a + bi$	***After #9:*** ***Convert to polar form first, then follow polar form instructions.***
polar form r cis θ	***After #9:*** ***Raise modulus to power.*** ***Multiply argument by power.*** ****add/subtract multiples of 360° if the angle is too large/small.***

← We will complete this part of the table after the next part of our task.

Now let's take a look at how we can find powers of complex numbers.

6. Use your summary table from #5 to help you multiply (5 cis 20°)(5 cis 20°)(5 cis 20°).

   (5 · 5 · 5) cis (20° + 20° + 20°) = 125 cis 60°

7. Just as we can write (5)(5)(5) as 5^3, we can write (5 cis 20°)(5 cis 20°)(5 cis 20°) as $(5 \text{ cis } 20°)^3$. Using patterns you notice, explain how to find $(2 \text{ cis } 10°)^{17}$ without actually multiplying everything out.

   Find the modulus by raising 2^{17}, then multiply 10° · 17 to find the angle.

8. Explain why this process for finding powers of complex numbers in polar form makes sense based on the processes you wrote in #5 for adding and multiplying complex numbers.

   Instead of multiplying 17 2's, we can use exponents. Instead of adding 17 10's, we can use multiplication… OR exponentiation is repeated multiplication and multiplication is repeated addition.

9. Go back to your table in #5 and complete the last column: exponentiation.

   These cells have been completed in the table.

We know how to write radicals using exponent notation. For example, $\sqrt[4]{5} = 5^{1/4}$. We can therefore use what we summarized in #8 to find n^{th} roots of complex numbers:

10. Write $\sqrt[3]{3+4i}$ using exponent notation. Use the processes you summarized in #9 to find $\sqrt[3]{3+4i}$.

    $(3 + 4i)^{1/3} = (5 \text{ cis } 53.1°)^{1/3} = \sqrt[3]{5} \text{ cis } 17.7°$

11. Use your summary table from #5 to help you perform the following operations:

a. $(3 + 6i) + (4 – 5i)$

(3 + 4) + (6 – 5)i
7 + i

b. (6 cis 30°) – (4 cis 120°)

(6 cos 30° + 6i sin 30°) – (4 cos 120° + 4i sin 120°)
(5.20 + 3i) – (-2 + 3.46i)
(5.20 – -2) + (3 – 3.46)i
7.20 – 0.46i

c. (8 cis 135°)(10 cis 210°)

(8·10) cis (135° + 210°)
80 cis 345°

d. $(4 – 3i)(-7 – 24i)$

-28 – 96i + 21i + 72i²
-28 – 96i + 21i – 72
-100 – 75i

e. $\dfrac{10\,\text{cis}\,330°}{2\,\text{cis}\,135°}$

(10/2) cis (330° – 135°)
5 cis 195°

f. $\dfrac{6 + 8i}{3 - 4i}$

[(6 + 8i)(3 + 4i)] / [(3 – 4i)(3 + 4i)]
[18 + 24i + 24i + 32i²] / [9 + 12i – 12i – 16i²]
[-14 + 48i] / [25]
(-14/25) + (48/25)i

Multiplication, division, and exponentiation can require an extra step to make the argument between 0° and 360°.

12. Use your summary table from #5 to help you multiply (6 cis 150°)(3 cis 240°) quickly. Why is your answer not in typical polar form?

    (6·3) cis (150° + 240°) = 18 cis 390°
    Polar form usually has angles between 0° and 360°; this angle is larger than 360°.

13. Adjust your answer to make the argument an angle between 0° and 360°. Explain your process.

    390° – 360° = 30° (rotate by 360° to find angle in standard position OR find a coterminal angle.)
    18 cis 30°

14. Use your summary table from #5 to divide $\frac{6\,\text{cis}\,150°}{3\,\text{cis}\,240°}$.
    Why is your answer not in typical polar form?

    (6/3) cis (150° – 240°) = 2 cis (-90°)
    Polar form usually has angles between 0° and 360°; this angle is less than 0°.

15. Adjust your answer to make the argument an angle between 0° and 360°. Explain your process.

    -90° + 360° = 270° (rotate by 360° to find angle in standard position)
    2 cis 270°

16. Use your summary table from #5 to find $(2 \text{ cis } 120°)^{14}$.
    Why is your answer not in typical polar form?

    (2^{14}) cis (120° · 14) = 16384 cis 1680°
    Polar form usually has angles between 0° and 360°; this angle is greater than 360°.

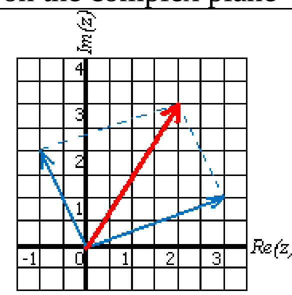
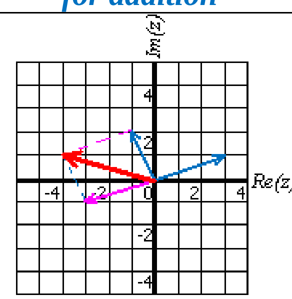
17. Adjust your answer to make the argument an angle between 0° and 360°. Explain your process.

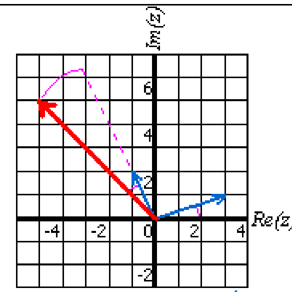
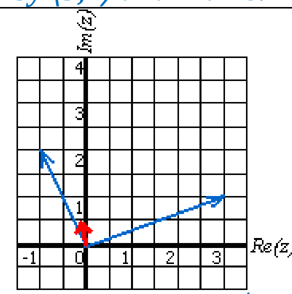
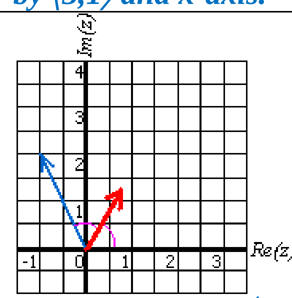
1680° – 4 · 360° = 240° (rotate four full rotations to find angle in standard position)
16384 cis 240°

18. In your table from #5, write the additional step you had to perform and when it would be necessary. This will make your summary tables complete.

This has been done on the table in red.

19. Use your summary tables to represent each of the following problems and their answers on the complex plane:

question	show your work here	answer	on the complex plane
parts in standard form: (-1 + 2*i*) + (3 + *i*) parts in polar form: ***($\sqrt{5}$ cis 116.6°) + ($\sqrt{10}$ cis 18.4°)***	***(-1 + 3) + (2 + 1)i*** ***2 + 3i*** ***mag: $\sqrt{(2^2 + 3^2)} = \sqrt{13}$*** ***arg: arctan(3/2) = 56.3°***	in standard form: ***2 + 3i*** in polar form: ***$\sqrt{13}$ cis 56.3°***	 ***parallelogram rule or end-to-tail rule for addition***
parts in standard form: (-1 + 2*i*) – (3 + *i*) parts in polar form: ***($\sqrt{5}$ cis 116.6°) – ($\sqrt{10}$ cis 18.4°)***	***(-1 – 3) + (2 – 1)i*** ***-4 + i*** ***mag: $\sqrt{(4^2 + 1^2)} = \sqrt{17}$*** ***arg: 180° – arctan(1/4)*** ***166.0°***	in standard form: ***-4 + i*** in polar form: ***$\sqrt{17}$ cis 166.0°***	 ***parallelogram rule or end-to-tail rule for subtraction***

parts in standard form: $(-1 + 2i)(3 + i)$ parts in polar form: $(\sqrt{5}\text{ cis }116.6°)$ $(\sqrt{10}\text{ cis }18.4°)$	$-3 - i + 6i + 2i^2$ $-3 - i + 6i - 2$ $-5 + 5i$ $(\sqrt{5}\cdot\sqrt{10})\text{ cis }(116.6°+18.4°)$ $5\sqrt{2}\text{ cis }135.0°$ *or do one and convert*	in standard form: $-5 + 5i$ in polar form: $5\sqrt{2}\text{ cis }135.0°$	 *Scale ⟨-1,2⟩ by √10, then rotate counterclockwise by the angle formed by ⟨3,1⟩ and x-axis.*
parts in standard form: $(-1 + 2i) \div (3 + i)$ parts in polar form: $(\sqrt{5}\text{ cis }116.6°)$ $\div\ (\sqrt{10}\text{ cis }18.4°)$	$[(-1+2i)(3-i)]/[(3+i)(3-i)]$ $[-3+i+6i-2i^2]/[9-3i+3i-i^2]$ $[-1 + 7i] / [10]$ $(-1/10) + (7/10)i$ $(\sqrt{5} / \sqrt{10})\text{ cis}(116.6°-18.4°)$ $(1 / \sqrt{2})\text{ cis }98.2°$ $(\sqrt{2} / 2)\text{ cis }98.2°$ *or do one and convert*	in standard form: $(-1/10) + (7/10)i$ in polar form: $(\sqrt{2} / 2)\text{ cis }98.2°$	 *Scale ⟨-1,2⟩ by 1/√10, then rotate clockwise by the angle formed by ⟨3,1⟩ and x-axis.*
parts in standard form: $\sqrt{-1+2i}$ parts in polar form: $(\sqrt{5}\text{cis}116.6°)^{1/2}$	$(\sqrt{5})^{1/2}\text{ cis }(116.6° / 2)$ $5^{1/4}\text{ cis }58.3°$ or $1.5\text{ cis }58.3°$ $5^{1/4}\cos 58.3° = 0.79$ $5^{1/4}\sin 58.3° = 1.27$	in standard form: $0.79 + 1.27i$ in polar form: $1.5\text{ cis }58.3°$	 *New magnitude = √old Rotate halfway back to positive x-axis. (bisect angle)*

Complex Operations

Mathematical Goals:

- Use rectangular and polar coordinates to represent complex numbers.
- Discuss advantages and disadvantages of each form for operations on complex numbers.
- Add, subtract, multiply, and divide complex numbers using geometric representations.
- Find powers and roots of complex numbers and represent this geometrically.

GSE Standards:

- **MGSE9-12.N.CN.5** Represent addition, subtraction, multiplication, and conjugation of complex numbers geometrically on the complex plane; use properties of this representation for computation.

Standards for Mathematical Practice:

- Reason abstractly and quantitatively.
- Construct viable arguments and critique the reasoning of others.
- Attend to precision.
- Look for and make use of structure.
- Look for and express regularity in repeated reasoning.

Complex Operations

With vectors, each form (component form and magnitude-direction form) had advantages and disadvantages for operations such as addition, subtraction, and scalar multiplication. The same is true for complex numbers. Let's see what the advantages and disadvantages of each form are.

1. First, find the modulus and argument (magnitude and direction) of each of the following complex numbers, then write each in polar form. We will be using these complex numbers throughout the task, so be careful! Round angles to the nearest whole number.

	$3 + 4i$	$2i$	$-5 - 12i$	$4 - 4i$
modulus				
argument				
polar form				

2. Use what you know about complex numbers to simplify each of these **addition and subtraction** expressions in standard form. Complete the table. The first one has been done for you.

expression in standard form	$(3 + 4i) + (2i)$	$(3 + 4i) + (-5 - 12i)$	$(-5 - 12i) - (4 - 4i)$
answer in standard form $a + bi$	$3 + 6i$		
modulus of answer	$3\sqrt{5}$		
argument of answer	$63°$		
answer in polar form $r \text{ cis } \theta$	$3\sqrt{5} \text{ cis } 63°$		
original in polar form	$5 \text{ cis } 53° + 2 \text{ cis } 90°$		

3. Use what you know about complex numbers to simplify each of these **multiplication** expressions in standard form. Complete the table. The first one has been done for you.

expression in standard form	$(3 + 4i)(2i)$	$(3 + 4i)(-5 - 12i)$	$(-5 - 12i)(4 - 4i)$
answer in standard form $a + bi$	$-8 + 6i$		
modulus of answer	10		
argument of answer	143°		
answer in polar form r cis θ	10 cis 143°		
original in polar form	(5 cis 53°)(2 cis 90°)		

4. Use what you know about complex numbers to simplify each of these **division** expressions in standard form. Complete the table. The first one has been done for you.

expression in standard form	$\frac{2i}{3 + 4i}$	$\frac{-5 - 12i}{3 + 4i}$
answer in standard form $a + bi$	$\frac{8}{25} + \frac{6}{25}i$	
modulus of answer	$\frac{2}{5}$	
argument of answer	37°	
answer in polar form r cis θ	$\frac{2}{5}$ cis 37°	
original in polar form	$\frac{2\operatorname{cis}90°}{5\operatorname{cis}53°}$	

As we saw with vectors, there are advantages and disadvantages to each form—component and magnitude-direction form. The same is true with complex numbers: some operations are easy to perform in standard (rectangular) form, whereas others are easy in polar form.

5. Investigate your tables from #2, #3, and #4, and see if you can find quick methods in either (or both) forms to perform each operation. Write instructions in each box explaining how to perform each operation. The first column has been completed for you.

	addition (#2)	subtraction (#2)
standard form $a + bi$	Easy! :) Add real parts. Add imaginary parts.	
polar form r cis θ	Convert to standard form first, then follow standard form instructions.	

	multiplication (#3)	division (#4)
standard form $a + bi$		
polar form r cis θ		

	exponentiation (#6-10)
standard form $a + bi$	
polar form r cis θ	

← We will complete this part of the table after the next part of our task.

Now let's take a look at how we can find powers of complex numbers.

6. Use your summary table from #5 to help you multiply (5 cis 20°)(5 cis 20°)(5 cis 20°).

7. Just as we can write (5)(5)(5) as 5^3, we can write (5 cis 20°)(5 cis 20°)(5 cis 20°) as $(5 \text{ cis } 20°)^3$. Using patterns you notice, explain how to find $(2 \text{ cis } 10°)^{17}$ without actually multiplying everything out.

8. Explain why this process for finding powers of complex numbers in polar form makes sense based on the processes you wrote in #5 for adding and multiplying complex numbers.

9. Go back to your table in #5 and complete the last column: exponentiation.

We know how to write radicals using exponent notation. For example, $\sqrt[4]{5} = 5^{1/4}$. We can therefore use what we summarized in #8 to find n^{th} roots of complex numbers:

10. Write $\sqrt[3]{3+4i}$ using exponent notation. Use the processes you summarized in #9 to find $\sqrt[3]{3+4i}$.

11. Use your summary table from #5 to help you perform the following operations:

    a. $(3 + 6i) + (4 - 5i)$

    b. (6 cis 30°) – (4 cis 120°)

c. (8 cis 135°)(10 cis 210°)

d. $(4 - 3i)(-7 - 24i)$

e. $\dfrac{10\,\text{cis}\,330^\circ}{2\,\text{cis}\,135^\circ}$

f. $\dfrac{6+8i}{3-4i}$

Multiplication, division, and exponentiation can require an extra step to make the argument between 0° and 360°.

12. Use your summary table from #5 to help you multiply (6 cis 150°)(3 cis 240°) quickly. Why is your answer not in typical polar form?

13. Adjust your answer to make the argument an angle between 0° and 360°. Explain your process.

14. Use your summary table from #5 to divide $\dfrac{6\,\text{cis}\,150^\circ}{3\,\text{cis}\,240^\circ}$.
Why is your answer not in typical polar form?

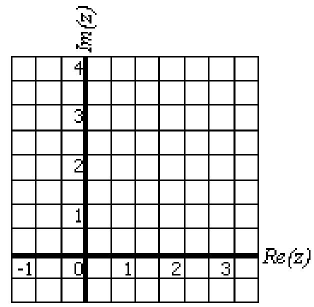
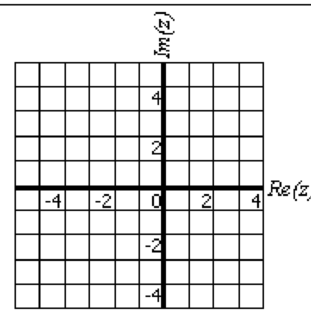
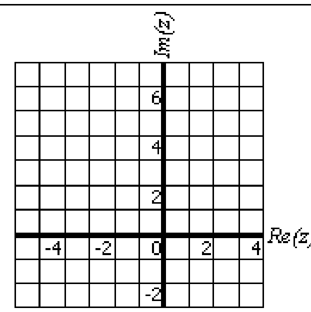
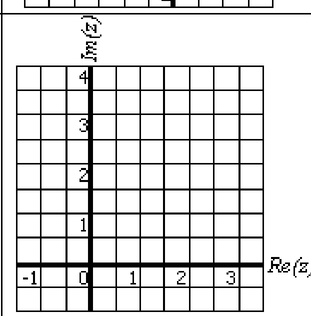
15. Adjust your answer to make the argument an angle between 0° and 360°.
Explain your process.

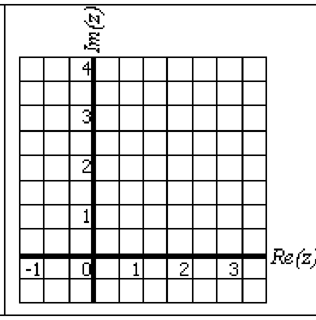
16. Use your summary table from #5 to find $(2 \text{ cis } 120°)^{14}$.
Why is your answer not in typical polar form?

17. Adjust your answer to make the argument an angle between 0° and 360°.
Explain your process.

18. In your table from #5, write the additional step you had to perform and when it would be necessary. This will make your summary tables complete.

19. Use your summary tables to represent each of the following problems and their answers on the complex plane:

question	show your work here	answer	on the complex plane
parts in standard form: $(-1 + 2i) + (3 + i)$ parts in polar form:		in standard form: in polar form:	
parts in standard form: $(-1 + 2i) – (3 + i)$ parts in polar form:		in standard form: in polar form:	
parts in standard form: $(-1 + 2i)(3 + i)$ parts in polar form:		in standard form: in polar form:	
parts in standard form: $(-1 + 2i) \div (3 + i)$ parts in polar form:		in standard form: in polar form:	

parts in standard form: $\sqrt{-1+2i}$ parts in polar form:		in standard form: in polar form:	

How Far and Halfway in Hogsmeade

Back to Task Table

Mathematical Goals:
- Find the distance between two complex numbers.
- Recognize that the distance between two complex numbers is the modulus of their difference.
- Find the average of two complex numbers.
- Recognize that the average of two complex numbers is the midpoint of a segment in the complex plane with those numbers as endpoints.

GSE Standards:
- **MGSE9-12.N.CN.6** Calculate the distance between numbers in the complex plane as the modulus of the difference, and the midpoint of a segment as the average of the numbers at its endpoints.

Standards for Mathematical Practice:
- Make sense of problems and persevere in solving them.
- Reason abstractly and quantitatively.
- Model with mathematics.
- Attend to precision.
- Look for and make use of structure.

Introduction:

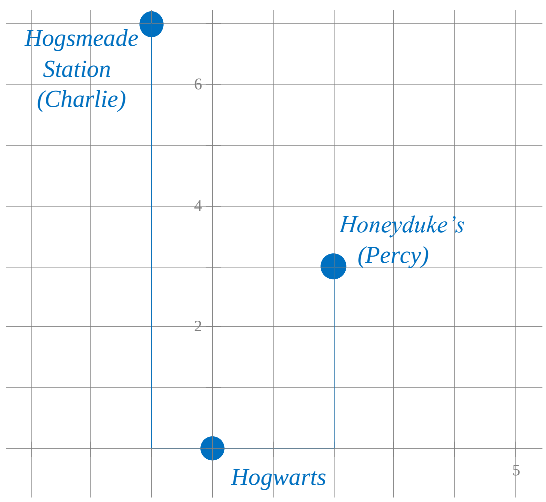
In this task, students explore distance and midpoint in the complex plane, relating these quantities to the distance between two complex numbers and the average of two complex numbers, respectively.

How Far and Halfway in Hogsmeade

We have seen that complex numbers can be represented in the two-dimensional complex plane, and that they have two-dimensional analogs for many of the characteristics of real numbers, such as absolute value. Let's continue looking for these connections, relating it back to our work with vectors.

From Hogwarts, Percy walks two blocks East and three blocks North to get to Honeyduke's Sweetshop. Also starting from Hogwarts, his brother Charlie walks one block West and seven blocks North to get to Hogsmeade Station.

1. Draw Percy and Charlie's paths on graph paper, with Hogwarts at the origin.



2. Represent Percy and Charlie's paths as vectors. Is it easier to write them in component form or in magnitude-direction form? Why?

Percy: ⟨2,3⟩ Charlie: ⟨-1,7⟩
Component form is easier because we were given horizontal and vertical components of their paths.

3. If Percy wants to join Charlie at Hogsmeade Station, what path should he take? Represent you answer in words and as a vector.

Words: ***West 3 blocks, North 4 blocks***
Vector: ***⟨-3, 4⟩***

4. Write an equation involving vectors to represent the process you used to answer #3.

⟨-1, 7⟩ – ⟨2, 3⟩ = ⟨-3, 4⟩

5. How far apart are Honeyduke's and Hogsmeade Station as the crow (or owl) flies?

$\sqrt{(3^2 + 4^2)} = \sqrt{25} = 5$ blocks

A similar method can be used to determine the distance between two complex numbers on the complex number plane.

6. Determine the distance between $2 + 3i$ and $-1 + 7i$.

   a. How far are they apart are the numbers in component form?

   $(-1 + 7i) - (2 + 3i) = -3 + 4i$

   b. What is the modulus of this complex number?

$$\sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

   c. How many units are they apart?

   5 blocks

7. Use a process similar to the process you used in #3-6 to determine the distance between $5 - 3i$, and $-2 - i$ on this complex number plane. Describe your steps.

   How far apart are they in component form? Subtract:
   $(5 - 3i) - (-2 - i) = (5 - -2) + (-3 - -1)i = 7 - 2i$

   What is the modulus of this complex number? Pythagorean Theorem:
   $\sqrt{(7^2 + 2^2)} = \sqrt{53}$

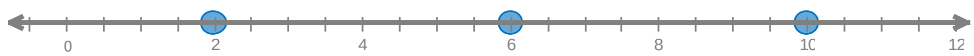
   They are $\sqrt{53}$ units apart.

   OR 5 – 3i can be represented by the vector <5, -3> and -2 – i can be represented by <-2, -1>. Subtracting these vectors yields a difference of <7, -2> and the length of this vector is $\sqrt{(7^2 + 2^2)} = \sqrt{53}$

   <u>Note:</u> The most efficient way to find the distance is to realize that 5 – 3i on the complex number plane is analogous to(5, -3) on the Cartesian number plane and -2 – i on the complex number plane is analogous to (-2, -1). The distance between these 2 points is $\sqrt{(7^2 + 2^2)} = \sqrt{53}$.

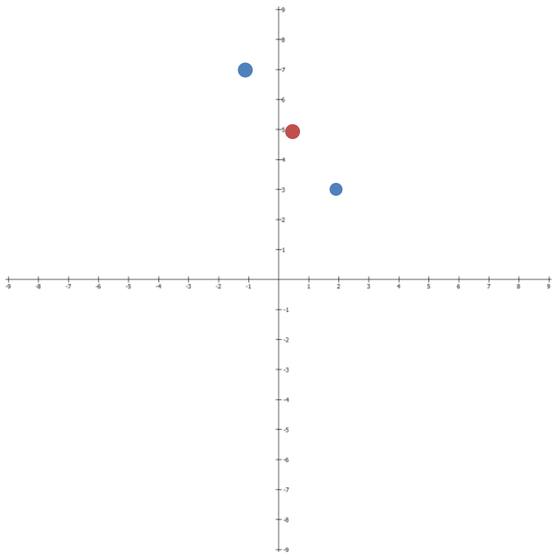
So far, we have found the **distance** between two complex numbers in the complex plane. We can also find the **average** of two complex numbers.

8. On a one-dimensional number line, plot the numbers 2, 10, and their average. What do you notice about the location of the average of the two numbers?



The average, 6, lies halfway between 2 and 10.

9. On the complex plane, plot the points $2 + 3i$ and $-1 + 7i$.



10. We know that we can find the average of two *real* numbers by adding them and dividing by 2. Apply the same process to the complex numbers from #9 to find their average.

Add them: ***(2 + 3i) + (-1 + 7i) = (2 + -1) + (3 + 7)i = 1 + 10i***
Divide by 2: ***(1 + 10i)/2 = (1/2) + (10/2)i = 0.5 + 5i***

11. Plot the average on your graph. What do you notice?

Point is plotted in red on the graph in #9 This point appears to be halfway between the two given points.

12. Is your point exactly halfway between the points $2 + 3i$ and $-1 + 7i$.? How can you tell?

Comment:
The points have been chosen specifically so the midpoint does not have integer coordinates, so students have to think more about how to "count" distances.

Solution:
Methods will vary, but most students will count horizontal and vertical distances to check that the components are the same. The given points are each 1.5 units from the midpoint horizontally, and 2 units from the midpoint vertically.

13. Use a process similar to the process you used in #10 to find a point halfway between $5 – 3i$, and $-2 – i$. Describe your steps.

Average them: $[(5 – 3i) + (-2 – i)] / 2 = [(5 + -2) + (-3 + -1)i] / 2 = [3 – 4i] / 2$
$= (3/2) + (-4/2)i$
$= 1.5 – 2i$

14. Find the distance between $5 – 3i$ and the midpoint from #13, and the distance between $-2 – i$ and the midpoint from #13. What do you notice? Discuss the modulus and argument of any complex numbers you find.

Comment:
Students should see a) both distances are the same, b) the complex numbers are opposites—180° apart, and c) the two distances (moduli) are half of the total distance found in #7. a) and b) together, or c) on its own, shows us that the average is equidistant from the two points and that it is on the segment connecting the two numbers. There are infinitely many points that are equidistant from both points (any point on the perpendicular bisector of the segment with those endpoints), but this particular point is also on the segment connecting the points. This is the definition of "between," so we can say the average is halfway between the two points, not just that it is equidistant from the two points.

Solution:
$5 – 3i$ to midpoint:
Subtract: $(5 – 3i) – (1.5 – 2i) = (5 – 1.5) + (-3 – 2)i = 3.5 – i$
Modulus: $\sqrt{3.5^2 + 1^2} = \sqrt{13.25} = \sqrt{53 / 4} = \sqrt{53} / \sqrt{4} = \sqrt{53} / 2$
Argument: $360° – \arctan(1 / 3.5) = 344.05°$

$-2 – i$ to midpoint:
Subtract: $(-2 – i) – (1.5 – 2i) = (-2 – 1.5) + (-1 – -2)i = -3.5 + i$
Modulus: $\sqrt{3.5^2 + 1^2} = \sqrt{13.25} = \sqrt{53 / 4} = \sqrt{53} / \sqrt{4} = \sqrt{53} / 2$
Argument: $180° – \arctan(1 / 3.5) = 164.05°$

How Far and Halfway in Hogsmeade

Mathematical Goals:

- Find the distance between two complex numbers.
- Recognize that the distance between two complex numbers is the modulus of their difference.
- Find the average of two complex numbers.
- Recognize that the average of two complex numbers is the midpoint of a segment in the complex plane with those numbers as endpoints.

GSE Standards:

- **MGSE9-12.N.CN.6** Calculate the distance between numbers in the complex plane as the modulus of the difference, and the midpoint of a segment as the average of the numbers at its endpoints.

Standards for Mathematical Practice:

- Make sense of problems and persevere in solving them.
- Reason abstractly and quantitatively.
- Model with mathematics.
- Attend to precision.
- Look for and make use of structure.

How Far and Halfway in Hogsmeade

We have seen that complex numbers can be represented in the two-dimensional complex plane, and that they have two-dimensional analogs for many of the characteristics of real numbers, such as absolute value. Let's continue looking for these connections, relating it back to our work with vectors.

From Hogwarts, Percy walks two blocks East and three blocks North to get to Honeyduke's Sweetshop. Also starting from Hogwarts, his brother Charlie walks one block West and seven blocks North to get to Hogsmeade Station.

1. Draw Percy and Charlie's paths on graph paper, with Hogwarts at the origin.

2. Represent Percy and Charlie's paths as vectors. Is it easier to write them in component form or in magnitude-direction form? Why?

3. If Percy wants to join Charlie at Hogsmeade Station, what path should he take? Represent you answer in words and as a vector.

4. Write an equation involving vectors to represent the process you used to answer #3.

5. How far apart are Honeyduke's and Hogsmeade Station as the crow (or owl) flies?

A similar method can be used to determine the distance between two complex numbers on the complex number plane.

6. Determine the distance between $2 + 3i$ and $-1 + 7i$.

   a. How far are they apart are the numbers in component form?

   b. What is the modulus of this complex number?

   c. How many units are they apart?

7. Use a process similar to the process you used in #3-6 to determine the distance between $5 – 3i$, and $-2 – i$ on this complex number plane. Describe your steps.

So far, we have found the **distance** between two complex numbers in the complex plane. We can also find the **average** of two complex numbers.

8. On a one-dimensional number line, plot the numbers 2, 10, and their average. What do you notice about the location of the average of the two numbers?

9. On the complex plane, plot the points $2 + 3i$ and $-1 + 7i$.

10. We know that we can find the average of two *real* numbers by adding them and dividing by 2. Apply the same process to the complex numbers from #9 to find their average.

11. Plot the average on your graph. What do you notice?

12. Is your point exactly halfway between the points $2 + 3i$ and $-1 + 7i$.? How can you tell?

13. Use a process similar to the process you used in #10 to find a point halfway between $5 – 3i$, and $-2 – i$. Describe your steps.

14. Find the distance between $5 – 3i$ and the midpoint from #13, and the distance between $-2 – i$ and the midpoint from #13. What do you notice? Discuss the modulus and argument of any complex numbers you find.

Culminating Task – Putting It All Together

Back to Task Table

Teacher Commentary:

This culminating task assesses all standards from this unit. Connections between complex numbers and vectors are drawn by putting both topics together in the "map" set of problems (beginning with #1). Students must understand graphical representations of operations on complex numbers—which are virtually identical to graphical representations of operations on vectors—in order to complete problems #12-15.

Major standards addressed by each question are listed in blue before the solution.

Mathematical Goals:

- This task assesses students' ability to apply everything they have learned in this unit.

GSE Standards:

- All standards from the unit.

Standards for Mathematical Practice:

- Make sense of problems and persevere in solving them.
- Reason abstractly and quantitatively.
- Construct viable arguments and critique the reasoning of others.
- Model with mathematics.
- Use appropriate tools strategically.
- Attend to precision.
- Look for and make use of structure.

Culminating Task – Putting It All Together

From school, four friends—Aliyah, Ben, Charlie, and Diane—recorded the paths they take to get home. Each student recorded his or her pathway in a different form, as shown in the table below. Aliyah and Ben represented their paths in the complex plane, but Charlie and Diane wrote their paths as vectors in the *x*-*y* plane. Fortunately, they did do some things the same:

- Each unit represents one kilometer.
- Right represents East, up represents North, etc.
- Complex numbers are represented as lowercase italicized letters: *a*.
- Vectors are represented as italicized, bold, uppercase letters: ***A***.
  Note: Vectors are not normally denoted with capital letters. This is done for this task in order to differentiate between complex numbers and vectors.

Throughout this task, you should:

- Give square roots in simplest radical form.
- Round decimals to the nearest hundredth when exact answers are impossible.

1. Complete the table below by converting each person's path into the three other forms.

Standards:
(strikethroughs indicate part of the standard not addressed in this question)

MGSE9-12.N.CN.4 *Represent complex numbers on the complex plane in rectangular and polar form (including real and imaginary numbers), and explain why the rectangular and polar forms of a given complex number represent the same number.*

MGSE9-12.N.VM.1 *Recognize vector quantities as having both magnitude and direction. Represent vector quantities by directed line segments,* ~~*and use appropriate symbols for vectors and their magnitudes (e.g.,* **v***,* |**v**|*,* ||**v**||*, v).*~~

Solution:

	complex number		vector	
	rectangular form	polar form	component form	mag.-dir. form
Aliyah	a = -5 + 2i	$\sqrt{(5^2 + 2^2)} = \sqrt{29}$ 180° – arctan(2/5) ≈ 158.20° $\sqrt{29}$ cis 158.20°	⟨-5, 2⟩	mag.: $\sqrt{29}$ dir.: 158.20°
Ben	4 cos 90° = 0 4 sin 90° = 4 0 + 4i or 4i	b = 4 cis 90°	⟨0, 4⟩	mag.: 4 dir.: 90°
Charlie	3 – 3i	$\sqrt{(3^2+3^2)}=\sqrt{18}=3\sqrt{2}$ 360° – arctan(3/3) = 315° $3\sqrt{2}$ cis 315°	C = ⟨3, -3⟩	mag.: $3\sqrt{2}$ dir.: 315°
Diane	3 + 5.20i	6 cis 60°	6 cos 60° = 3 6 sin 60° ≈ 5.20 ⟨3, 5.20⟩	***D:*** mag.: 6 dir.: 60°

2. Who lives farthest from the school? Justify your answer.

Standards:
(strikethroughs indicate part of the standard not addressed in this question)

MGSE9-12.N.VM.2 *Find the components of a vector by subtracting the coordinates of an initial point from the coordinates of a terminal point.*

MGSE9-12.N.CN.6 *Calculate the distance between numbers in the complex plane as the modulus of the difference~~, and the midpoint of a segment as the average of the numbers at its endpoints.~~*

MGSE9-12.N.CN.3 Find the conjugate of a complex number; use the conjugate to find the absolute value (modulus) and quotient of complex numbers.

Solution:

Diane. The magnitude of her vector (modulus of her complex number)—$\sqrt{36}$—is larger than the magnitudes of the others: $\sqrt{16}$, $\sqrt{18}$, and $\sqrt{29}$.

3. Charlie walks home from school, then he walks 2 kilometers East and 1 kilometer South to the gym. Write an **equation with vectors** that shows how to determine where the gym is located (with respect to school), then write your answer as a sentence.

Standards:

MGSE9-12.N.VM.4 Add and subtract vectors.

Solution:

$\langle 3, -3\rangle + \langle 2, -1\rangle = \langle 3 + 2, -3 + -1\rangle = \langle 5, -4\rangle$
The gym is 5 km East and 4 km South of the school.

4. If Aliyah wants to get to the gym from her house, what path should she take? Write an **equation with complex numbers** that shows how to get your answer, then write your answer as a sentence.

Standards:
(strikethroughs indicate part of the standard not addressed in this question)

MGSE9-12.N.CN.5 *Represent addition, ~~subtraction, multiplication, and conjugation of complex numbers geometrically on the complex plane~~; use properties of this representation for computation.*

Solution:

$(5 - 4i) - (-5 + 2i) = (5 - -5) + (-4 - 2)i = 10 - 6i$
Aliyah should walk 10 km East and 6 km South from her home.

5. What does $|a - c|$ represent in the context of the problem? Find $|a - c|$.

Standards:
(strikethroughs indicate part of the standard not addressed in this question)

MGSE9-12.N.CN.6 *Calculate the distance between numbers in the complex plane as the modulus of the difference, ~~and the midpoint of a segment as the average of the numbers at its endpoints.~~*

Solution:

This represents the distance between Aliyah and Charlie's house.
$|a - c| = |(-5 + 2i) - (3 - 3i)| = |(-5 - 3) + (2 - -3)i| = |-8 + 5i| = \sqrt{8^2 + 5^2} = \sqrt{89}$ km

6. Write a one-sentence "story" that could be represented by the vector ½ ***D***, then find ½ ***D*** and its magnitude.

Standards:
(strikethroughs indicate part of the standard not addressed in this question)

MGSE9-12.N.VM.5 *Multiply a vector by a scalar.*

MGSE9-12.N.VM.5a *Represent scalar multiplication graphically by scaling vectors and possibly reversing their direction; perform scalar multiplication component-wise, e.g., as* $c(\mathbf{v}_x, \mathbf{v}_y) = (c\cdot\mathbf{v}_x, c\cdot\mathbf{v}_y)$.

MGSE9-12.N.VM.5b *Compute the magnitude of a scalar multiple c**v** using* $||c\mathbf{v}|| = |c|\mathbf{v}$. ~~*Compute the direction of c**v** knowing that when |c|**v** ≠ 0, the direction of c**v** is either along **v** (for c > 0) or against **v** (for c < 0).*~~

Solution:

Answers will vary. Two examples are below.

Diane gets tired halfway into her walk home and decides to rest. If she has been walking in a straight line from school to home, where is she resting?

Eli wants to meet Diane at a point halfway between school and home. What path should Eli take to get to the meeting spot?

½D = the vector with magnitude ½(6) = 3 and direction 60° OR the vector ⟨1.5, 2.60⟩

7. Find the average of the complex numbers *a* and *c*. What does this represent in the context of the problem?

Standards:
(strikethroughs indicate part of the standard not addressed in this question)

MGSE9-12.N.CN.6 *Calculate* ~~*the distance between numbers in the complex plane as the modulus of the difference, and*~~ *the midpoint of a segment as the average of the numbers at its endpoints.*

Solution:

[(-5 + 2i) + (3 – 3i)] / 2 = [(-5 + 3) + (2 + -3)i] / 2 = [-2 – i] / 2 = (-2/2) + (-1/2)i = -1 + 0.5i
This represents a point that is halfway between Aliyah and Charlie's houses.

8. Eric lives at $\bar{b}$. What does this mean, and where does Eric live?

Standards:
(strikethroughs indicate part of the standard not addressed in this question)

MGSE9-12.N.CN.3 Find the conjugate of a complex number; use the conjugate to find the absolute value (modulus) and quotient of complex numbers.

MGSE9-12.N.CN.5 *Represent ~~addition, subtraction, multiplication, and~~ conjugation of complex numbers geometrically on the complex plane; ~~use properties of this representation for computation.~~*

Solution:

This is the conjugate of b. Thus, the real (horizontal) component of his location is the same as Ben's, but the imaginary (vertical) component is the opposite of Ben's. Eric lives at -4i. That is, Ben lives 4 km due South of school.

9. If all the vectors were multiplied by the transformation matrix $\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$, what effect would this have on the "map" of the town?

Standards:

MGSE9-12.N.VM.11 *Multiply a vector (regarded as a matrix with one column) by a matrix of suitable dimensions to produce another vector. Work with matrices as transformations of vectors.*

Solution:

This would rotate all of the points 90° clockwise and scale everything by a factor of 2. In other words, the new argument will be 90° more than the old argument, and the new modulus will be twice as large as the old argument.

Ben and Diane meet up to fly a model airplane they have built together. At full power, the airplane can fly 160 kilometers per hour in calm air. Ben has the controls, and he makes the plane take off heading 25° North of East. After he feels comfortable with the controls, he turns on full power.

9. A steady wind begins to blow from North to South at a speed of 32 kilometers per hour. In what direction and at what speed is the plane traveling now?

Standards:
(strikethroughs indicate part of the standard not addressed in this question)

MGSE9-12.N.VM.3 *Solve problems involving velocity and other quantities that can be represented by vectors.*

MGSE9-12.N.VM.4 *Add and subtract vectors.*

MGSE9-12.N.VM.4b *Given two vectors in magnitude and direction form, determine the magnitude and direction of their sum.*

MGSE9-12.N.VM.4a *Add vectors ~~end-to-end,~~ component-wise, ~~and by the parallelogram rule. Understand that the magnitude of a sum of two vectors is typically not the sum of the magnitudes.~~*

Solution:

mag	dir	x	y
160 km/hr	25°	160cos25° ≈ 145.01	160sin25° ≈ 67.62
32 km/hr	270°	32cos270° = 0	32sin270° = -32
	SUM:	145.01 km/hr	35.62 km/hr

speed: $\sqrt{(145.01^2 + 35.62^2)} \approx 149.32$ km/hr
direction: $\arctan(35.62/145.01) \approx 13.80°$

10. Diane starts to get frustrated that the plane is off-course, so she takes the controls. She wants to get the plane to fly at an angle of 25° North of East, with a resultant groundspeed of 140 kilometers per hour. She knows she must "aim" the airplane at an angle that compensates for wind. At what angle should she direct the airplane to fly?

Standards:
(strikethroughs indicate part of the standard not addressed in this question)

MGSE9-12.N.VM.3 *Solve problems involving velocity and other quantities that can be represented by vectors.*

MGSE9-12.N.VM.4 *Add and subtract vectors.*

MGSE9-12.N.VM.4b *Given two vectors in magnitude and direction form, determine the magnitude and direction of their sum.*

MGSE9-12.N.VM.4c *Understand vector subtraction **v – w** as **v + (–w)**, where **(–w)** is the additive inverse of **w**, with the same magnitude as **w** and pointing in the opposite direction. ~~Represent vector subtraction graphically by connecting the tips in the appropriate order, and~~ perform vector subtraction component-wise.*

Solution:

We want to figure out d so that d + 32 cis 270° = 140 cis 25°.
Thus, d = 140 cis 25° – 32 cis 270°.

mag	dir	x	y
140 km/hr	25°	140cos25° ≈ 126.88	140sin25° ≈ 59.17
32 km/hr	270°	32cos270° = 0	32sin270° = -32
DIFFERENCE:		126.88 km/hr	91.17 km/hr

speed: $\sqrt{(126.88^2 + 91.17^2)} \approx 156.23$ ***km/hr***
direction: $\arctan(91.17/126.88) \approx 54.30°$

OR $\langle x, y\rangle + \langle 0, -32\rangle = \langle 140\cdot\cos 25°, 140\cdot\sin 25°\rangle \rightarrow \langle x, y\rangle = \langle 126.88, 91.17\rangle$.
Therefore: speed: $\sqrt{(126.88^2 + 91.17^2)} \approx 156.23$ ***km/hr***
direction: $\arctan(91.17/126.88) \approx 54.30°$

11. Based on the limitations of the power of Diane's plane, is her plan from #10 possible? Explain your answer.

Yes. Maximum power will get the plane to travel 160 km/hr; we just need it to go 156.23 km/hr (in still air).

12. *p* and *q* represent complex numbers, as shown below. Given the description of four new numbers—*r*, *s*, *t*, and *u*—use what you know about complex operations to plot those numbers on the complex plane. Show any segments, angles, etc., you use to find your answer. Clearly identify your final answer.

Standards:
(strikethroughs indicate part of the standard not addressed in this question)

MGSE9-12.N.CN.5 *Represent addition, subtraction, ~~multiplication, and conjugation~~ of complex numbers geometrically on the complex plane; use properties of this representation for computation.*

MGSE9-12.N.CN.6 *Calculate ~~the distance between numbers in the complex plane as the modulus of the difference, and~~ the midpoint of a segment as the average of the numbers at its endpoints.*

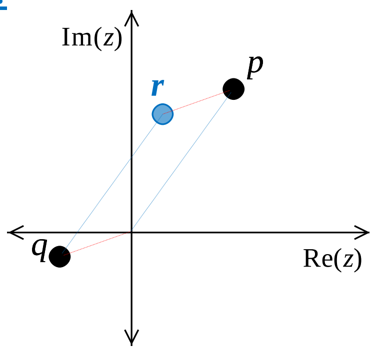
MGSE9-12.N.VM.4a *Add vectors end-to-end, component-wise, and by the parallelogram rule. ~~Understand that the magnitude of a sum of two vectors is typically not the sum of the magnitudes.~~ [applied to the complex plane]*

MGSE9-12.N.VM.4c *Understand vector subtraction* $\mathbf{v} - \mathbf{w}$ *as* $\mathbf{v} + (-\mathbf{w})$*, where* $(-\mathbf{w})$ *is the additive inverse of* $\mathbf{w}$*, with the same magnitude as* $\mathbf{w}$ *and pointing in the opposite direction. Represent vector subtraction graphically by connecting the tips in the appropriate order, ~~and perform vector subtraction component-wise.~~*

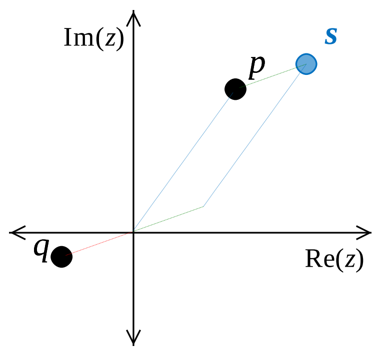
MGSE9-12.N.VM.5a *Represent scalar multiplication graphically by scaling vectors and possibly reversing their direction; ~~perform scalar multiplication component-wise, e.g., as~~* $\cancel{c(v_x, v_y) = (cv_x, cv_y)}$.

MGSE9-12.N.VM.5b ~~*Compute the magnitude of a scalar multiple* $c\mathbf{v}$ *using* $||c\mathbf{v}|| = |c|\mathbf{v}$.~~ *Compute the direction of* $c\mathbf{v}$ *knowing that when* $|c|\mathbf{v} \neq 0$*, the direction of* $c\mathbf{v}$ *is either along* $\mathbf{v}$ *(for* $c > 0$*) or against* $\mathbf{v}$ *(for* $c < 0$*).*

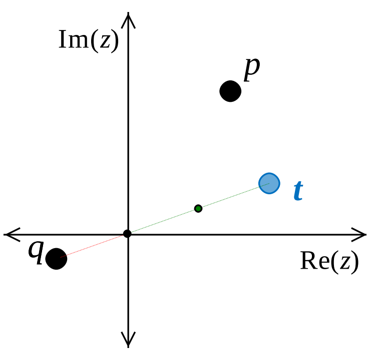
<u>Solution:</u>



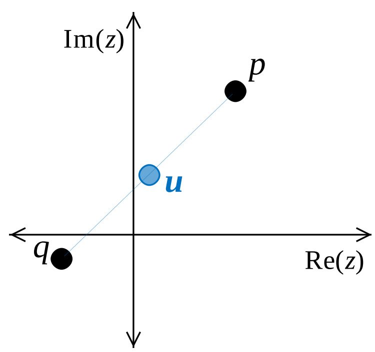
Use parallelogram rule or tail-to-end process for addition.
$r = p + q$



Parallelogram or tail-to-end process for subtraction. (Flip q vector first)
$s = p - q$



Rotate q vector by 180°. Scale by factor of 2.
$t = -2q$



Midpoint of segment that has p and q as endpoints.
u is the average of

p and *q*

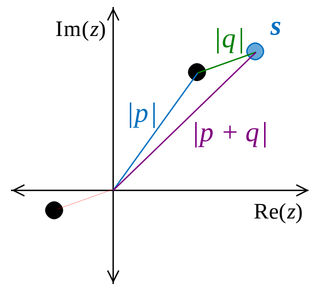
13. Is $|p + q|$ greater than, less than, or equal to $|p| + |q|$? Justify your answer.

Standards:

MGSE9-12.N.VM.4a *Add vectors end-to-end, component-wise, and by the parallelogram rule. Understand that the magnitude of a sum of two vectors is typically not the sum of the magnitudes. [applied to moduli of complex numbers]*

Solution:

Less than. Since p and q have different angles this must be true. (A triangle is formed, so the sum of the lengths of the sides marked $|p|$ and $|q|$ must be greater than the length of the side marked $|p + q|$ by the triangle inequality).



14. If $|p| > 1$ and $|q| < 1$, will $|pq|$ be greater than, less than, or equal to $|p|$? Justify your answer.

Standards:
(strikethroughs indicate part of the standard not addressed in this question)

MGSE9-12.N.CN.5 *Represent ~~addition, subtraction,~~ multiplication, ~~and conjugation~~ of complex numbers geometrically on the complex plane; use properties of this representation for computation.*

MGSE9-12.N.VM.1 *Recognize vector quantities as having both magnitude and direction. Represent vector quantities by directed line segments, and use appropriate symbols for vectors and their magnitudes (e.g.,* **v**, |**v**|, ||**v**||, *v).*

Solution:

Less than. $|pq| = |p||q|$ for all complex numbers. Since $|q| < 1$, multiplying $|p|$ by $|q|$ makes it smaller.

15. If $\arg(p) = 61°$ and $\arg(q) = 205°$, what is…

Standards:
(strikethroughs indicate part of the standard not addressed in this question)

MGSE9-12.N.CN.5 *Represent ~~addition, subtraction,~~ multiplication, ~~and conjugation~~ of complex numbers geometrically on the complex plane; use properties of this representation for computation.*

Solution:

…arg(pq)? **$61° + 205° = 266°$**

…arg(p / q)? **$61° – 205° = -144° \rightarrow -144° + 360° = 216°$**

Culminating Task – Putting It All Together

Mathematical Goals:
- This task assesses students' ability to apply everything they have learned in this unit.

GSE Standards:
- All standards from the unit.

Standards for Mathematical Practice:
- Make sense of problems and persevere in solving them.
- Reason abstractly and quantitatively.
- Construct viable arguments and critique the reasoning of others.
- Model with mathematics.
- Use appropriate tools strategically.
- Attend to precision.
- Look for and make use of structure.

Culminating Task – Putting It All Together

From school, four friends—Aliyah, Ben, Charlie, and Diane—recorded the paths they take to get home. Each student recorded his or her pathway in a different form, as shown in the table below. Aliyah and Ben represented their paths in the complex plane, but Charlie and Diane wrote their paths as vectors in the *x*-*y* plane. Fortunately, they did do some things the same:

- Each unit represents one kilometer.
- Right represents East, up represents North, etc.
- Complex numbers are represented as lowercase italicized letters: *a*.
- Vectors are represented as italicized, bold, uppercase letters: ***A***.

Throughout this task, you should:

- Give square roots in simplest radical form.
- Round decimals to the nearest hundredth when exact answers are impossible.

1. Complete the table below by converting each person's path into the three other forms.

	complex number		vector	
	rectangular form	polar form	component form	mag.-dir. form
Aliyah	$a = -5 + 2i$			
Ben		$b = 4 \text{ cis } 90°$		
Charlie			$\boldsymbol{C} = \langle 3, -3 \rangle$	
Diane				***D:*** mag.: 6 dir.: 60°

2. Who lives farthest from the school? Justify your answer.

3. Charlie walks home from school, then he walks 2 kilometers East and 1 kilometer South to the gym. Write an **equation with vectors** that shows how to determine where the gym is located (with respect to school), then write your answer as a sentence.

4. If Aliyah wants to get to the gym from her house, what path should she take? Write an **equation with complex numbers** that shows how to get your answer, then write your answer as a sentence.

5. What does $|a - c|$ represent in the context of the problem? Find $|a - c|$.

6. Write a one-sentence "story" that could be represented by the vector ½ ***D***, then find ½ ***D*** and its magnitude.

7. Find the average of the complex numbers *a* and *c*. What does this represent in the context of the problem?

8. Eric lives at $\bar{b}$. What does this mean, and where does Eric live?

9. If all the vectors were multiplied by the transformation matrix $\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$, what effect would this have on the "map" of the town?

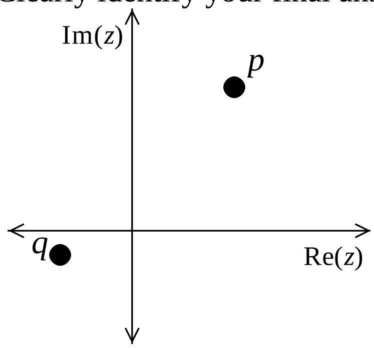
Ben and Diane meet up to fly a model airplane they have built together. At full power, the airplane can fly 160 kilometers per hour in calm air. Ben has the controls, and he makes the plane take off heading 25° North of East. After he feels comfortable with the controls, he turns on full power.

9. A steady wind begins to blow from North to South at a speed of 32 kilometers per hour. In what direction and at what speed is the plane traveling now?

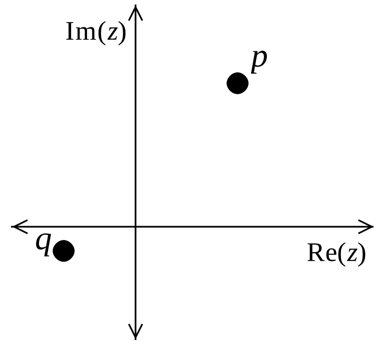
10. Diane starts to get frustrated that the plane is off-course, so she takes the controls. She wants to get the plane to fly at an angle of 25° North of East, with a resultant groundspeed of 140 kilometers per hour. She knows she must "aim" the airplane at an angle that compensates for wind. At what angle should she direct the airplane to fly?

11. Based on the limitations of the power of Diane's plane, is her plan from #10 possible? Explain your answer.

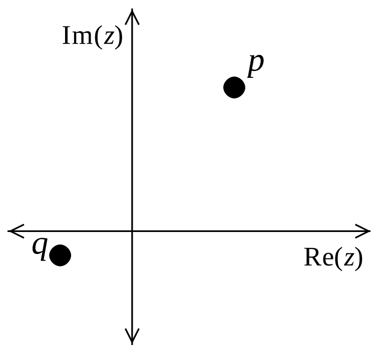
12. p and q represent complex numbers, as shown below. Given the description of four new numbers—r, s, t, and u—use what you know about complex operations to plot those numbers on the complex plane. Show any segments, angles, etc., you use to find your answer. Clearly identify your final answer.



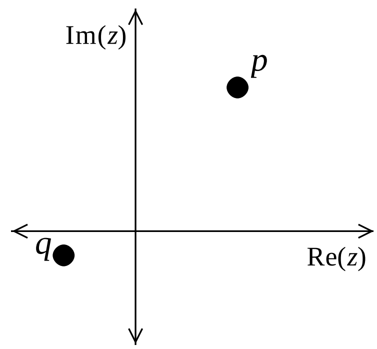
$r = p + q$



$s = p - q$



$t = -2q$



u is the average of p and q

13. Is $|p + q|$ greater than, less than, or equal to $|p| + |q|$? Justify your answer.

14. If $|p| > 1$ and $|q| < 1$, will $|pq|$ be greater than, less than, or equal to $|p|$? Justify your answer.

15. If $\arg(p) = 61°$ and $\arg(q) = 205°$, what is…

…$\arg(pq)$?

…$\arg(p / q)$?