



# Georgia Standards of Excellence Curriculum Frameworks

## Mathematics

GSE Algebra II/Advanced Algebra

Unit 1: Quadratics Revisited



Richard Woods, Georgia's School Superintendent  
"Educating Georgia's Future"

**Unit 1**  
**Quadratics Revisited**

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## **OVERVIEW**

In this unit students will:

- Define rational exponents
- Rewrite expression involving radicals and rational exponents
- Define the imaginary number  $i$
- Define complex numbers
- Operate with complex numbers
- Understand that the basic properties of numbers continue to hold with expressions involving exponents.

During the school-age years, students must repeatedly extend their conception of numbers. From counting numbers to fractions, students are continually updating their use and knowledge of numbers. In Grade 8, students extend this system once more by differentiating between rational and irrational numbers. In high school, students' number knowledge will be supplemented by the addition of the imaginary number system. The basic properties of numbers continue to hold as exponential simplifications are explored.

Although the units in this instructional framework emphasize key standards and big ideas at specific times of the year, routine topics such as estimation, mental computation, and basic computation facts should be addressed on an ongoing basis. Ideas related to the eight practice standards should be addressed constantly as well. This unit provides much needed content information and excellent learning activities. However, the intent of the framework is not to provide a comprehensive resource for the implementation of all standards in the unit. A variety of resources should be utilized to supplement this unit. The tasks in this unit framework illustrate the types of learning activities that should be utilized from a variety of sources. To assure that this unit is taught with the appropriate emphasis, depth, and rigor, it is important that the “**Strategies for Teaching and Learning**” in the Comprehensive course Overview and the tasks listed under “**Evidence of Learning**” be reviewed early in the planning process.

## **STANDARDS ADDRESSED IN THIS UNIT**

Mathematical standards are interwoven and should be addressed throughout the year in as many different units and activities as possible in order to emphasize the natural connections that exist among mathematical topics.

## **KEY STANDARDS**

### **Perform arithmetic operations with complex numbers.**

**MGSE9-12.N.CN.1** Understand there is a complex number  $i$  such that  $i^2 = -1$ , and every complex number has the form  $a + bi$  where  $a$  and  $b$  are real numbers.

**MGSE9-12.N.CN.2** Use the relation  $i^2 = -1$  and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.

**MGSE9-12.N.CN.3** Find the conjugate of a complex number; use the conjugate to find the absolute value (modulus) and quotient of complex numbers.

**Use complex numbers in polynomial identities and equations.**

**MGSE9-12.N.CN.7** Solve quadratic equations with real coefficients that have complex solutions by (but not limited to) square roots, completing the square, and the quadratic formula.

**MGSE9-12.N.CN.8** Extend polynomial identities to include factoring with complex numbers. *For example, rewrite  $x^2 + 4$  as  $(x + 2i)(x - 2i)$ .*

**Solve equations and inequalities in one variable**

**MGSE9-12.A.REI.4** Solve quadratic equations in one variable.

**MGSE9-12.A.REI.4b** Solve quadratic equations by inspection (e.g., for  $x^2 = 49$ ), taking square roots, factoring, completing the square, and the quadratic formula, as appropriate to the initial form of the equation (~~limit to real number solutions~~).

**Extend the properties of exponents to rational exponents.**

**MGSE9-12.N.RN.1.** Explain how the meaning of rational exponents follows from extending the properties of integer exponents to rational numbers, allowing for a notation for radicals in terms of rational exponents. *For example, we define  $5^{(1/3)}$  to be the cube root of 5 because we want  $[5^{(1/3)}]^3 = 5^{[(1/3) \times 3]}$  to hold, so  $[5^{(1/3)}]^3$  must equal 5.*

**MGSE9-12.N.RN.2** Rewrite expressions involving radicals and rational exponents using the properties of exponents.

**STANDARDS FOR MATHEMATICAL PRACTICE**

Refer to the Comprehensive Course Overview for more detailed information about the Standards for Mathematical Practice.

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

*SMP = Standards for Mathematical Practice*

## **ENDURING UNDERSTANDINGS**

- *Nth* roots are inverses of power functions. Understanding the properties of power functions and how inverses behave explains the properties of *nth* roots.
- Real-life situations are rarely modeled accurately using discrete data. It is often necessary to introduce rational exponents to model and make sense of a situation.
- Computing with rational exponents is no different from computing with integral exponents.
- The complex numbers are an extension of the real number system and have many useful applications.
- Addition and subtraction of complex numbers are similar to polynomial operations.

## **ESSENTIAL QUESTIONS**

- How are rational exponents and roots of expressions similar?
- Why is it important to allow solutions for  $x^2 + 1 = 0$ ?
- How are complex- and real numbers related?

## **CONCEPTS/SKILLS TO MAINTAIN**

It is expected that students will have prior knowledge/experience related to the concepts and skills identified below. It may be necessary to pre-assess in order to determine if time needs to be spent on conceptual activities that help students develop a deeper understanding of these ideas.

- number sense
- computation with whole numbers and integers, including application of order of operations
- operations with algebraic expressions
- simplification of radicals
- measuring length and finding perimeter and area of rectangles and squares
- laws of exponents, especially the power rule

## **SELECTED TERMS AND SYMBOLS**

According to Dr. Paul J. Riccomini, Associate Professor at Penn State University,

*“When vocabulary is not made a regular part of math class, we are indirectly saying it **isn’t important!**”* (Riccomini, 2008) Mathematical vocabulary can have significant positive and/or negative impact on students’ mathematical performance.

- ☉ Require students to use mathematically correct terms.
- ☉ Teachers must use mathematically correct terms.

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- ⦿ Classroom tests must regularly include math vocabulary.
- ⦿ Instructional time must be devoted to mathematical vocabulary.

<http://www.nasd.k12.pa.us/pubs/SpecialED/PDEConference//Handout%20Riccomini%20Enhancing%20Math%20InstructionPP.pdf>

The following terms and symbols are often misunderstood. Students should explore these concepts using models and real life examples. Students should understand the concepts involved and be able to recognize and/or demonstrate them with words, models, pictures, or numbers. For help in teaching vocabulary, one technique which could be used is as follows:

*Systematic Vocabulary Instruction; McREL 2008*

- Step 1: Present students with a brief explanation or description of the new term or phrase.
- Step 2: Present students with a nonlinguistic representation of the new term or phrase.
- Step 3: Ask students to generate their own explanation or description of the term or phrase.
- Step 4: Ask students to create their own nonlinguistic representations of the term or phrase.
- Step 5: Periodically ask students to review the accuracy of their explanations and representation.

<http://www.mcrel.org/topics/products/340/>

- **Binomial Expression:** An algebraic expression with two unlike terms.
- **Complex Conjugate:** A pair of [complex numbers](#), both having the same [real](#) part, but with [imaginary](#) parts of equal magnitude and opposite [signs](#). For example,  $3 + 4i$  and  $3 - 4i$  are complex conjugates.
- **Complex number:** A complex number is the sum of a real number and an imaginary number (a number whose square is a real number less than zero), i.e. an expression of the form

$a + bi$ , where  $a$  and  $b$  are real numbers and  $i$  is the *imaginary unit*, satisfying  $i^2 = -1$ .

- **Exponential functions:** A function of the form  $y = a \cdot b^x$  where  $a > 0$  and either  $0 < b < 1$  or  $b > 1$ .
- **Expression:** A mathematical phrase involving at least one variable and sometimes numbers and operation symbols.

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- **Nth roots:** The number that must be multiplied by itself  $n$  times to equal a given value. The  $n$ th root can be notated with radicals and indices or with rational exponents, i.e.  $x^{1/3}$  means the cube root of  $x$ .
- **Polynomial function:** A *polynomial function* is defined as a function,  $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-2}x^2 + a_{n-3}x^1 + a_n$ , where the coefficients are real numbers.
- **Rational exponents:** For  $a > 0$ , and integers  $m$  and  $n$ , with  $n > 0$ ,  $a^{\frac{m}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$ ;  $a^{m/n} = (a^{1/n})^m = (a^m)^{1/n}$ .
- **Rational expression:** A quotient of two polynomials with a non-zero denominator.
- **Rational number:** A number expressible in the form  $a/b$  or  $-a/b$  for some fraction  $a/b$ . The rational numbers include the integers.
- **Standard Form of a Polynomial:** To express a polynomial by putting the terms in descending exponent order.
- **Trinomial:** An algebraic expression with three unlike terms.
- **Whole numbers.** The numbers 0, 1, 2, 3, ....

**The properties of operations:** Here  $a$ ,  $b$  and  $c$  stand for arbitrary numbers in a given number system. The properties of operations apply to the rational number system, the real number system, and the complex number system.

*Associative property of addition*  $(a + b) + c = a + (b + c)$

*Commutative property of addition*  $a + b = b + a$

*Additive identity property of 0*  $a + 0 = 0 + a = a$

*Existence of additive inverses* For every  $a$  there exists  $-a$  so that  $a + (-a) = (-a) + a = 0$ .

*Associative property of multiplication*  $(a \times b) \times c = a \times (b \times c)$

*Commutative property of multiplication*  $a \times b = b \times a$

*Distributive property of multiplication over addition*  $a \times (b + c) = a \times b + a \times c$

This web site has activities to help students more fully understand and retain new vocabulary (i.e. the definition page for *dice* actually generates rolls of the dice and gives students an opportunity to add them). <http://www.teachers.ash.org.au/jeather/maths/dictionary.html>

Definitions and activities for these and other terms can be found on the InterMath website <http://intermath.coe.uga.edu/dictionary/homepg.asp>

## **TECHNOLOGY RESOURCES**

- <http://brightstorm.com/search/?k=polynomials>
- <http://brightstorm.com/search/?k=rational+exponents>
- <http://brightstorm.com/search/?k=complex+numbers>
- [http://www.khanacademy.org/search?page\\_search\\_query=complex+numbers](http://www.khanacademy.org/search?page_search_query=complex+numbers)
- [http://www.khanacademy.org/search?page\\_search\\_query=polynomials](http://www.khanacademy.org/search?page_search_query=polynomials)
- For mathematical applications
  - <http://www.thefutureschannel.com/>

## **EVIDENCE OF LEARNING**

By the conclusion of this unit, students should be able to

- Make connections between radicals and fractional exponents
- Distinguish between real and imaginary numbers. For example,  $(-81)^{3/4}$  as opposed to  $-(81)^{3/4}$
- Results of operations performed between numbers from a particular number set does not always belong to the same set. For example, the sum of two irrational numbers  $(2 + \sqrt{3})$  and  $(2 - \sqrt{3})$  is 4, which is a rational number; however, the sum of a rational number 2 and irrational number  $\sqrt{3}$  is an irrational number  $(2 + \sqrt{3})$
- Realize that irrational numbers are not the same as complex numbers

### **Imaginary Numbers**

The  $\sqrt{-1}$  is called a complex number or an imaginary number. It is the imaginary unit. Its symbol is  $i$ .

$$i = \sqrt{-1}$$

Although Greek mathematician and engineer Heron of Alexandria is noted as the first to have observed these numbers, imaginary numbers were defined in 1572 by Bombelli. At the time, such numbers were regarded by some as fictitious or useless. However, they exist in so many beautiful images. <http://www.ima.umn.edu/~arnold/complex-j.html>

### **Mandelbrot Set**

The Mandelbrot set is a set of complex numbers that can be graphed on the complex number plane. The resulting images can be magnified; thereby, making a variety of images. Such graphs can be found at <http://home.olympus.net/~dewey/mandelbrot.html>. There are many books that discuss similar theories including Jurassic Park by Michael Crichton.



## **FORMATIVE ASSESSMENT LESSONS (FAL)**

**Formative Assessment Lessons** are intended to support teachers in formative assessment. They reveal and develop students' understanding of key mathematical ideas and applications. These lessons enable teachers and students to monitor in more detail their progress towards the targets of the standards. They assess students' understanding of important concepts and problem solving performance, and help teachers and their students to work effectively together to move each student's mathematical reasoning forward.

More information on Formative Assessment Lessons may be found in the Comprehensive Course Overview.

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**TASKS**

The following tasks represent the level of depth, rigor, and complexity expected of all Algebra II/Advanced Algebra students. These tasks, or tasks of similar depth and rigor, should be used to demonstrate evidence of learning. It is important that all elements of a task be addressed throughout the learning process so that students understand what is expected of them. While some tasks are identified as a performance task, they may also be used for teaching and learning (learning/scaffolding task).

| <b>Task Name</b>                           | <b>Task Type</b><br><i>Grouping Strategy</i>      | <b>Content Addressed</b>                                      | <b>SMPs Addressed</b> |
|--|---|---|-----------------------|
| How Long Does It Take?                     | Learning Task<br><i>Partner/Small Group Task</i>  | Exploring phenomena related to exponential                    | 2, 3, 7               |
| <i>i</i> -magine That! (FAL)               | Formative Assessment Lesson                       | Operating with Complex Numbers                                | 6, 8                  |
| The Real Number System                     | Short Cycle Task                                  | Selecting and applying knowledge from the real number system. | 2, 6                  |
| Power of Roots                             | Assessment Task<br><i>Individual/Partner Task</i> | Converting rational powers to roots                           | 2, 3, 7               |
| Not as Complex as You Might Imagine        | Learning Task<br><i>Individual/Partner Task</i>   | Complex roots to quadratics and the role of the discriminant  | 2, 3, 7               |
| Culminating Task: Amusement Park Revisited | Culminating Task<br><i>Partner/Group</i>          | Operations on Complex Numbers                                 | 2, 3, 6, 7, 8         |

## **How Long Does It Take?**

### **Standards Addressed in this Task**

**MGSE9-12.N.RN.1.** Explain how the meaning of rational exponents follows from extending the properties of integer exponents to rational numbers, allowing for a notation for radicals in terms of rational exponents. *For example, we define  $5^{(1/3)}$  to be the cube root of 5 because we want  $[5^{(1/3)}]^3 = 5^{[(1/3) \times 3]}$  to hold, so  $[5^{(1/3)}]^3$  must equal 5.*

**MGSE9-12.N.RN.2** Rewrite expressions involving radicals and rational exponents using the properties of exponents.

**Use properties of rational and irrational numbers.**

**MGSE9-12.N.RN.3** Explain why the sum or product of rational numbers is rational; that the sum of a rational number and an irrational number is irrational; and that the product of a nonzero rational number and an irrational number is irrational.

### **Standards of Mathematical Practice**

**2. Reason abstractly and quantitatively** by requiring students to make sense of quantities and their relationships to one another in problem situations.

**3. Construct viable arguments and critique the reasoning of others** by engaging students on discussion of why they agree or disagree with responses, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

**7. Look for and make use of structure** by expecting students to apply rules, look for patterns and analyze structure.

### **Common Student Misconceptions**

1. Students sometimes misunderstand the meaning of exponential operations, the way powers and roots relate to one another, and the order in which they should be performed. Attention to the base is very important. Consider examples:  $(-81)^{3/4}$  and  $(-81)^{3/4}$ . The position of a negative sign of a term with a rational Exponent can mean that the rational exponent should be either applied first to the base, 81, and then the opposite of the result is taken,  $(-81)^{3/4}$ , or the rational exponent should be applied to a negative term  $(-81)^{3/4}$ . The answer of the fourth root of  $\sqrt[4]{-}$  will be not real if the denominator of the exponent is even. If the root is odd, the answer will be a negative number. Students should be able to make use of estimation when incorrectly using multiplication instead of exponentiation. Students may believe that the fractional exponent in the expression  $36^{1/3}$  means the same as a factor of 1/3 in multiplication expression,  $36 \cdot 1/3$  and multiple the base by the exponent.

2. Some students may believe that both terminating and repeating decimals are rational numbers, without considering nonrepeating and nonterminating decimals as irrational numbers.

3. Students may also confuse irrational numbers and complex numbers, and therefore mix their properties. In this case, student should encounter examples that support or contradict properties and relationships between number sets (i.e., irrational numbers are real numbers and complex numbers are non-real numbers. The set of real numbers is a subset of the set of complex numbers).

4. By using false extensions of properties of rational numbers, some students may assume that the sum of any two irrational numbers is also irrational. This statement is not always true (e.g.,  $(2+\sqrt{3}) + (2-\sqrt{3})=4$ , rational numbers), and therefore, cannot be considered as a property.

Supplies Needed:

- Graphing Calculator
- Graph paper

Before sending astronauts to investigate the new planet of Exponentia, NASA decided to run a number of tests on the astronauts.

1. A specific multi-vitamin is eliminated from an adult male’s bloodstream at a rate of about 20% per hour. The vitamin reaches peak level in the bloodstream of 300 milligrams.
  - a. How much of the vitamin remains 2 hours after the peak level? 5 hours after the peak level? Make a table of values to record your answers. Write expressions for how you obtain your answers.

*Solution:*

| Time (hours) since peak                   | 0   | 1   | 2   | 3     | 4      | 5      |
|---|-----|-----|-----|-------|--------|--------|
| Vitamin concentration in bloodstream (mg) | 300 | 240 | 192 | 153.6 | 122.88 | 98.304 |

$300*(1 -.2) = 240; 240 * .8 = 192; 192 * .8 = 153.6; 153.6 * .8 = 122.88; 122.88 * .8 = 98.304$

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- b. Using your work from (a), write expressions for each computed value using the original numbers in the problem (300 and 20%). For example, after 2 hours, the amount of vitamin left is  $[300 * (1 - .2)] * (1 - .2)$ .

**Solution:**

*After 1 hour:  $300 * (1 - .2) = 240$ ; after 2 hours:  $300 * (1 - .2)^2 = 192$ ;*

*After 3 hours:  $300 * (1 - .2)^3 = 153.6$ ; after 4 hours:  $300 * (1 - .2)^4 = 122.88$ ;*

*After 5 hours:  $300 * (1 - .2)^5 = 98.304$*

- c. Using part (b), write a function for the vitamin level with respect to the number of hours after the peak level,  $x$ .

**Solution:**

*$f(x) = 300(1 - .2)^x$  or  $f(x) = 300(.8)^x$*

- d. How would use the function you wrote in (c) to answer the questions in (a)? Use your function and check to see that you get the same answers as you did in part (a).

**Solution:**

*$f(0) = 300(.8)^0 = 300$*

*$f(1) = 300(.8)^1 = 240$*

*$f(2) = 300(.8)^2 = 192$*

*$f(3) = 300(.8)^3 = 153.6$*

*$f(4) = 300(.8)^4 = 122.88$*

*$f(5) = 300(.8)^5 = 98.304$*

- e. After how many hours will there be less than 10 mg of the vitamin remaining in the bloodstream? Explain how you would determine this answer using both a table feature and a graph.

**Solution:**

*Between 15 and 16 hours after peak, the concentration dips below 10 mg. Using the table feature, at 15 hours, the concentration is 10.555 mg whereas at 16 hours, it is 8.4442. Using a graph, we can trace the graph to determine where the vitamin concentration dips below 10 mg. This occurs around 15.3 hours.*

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- f. Write an equation that you could solve to determine when the vitamin concentration is exactly 10 mg. Could you use the table feature to solve this equation? Could you use the graph feature? How could you use the intersection feature of your calculator? Solve the problem using one of these methods.

*Solution:*

*$300(.8)^x = 10$ ; You could use the table feature to approximate the solution by making smaller and smaller table steps. The trace feature could also be used to approximate the solution. If the original equation was graphed as  $Y1$  and you let  $Y2 = 10$ , we could look for the intersection. Using this last method, the solution is 15.242194 hours or approximately 15 hours, 14 minutes, 32 seconds.*

- g. How would you solve the equation you wrote in (f) algebraically? What is the first step?

*Solution:* *The first step would be to divide by 300.  $\rightarrow .8^x = 1/30$ .*

2. A can of Instant Energy, a 16-ounce energy drink, contains 80 mg of caffeine. Suppose the caffeine in the bloodstream peaks at 80 mg. If  $\frac{1}{2}$  of the caffeine has been eliminated from the bloodstream after 5 hours (the half-life of caffeine in the bloodstream is 5 hours), complete the following:
- a. How much caffeine will remain in the bloodstream after 5 hours? 10 hours? 1 hour? 2 hours? Make a table to organize your answers. Explain how you came up with your answers. (Make a conjecture. You can return to your answers later to make any corrections.)

*Solution:*

|                                     |           |               |               |           |           |
|-------------------------------------|-----------|---------------|---------------|-----------|-----------|
| <i>Time (hours) since peak</i>      | <i>0</i>  | <i>1</i>      | <i>2</i>      | <i>5</i>  | <i>10</i> |
| <i>Caffeine in bloodstream (mg)</i> | <i>80</i> | <i>69.644</i> | <i>60.629</i> | <i>40</i> | <i>20</i> |

*(Explanations will vary. At this point, students may not have the answers for 1 hour and 2 hours correct. They will be able to go back and correct it.)*

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- b. Unlike problem (1), in this problem in which 80% remained after each hour, in this problem, 50% remains after each **5 hours**.

- i. In problem (1), what did the exponent in your equation represent?

*Solution:* *The exponent represented the number of hours that had passed.*

- ii. In this problem, our exponent needs to represent the number of 5-hour time periods that elapsed. If you represent 1 hour as  $1/5$  of a 5-hour time period, how do you represent 2 hours? 3 hours? 10 hours?  $x$  hours?

*Solution:* *2 hours is  $2/5$  of a time period; 3 hours is  $3/5$ ; 10 hours is 2 5-hour time periods;  $x$  hours is  $x/5$  of a time period.*

- c. Using your last answer in part (b) as your exponent, write an exponential function to model the amount of caffeine remaining in the blood stream  $x$  hours after the peak level.

*Solution:*  $f(x) = 80(.5)^{x/5}$

- d. How would use the function you wrote in (c) to answer the questions in (a)? Use your function and check to see that you get the same answers as you did in part (a). (Be careful with your fractional exponents when entering in the calculator. Use parentheses.) If you need to, draw a line through your original answers in part (a) and list your new answers.

*Solution:*  $f(0) = 80(.5)^{0/5} = 80$ ;  $f(1) = 80(.5)^{1/5} = 69.644$ ;  $f(2) = 80(.5)^{2/5} = 60.629$ ;  
 $f(5) = 80(.5)^1 = 40$ ;  $f(10) = 80(.5)^2 = 20$

- e. Determine the amount of caffeine remaining in the bloodstream 3 hours after the peak level? What about 8 hours after peak level? 20 hours? (Think about how many 5-hour intervals are in the number of hours you're interested in.)

*Solution:*  $f(3) = 80(.5)^{3/5} = 53.78$ ;  $f(8) = 80(.5)^{8/5} = 26.39$ ;  $f(20) = 80(.5)^4 = 5$

- f. Suppose the half-life of caffeine in the bloodstream was 3 hours instead of 5.
- i. Write a function for this new half-life time.

*Solution:*  $f(x) = 80(.5)^{x/3}$

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- ii. Determine the amount of caffeine in the bloodstream after 1 hour, 2 hours, 5 hours, and 10 hours. (You need to consider how many 3-hour time intervals are used in each time value.)

**Solution:**  $f(0) = 80(.5)^{0/3} = 80$ ;  $f(1) = 80(.5)^{1/3} = 63.496$ ;  $f(2) = 80(.5)^{2/3} = 50.397$ ;  $f(5) = 80(.5)^{5/3} = 25.198$ ;  $f(10) = 80(.5)^{10/3} = 7.937$

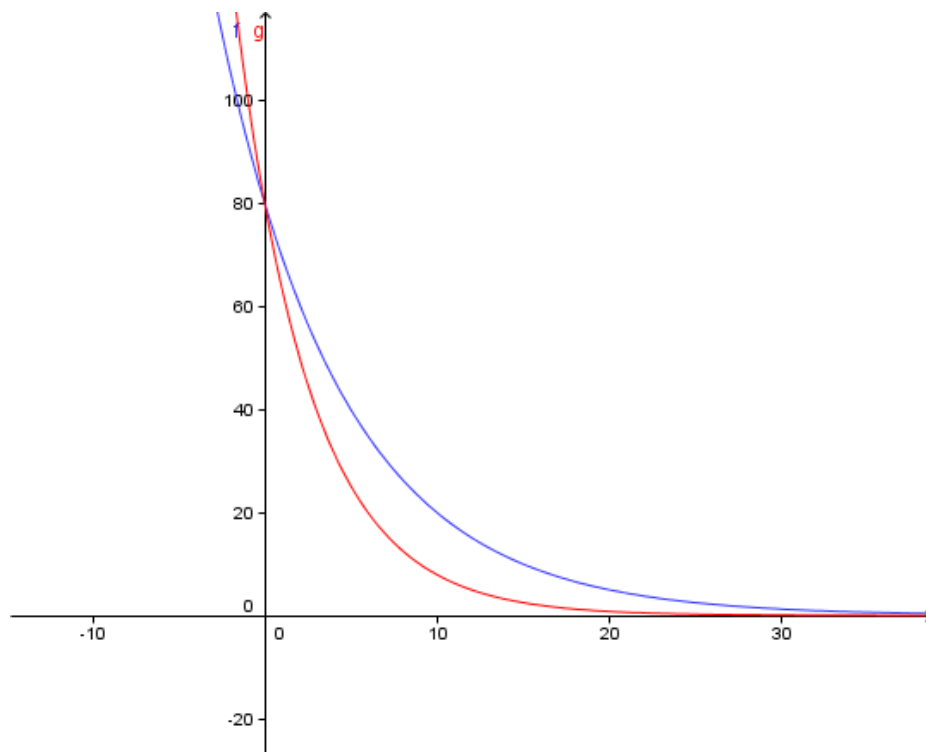
- iii. Which half-life time results in the caffeine being eliminated faster? Explain why this makes sense.

**Solution:** *Since the half-life is less in (f), caffeine is eliminated faster. It makes sense that the answers in (f, ii) are smaller than those in (d). Because the caffeine is eliminated faster in (f), there is less remaining in the bloodstream after the same number of hours than in (d).*

- g. Graph both equations (from d and f) on graph paper. How are the graphs similar? Different? What are the intercepts? When are they increasing/decreasing? Are there asymptotes? If so, what are they? Do the graphs intersect? Where?

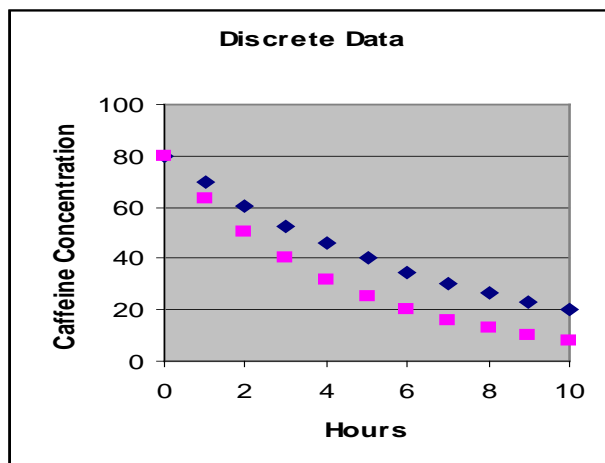
**Solution:**

*The graphs have the same shape; the same y-intercept (0, 80); they are both always decreasing, never increasing; they are both asymptotic to the x-axis. They only intersect at the y-intercept. After that, because the second (half-life of 3 – red graph) decreases faster than the first, they will not intersect again.*





*Note that if we could only use integer exponents; e.g. 1, 2, 3, etc; our graphs would be discontinuous. We would have points (see right), rather than the smooth, continuous curve you graphed above.*



*It makes sense, in thinking about time, that we need all rational time values, e.g. 1/3 hour, 5/8 hour, etc. This raises the idea of rational exponents, that is, computing values such as  $3^{3/4}$  or  $(1/2)^{7/3}$ .*

3. **Rational Exponents.** In previous courses, you learned about different types of numbers and lots of rules of exponents.

- a. What are integers? Rational numbers? Which set of numbers is a subset of the other? Explain why this is true.

*Solution:*

*Integers are whole numbers and their opposites. Rational numbers are quotients of integers, where the denominator is not equal to zero. Integers are a subset of rational numbers. Every integer can be written as a rational number by writing it as a fraction of the integer over 1.*

- b. Based on (a), what is the difference between integer exponents and rational exponents?

*Solution:*

*An integer exponent means that the exponent is an integer value; a rational exponent means that the exponent can be a rational number, including fractional or decimal values.*

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- c. Complete the following exponent rules. (If you don't remember the rules from your previous classes, try some examples to help you.)

For  $a > 0$  and  $b > 0$ , and all values of  $m$  and  $n$ ,

$$a^0 = \underline{1} \qquad a^1 = \underline{a} \qquad a^n = \underline{a \times a \times \dots \times a} \text{ (} a \text{ times itself } n \text{ times)}$$

$$(a^m)(a^n) = \underline{a^{m+n}} \qquad (a^m)/(a^n) = \underline{a^{m-n}} \qquad a^{-n} = \underline{1/a^n}$$

$$(a^m)^n = \underline{a^{mn}} \qquad (ab)^m = \underline{(a^m)(b^m)} \qquad (a/b)^m = \underline{(a^m)/(b^m)}$$

$$\text{If } a^m = a^n, \text{ then } m \underline{=} n.$$

**The same rules you use for integer exponents also apply to rational exponents.**

- d. You have previously learned that the  $n$ th root of a number  $x$  can be represented as  $x^{1/n}$ .

- i. Using your rules of exponents, write another expression for  $(x^{1/n})^m$ .

*Solution:*

$$x^{m/n}$$

- ii. Using your rules of exponents, write another expression for  $(x^m)^{1/n}$ .

*Solution:*

$$x^{m/n}$$

- iii. What do you notice about the answers in (ii) and (iii)? What does this tell you about rational exponents?

*Solution:*

*The answers are the same. So  $(x^{1/n})^m = (x^m)^{1/n} = x^{m/n}$ .*

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e. Rewrite the following using simplified rational exponents.

i.  $\sqrt[7]{x^3}$     ii.  $\left(\frac{1}{x}\right)^{-5}$     iii.  $(\sqrt{x})^6$     iv.  $\frac{1}{\sqrt[3]{x^5}}$

**Solution:**

i.  $\sqrt[7]{x^3} = x^{3/7}$     ii.  $\left(\frac{1}{x}\right)^{-5} = x^5$     iii.  $(\sqrt{x})^6 = x^{6/2} = x^3$

iv.  $\frac{1}{\sqrt[3]{x^5}} = x^{-5/3}$

f. Simplify each of the following. Show your steps. For example,  $27^{2/3} = (27^{1/3})^2 = 3^2 = 9$ .

ii.  $16^{3/4}$     ii.  $36^{3/2}$     iii.  $81^{7/4}$

**Solution:**

i.  $16^{3/4} = (16^{1/4})^3 = 2^3 = 8$     ii.  $36^{3/2} = (36^{1/2})^3 = 6^3 = 216$

iii.  $81^{7/4} = (81^{1/4})^7 = 3^7 = 2187$

Problems involving roots and rational exponents can sometimes be solved by rewriting expressions or by using inverses.

g. Consider the caffeine situation from above. The half-life of caffeine in an adult's bloodstream is 5 hours. How much caffeine remains in the bloodstream each hour? Our original equation was  $f(x) = 80(.5)^{x/5}$ . Use the rules of rational exponents to rewrite the equation so that the answer to the above question is given in the equation. What percent of the caffeine remains in the bloodstream each hour?

**Solution:**

$f(x) = 80(.5)^{x/5} = 80(.5^{1/5})^x = 80(.8705505633)^x$ ; *approximately 87% remains after an hour.*

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- h. Rewrite each of the following using rational exponents and use inverses to solve for the variable. (You may need to use a calculator for some of them. Be careful!)

iii.  $\sqrt[5]{b} = 2$       ii.  $\sqrt[5]{c^3} = 4.2$       iii.  $\frac{1}{\sqrt[4]{d}} = \frac{1}{5}$

**Solution:**

i.  $b^{1/5} = 2 \rightarrow (b^{1/5})^5 = 2^5 \rightarrow b = 32$

ii.  $c^{3/5} = 4.2 \rightarrow (c^{3/5})^{5/3} = 4.2^{5/3} \rightarrow c = 10.9323838$

iii.  $d^{-1/4} = 1/5 \rightarrow (d^{-1/4})^{-4} = (1/5)^{-4} \rightarrow d = 5^4 = 625$

4. Let's use a calculator to model bacteria growth. Begin with 25 bacteria.

- a. If the number of bacteria doubles each hour, how many bacteria are alive after 1 hour? 2 hours?

**Solution:**

*After 1 hour – 50; after 2 hours – 100*

- b. Complete the chart below.

**Solution:**

| Time (hours) | 0  | 1  | 2   | 3   | 4   | 5   | 6    |
|--------------|----|----|-----|-----|-----|-----|------|
| Population   | 25 | 50 | 100 | 200 | 400 | 800 | 1600 |

- c. Write a function that represents the population of bacteria after  $x$  hours. (Check that your function gives you the same answers you determined above. Think about what it means if the base number is 1. What type of base number is needed if the population is increasing?)

**Solution:**

*$f(x) = 25(1 + 1)^x = 25(2^x)$ . (100% of the population remains after an hour plus an additional 100 %.) The base number must be greater than 1 to indicate an increasing population.*

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- d. Use this expression to find the number of bacteria present after 7 1/2 and 15 hours.

**Solution:**

$$f(7.5) = 25(2^{7.5}) = 4525.4834 \text{ bacteria}; f(15) = 25(2^{15}) = 819200 \text{ bacteria}$$

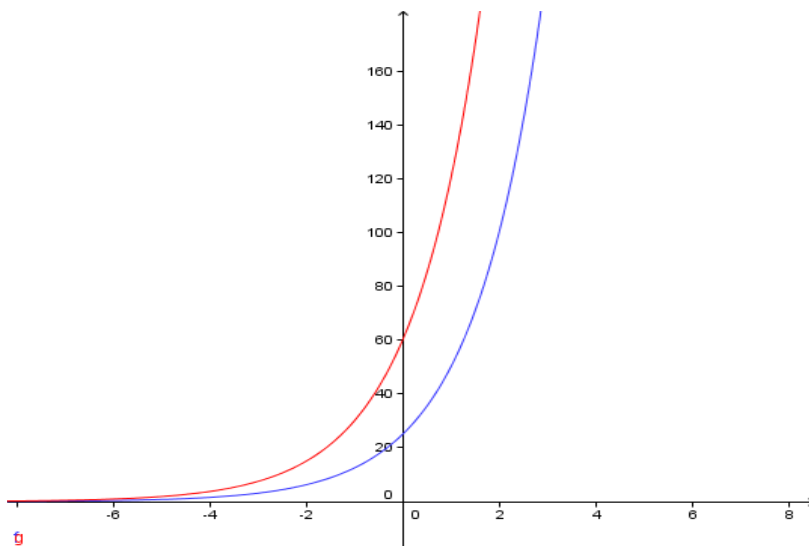
- e. Suppose the initial population was 60 instead of 25. Write a function that represents the population of bacteria after  $x$  hours. Find the population after 7 1/2 hours and 15 hours.

**Solution:**

$$f(x) = 60(2^x) \quad f(7.5) = 10861.16016 \quad f(15) = 1966080 \text{ bacteria}$$

- f. Graph the functions in part (c) and (e). How are the graphs similar or different? What are the intercepts? What do the intercepts indicate? When are they increasing/decreasing? Are there asymptotes? If so, what are they? Do the graphs intersect? Where?

**Solution:**



***Both graphs are increasing and are asymptotic to the x-axis. The two graphs do not intersect. The intercept of the first is (0, 25) and the second is (0, 60). The intercepts indicate the original population, the population at time 0 hours.***

- g. Revisit the graphs in problem (2). Compare with the graphs above. How are they similar and different? How do the equations indicate if the graphs will be increasing or decreasing?

Solution:

*The graphs in number 2 are always decreasing and the bases in the equations are less than 1. The graphs above are always increasing and the bases in the equations are greater than 1. All the graphs are asymptotic to the x-axis. All the graphs have y-intercepts that correspond to the “original” number/concentration.*

- h. Consider the following: Begin with 25 bacteria. The number of bacteria doubles every 4 hours. Write a function, using a rational exponent, for the number of bacteria present after  $x$  hours.

Solution:

$$f(x) = 25(2)^{x/4}$$

- i. Rewrite the function in (g), using the properties of exponents, so that the exponent is an integer. What is the rate of growth of the bacteria each hour?

Solution:

$$f(x) = 25(2^{1/4})^x = 25(1.189207115)^x; 18.92\% \text{ growth rate}$$

- j. If there are originally 25 bacteria, at what rate are they growing if the population of the bacteria doubles in 5 hours? (Hint: Solve  $50 = 25(1 + r)^5$ .) What about if the population triples in 5 hours? (Write equations and solve.)

Solution:

$$50 = 25(1 + r)^5 \rightarrow 2 = (1 + r)^5 \rightarrow 2^{1/5} = (1 + r) \rightarrow 2^{1/5} - 1 = .148698355; 14.87\%$$

*If the population triples in 5 hours, we solve  $75 = 25(1 + r)^5$ . The rate is 24.57%.*

- k. If there are originally 25 bacteria and the population doubles each hour, how long will it take the population to reach 100 bacteria? Solve the problem algebraically.

Solution:

$$100 = 25(2^x) \rightarrow 4 = 2^x \rightarrow x = 2$$

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1. If there are originally 60 bacteria and the population doubles each hour, how long will it take the population to reach 100 bacteria? Explain how you solved the problem. (Solving the problem algebraically will be addressed later in the unit.)

*Solution:*

*$100 = 60(2^x) \rightarrow 5/3 = 2^x \rightarrow$  Using the intersecting graphs, we find that the population reaches 100 approximately at .73696559 hours or 44 minutes, 13 seconds.*

## **How Long Does It Take?**

### **Standards Addressed in this Task**

**MGSE9-12.N.RN.1.** Explain how the meaning of rational exponents follows from extending the properties of integer exponents to rational numbers, allowing for a notation for radicals in terms of rational exponents. *For example, we define  $5^{(1/3)}$  to be the cube root of 5 because we want  $[5^{(1/3)}]^3 = 5^{[(1/3) \times 3]}$  to hold, so  $[5^{(1/3)}]^3$  must equal 5.*

**MGSE9-12.N.RN.2** Rewrite expressions involving radicals and rational exponents using the properties of exponents.

### **Use properties of rational and irrational numbers.**

**MGSE9-12.N.RN.3** Explain why the sum or product of rational numbers is rational; that the sum of a rational number and an irrational number is irrational; and that the product of a nonzero rational number and an irrational number is irrational.

### **Standards of Mathematical Practice**

**2. Reason abstractly and quantitatively** by requiring students to make sense of quantities and their relationships to one another in problem situations.

**3. Construct viable arguments and critique the reasoning of others** by engaging students on discussion of why they agree or disagree with responses, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

**7. Look for and make use of structure** by expecting students to apply rules, look for patterns and analyze structure.

Supplies Needed:

- Graphing Calculator
- Graph paper

Before sending astronauts to investigate the new planet of Exponentia, NASA decided to run a number of tests on the astronauts.

1. A specific multi-vitamin is eliminated from an adult male's bloodstream at a rate of about 20% per hour. The vitamin reaches peak level in the bloodstream of 300 milligrams.
  - a. How much of the vitamin remains 2 hours after the peak level? 5 hours after the peak level? Make a table of values to record your answers. Write expressions for how you obtain your answers.



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|   |     |   |   |   |   |   |
|---|-----|---|---|---|---|---|
| Time (hours) since peak                   | 0   | 1 | 2 | 3 | 4 | 5 |
| Vitamin concentration in bloodstream (mg) | 300 |   |   |   |   |   |

- b. Using your work from (a), write expressions for each computed value using the original numbers in the problem (300 and 20%). For example, after 2 hours, the amount of vitamin left is  $[300 * (1 - .2)] * (1 - .2)$ .
- c. Using part (b), write a function for the vitamin level with respect to the number of hours after the peak level,  $x$ .
- d. How would use the function you wrote in (c) to answer the questions in (a)? Use your function and check to see that you get the same answers as you did in part (a).
- e. After how many hours will there be less than 10 mg of the vitamin remaining in the bloodstream? Explain how you would determine this answer using both a table feature and a graph.
- f. Write an equation that you could solve to determine when the vitamin concentration is exactly 10 mg. Could you use the table feature to solve this equation? Could you use the graph feature? How could you use the intersection feature of your calculator? Solve the problem using one of these methods.
- g. How would you solve the equation you wrote in (f) algebraically? What is the first step?

To finish solving the problem algebraically, we must know how to find inverses of exponential functions. This topic will be explored in more detail later.

2. A can of Instant Energy, a 16-ounce energy drink, contains 80 mg of caffeine. Suppose the caffeine in the bloodstream peaks at 80 mg. If  $\frac{1}{2}$  of the caffeine has been eliminated from the bloodstream after 5 hours (the half-life of caffeine in the bloodstream is 5 hours), complete the following:
  - a. How much caffeine will remain in the bloodstream after 5 hours? 10 hours? 1 hour? 2 hours? Make a table to organize your answers. Explain how you came up with your answers. (Make a conjecture. You can return to your answers later to make any corrections.)

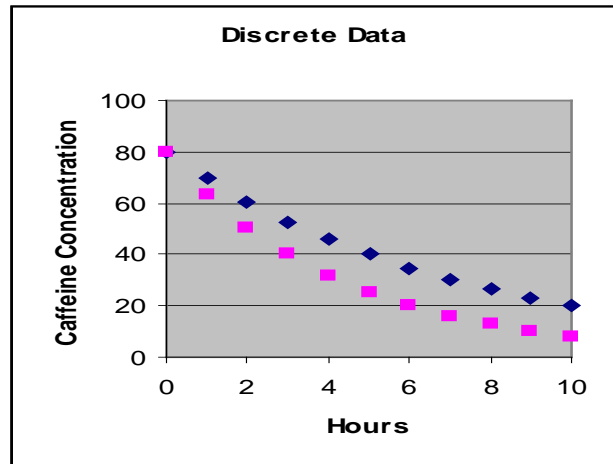
|                         |   |   |   |   |    |
|-------------------------|---|---|---|---|----|
| Time (hours) since peak | 0 | 1 | 2 | 5 | 10 |
|-------------------------|---|---|---|---|----|

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|                              |    |  |  |  |  |
|------------------------------|----|--|--|--|--|
| Caffeine in bloodstream (mg) | 80 |  |  |  |  |
|------------------------------|----|--|--|--|--|

- b. Unlike problem (1), in this problem in which 80% remained after each hour, in this problem, 50% remains after each **5 hours**.
- i. In problem (1), what did the exponent in your equation represent?
  - ii. In this problem, our exponent needs to represent the number of 5-hour time periods that elapsed. If you represent 1 hour as  $1/5$  of a 5-hour time period, how do you represent 2 hours? 3 hours? 10 hours?  $x$  hours?
- c. Using your last answer in part (b) as your exponent, write an exponential function to model the amount of caffeine remaining in the blood stream  $x$  hours after the peak level.
- d. How would use the function you wrote in (c) to answer the questions in (a)? Use your function and check to see that you get the same answers as you did in part (a). (Be careful with your fractional exponents when entering in the calculator. Use parentheses.) If you need to, draw a line through your original answers in part (a) and list your new answers.
- e. Determine the amount of caffeine remaining in the bloodstream 3 hours after the peak level? What about 8 hours after peak level? 20 hours? (Think about how many 5-hour intervals are in the number of hours you're interested in.)
- f. Suppose the half-life of caffeine in the bloodstream was 3 hours instead of 5.
- i. Write a function for this new half-life time.
  - ii. Determine the amount of caffeine in the bloodstream after 1 hour, 2 hours, 5 hours, and 10 hours. (You need to consider how many 3-hour time intervals are used in each time value.)
  - iii. Which half-life time results in the caffeine being eliminated faster? Explain why this makes sense.
- g. Graph both equations (from d and f) on graph paper. How are the graphs similar? Different? What are the intercepts? When are they increasing/decreasing? Are there asymptotes? If so, what are they? Do the graphs intersect? Where?

Note that if we could only use integer exponents; e.g. 1, 2, 3, etc; our graphs would be discontinuous. We would have points (see right), rather than the smooth, continuous curve you graphed above.



It makes sense, in thinking about time, that we need all rational time values, e.g. 1/3 hour, 5/8 hour, etc. This raises the idea of rational exponents, that is, computing values such as  $3^{3/4}$  or  $(1/2)^{7/3}$ .

3. **Rational Exponents.** In previous courses, you learned about different types of numbers and lots of rules of exponents.
  - a. What are integers? Rational numbers? Which set of numbers is a subset of the other? Explain why this is true.
  - b. Based on (a), what is the difference between integer exponents and rational exponents?
  - c. Complete the following exponent rules. (If you don't remember the rules from your previous classes, try some examples to help you.)

For  $a > 0$  and  $b > 0$ , and all values of  $m$  and  $n$ ,

$a^0 = \underline{\hspace{2cm}}$                        $a^1 = \underline{\hspace{2cm}}$                        $a^n = \underline{\hspace{4cm}}$

$(a^m)(a^n) = \underline{\hspace{3cm}}$                        $(a^m)/(a^n) = \underline{\hspace{3cm}}$                        $a^{-n} = \underline{\hspace{2cm}}$

$(a^m)^n = \underline{\hspace{3cm}}$                        $(ab)^m = \underline{\hspace{3cm}}$                        $(a/b)^m = \underline{\hspace{3cm}}$

If  $a^m = a^n$ , then  $m \underline{\hspace{1cm}} n$ .

**The same rules you use for integer exponents also apply to rational exponents.**

- d. You have previously learned that the  $n$ th root of a number  $x$  can be represented as  $x^{1/n}$ .
  - i. Using your rules of exponents, write another expression for  $(x^{1/n})^m$ .
  - ii. Using your rules of exponents, write another expression for  $(x^m)^{1/n}$ .

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- iii. What do you notice about the answers in (ii) and (iii)? What does this tell you about rational exponents?

This leads us to the definition of **rational exponents**.

For  $a > 0$ , and integers  $m$  and  $n$ , with  $n > 0$ ,

$$a^{\frac{m}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m; a^{m/n} = (a^{1/n})^m = (a^m)^{1/n}.$$

- e. Rewrite the following using simplified rational exponents.

i.  $\sqrt[7]{x^3}$     ii.  $\left(\frac{1}{x}\right)^{-5}$     iii.  $(\sqrt{x})^6$     iv.  $\frac{1}{\sqrt[3]{x^5}}$

- f. Simplify each of the following. Show your steps. For example,  $27^{2/3} = (27^{1/3})^2 = 3^2 = 9$ .

i.  $16^{3/4}$     ii.  $36^{3/2}$     iii.  $81^{7/4}$

Problems involving roots and rational exponents can sometimes be solved by rewriting expressions or by using inverses.

- g. Consider the caffeine situation from above. The half-life of caffeine in an adult's bloodstream is 5 hours. How much caffeine remains in the bloodstream each hour? Our original equation was  $f(x) = 80(.5)^{x/5}$ . Use the rules of rational exponents to rewrite the equation so that the answer to the above question is given in the equation. What percent of the caffeine remains in the bloodstream each hour?

To solve equations such as  $x^3 = 27$ , we take the cube root of both sides. Alternately, we can raise both sides of the equation to the  $1/3$  power. That is, we raise both sides of the equation to the power that is the inverse (or reciprocal) of the power in the problem. To solve  $x^{3/2} = 27$ , we can either square both sides and then take the cube root, we can take the cube root of both sides and then square them, or we can raise both sides to the  $2/3$  power.

$$x^{3/2} = 27 \rightarrow (x^{3/2})^{2/3} = 27^{2/3} \rightarrow x = (27^{1/3})^2 \rightarrow x = 3^2 = 9$$

- h. Rewrite each of the following using rational exponents and use inverses to solve for the variable. (You may need to use a calculator for some of them. Be careful!)

i.  $\sqrt[5]{b} = 2$     ii.  $\sqrt[5]{c^3} = 4.2$     iii.  $\frac{1}{\sqrt[4]{d}} = \frac{1}{5}$

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Let's look at some more problems that require the use of rational exponents.

4. Let's use a calculator to model bacteria growth. Begin with 25 bacteria.
- If the number of bacteria doubles each hour, how many bacteria are alive after 1 hour? 2 hours?
  - Complete the chart below.

|              |    |    |   |   |   |   |   |
|--------------|----|----|---|---|---|---|---|
| Time (hours) | 0  | 1  | 2 | 3 | 4 | 5 | 6 |
| Population   | 25 | 50 |   |   |   |   |   |

- Write a function that represents the population of bacteria after  $x$  hours. (Check that your function gives you the same answers you determined above. Think about what it means if the base number is 1. What type of base number is needed if the population is increasing?)
- Use this expression to find the number of bacteria present after  $7\frac{1}{2}$  and 15 hours.
- Suppose the initial population was 60 instead of 25. Write a function that represents the population of bacteria after  $x$  hours. Find the population after  $7\frac{1}{2}$  hours and 15 hours.
- Graph the functions in part (c) and (e). How are the graphs similar or different? What are the intercepts? What do the intercepts indicate? When are they increasing/decreasing? Are there asymptotes? If so, what are they? Do the graphs intersect? Where?
- Revisit the graphs in problem (2). Compare with the graphs above. How are they similar and different? How do the equations indicate if the graphs will be increasing or decreasing?
- Consider the following: Begin with 25 bacteria. The number of bacteria doubles every 4 hours. Write a function, using a rational exponent, for the number of bacteria present after  $x$  hours.

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- i. Rewrite the function in (g), using the properties of exponents, so that the exponent is an integer. What is the rate of growth of the bacteria each hour?
  
  
  
  
  
  
  
  
  
  
- j. If there are originally 25 bacteria, at what rate are they growing if the population of the bacteria doubles in 5 hours? (Hint: Solve  $50 = 25(1 + r)^5$ .) What about if the population triples in 5 hours? (Write equations and solve.)
  
  
  
  
  
  
  
  
  
  
- k. If there are originally 25 bacteria and the population doubles each hour, how long will it take the population to reach 100 bacteria? Solve the problem algebraically.
  
  
  
  
  
  
  
  
  
  
- l. If there are originally 60 bacteria and the population doubles each hour, how long will it take the population to reach 100 bacteria? Explain how you solved the problem. (Solving the problem algebraically will be addressed later in the unit.)

## Imagine That!

### Standards Addressed in this Task

**MGSE9-12.N.CN.1** Understand there is a complex number  $i$  such that  $i^2 = -1$ , and every complex number has the form  $a + bi$  where  $a$  and  $b$  are real numbers.

**MGSE9-12.N.CN.2** Use the relation  $i^2 = -1$  and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.

**MGSE9-12.N.CN.3** Find the conjugate of a complex number; use the conjugate to find the absolute value (modulus) and quotient of complex numbers.

### Standards for Mathematical Practice

**2. Reason abstractly and quantitatively** by requiring students to make sense of quantities and their relationships to one another in problem situations.

**6. Attend to precision** by requiring students to calculate efficiently and accurately; and to communicate precisely with others by using clear mathematical language to discuss their reasoning.

**7. Look for and make use of structure** by expecting students to apply rules, look for patterns and analyze structure.

**8. Look for and express regularity in repeated reasoning** by expecting students to understand broader applications and look for structure and general methods in similar situations.

### *Complex Number System*

The basic algebraic property of  $i$  is the following:

$$i^2 = -1$$

Let us begin with  $i^0$ , which is 1. Each power of  $i$  can be obtained from the previous power by multiplying it by  $i$ . We have:

$$i^0 = 1$$

$$i^2 = -1$$

$$i^3 = i^2 * i = -1 * i = -i$$

$$i^4 = i^2 * i^2 = -1 * -1 = 1$$

And we are back at 1 -- the cycle of powers will repeat! Any power of  $i$  will be either 1,  $i$ ,  $-1$ , or  $-i$

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-- According to the remainder upon dividing the exponent  $n$  by 4.

$i^n$  is

- 1 if the remainder is 0
- $i$  if the remainder is 1
- $-1$  if the remainder is 2
- $-i$  if the remainder is 3

To avoid having to memorize this algorithm,  $i$  to powers higher than two can be mathematically calculated using laws of exponents. The key is to create  $i$  squares.

$$i^3 = i^2 * i = -1 * i = -i$$

$$i^4 = i^2 * i^2 = -1 * -1 = 1$$

Examples Using Both Methods

$i^9 = i$ , because when dividing 9 by 4, the remainder is 1.  $i^9 = i^1$

OR,  $i^9 = i^2 * i^2 * i^2 * i^2 * i = -1 * -1 * -1 * -1 * i = i$

$i^{18} = -1$ , because when dividing 18 by 4, the remainder is 2.  $i^{18} = i^2$

OR,  $i^{18} = (i^2)^9 = (-1)^9 = -1$ . Using the exponent power rule, avoids having to write so many multiplications.

$i^{35} = -i$ , because dividing 35 by 4, the remainder is 3.  $i^{35} = i^3$

OR,  $i^{35} = (i^{34})(i) = (i^2)^{17}(i) = (-1)^{17}(i) = -i$

$i^{40} = 1$ , because on dividing 40 by 4, the remainder is 0.  $i^{40} = i^0$ .

OR,  $i^{40} = (i^2)^{20} = (-1)^{20} = 1$

Note: Even powers of  $i$  will be either 1 or  $-1$ , since the exponent is a multiple of 4 or 2 more than a multiple of 4. Odd powers will be either  $i$  or  $-i$ .

Example 1:  $3i \cdot 4i = 12i^2 = 12(-1) = -12$ .

Example 2:  $-5i \cdot 6i = -30i^2 = 30$ .

We can see that the factor  $i^2$  changes the sign of a product.



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Evaluate the following.

1.  $i * 2i$

2.  $-5i * 4i$

3.  $(3i)^2$

The complex number  $i$  is purely algebraic. That is, we call it a "number" because it will obey all the rules we normally associate with a number. We may add it, subtract it, multiply it, and divide it. In fact, adding and subtracting are just like the basic algebra operations with variables. For example,

$(6 + 3i) - (2 - 4i) = 6 + 3i - 2 + 4i = 4 + 7i$ . Notice that the negative sign or  $-1$  was distributed through the second expression.

For multiplication, remember that the factor  $i^2 = -1$

$$(3 + 2i)(1 + 4i) = 3*1 + 3 * 4i + 2i * 1 + 2i * 4i = 3 + 12i + 2i + 8i^2 = 3 + 14i + 8(-1) = -5 + 14i.$$

$$\begin{aligned}(2 + 3i) \cdot (4 + 5i) &= 2(4 + 5i) + 3i(4 + 5i) \\ &= 8 + 10i + 12i + 15i^2 \\ &= 8 + 22i + 15(-1) \\ &= 8 + 22i - 15 \\ &= -7 + 22i\end{aligned}$$

Try:  $(2 + 3i)(5 + 2i) = ?$

Did you get  $4 + 19i$ ?

The operation of division reviews all of the above operations

Complex conjugates

The complex conjugate of  $a + bi$  is  $a - bi$ . The main point about a conjugate pair is that when they are multiplied --

$$(a + bi)(a - bi)$$

-- A positive real number is produced. That form is the difference of two squares:

$$(a + bi)(a - bi) = a * a + a * (-bi) + bi * a + bi * (-bi) = a^2 - b^2i^2 = a^2 + b^2$$

The product of a conjugate pair is equal to the sum of the squares of the components. Notice that the middle terms cancel. This fact helps us rationalize denominators, which is a form of division.

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Example:

$$\begin{aligned}\frac{(4+2i)}{(3-i)} &= \frac{(4+2i) \cdot (3+i)}{(3-i)(3+i)} \\ &= \frac{12+4i+6i+2i^2}{9+3i-3i-i^2} = \frac{12+10i+2(-1)}{9-(-1)} \\ &= \frac{10+10i}{10} = \frac{1+i}{1} = 1+i\end{aligned}$$

Please notice that the  $i$ 's are removed from the denominators because they are square roots!

Please follow this link for kutasoftware worksheets on imaginary numbers:

<http://www.kutasoftware.com/FreeWorksheets/Alg2Worksheets/Operations%20with%20Complex%20Numbers.pdf>

For division:

<http://www.kutasoftware.com/FreeWorksheets/Alg2Worksheets/Rationalizing%20Imaginary%20Denominators.pdf>

If you really enjoy the imaginary number system, you can always have a t-shirt made with an  $i$  on it; in this way, you are truly using your imagination!

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**MORE REVIEW**

Match the items.

- a.  $29/13 - 54i/13$
- b.  $-9 + 46i$
- c.  $-12 + 10i$
- d.  $i$
- e.  $7i$
- f.  $18 - 6i$
- g.  $-10$
- h.  $61 + 6i$
- i.  $5/2$

1.  $i^5, i^{17}, i^{265}$
2.  $2i + 5i$
3.  $2i \times 5i$
4.  $5i/2i$
  
5.  $(3 + 2i) + (15 - 8i)$
6.  $(3 + 2i) - (15 - 8i)$
7.  $(3 + 2i)(15 - 8i)$
8.  $(15 - 8i)/(3 + 2i)$
9.  $(3 + 2i)^3$

## **Formative Assessment Lesson: i-magine That!**

*Source: Georgia Mathematics Design Collaborative*

This lesson is intended to help you assess how well students are able to:

- Perform arithmetic operations with complex numbers
- To use the relation  $i^2 = -1$  to simplify complex numbers

### **STANDARDS ADDRESSED IN THIS TASK:**

**MGSE9-12.N.CN.1** Understand there is a complex number  $i$  such that  $i^2 = -1$ , and every complex number has the form  $a + bi$  where  $a$  and  $b$  are real numbers.

**MGSE9-12.N.CN.2** Use the relation  $i^2 = -1$  and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.

### **STANDARDS FOR MATHEMATICAL PRACTICE:**

This lesson uses all of the practices with emphasis on:

6. Attend to precision
8. Look for and express regularity in repeated reasoning

### **TASK DESCRIPTION, DEVELOPMENT AND DISCUSSION:**

Tasks and lessons from the Georgia Mathematics Design Collaborative are specifically designed to help teachers effectively formatively assess their students. The way the tasks and lessons are designed gives the teacher a clear understanding of what the students are able to do and not do. Within the lesson, teachers will find suggestions and question prompts that will help guide students towards understanding.

The task, *i-magine That!*, is a Formative Assessment Lesson (FAL) that can be found at:

<http://ccgpsmathematics9-10.wikispaces.com/Georgia+Mathematics+Design+Collaborative+Formative+Assessment+Lessons>

## **The Real Number System (Short Cycle Task)**

*Source: Balanced Assessment Materials from Mathematics Assessment Project*  
<http://www.map.mathshell.org/materials/download.php?fileid=830>

### **ESSENTIAL QUESTIONS:**

- How do you select and apply knowledge from the real number system?

### **TASK COMMENTS:**

This task has been used in previous courses so your students may be familiar with it though the focus in previous classes has been on the first and second exercises. It is included here for the value of the third and fourth exercises. The first and second will prove to be good review problems.

Tasks and lessons from the Mathematics Assessment Project are specifically designed to help teachers effectively formatively assess their students. The way the tasks and lessons are designed gives the teacher a clear understanding of what the students are able to do and not do. Within the lesson, teachers will find suggestions and question prompts that will help guide students towards understanding. For more information access the MAP website:  
<http://www.map.mathshell.org/materials/background.php?subpage=summative>

The task, *The Real Number System*, is a Mathematics Assessment Project Assessment Task that can be found at the website:

<http://www.map.mathshell.org/materials/tasks.php?taskid=289&subpage=novice>

The PDF version of the task can be found at the link below:

<http://www.map.mathshell.org/materials/download.php?fileid=830>

The scoring rubric can be found at the following link:

<http://www.map.mathshell.org/materials/download.php?fileid=831>

### **STANDARDS ADDRESSED IN THIS TASK:**

#### **Extend the properties of exponents to rational exponents.**

**MGSE9-12.N.RN.1.** Explain how the meaning of rational exponents follows from extending the properties of integer exponents to rational numbers, allowing for a notation for radicals in terms of rational exponents. *For example, we define  $5^{(1/3)}$  to be the cube root of 5 because we want  $[5^{(1/3)}]^3 = 5^{[(1/3) \times 3]}$  to hold, so  $[5^{(1/3)}]^3$  must equal 5.*

**MGSE9-12.N.RN.2** Rewrite expressions involving radicals and rational exponents using the properties of exponents.

#### **Use properties of rational and irrational numbers.**

**MGSE9-12.N.RN.3** Explain why the sum or product of rational numbers is rational; that the sum of a rational number and an irrational number is irrational; and that the product of a nonzero rational number and an irrational number is irrational.

**Standards for Mathematical Practice**

This task uses all of the practices with emphasis on:

**2. Reason abstractly and quantitatively** by requiring students to make sense of quantities and their relationships to one another in problem situations.

**6. Attend to precision** by requiring students to calculate efficiently and accurately; and to communicate precisely with others by using clear mathematical language to discuss their reasoning.

## **Power of Roots**

### **Mathematical Goals**

This lesson unit is intended to help you assess how well students are able to recognize radicals in terms of rational exponents

### **Standards Addressed in this Task**

**MGSE9-12.N.RN.1.** Explain how the meaning of rational exponents follows from extending the properties of integer exponents to rational numbers, allowing for a notation for radicals in terms of rational exponents. *For example, we define  $5^{(1/3)}$  to be the cube root of 5 because we want  $[5^{(1/3)}]^3 = 5^{[(1/3) \times 3]}$  to hold, so  $[5^{(1/3)}]^3$  must equal 5.*

**MGSE9-12.N.RN.2** Rewrite expressions involving radicals and rational exponents using the properties of exponents.

### **Standards for Mathematical Practice**

**2. Reason abstractly and quantitatively** by requiring students to make sense of quantities and their relationships to one another in problem situations.

**3. Construct viable arguments and critique the reasoning of others** by engaging students on discussion of why they agree or disagree with responses, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

**7. Look for and make use of structure** by expecting students to apply rules, look for patterns and analyze structure.

### **Common Student Misconception**

Students sometimes misunderstand the meaning of exponential operations, the way powers and roots relate to one another, and the order in which they should be performed. Attention to the base is very important. Consider examples:  $(-81)^{3/4}$  and  $(-81)^{3/4}$ . The position of a negative sign of a term with a rational exponent can mean that the rational exponent should be either applied first to the base, 81, and then the opposite of the result is taken,  $(-81)^{3/4}$ , or the rational exponent should be applied to a negative term  $(-81)^{3/4}$ . The answer of the fourth root of  $\sqrt[4]{-81}$  will be not real if the denominator of the exponent is even. If the root is odd, the answer will be a negative number. Students should be able to make use of estimation when incorrectly using multiplication instead of exponentiation. Students may believe that the fractional exponent in the expression  $36^{1/3}$  means the same as a factor of  $1/3$  in multiplication expression,  $36 \cdot 1/3$  and multiply the base by the exponent.

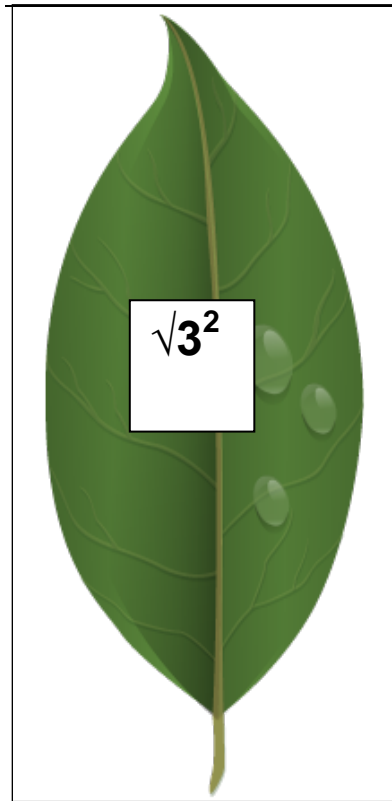
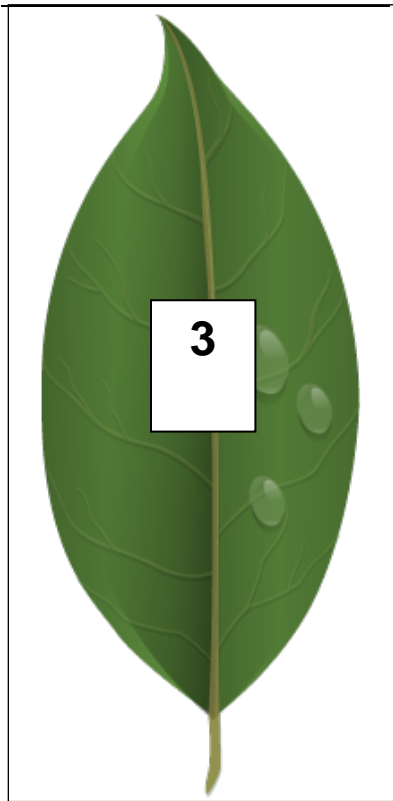
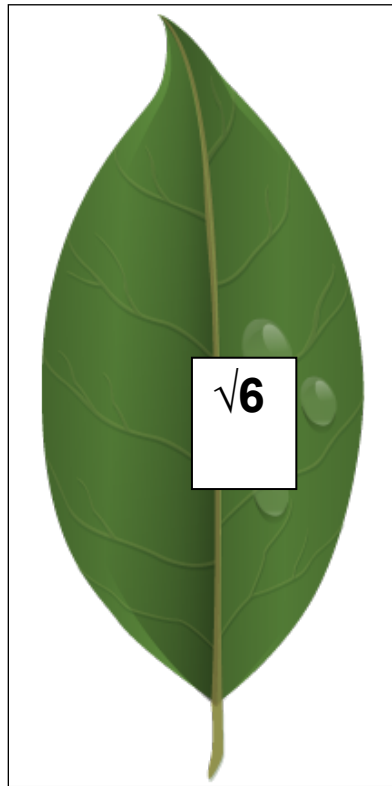
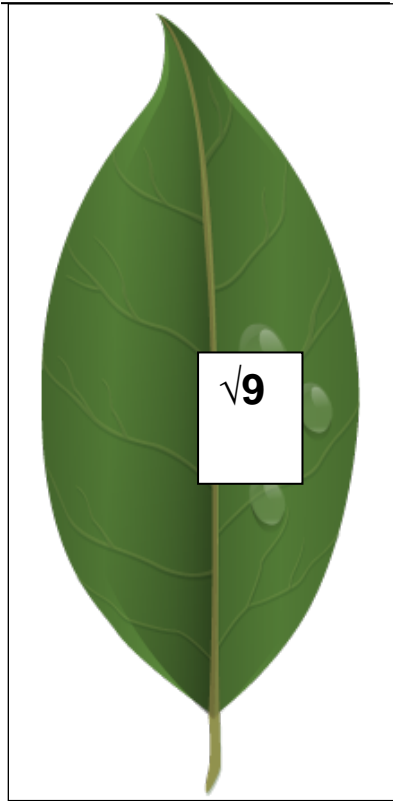
### **Materials required**

Pretest/Posttest

Scientific Calculator

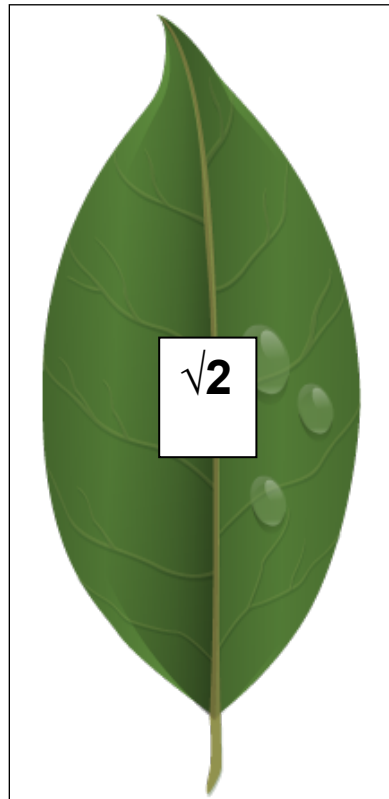
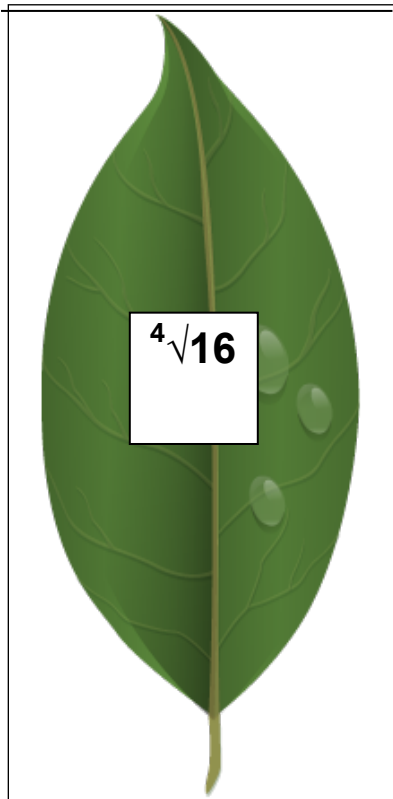
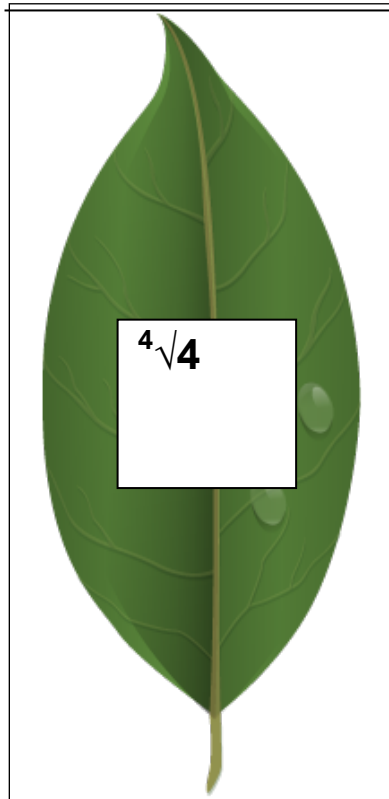
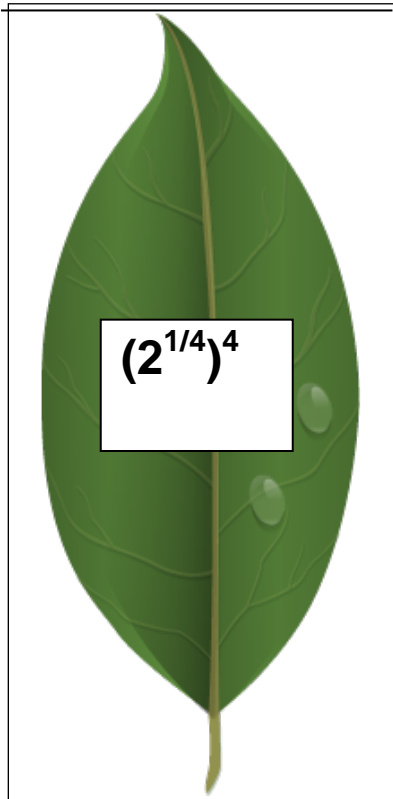
Leaf Cards (1 set per pair of students)

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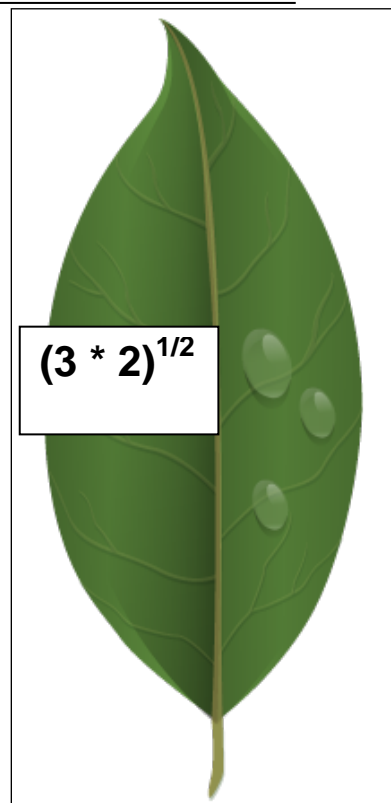
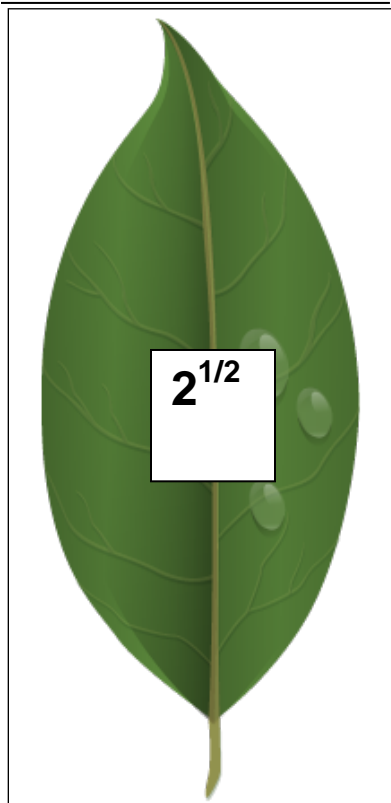
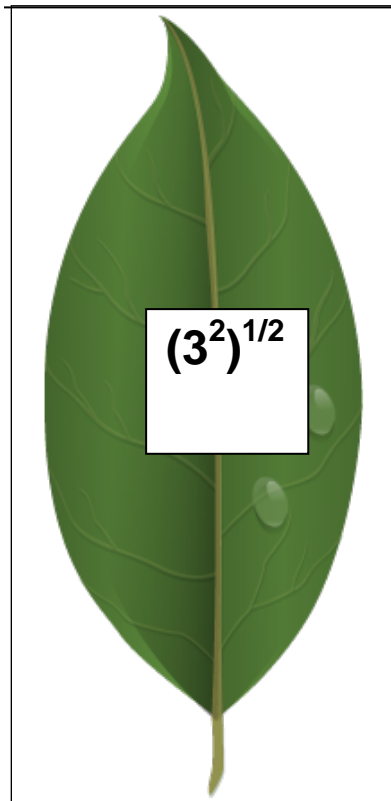
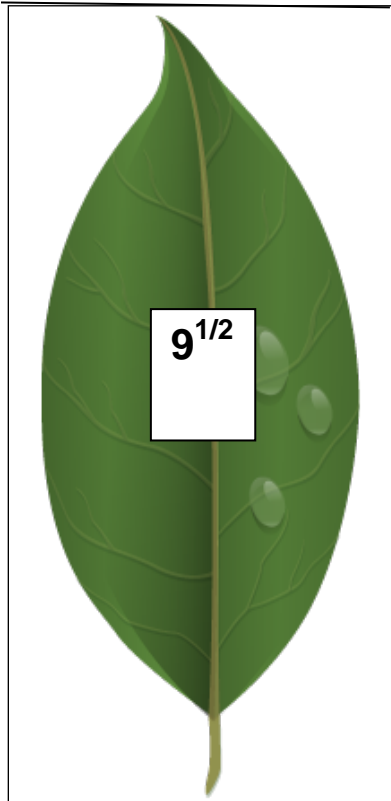


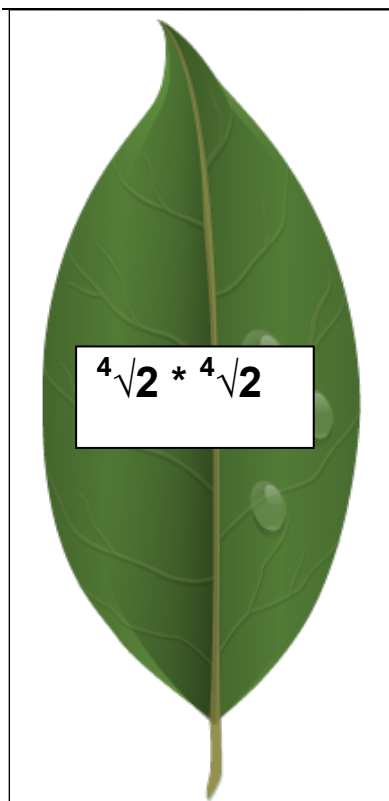
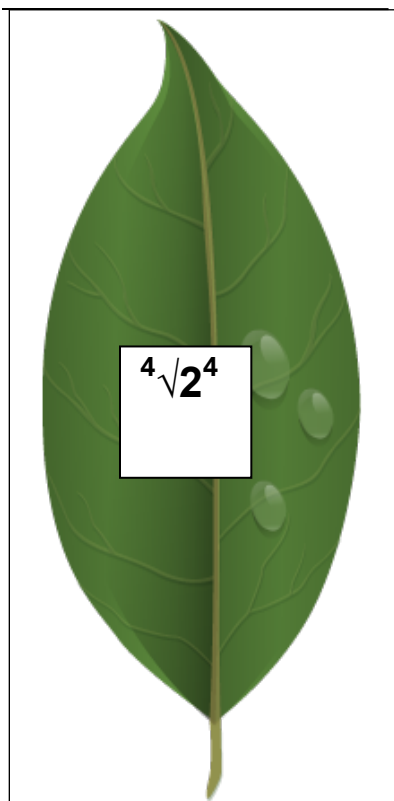


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### Before the lesson

#### Assessment task: (20 minutes)

Students can do this task in class or for homework a day or more before the formative assessment lesson. This will give the teacher an opportunity to assess the work and to find out the kinds of difficulties students have with it. The difficulties will be targeted more effectively in the follow-up lesson.

**Pretest** (Print on regular paper and have students use a PEN for answers)

A scientific calculator may be used.

Directions to read aloud:

Read through the questions and try to answer them as carefully as you can.  
Do not worry too much if you can't understand and do everything.  
I will teach a lesson with a task like this later in the week.  
By the end of that lesson your goal is to answer the questions with confidence.

1. Find the exact value of  $4^{1/2}$   
2

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2. The  $\sqrt{3}$  can be written as 3 to what power?

$3^{1/2}$

3. Compare the decimal approximations of  $\sqrt[3]{3}$  to  $3^{1/3}$

$1.442$

4. Change  $\sqrt[4]{16}$  to an exponential expression.

$16^{1/4}$

5. Evaluate the following:

a.  $\sqrt{8} * \sqrt{2}$

$4$

b.  $2^{2/3} * 2^{1/3}$

$2$

**The rest of the lesson will be completed using the directions from the teacher’s edition.**

**Assessing students’ responses**

*Collect students’ responses to the task. Make some notes on what their work reveals about their current levels of understanding. The purpose of doing this is to forewarn you of issues that will arise during the lesson itself, so that you may prepare carefully.*

*Do not score students’ work. Research suggests this will be counterproductive, as it encourages students to compare their scores, and distracts their attention from what they can do to improve their mathematics.*

*Instead, help students to make further progress by summarizing their difficulties as a series of questions. Some suggestions for these are given on the next page. These have been drawn from common difficulties observed in trials of this lesson unit.*

*Suggestion: Write a list of your own questions, based on students’ work, using the ideas that follow. You may choose to write questions on each student’s work. If you do not have time to do this, just select a few questions that will be of help to the majority of students. These can be written on the board at the end of the lesson.*

*Common issues:*

*Suggested questions and prompts:*

|  |  |
|--|--|
| <i>Student has difficulty using their calculator for exponents or roots.</i> | <i>What key is used for square roots? What key is used for <math>n</math>th roots?</i>                           |
| <i>Student makes an incorrect interpretation of fractional exponents.</i>    | <i>Is a fractional exponents smaller or larger than an integer exponent?</i>                                     |
| <i>Student has technical difficulties using ( )</i>                          | <i>Use your calculator to evaluate <math>\frac{1}{2} \times \frac{1}{4}</math> without using a fraction key.</i> |
| <i>Student cannot find the power key on the calculator.</i>                  | <i>How many different ways can a power key be found on a calculator or computer?</i>                             |

## Suggested lesson outline

### Whole class introduction (10 minutes)

*The teacher will read the following passage about trees and their roots. Students should be told to think about tree roots as they are working through computations involving the roots of numbers. In most trees, roots grown on the bottom of the trees just like the “bottom” /denominator of the fraction is a root. <http://en.wikipedia.org/wiki/Root>*

**Note:** If this unit is near Earth Day, students might read Dr. Seuss’ [The Lorax](http://video.google.com/videoplay?docid=6650219631867189375#) or watch the video at <http://video.google.com/videoplay?docid=6650219631867189375#>

### Working in pairs OR Collaborative small-group work (15 minutes)

Students will work with their partner to match leaves. (*The teachers should cut these out, shuffle, and place together with a clip.*) Some will have more than one match. After matching the leaves, students will explain why some of the leaves have the same value. If the leaves are laminated, students may write on the back of the leaves. After explaining, students will tape the leaves on the wall forming a pyramid. (The correct shape should have the base with 5 leaves, the next row with 4, then 3 and finally 2)

Note different student approaches to the task or note student difficulties

*The activity could be used as a whole-class introduction with calculator usage being taught during the process.*

### Plenary whole-class discussion (10 minutes)

*After all of the leaf pyramids are displayed, the teacher will lead the class into a discussion of any that are different. Students’ ideas will be posted on a board or large sheet of paper with the names of the students beside their ideas.*

*After all ideas are posted, student pairs should be allowed to make any necessary corrections.*

### Improve individual solutions to the assessment task (10 minutes)

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Give students back their work on the pre-test along with a fresh copy of the test. Directions are as follows:

*Work on your own for ten minutes.*

*I'm giving you your own answers, and a new sheet to work on.*

*Read through your original solution and think about what you learned during the lesson.*

*I want you to use what you learned to improve your solutions to the pretest. Then compare your answers to see what progress you made.*

*After everyone is finished, the teacher will collect the student work and analyze for future spiraling lessons.*

*More formative assessments as well as more detail instructions can be found at*

*<http://www.map.mathshell.org/materials/lessons.php>*

## Assessment Task: Power of Roots

### Pretest

1. Find the exact value of  $4^{1/2}$
  
2. The  $\sqrt{3}$  can be written as 3 to what power?
  
3. Compare the decimal approximations of  $\sqrt[3]{3}$  to  $3^{1/3}$
  
4. Change  $\sqrt[4]{16}$  to an exponential expression.
  
5. Evaluate the following:
  - a.  $\sqrt{8} * \sqrt{2}$
  
  - b.  $2^{2/3} * 2^{1/3}$



## **Not as Complex as You Might Imagine...**

### **Math Goals**

- Students will be able to identify solution types of quadratic equations through various means including by inspection, by identifying zeros, and by using the discriminant.

### **Georgia Standards of Excellence**

#### **Solve equations and inequalities in one variable**

**MGSE9-12.A.REI.4** Solve quadratic equations in one variable.

**MGSE9-12.A.REI.4b** Solve quadratic equations by inspection (e.g., for  $x^2 = 49$ ), taking square roots, factoring, completing the square, and the quadratic formula, as appropriate to the initial form of the equation (~~limit to real number solutions~~).

**MGSE9-12.N.CN.7** Solve quadratic equations with real coefficients that have complex solutions by (but not limited to) square roots, completing the square, and the quadratic formula.

### **Standards for Mathematical Practice**

1. Attend to precision
2. Construct viable arguments and critique the reasoning of others
3. Look for and make use of structure
4. Look for and express regularity in repeated reasoning

### **Introduction**

This task explores complex solutions to quadratic equations and was to distinguish quadratics with real solutions from quadratics with imaginary solutions. Students will be encouraged to draw on their knowledge of quadratics from previous classes, especially factoring and zeros, vertex-form equations, and graphs of quadratic equations. A cooperative card-sorting activity is followed by an exploration of the discriminant. The skills learned are reinforced through a problem set.

### **Teacher Notes**

This task is comprised of four parts and should be administered in the following order:

- cooperative card-matching task
- exploration activity
- class discussion
- individual problem set

The 12 cards are divided into three groups: those with 2 real solutions, those with one real solution, and those with no real solutions (2 imaginary). Have partners work together sorting cards into the three different groups. When the work has been checked, direct them to begin the exploration. Lead the class in a group discussion to summarize findings. Finally, allow students to complete the practice problem set.

### **Not as Complex as You Might Imagine...**

Allow students to work in pairs to create three groups of cards based on the number of zeros the equation has (either one, two, or none). The cards divide into three groups identifiable by their letters. If student arrange them in alphabetical order they will spell the words “BEGIN” (for quadratics with two solutions), “AH” (for quadratics with one solution), and “GOUT” (for quadratics with two imaginary solutions). Many of the quadratics can be classified without resorting to the discriminant. The cards are designed is such a way as to let students know that when the number and type of solutions can’t be discovered by inspection, a different method must be employed (the discriminant).

#### Exploration

After completing the cards activity....

Choose one of the following questions and ask your partner. After they have answered, let them choose one to ask you.

Which quadratic was easier to classify, letter B or letter G? Explain your choice.

Students will likely say letter B since the factors are given and so the number of solutions (2) is easy to spot. Letter G is not difficult to classify - it has a positive y intercept and opens downward and so must have 2 solutions also.

Which of the quadratics did you find most difficult to classify? Did those quadratics have anything in common that made them difficult?

Students who are familiar with both vertex- and factored form of a quadratic should be able to classify those types easily. It will be the standard form quadratics that will prove most difficult for students, particularly O, T and N. These quadratics have a discriminant close to zero.

Letter H has exactly *one* solution. What changes could be made to the term coefficients to make H have *no* solutions?

It is important to point out to students here that *all three coefficients* can be changed in a way to alter the number of solutions. This foreshadows the fact that the discriminant contains all three coefficients – that any one of the three can force the value of the discriminate negative or positive (or zero!).

Which quadratic was easier to classify, letter O or letter E? Explain your choice.

E is easier to classify since it can be done so by inspection – the y intercept is negative and it opens upward (2 solutions). Letter O is difficult to classify by inspection and is a good candidate for the discriminant test  
( $D = -7$ )

Letter T turned out to have NO real solutions though it comes pretty close to having one or two. How were you able to classify letter T correctly?

Resourceful students may graph the quadratic or apply the quadratic formula.

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Now work on these questions together:

$$y_1 = \frac{2 + \sqrt{x}}{3}$$

$$y_2 = \frac{5 - \sqrt{6}}{x}$$

$$y_3 = \frac{x - \sqrt{2}}{5}$$

What is the domain for  $y_1$ ? *All real numbers except...* Write a complete sentence here explaining why you chose to exclude the numbers you did.

The domain for  $y_1$  is all real numbers except negative values. Square roots of negative numbers are not real. Use the structure of this expression to point out that the quadratic formula cannot have negatives under the radical if we hope to have real solutions.

What is the domain for  $y_2$ ? *All real numbers except...* Write a complete sentence here explaining why you chose to exclude the numbers you did.

The domain for  $y_2$  is all real numbers except zero. Division by zero is undefined. Use this expression to point out that the quadratic formula cannot have zeros in the denominator – if it did, that would mean  $a$  equaled zero which would mean that the expression wasn't a quadratic! (The student is asked this same question on the next page).

What is the domain for  $y_3$ ? *All real numbers except...* Write a complete sentence here explaining why you chose to exclude the numbers you did.

(The wording here is a bit misleading...) There are no domain restrictions for  $y_3$ . Use this expression to point out that the quadratic formula can have *any* number appear in this slot (the value of  $b$ ).

With the answers from the previous three questions in mind, let's look at the quadratic formula:

$$\text{If } ax^2 + bx + c = 0 \text{ then } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Just like  $y_1$ ,  $y_2$ , and  $y_3$  the quadratic formula has members in three different parts:

1. in the numerator outside the radical,  $(-b)$
2. in the numerator inside the radical,  $(b^2 - 4ac)$  and
3. in the denominator  $(2a)$

There are no problems with 1 since there were no problems with  $y_3$  (study the similarities).

There could be problems with 2 if  $a = 0$ . *Why will a not equal zero for a quadratic expression?*

There *will* be problems with 3, just like there were problems with  $y_1$ .

Answer with you partner:

Why do quadratics like letter O cause problems in the quadratic formula?

They cause problems because they have negative numbers under the radical.

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The part of the quadratic formula you're examining in the question above (specifically  $b^2 - 4ac$ ) plays an important part in determining which group a quadratic expression's card will fall in. It turns out that  $b^2 - 4ac$  is so important that mathematicians give it a name – the “discriminant” – because it can discriminate between quadratics with 2, 1, or no real solutions.

Complete these sentences:

If the discriminant is positive then... *...the quadratic will have two real solutions*

If the discriminant is zero then... *...the quadratic will have one real solution.*

If the discriminant is negative then... *...the quadratic will have no real solutions.*

Use the quadratic equations below to practice what you have just learned. Your teacher will give specific instructions. *Students could complete these in several ways: they could find the discriminant and classify the solutions, they could find the solutions, they could factor into the product of two binomials with complex terms, or they could repeat the cards activity a second time. Notice, also that the form of the equation changes slightly in numbers 9 through 12.*

1.  $y = x^2 + 3x + 8$   
 $y = \frac{-3 \pm i\sqrt{23}}{2} \quad D = -23$

5.  $y = 4x^2 + 12x + 9$   
 $y = \frac{3}{2} \quad D = 0$

9.  $0 = 0.25x^2 - x + 1$   
 $y = 2 \quad D = 0$

2.  $y = 2x^2 + 5x - 1$   
 $y = \frac{-5 \pm \sqrt{33}}{4} \quad D = 33$

6.  $y = x^2 + 4x + 4$   
 $y = -2 \quad D = 0$

10.  $0 = \frac{3}{4}x^2 + 2x + \frac{1}{2}$   
 $y = \frac{-4 \pm \sqrt{10}}{3} \quad D = \frac{5}{2}$

3.  $y = 3x^2 - 6x - 2$   
 $y = \frac{3 \pm \sqrt{15}}{3} \quad D = 60$

7.  $y = x^2 + 4x + 3$   
 $y = -1, -3 \quad D = 4$

11.  $0 = x^2 + 2x + 2$   
 $y = -1 \pm i \quad D = -4$

4.  $y = x^2 - 14x + 50$   
 $y = 7 \pm i \quad D = -4$

8.  $y = x^2 + 4x + 5$   
 $y = -2 \pm i \quad D = -4$

12.  $0 = -x^2 + 10x + 9$   
 $y = 5 \pm \sqrt{34} \quad D = 136$

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|                           |                           |
|---------------------------|---------------------------|
| I<br>$y = x^2 + 3x + 2$   | O<br>$y = 2x^2 + 5x + 4$  |
| G<br>$y = -x^2 - 2x + 10$ | T<br>$y = x^2 + 5x + 7$   |
| A<br>$y = (x - 3)^2$      | G<br>$y = 2(x + 1)^2 + 1$ |
| H<br>$y = x^2 + 6x + 9$   | S<br>$y = 5x^2 - 8x - 1$  |
| N<br>$y = x^2 + 9x + 2$   | E<br>$y = 2x^2 + 5x - 4$  |
| B<br>$y = (x + 2)(3 - x)$ | U<br>$y = x^2 + 1$        |

**Not as Complex as You Might Imagine...**

Exploration

After completing the cards activity....

Choose one of the following questions and ask your partner. After they have answered, let them choose one to ask you.

Which quadratic was easier to classify, letter B or letter G? Explain your choice.

Which quadratic was easier to classify, letter O or letter E? Explain your choice.

Which of the quadratics did you find most difficult to classify? Did those quadratics have anything in common that made them difficult?

Letter T turned out to have NO real solutions though it comes pretty close to having one or two. How were you able to classify letter T correctly?

Letter H has exactly *one* solution. What changes could be made to the term coefficients to make H have *no* solutions?

Now work on these questions together:

$$y_1 = \frac{2 + \sqrt{x}}{3}$$

$$y_2 = \frac{5 - \sqrt{6}}{x}$$

$$y_3 = \frac{x - \sqrt{2}}{5}$$

What is the domain for  $y_1$ ? *All real numbers except...* Write a complete sentence here explaining why you chose to exclude the numbers you did.

What is the domain for  $y_2$ ? *All real numbers except...* Write a complete sentence here explaining why you chose to exclude the numbers you did.

What is the domain for  $y_3$ ? *All real numbers except...* Write a complete sentence here explaining why you chose to exclude the numbers you did.

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With the answers from the previous three questions in mind, let's look at the quadratic formula:

$$\text{If } ax^2 + bx + c = 0 \text{ then } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

That, just like  $y_1$ ,  $y_2$ , and  $y_3$  the quadratic formula has members in three different parts:

1. in the numerator outside the radical,  $(-b)$
2. in the numerator inside the radical,  $(b^2 - 4ac)$  and
3. in the denominator  $(2a)$

There are no problems with 1 since there were no problems with  $y_3$  (study the similarities).

There could be problems with 2 if  $a = 0$ . *Why will  $a$  not equal zero for a quadratic expression?*

There *will* be problems with 3, just like there were problems with  $y_1$ .

Answer with you partner:  
Why do quadratics like letter O cause problems in the quadratic formula?

The part of the quadratic formula you're examining (specifically  $b^2 - 4ac$ ) plays an important part in determining which group a quadratic expression's card will fall in. It turns out that  $b^2 - 4ac$  is so important that mathematicians give it a name – the “discriminant” – because it can discriminate between quadratics with 2, 1, or no real solutions.

Complete these sentences:

If the discriminant is positive then...

If the discriminant is zero then...

If the discriminant is negative then...

Use the quadratic equations below to practice what you have just learned. Your teacher will give specific instructions.

- |                         |                         |  |
|-------------------------|-------------------------|--|
| 1. $y = x^2 + 3x + 8$   | 5. $y = 4x^2 + 12x + 9$ | 9. $0 = 0.25x^2 - x + 1$                   |
| 2. $y = 2x^2 + 5x - 1$  | 6. $y = x^2 + 4x + 4$   | 10. $0 = \frac{3}{4}x^2 + 2 + \frac{1}{2}$ |
| 3. $y = 3x^2 - 6x - 2$  | 7. $y = x^2 + 4x + 3$   | 11. $0 = x^2 + 2x + 2$                     |
| 4. $y = x^2 - 14x + 50$ | 8. $y = x^2 + 4x + 5$   | 12. $0 = -x^2 + 10x + 9$                   |

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|                           |                           |
|---------------------------|---------------------------|
| I<br>$y = x^2 + 3x + 2$   | O<br>$y = 2x^2 + 5x + 4$  |
| G<br>$y = -x^2 - 2x + 10$ | T<br>$y = x^2 + 5x + 7$   |
| A<br>$y = (x - 3)^2$      | G<br>$y = 2(x + 1)^2 + 1$ |
| H<br>$y = x^2 + 6x + 9$   | S<br>$y = 5x^2 - 8x - 1$  |
| N<br>$y = x^2 + 9x + 2$   | E<br>$y = 2x^2 + 5x - 4$  |
| B<br>$y = (x + 2)(3 - x)$ | U<br>$y = x^2 + 1$        |



## **Culminating Task: Amusement Park Problem Revisited**

The teacher should note that the bulk of this Culminating Task has been used in a previous course. It appears here with some extensions pertaining to the standards particularly relevant to Advanced Algebra.

### **Standards Addressed in this Task**

#### **Extend the properties of exponents to rational exponents.**

**MGSE9-12.N.RN.1.** Explain how the meaning of rational exponents follows from extending the properties of integer exponents to rational numbers, allowing for a notation for radicals in terms of rational exponents. *For example, we define  $5^{(1/3)}$  to be the cube root of 5 because we want  $[5^{(1/3)}]^3 = 5^{[(1/3) \times 3]}$  to hold, so  $[5^{(1/3)}]^3$  must equal 5.*

**MGSE9-12.N.RN.2** Rewrite expressions involving radicals and rational exponents using the properties of exponents.

**MGSE9-12.N.RN.3** Explain why the sum or product of rational numbers is rational; that the sum of a rational number and an irrational number is irrational; and that the product of a nonzero rational number and an irrational number is irrational.

#### **Perform arithmetic operations with complex numbers.**

**MGSE9-12.N.CN.1** Understand there is a complex number  $i$  such that  $i^2 = -1$ , and every complex number has the form  $a + bi$  where  $a$  and  $b$  are real numbers.

**MGSE9-12.N.CN.2** Use the relation  $i^2 = -1$  and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.

**MGSE9-12.N.CN.3** Find the conjugate of a complex number; use the conjugate to find the absolute value (modulus) and quotient of complex numbers.

#### **Perform arithmetic operations on polynomials**

### **Standards for Mathematical Practice**

**2. Reason abstractly and quantitatively** by requiring students to make sense of quantities and their relationships to one another in problem situations.

**3. Construct viable arguments and critique the reasoning of others** by engaging students on discussion of why they agree or disagree with responses, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

**6. Attend to precision** by requiring students to calculate efficiently and accurately; and to communicate precisely with others by using clear mathematical language to discuss their reasoning.

**7. Look for and make use of structure** by expecting students to apply rules, look for patterns and analyze structure.

**8. Look for and express regularity in repeated reasoning** by expecting students to understand broader applications and look for structure and general methods in similar situations.

The Radical World of Math is reviewing the master plan of a proposed amusement park coming to your area. Your help is needed with the land space and with park signage.

**First**, the planners need help in designing the land space. The parameters are as follows:

- 15 rows of parking are required
- The rows will be the same length as the park
- The park size will be square with a length of  $X$  so that expansion is possible.
- “Green Space” for planting, sitting, or picnicking is a must.
- Parking will be adjacent to only two sides of the park

Your task is to choose 3 possible configurations of land use with 15 rows of parking. Find the area of the picnic (green space) for each configuration. There is more than one way to solve the problem. For your maximum picnic space, write an equation for the total AREA of the park.

**Extension:** The Park is expected to be successful and the planners decide to expand the parking lot by adding 11 more rows. Assume the new plan will add not only 11 rows of parking but will also triple the maximum original green space (approximately). Choose 1 of your park configurations (your best) to complete this section and redraw your park configuration. What is the percentage increase in area that was created by expanding to 26 rows of parking?

**Second**, signs have to be designed for the park. For one of the areas called “Radical Happenings”, the signs must show conversions between radical expressions and exponential expressions. There must be at least 10 signs in all that reflect square roots, cube roots, and fourth roots. Create 10 unique signs for use in the park. Would there be appropriate areas for these values to be placed?

**Third**, another area will be titled “Imaginary World”. One of the movies shown in this area can be chosen from You Tube dealing with the Mandelbrot Set.

Park guests will progress through a series of obstacles by demonstrating their ability to

- Add complex numbers
- Subtract complex numbers
- Multiply complex numbers
- Divide complex numbers
- Evaluate powers of imaginary numbers

You must create problems demonstrating each concept and have another team try to “get through” your obstacles.

Extension: Using fractals create advertising for this park. This link may help:  
<http://mathworld.wolfram.com/Fractal.html>