Georgia
Standards of Excellence
Curriculum Frameworks

Mathematics

GSE Algebra II/Advanced Algebra

Unit 5: Exponential and Logarithmic Functions

Georgia Department of Education

Richard Woods, Georgia's School Superintendent

"Educating Georgia's Future"

Unit 5
Exponential and Logarithmic Functions

Table of Contents

OVERVIEW

In this unit students will:

- Review exponential functions and their graphs
- Explore exponential growth
- Develop the concept of a logarithm as an exponent along with the inverse relationship with exponents
- Define logarithms and natural logarithms
- Develop the change of base formula
- Develop the concept of logarithmic function
- Solving problems relating to exponential functions and logarithms

Although the units in this instructional framework emphasize key standards and big ideas at specific times of the year, routine topics such as estimation, mental computation, and basic computation facts should be addressed on an ongoing basis. Ideas related to the eight process standards should be addressed constantly as well. To assure that this unit is taught with the appropriate emphasis, depth, and rigor, it is important that the tasks listed under "Evidence of Learning" be reviewed early in the planning process. A variety of resources should be utilized to supplement this unit. This unit provides much needed content information, but excellent learning activities as well. The tasks in this unit illustrate the types of learning activities that should be utilized from a variety of sources.

STANDARDS ADDRESSED IN THIS UNIT

Mathematical standards are interwoven and should be addressed throughout the year in as many different units and activities as possible in order to emphasize the natural connections that exist among mathematical concept and topics.

KEY STANDARDS

Write expressions in equivalent forms to solve problems

MGSE9-12.A.SSE.3 Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression. *(Limit to exponential and logarithmic functions.)*

MGSE9-12.A.SSE.3c Use the properties of exponents to transform expressions for exponential functions. *For example, the expression* 1.15^t *, where t is in years, can be rewritten as* $[1.15^{(1/12)}]^{(12t)} \approx 1.012^{(12t)}$ *to reveal the approximate equivalent monthly interest rate if the annual rate is 15%.*

Analyze functions using different representations

MGSE9-12.F.IF.7 Graph functions expressed algebraically and show key features of the graph both by hand and by using technology. *(Limit to exponential and logarithmic functions.)*

MGSE9-12.F.IF.7e Graph exponential and logarithmic functions, showing intercepts and end behavior, ~~and trigonometric functions, showing period, midline, and amplitude.~~

MGSE9-12.F.IF.8 Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function. *(Limit to exponential and logarithmic functions.)*

MGSE9-12.F.IF.8b Use the properties of exponents to interpret expressions for exponential functions. *For example, identify percent rate of change in functions such as* $y = (1.02)^t$, $y = (0.97)^t$, $y = (1.01)^{(12t)}$, $y = (1.2)^{(t/10)}$, *and classify them as representing exponential growth and decay. (Limit to exponential and logarithmic functions.)*

Build new functions from existing functions

MGSE9-12.F.BF.5 Understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents.

Construct and compare linear, quadratic, and exponential models and solve problems

MGSE9-12.F.LE.4 For exponential models, express as a logarithm the solution to $ab^{(ct)} = d$ where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology.

Standards for Mathematical Practice

Refer to the Comprehensive Course Overview for more detailed information about the Standards for Mathematical Practice.

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.

7. Look for and make use of structure.

8. Look for and express regularity in repeated reasoning.

ENDURING UNDERSTANDINGS

- There is an inverse relationship between exponents and logarithms.
- Logarithms can be used to solve exponential equations.
- An exponential equation can be written as a logarithmic equation; a logarithmic equation can be written as an exponential equation.
- Two special logarithmic functions are the common logarithmic function and the natural logarithmic function. These special functions occur often in nature.
- Common logarithms and natural logarithms can be used to evaluate logarithms with bases other than 10 or e.

ESSENTIAL QUESTIONS

- What does exponential growth mean? What does exponential decay mean?
- What are the characteristics of the graph of an exponential function?
- Why are logarithms important?
- What are the characteristics of the graph of a logarithmic function?
- How are logarithms used to solve equations?
- What is the meaning of half-life?
- What kinds of situations are represented by an exponential function?
- What are some real-world applications of logarithmic functions?

CONCEPTS/SKILLS TO MAINTAIN

It is expected that students will have prior knowledge/experience related to the concepts and skills identified below. It may be necessary to pre-assess in order to determine if time needs to be spent on conceptual activities that help students develop a deeper understanding of these ideas.

1. The concept of a function
2. Various representations of functions
3. Exponential functions and characteristics of their graphs
4. The solution of linear equations using algebra and graphing approaches
5. Familiarity with graphing technology
6. Use patterns to write a function to model a situation

***It is MUCH EASIER to convince students of exponential and logarithmic functions being inverses of one another if students have a firm grasp and quick recall of some integer bases and exponents. A chart and some quick quizzes are included in this unit for teachers to use (in order) to allow the meaning of a logarithm to be a slow dawning on students rather than throwing the idea out all at once.

NOTE TO TEACHER:

The development of a logarithm as the inverse of an exponential equation is typically very difficult-just as subtraction is harder than addition and multiplication is easier than division. A full and complete understanding of addition facts is needed to be able to understand subtraction. Furthermore, quick access to addition facts enhances understanding of subtraction. The same is true for multiplication and division-quick recall of multiplication facts is extremely helpful to learn division. While there is no "magic bullet" for teaching logarithms, the development of the concept will be much easier if students have at their command quick reference and recall to the exponential facts. At the conclusion of this unit, you will find examples of exponent charts and exponent quizzes that slowly build the logarithm understanding without even using the word, "logarithm." These are provided for your convenience and are suggested as openers or tickets out the door for at least the first half of this unit, if not all the way through.

SELECTED TERMS AND SYMBOLS

The following terms and symbols are often misunderstood. These concepts are not an inclusive list and should not be taught in isolation. However, due to evidence of frequent difficulty and misunderstanding associated with these concepts, instructors should pay particular attention to them and how their students are able to explain and apply them.

The definitions below are for teacher reference only and are not to be memorized by the students. Students should explore these concepts using models and real life examples. Students should understand the concepts involved and be able to recognize and/or demonstrate them with words, models, pictures, or numbers.

The websites below are interactive and include a math glossary suitable for middle school children. Note – At the high school level, different sources use different definitions. Please preview any website for alignment to the definitions given in the frameworks.

http://intermath.coe.uga.edu/dictnary/homepg.asp
Definitions and activities for these and other terms can be found on the Intermath website. Intermath is geared towards middle and high school students.

- **Asymptote:** An asymptote is a line or curve that approaches a given curve arbitrarily closely. A graph never crosses a vertical asymptote, but it may cross a horizontal or oblique asymptote.

- **Common logarithm:** A logarithm with a base of 10. A common logarithm is the exponent, a, such that $10^a = b$. The common logarithm of x is written $\log x$. For example, $\log 100 = 2$ because $10^2 = 100$.

- **Continuously compounded interest:** Interest that is, theoretically, computed and added to the balance of an account each instant. The formula is $A = Pe^{rt}$, where A is the ending amount, P is the principal or initial amount, r is the annual interest rate, and t is the time in years.

- **Compounded interest:** A method of computing the interest, after a specified time, and adding the interest to the balance of the account. Interest can be computed as little as once a year to as many times as one would like. The formula is $A = P\left(1+\frac{r}{n}\right)^{nt}$ where A is the ending amount, P is the principal or initial amount, r is the annual interest rate, n is the number of times compounded per year, and t is the number of years.

- **Exponential functions:** A function of the form $y = a^x$ where $a > 0$ and $a \neq 1$.

- **Logarithmic functions:** A function of the form $y = \log_b x$ with $b \neq 1$ and b and x both positive. A logarithmic function is the inverse of an exponential function. The inverse of $y = b^x$ is $y = \log_b x$.

- **Logarithm:** The logarithm base b of a number x, $\log_b x$, is the exponent to which b must be raised to equal x.

- **Natural exponential:** Exponential expressions or functions with a base of e; i.e., $y = e^x$.

- **Natural logarithm:** A logarithm with a base of e. $\ln b$ is the exponent, a, such that $e^a = b$. The natural logarithm of x is written $\ln x$ and represents $\log_e x$. For example, $\ln 8 = 2.0794415\ldots$ because $e^{2.0794415\ldots} = 8$.

EVIDENCE OF LEARNING

By the conclusion of this unit, students should be able to demonstrate the following competencies:

1. Identify the characteristics of the graphs of exponential functions and logarithmic functions.
2. Use logarithms to solve exponential equations.
3. Understand that logarithms and exponential equations connect the same numeric data inversely.
4. Given an exponential equation, write the corresponding logarithmic equation; given a logarithmic equation, write the corresponding exponential equation.
5. Given a situation that can be modeled with an exponential function or logarithmic function, write the appropriate function and use it to answer questions about the situation.

FORMATIVE ASSESSMENT LESSONS (FAL)

Formative Assessment Lessons are intended to support teachers in formative assessment. They reveal and develop students' understanding of key mathematical ideas and applications. These lessons enable teachers and students to monitor in more detail their progress towards the targets of the standards. They assess students' understanding of important concepts and problem solving performance, and help teachers and their students to work effectively together to move each student's mathematical reasoning forward.

More information on Formative Assessment Lessons may be found in the Comprehensive Course Overview.

SPOTLIGHT TASKS

A Spotlight Task has been added to each GSE mathematics unit in the Georgia resources for middle and high school. The Spotlight Tasks serve as exemplars for the use of the Standards for Mathematical Practice, appropriate unit-level Georgia Standards of Excellence, and research-based pedagogical strategies for instruction and engagement. Each task includes teacher commentary and support for classroom implementation. Some of the Spotlight Tasks are revisions of existing Georgia tasks and some are newly created. Additionally, some of the Spotlight Tasks are 3-Act Tasks based on 3-Act Problems from Dan Meyer and Problem-Based Learning from Robert Kaplinsky.

TASKS

The following tasks represent the level of depth, rigor, and complexity expected of all Algebra II/Advanced Algebra students. These tasks, or tasks of similar depth and rigor, should be used to demonstrate evidence of learning. It is important that all elements of a task be addressed throughout the learning process so that students understand what is expected of them. While some tasks are identified as a performance task, they may also be used for teaching and learning (learning/scaffolding task).

Task Name	Task Type *Grouping Strategy*	Content Addressed	SMPs Addressed
Investigating Exponential Growth and Decay	Scaffolding Task Individual/Partner	Understand the concept of a function Interpret functions that arise in applications in terms of the context Analyze functions using different representations	1,2,3,4,5
Graphs of Exponential Functions	Scaffolding Task Partner/Small Group	Analyze functions using different representations Graph exponential functions expressed symbolically and show key features of the graph	2,3,5,8
Zombie Apocalypse Simulation (Spotlight Task)	Scaffolding Task Partner	Understand the nature of exponential growth, and what makes a base of a function, write the equation of the exponential function and explore limits to exponential growth	1-8
Bacteria in the Swimming Pool	Scaffolding Task Partner/Small Group	Interpret functions that arise in applications in terms of context Analyze functions using different representations Build a function that models a relationship between two quantities	2,3,4,5,8
What is a Logarithm? (Spotlight Task)	Scaffolding Task Partner/Small Group	Write expressions in equivalent forms	2,3,4,5,6,7

Evaluating Logarithms that are not Common or Natural	Scaffolding Task Partner/Small Group	Write expressions in equivalent forms	1-8
The Logarithmic Function	Scaffolding Task Partner/Small Group	Analyze functions using different representations Graph logarithmic functions expressed symbolically and show key features of the graph	2,3,4,5,7
How Long Does It Take?	Performance Task Partner/Small Group	Write expressions in equivalent forms Understand the concept of a function and use function notation Interpret functions that arise in applications in terms of the context Analyze functions using different representations Construct exponential models and solve problems	1-8
Zombies Revisited (Spotlight Task)	Scaffolding Task	Represent "real-life" situation with exponential equation and then solve these equations graphically.	1-8
Half-Life	Scaffolding Task Partner/Small Group	Write expressions in equivalent forms Understand the concept of a function and use function notation Interpret functions that arise in applications in terms of the context Analyze functions using different representations Construct exponential models and solve problems	1-8
How Does Your Money Grow?	Performance Task Partner/Small Group	Write expressions in equivalent forms Understand the concept of a	1-8

		function and use function notation Interpret functions that arise in applications in terms of the context Analyze functions using different representations Construct exponential models and solve problems	
Applications of Logarithmic Functions	Performance Task Partner/Small Group	Write expressions in equivalent forms Understand the concept of a function and use function notation Interpret functions that arise in applications in terms of the context Analyze functions using different representations Construct logarithmic models and solve problems	1-8
Newton's Law of Cooling-Coffee, Donuts, and Corpses (Spotlight Task)	Performance (Spotlight Task) Partner/Small Group Tasks	Write expressions in equivalent forms Interpret functions that arise in applications in terms of context Analyze functions using different representations Construct logarithmic models and solve problems	1-8
Graphing Logarithmic and Exponential Functions (FAL)	Formative Assessment Lesson Pairs	Graph Exponential and Logarithmic Equations and Identify the Key features of each.	1,2,3,6
John and Leonhard at the Café Mathematica	Learning/Performance Individual/Partner	Create and analyze models of scenarios using exponential and logarithmic functions	1-8
Culminating Task: Jason's Graduation	Culminating Task Partner/Small Group	Write expressions in equivalent forms Understand the concept of a	1-8

Present		function and use function notation Interpret functions that arise in applications in terms of the context Analyze functions using different representations Construct exponential models and solve problems	

An end of unit balanced assessment with some constructed response, multiple choice, and technology modeling activities should be administered to thoroughly assess mastery of the standards in this unit.

Investigating Exponential Growth and Decay

Back to Task Table

Mathematical Goals

Develop the concepts of exponential growth and decay through a visual model

Georgia Standards of Excellence

MGSE9-12.A.SSE.3c Use the properties of exponents to transform expressions for exponential functions. *For example, the expression* 1.15^t *, where t is in years, can be rewritten as* $[1.15^{(1/12)}]^{(12t)} \approx 1.012^{(12t)}$ *to reveal the approximate equivalent monthly interest rate if the annual rate is 15%.*

MGSE9-12.F.IF.8b Use the properties of exponents to interpret expressions for exponential functions. *For example, identify percent rate of change in functions such as* $y = (1.02)^t$, $y = (0.97)^t$, $y = (1.01)^{(12t)}$, $y = (1.2)^{(t/10)}$, *and classify them as representing exponential growth and decay. (Limit to exponential and logarithmic functions.)*

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.

Introduction

This task gives students a visual way to experience exponential growth and decay. Two things are going on in this task: As you continue to fold the paper, the process shows a simultaneous exponential growth (the number of sections) and exponential decay (the area of one of the sections). However, this task is also located in Coordinate Algebra, Unit 1. For that reason, another exponential growth/decay task is included after this task. Do not hesitate to ask students if they recall doing the Paper Folding task. If they didn't do it or don't remember it, it is elegant in its simplicity. But if they do remember it still, moving to the other demonstration of exponential functions would be advisable.

Materials

- Large sheet of rectangular paper for folding

Investigating Exponential Growth and Decay

1. Take a large rectangular sheet of paper and fold it in half. You now have two equal sized sections each with an area that is half the original area.

2. Fold the paper in half again. How many sections of paper do you have? What is the area of each section compared to the area of the original piece of paper?

Area is ¼ the area of the original piece of paper.

Continue this process until you cannot fold the paper anymore. Fill in the table below as you go.

Number of Folds	0	1	2	3	4	5
Number of Sections	1	2	4	8	16	32
Area of each section compared to area of original paper	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$

3. The relationship between the number of folds and the number of sections is a function. Why? *For each number of folds there is a unique number of sections.*

   What is the domain of this function? *{0, 1, 2, 3, 4, …}*

   On graph paper let the horizontal axis represent the number of folds. Let the vertical axis represent the number of sections. Plot the points (# of folds, # of sections).

   Does it make sense to connect these points with a smooth curve? Why or why not?

   In the context of this problem, it does not make sense to connect the points. The domain is the set of whole numbers so only points that have x-coordinates that are whole numbers will be included.

   Write the function f for the number of sections of paper you will have after *x* folds.

$$f(x) = 2^x$$

   Use your function to determine the number of sections you would have if you were able to fold the paper 15 times.

   $f(15) = 2^{15}$ *= 32768 sections*

The function f is an example of exponential growth. What do you notice about the table, equation, and graph of an exponential growth function?

4. The relationship between the number of folds and the area of a section is a function. Why? *For each number of folds there is a unique number that represents the area of a section.*

What is the domain of this function? *{0, 1, 2, 3, 4, …}*

Now plot the points (# of folds, section area). Let the horizontal axis represent the number of folds; let the vertical axis represent the area of the section created.

Does it make sense to connect these points with a smooth curve? Why or why not?

In the context of this problem, it does not make sense to connect the points. The domain is the set of whole numbers so only points that have x-coordinates that are whole numbers will be included.

Write the function g for the section area you will have after x folds.

$$g(x) = \left(\frac{1}{2}\right)^x$$

Use your function to determine the area of a section as compared to the area of the original paper if you were able to fold the paper 15 times.

The area is $\frac{1}{2^{15}}$ *of the area of the original piece of paper.*

The function g for the area of a section is an example of exponential decay. What do you notice about the table, equation, and graph of an exponential decay function?

As the independent variable increases in value the dependent variable decreases in value but not at a constant rate. Furthermore, as the independent variable approaches infinity, the dependent variable approaches a constant value.

Investigating Exponential Growth and Decay

1. Take a large rectangular sheet of paper and fold it in half. You now have two equal sized sections each with an area that is half the original area.

2. Fold the paper in half again. How many sections of paper do you have? What is the area of each section compared to the area of the original piece of paper?

   Continue this process until you cannot fold the paper anymore. Fill in the table below as you go.

Number of Folds	0	1	2	3	4	5
Number of Sections						
Area of each section compared to area of original paper						

3. The relationship between the number of folds and the number of sections is a function. Why?

   What is the domain of this function?

   On graph paper let the horizontal axis represent the number of folds. Let the vertical axis represent the number of sections. Plot the points (# of folds, # of sections).

   Does it make sense to connect these points with a smooth curve? Why or why not?

   Write the function f for the number of sections of paper you will have after x folds.

   Use your function to determine the number of sections you would have if you were able to fold the paper 15 times.

   The function f is an example of exponential growth. What do you notice about the table, equation, and graph of an exponential growth function?

4. The relationship between the number of folds and the area of a section is a function. Why?

   What is the domain of this function?

   Now plot the points (# of folds, section area). Let the horizontal axis represent the number of folds; let the vertical axis represent the area of the section created.

   Does it make sense to connect these points with a smooth curve? Why or why not?

   Write the function g for the section area you will have after x folds.

   Use your function to determine the area of a section as compared to the area of the original paper if you were able to fold the paper 15 times.

   The function g for the area of a section is an example of exponential decay. What do you notice about the table, equation, and graph of an exponential decay function?

Graphs of Exponential Functions

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Mathematical Goals
Graph exponential functions
Identify the characteristics of an exponential function

Georgia Standards of Excellence
MGSE9-12.F.IF.7 Graph functions expressed algebraically and show key features of the graph both by hand and by using technology. *(Limit to exponential and logarithmic functions.)*

Investigating Exponential Growth and Decay
Graph exponential and logarithmic functions, showing intercepts and end behavior.

Standards for Mathematical Practice
1. Reason abstractly and quantitatively.
2. Construct viable arguments and critique the reasoning of others.
3. Use appropriate tools strategically.
4. Look for and make use of structure.

Introduction
In this task students graph a variety of exponential functions to determine the characteristics of these functions.

Materials
Graphing calculator or some other graphing utility

In Coordinate Algebra and Algebra I one of the functions studied was the exponential function. By definition an exponential function is $y = a^x$ where $a > 0$ and $a \neq 1$. An exponential function returns powers of a base number a. The input of the exponential function is the exponent and the output is the number obtained when the base number is raised to that exponent.

1. For each function given, represent the function as a table and then use these points to graph the function on graph paper.

   a. $y = 2^x$
   b. $y = 3^x$
   c. $y = 4^x$
   d. $y = 10^x$

2. What common characteristics of these functions do you see? In particular, determine the domain and range of the functions and any intercepts. Also describe any characteristics of their graphs such as increasing/decreasing, asymptotes, end-behavior, etc.

   For each function the domain is all real numbers and the range is positive real numbers. The y-intercept for each is (0,1). There is no x-intercept. The functions are increasing throughout. There is an asymptote of y = 0. As x becomes infinitely small, y is positive but approaches 0 in value. As x becomes infinitely large, y also becomes infinitely large.

   How does the graph of the exponential function change as the base a changes?

   As a increases, the graph increases more quickly.

3. The symbol e represents the irrational number 2.718281828…. Recall an irrational number is represented by a non-terminating, non-repeating decimal number. e is one of those important numbers in mathematics like π that keeps showing up in all kinds of places. $y = e^x$ is the natural exponential function.

   Use graphing technology to graph $y = 2^x$, $y = 3^x$, and $y = e^x$. How do their graphs compare? What do you notice about the graph of $y = e^x$ in relationship to the graphs of $y = 2^x$ and $y = 3^x$?

   The graph of $y = e^x$ *lies between the graphs of* $y = 2^x$ *and* $y = 3^x$.

4. Use graphing technology to graph each function.
   a. $y = 2^{-x}$
   b. $y = 3^{-x}$
   c. $y = 4^{-x}$
   d. $y = 10^{-x}$

5. How do these graphs compare to those in part (1) above? Use what you know about

transformations of functions to explain the relationship between the graphs of $y = 2^x$ and $y = 2^{-x}$.
The graphs are reflections over the y-axis of the graphs in part (1).

6. Does the same relationship hold for $y = 3^x$ and $y = 3^{-x}$? For $y = 4^x$ and $y = 4^{-x}$? In general, what is the relationship between the graphs of $y = a^x$ and $y = a^{-x}$?
Yes, the same relationship holds. The graphs of $y = a^x$ and $y = a^{-x}$ are reflections over the y-axis of each other.

7. Graph $y = \left(\frac{1}{2}\right)^x$. Compare its graph to $y = 2^{-x}$. What do you observe?
These graphs are the same.
Use properties of exponents to explain the relationship between $\left(\frac{1}{2}\right)^x$ and 2^{-x}.

$\left(\frac{1}{2}\right)^x = [\,(2)^{-1}]^{\,x} = 2^{-x}$

Do your observations about the graphs of $y = \left(\frac{1}{2}\right)^x$ and $y = 2^{-x}$ now make sense?

Since $\left(\frac{1}{2}\right)^x$ and 2^{-x} are equal, then the graphs of $y = \left(\frac{1}{2}\right)^x$ and $y = 2^{-x}$ should certainly be equal.

8. Graph $y = 2^x + 3$. How does this graph compare to that of $y = 2^x$?

The graph of $y = 2^x + 3$ is shifted up 3 units from the graph of $y = 2^x$.

Based on what you know about transformations of functions, describe in words how $y = 2^x + 3$ transforms the graph of the parent function $y = 2^x$.

$y = 2^x + 3$ vertically translates $y = 2^x$ up 3 units .

Discuss what you notice about the domain, range, intercepts, and asymptote of $y = 2^x + 3$.
The domain is all real numbers but the range is $y > 3$. The asymptote is $y = 3$ and the y-intercept is $(0, 4)$.

9. Graph $y = 2^{x-5}$. How does this graph compare to that of $y = 2^x$?

Graph has been shifted 5 units to the right.

Based on what you know about transformations of functions, describe in words how $y = 2^{x-5}$ transforms the graph of the parent function $y = 2^x$.
$y = 2^{x-5}$ horizontally translates the graph o f $y = 2^x$ 5 units to the right.

Discuss what you notice about the domain, range, intercepts, and asymptote of $y = 2^{x-5}$.

Domain is all real numbers. Range is y > 0. Read from the graph we find the y-intercept is (0, .03125). Asymptote is y = 0.

10. The exponential function $y = a^x$ is defined for all real numbers $a > 0$ and $a \neq 1$.

   a. Why do you think the function is not defined for bases that are negative real numbers? Often to determine why something cannot be true, it helps to see what would happen if it were true!! So…explore what would happen for negative values of a; for example, see what would happen if $a = -2$. Set up a table of values to see if you can determine a reasonable explanation for why the base is not allowed to be negative in an exponential function.

   For a < 0, there would be x values for which the function is defined. For example, if x = ½, the value of y is undefined since the square root of a negative number is not a real number.

   b. Why do you think the function is not defined for a base of 0 or a base of 1? Explore the functions $y = a^x$ for $a = 0$ and $a = 1$. Can you offer a reasonable explanation for excluding values of 0 and 1 for the base of an exponential function?

   For a = 0, the function is the constant linear function y = 0 (for x ≠ 0) ; for a = 1, the function is the constant linear function y = 1.

Graphs of Exponential Functions

In Coordinate Algebra one of the functions studied was the exponential function. By definition an exponential function is $y = a^x$ where $a > 0$ and $a \neq 1$. An exponential function returns powers of a base number a. The input of the exponential function is the exponent and the output is the number obtained when the base number is raised to that exponent.

1. For each function given, represent the function as a table and then use these points to graph the function on graph paper.

   a. $y = 2^x$
   b. $y = 3^x$
   c. $y = 4^x$
   d. $y = 10^x$

2. What common characteristics of these functions do you see? In particular, determine the domain and range of the functions and any intercepts. Also describe any characteristics of their graphs such as increasing/decreasing, asymptotes, end-behavior, etc.

   How does the graph of the exponential function change as the base a changes?

3. The symbol e represents the irrational number 2.718281828…. Recall an irrational number is represented by a non-terminating, non-repeating decimal number. e is one of those important numbers in mathematics like π that keeps showing up in all kinds of places. $y = e^x$ is the natural exponential function.

   Use graphing technology to graph $y = 2^x$, $y = 3^x$, and $y = e^x$. How do their graphs compare? What do you notice about the graph of $y = e^x$ in relationship to the graphs of $y = 2^x$ and $y = 3^x$?

4. Use graphing technology to graph each function.
   a. $y = 2^{-x}$
   b. $y = 3^{-x}$
   c. $y = 4^{-x}$
   d. $y = 10^{-x}$

5. How do these graphs compare to those in part (1) above? Use what you know about transformations of functions to explain the relationship between the graphs of $y = 2^x$ and $y = 2^{-x}$.

6. Does the same relationship hold for $y = 3^x$ and $y = 3^{-x}$? For $y = 4^x$ and $y = 4^{-x}$? In general, what is the relationship between the graphs of $y = a^x$ and $y = a^{-x}$?

7. Graph $y = \left(\frac{1}{2}\right)^x$. Compare its graph to $y = 2^{-x}$. What do you observe?

   Use properties of exponents to explain the relationship between $\left(\frac{1}{2}\right)^x$ and 2^{-x}.

   Do your observations about the graphs of $y = \left(\frac{1}{2}\right)^x$ and $y = 2^{-x}$ now make sense?

8. Graph $y = 2^x + 3$. How does this graph compare to that of $y = 2^x$?

   Based on what you know about transformations of functions, describe in words how $y = 2^x + 3$ transforms the graph of the parent function $y = 2^x$.

   Discuss what you notice about the domain, range, intercepts, and asymptote of $y = 2^x + 3$.

9. Graph $y = 2^{x-5}$. How does this graph compare to that of $y = 2^x$?

   Based on what you know about transformations of functions, describe in words how $y = 2^{x-5}$ transforms the graph of the parent function $y = 2^x$.

   Discuss what you notice about the domain, range, intercepts, and asymptote of $y = 2^{x-5}$.

10. The exponential function $y = a^x$ is defined for all real numbers $a > 0$ and $a \neq 1$.

a. Why do you think the function is not defined for bases that are negative real numbers? Often to determine why something cannot be true, it helps to see what would happen if it were true!! So…explore what would happen for negative values of a; for example, see what would happen if $a = -2$. Set up a table of values to see if you can determine a reasonable explanation for why the base is not allowed to be negative in an exponential function.

b. Why do you think the function is not defined for a base of 0 or a base of 1? Explore the functions $y = a^x$ for $a = 0$ and $a = 1$. Can you offer a reasonable explanation for excluding values of 0 and 1 for the base of an exponential function?

Zombie Apocalypse Simulation (Spotlight Task)

Back to Task Table

Mathematical Goals

Develop the concepts of exponential growth through a visual model, as well as understand the magnitude of exponential growth and the idea of exponential compounding.

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.

Georgia Standards of Excellence

MGSE9-12.A.SSE.3 Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression. *(Limit to exponential and logarithmic functions.)*

Analyze functions using different representations

MGSE9-12.F.IF.7 Graph functions expressed algebraically and show key features of the graph both by hand and by using technology. *(Limit to exponential and logarithmic functions.)*

MGSE9-12.F.IF.8 Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function. *(Limit to exponential and logarithmic functions.)*

MGSE9-12.F.IF.8b Use the properties of exponents to interpret expressions for exponential functions. *For example, identify percent rate of change in functions such as* $y = (1.02)^t$, $y = (0.97)^t$, $y = (1.01)^{(12t)}$, $y = (1.2)^{(t/10)}$, *and classify them as representing exponential growth and decay. (Limit to exponential and logarithmic functions.)*

Introduction

This task gives students a visual way to experience exponential growth, as well as graphing exponential equations. The fast increase in the rate of growth is explored, as is a beginning of the "Compound Interest" idea in that the original amount (principal) remains PLUS additional. The comparison of large vs. small beans as "Zombies" allows them to speculate as to how the relative size of the "zombie"

Materials

- Dried Pinto Beans and Dried Beans of at least two other types

- Pan or Paper Box Lid (one per pair)
- Cups-2 per pair of students

Zombie Apocalypse Simulation

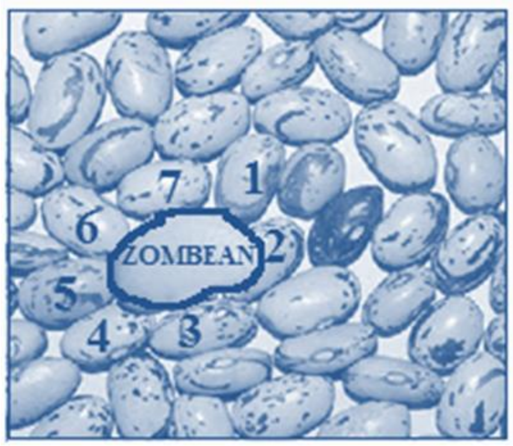
Say to students:
You have been placed into pairs for this activity. You should have a box lid/pan on your desk and two cups. In one of those cups, you should see normal dried pinto beans. Each of those beans represents a healthy member of the human population. Each group also has an empty cup. Take the empty cup and come choose a second type of bean. You can choose Giant Lima Beans, tiny Adzuki Beans, Lentils, or Kidney Beans. It doesn't matter. But your second cup will contain only one type.

For this activity, you will need at least two different types of dried beans. Kidney beans of two colors are good, but you may want to consider using several different types of beans. While each student or pair of students would only get two types, the different types of beans leads to richer discussion. For instance, if every group got pinto beans to represent the non-diseased population, but one group got giant lima beans to represent disease, another group got black beans to represent disease, and another group got kidney beans, the discussions would be far richer.
Materials:
Beans of at least two different types (not mixed together…separated by type)-enough for each group to get two cups –each of different type-would be easier for teacher if the beans are in containers that the beans can be easily poured into, and also that the cups fit into and can scoop from easily to facilitate quick setup and cleanup.

At exactly 8am, a Zombie staggers into a building of healthy, disease free people and begins to infect others. It takes a normal, disease-free person four hours to turn into a Zombie once bitten. *There is actually a Pentagon Action plan for "Zombie Apocalypse."Read below. The Zombie analogy was used by the Pentagot to counter any sort of exponentially-increasing threat. Zombiefication was a tongue-in-cheek look at a very real possibility in terms of disease, biological warfare, Social Media bandwagon threats (which arguably has played major role in the overthrow of many world governments very recently) or propagation of cyber attacks. The text is below-students will appreciate the levity!*

In this exploration, you will use a model to examine the spread of the Zombie pathogen. You will need a cup of pinto beans which represents people and a cup of another type of beans which represents zombies.
Procedure: You may change the bean types listed below if you happen to have different types.
Pour the dried pinto beans in a flat box. Each pinto bean represents a healthy, disease-free individual. *Try to encourage a variety of types of beans if you can to create more interesting discussion later.*
Place one dried white bean in the container. This bean represents a Zombie. *This is time $t=0$, where $b(x)=1$ and $z(x)=1$.* Gently shake the container. *They are stirring it up each time so that the Zombies are not bunched together.* Pat the beans down so there is only one layer. Any pinto bean that is within 1 mm of the white bean has been bitten by the Zombie. *The thickness of a dime is about 1.35 mm. So if they could not squeeze a dime in between the Zombie bean and the*

Pinto…the Pinto is bitten. Replace the newly-bitten pintos with a white Zombie bean, because that's what happens when you are bitten-you turn into a zombie. Count the number of white beans in the container. *The number of new white beans is the number of bites, and the total number of white beans is the number of zombies See the blue photo example below. Student answers will be different based on the type of beans and luck of the Shake.* Write this number in the appropriate column of Table 1.



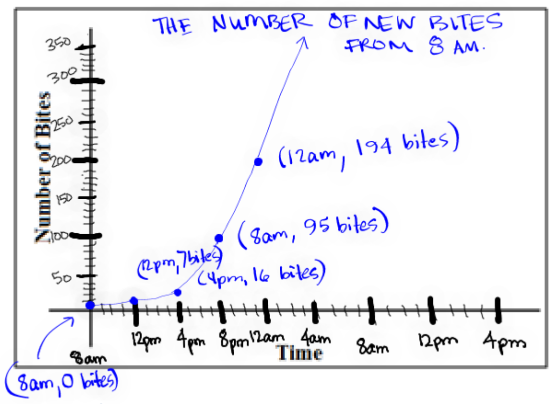
(This table is sample data. The data will vary…)

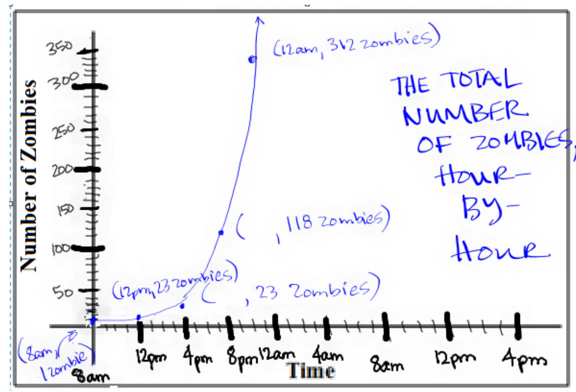
TABLE 1: Simulated Spread of a Disease

Shake Number (x)	Number of Bites (*b*(*x*))	Number of Zombies (*z*(*x*))
(8:00 am) 0	*0*	*1*
(12:00 pm) 1	*7*	*8*
(4:00 pm) 2	*16*	*23*
(8:00 pm) 3	*95*	*118*
(12:00 am) 4	*194*	*312*
(4:00 am) 5	*OUT OF BEANS*	*OUT OF BEANS*
(8:00am) 6	*OUT OF BEANS*	*OUT OF BEANS*

1. What times correlate to each Shake Number? Write those in the space to the left of the Shake Numbers in the chart. *See chart above.*

2. Graph *b(x)* and z(x) on their respective axes, below, being sure to label completely.





3. What is the difference between the meanings of *b(x)* and *z(x)* at any given time? *The difference is that the number of Zombies is the SUM of all of the Number of Bites, inclusive of the most recent ones. In a Calculus sense, the Number of Zombies graph is giving the integral (area under the curve) of the Number of Bites graph. Obviously that's beyond the scope of this course, but if you were to plot dots underneath the "Bites" curve, they would understand that every dot represents a new Zombie.*
4. About how many more shakes do you think it would take for your entire population of pinto bean people to be Zombies? *It only took five, total. (students would guess if it were longer)*

5. What does the y-intercept on your graph represent, as far as diseased/healthy people? *The y-intercept represents the one Zombie. If there were three zombies to start, the y-intercept would be (0,3).*

6. About how many healthy people did your diseased person infect in a single interaction? *Each Zombie bit 6-7 people each time.*

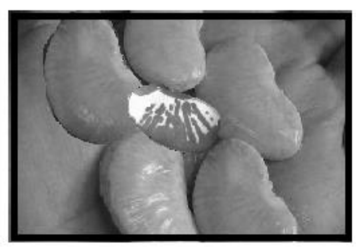
7. How does the size of the pinto bean vs. the Zombie bean affect the number of bites? *The Zombie was bigger, so he could touch more people. In real terms, it may be that the Zombie is bigger, but it could also mean he was quicker or that the environment gives people nowhere to go.*

8. Now come together with another group whose "diseased person" was a different type of bean than the one that you measured, preferably one with a different size, and compare your answers to 1-4 with theirs. Do you notice any differences? If so, why do you think one "diseased person" affects the healthy population differently? *The smaller the Zombie, the smaller the multiplier each time. The bigger the Zombie, the more people it could take out each shake.*

9. For each shake for your group and your neighbors, describe how fast the number of zombies changed mathematically compared with the last shake. What is the pattern or relationship? (It won't be exact because this is a simulation, but you should see a relationship between a shake and the one before it that is similar to the relationship between a shake and the one after). *For this one, the number of bites changed by a factor of around 7 times the previous number. But the total number of Zombies appeared to be a multiple of 8, which is generally not what students expect. It is because of the compounding nature of the Zombie Problem. This will be addressed later after Compound Interest is discussed, and we will revisit this.* $A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$ *. A(t) is the amount in an account at any given time, "t." P is the initial amount, r is the rate of return, n is the number of times the return is calculated, and t is the units of time you choose to use (money uses years but we can use others). So P in this case is 1 (there is one Zombie initially), r is the rate of return, and in this case, it's 700% so r=7. n=1. While this is beyond the scope of their knowledge right now, you can revisit this task after compound interest to determine the actual equation.*
10. Noting the relationships that you and your neighbors noticed #12, what do you think will happen from one shake of a box to another in the photos below, if the speckled bean is the "Zombie?" *You can tell the repeated multiplier by the number of beans surrounding it.*

a. The factor/repeated multiplier would be 4.

b. The factor/repeated multiplier would be about 11.



c. The factor/repeated multiplier would be 6.

d. The factor/repeated multiplier would be 7 or 8. It could be that the little beans at the top aren't close enough to count-but if students recognize this, they understand this concept.

Zombie Apocalypse Simulation (Spotlight Task)

Mathematical Goals
Develop the concepts of exponential growth through a visual model, as well as understand the magnitude of exponential growth and the idea of exponential compounding.

Standards for Mathematical Practice
1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.

Georgia Standards of Excellence

MGSE9-12.A.SSE.3 Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression. *(Limit to exponential and logarithmic functions.)*

<u>Analyze functions using different representations</u>

MGSE9-12.F.IF.7 Graph functions expressed algebraically and show key features of the graph both by hand and by using technology. *(Limit to exponential and logarithmic functions.)*

MGSE9-12.F.IF.8 Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function. *(Limit to exponential and logarithmic functions.)*

MGSE9-12.F.IF.8b Use the properties of exponents to interpret expressions for exponential functions. *For example, identify percent rate of change in functions such as $y = (1.02)^t$, $y = (0.97)^t$, $y = (1.01)^{(12t)}$, $y = (1.2)^{(t/10)}$, and classify them as representing exponential growth and decay. (Limit to exponential and logarithmic functions.)*

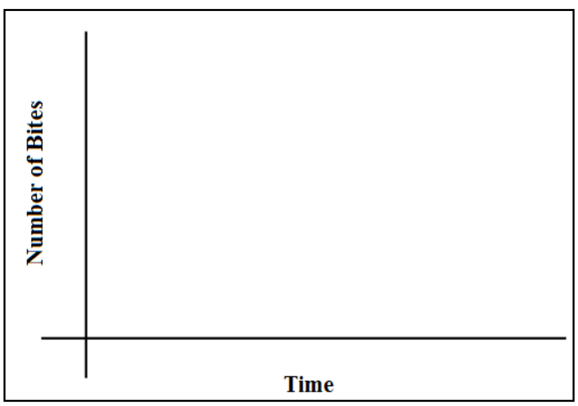
Introduction
This task gives students a visual way to experience exponential growth, as well as graphing exponential equations. The fast increase in the rate of growth is explored, as is a beginning of the "Compound Interest" idea in that the original amount (principal) remains PLUS additional. The comparison of large vs. small beans as "Zombies" allows them to speculate as to how the relative size of the "zombie"

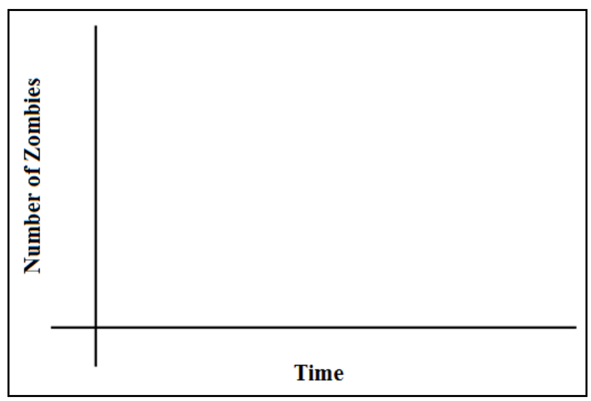
Materials
- Dried Pinto Beans and Dried Beans of at least two other types
- Pan or Paper Box Lid (one per pair)
- Cups-2 per pair of students

The situation:
At exactly 8am, a Zombie staggers into a school building of healthy, disease free people. It takes a normal, disease-free person four hours to turn into a Zombie once bitten. In this exploration, you will use a model to examine the spread of the Zombie pathogen. You will need a cup of pinto beans which represents people and a cup of another type of beans which represents zombies.
TABLE 1: Simulated Spread of a Disease

Shake Number (x)	Number of Bites (*b*(*x*))	Number of Zombies (*z*(*x*))
0		
1		
2		
3		
4		
5		
6		

1. What times correlate to each Shake Number? Write those in the space to the left of the Shake Numbers in the chart.
2. Graph *b(x)* and z(x) on their respective axes, below, being sure to label completely.





3. What is the difference between the meanings of *b(x)* and *z(x)* at any given time?

4. About how many more shakes do you think it would take for your entire population of pinto bean people to be Zombies?

5. What does the y-intercept on your graph represent, as far as diseased/healthy people?

6. About how many healthy people did your diseased person infect in a single interaction?

7. How does the size of the pinto bean vs. the Zombie bean affect the number of bites?

8. Now come together with another group whose “diseased person” was a different type of bean than the one that you measured, preferably one with a different size, and compare your answers to 1-4 with theirs. Do you notice any differences? If so, why do you think one “diseased person” affects the healthy population differently?

9. For each shake for your group and your neighbors, describe how fast the number of zombies changed mathematically compared with the last shake. What is the pattern or relationship? (It won’t be exact because this is a simulation, but you should see a relationship between a shake and the one before it that is similar to the relationship between a shake and the one after).
10. Noting the relationships that you and your neighbors noticed #12, what do you think will happen from one shake of a box to another in the photos below, if the speckled bean is the “Zombie?”

a. b. c. d.

Bacteria in the Swimming Pool[1]

Back to Task Table

Mathematical Goals
Represent a real-world situation with an exponential function
Solve an exponential equation using a numerical and a graphical approach

Georgia Standards of Excellence

MGSE9-12.F.IF.8 Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function. *(Limit to exponential and logarithmic functions.)*

MGSE9-12.F.IF.8b Use the properties of exponents to interpret expressions for exponential functions. *For example, identify percent rate of change in functions such as* $y = (1.02)^t$, $y = (0.97)^t$, $y = (1.01)^{(12t)}$, $y = (1.2)^{(t/10)}$, *and classify them as representing exponential growth and decay. (Limit to exponential and logarithmic functions.)*

Standards for Mathematical Practice
1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.

Introduction
This is a problem that models exponential growth. It is solved using a numerical approach and a graphical approach. The lack of an algebraic way to solve it motivates the need for a logarithm.

Materials
Graphing calculator or some other graphing utility

[1] Adapted from NCTM's Curriculum and Evaluation Standards for School Mathematics Addenda Series, Grades 9-12: Algebra in a Technological World.

Bacteria in the Swimming Pool

The bacteria count in a heated swimming pool is 1500 bacteria per cubic centimeter on Monday morning at 8 AM, and the count doubles each day thereafter.

1. What bacteria count can you expect on Wednesday at 8 AM?

2. Complete the table below.

Time (days) since Monday at 8 AM	0	1	2	3	4	5
Number of bacteria per cc	*1500*	*3000*	*6000*	*12000*	*24000*	*48000*

3. Suppose we want to know the expected bacteria count at 2 PM Thursday, 3.25 days after the initial count. Use the values in your table to estimate the number of bacteria. Explain your thinking.

Students should give a value between 12000 and 24000. Some may incorrectly predict 15000 since 15000 is ¼ of the way between 12000 and 24000 just as 3.25 is ¼ of the distance between 3 and 4. If this happens, come back to it after doing #6; point out the bacteria problem is not a direct variation.

4. To answer this question more precisely, it would be helpful if we can write a function for the bacteria count in terms of the number of days since Monday at 8 AM. To do this, it is helpful to look for a pattern. However, if you calculated the bacteria count for each number of days in the table, then the process you used to get the number of bacteria is probably camouflaged; this means it may be difficult to identify a pattern that can lead you to a generalized expression. Instead, consider writing the number of bacteria for each number of days in terms of 1500, the initial bacteria count. Ask yourself: If you began with a count of 1500 bacteria, how do you get the number of bacteria after 1 day? Do you see this is $1500 \cdot 2$ *? Then the number after 2 days is found by doubling the number of bacteria after 1 day so we now have* $(1500 \cdot 2) \cdot 2$*. Then the number after 3 days is found by doubling the number after 2 days so we have* $(1500 \cdot 2 \cdot 2) \cdot 2$ *. Do you see a pattern? As the number of days increases, what stays the same in the expressions for the number of bacteria? What is changing in the expressions for the number of bacteria?*

1500 stays the same. There is a factor of 2 but what changes is the number of factors of 2.

5. Use the pattern from Problem 3 to write a function *P* that represents the number of bacteria per cc after *t* days. (Be sure your function gives you the same data you wrote in the table of Problem 2.)

Solution: $\mathrm{P(t) = 1500(2)^t}$

6. How can you use the function to determine the number of bacteria present after 3.25 days?

Solution: $P(3.25) = 1500(2)^{3.25} \approx 14270$ *bacteria per cc*

7. Use graphing technology to graph the function. Explain how to use the graph to determine the bacteria count after 3.25 days.

We could trace the function to find the point that has an x-coordinate of approximately 3.25. The y-coordinate of this point is the number of bacteria after 3.25 days.

8. If nothing is done and the bacteria continue to double, how long will it take for the count to reach 3 million bacteria? Write an equation to represent this situation. Find at least 2 different ways to solve the equation.

$1500\,(2^t) = 3000000$

It could be solved graphically. Graph $y_1 = 1500\,(2^t)$ *and* $y_2 = 3000000$ *and find their point of intersection. The x-coordinate of this point is the number of days it will take for the population to reach 3000000.*

We could also use guess and check; just continue to substitute values in for t until you get $1500\,(2^t)$ *to equal 3000000 (or very close to it!).*

The number is 3000000 in just under 11 days.

To solve the equation you wrote in Problem 8 algebraically, we need a strategy to isolate the exponent *t*. This strategy requires logarithms that are defined in the next task.

Bacteria in the Swimming Pool

The bacteria count in a heated swimming pool is 1500 bacteria per cubic centimeter on Monday morning at 8 AM, and the count doubles each day thereafter.

1. What bacteria count can you expect on Wednesday at 8 AM?

2. Complete the table below.

Time (days) since Monday at 8 AM	0	1	2	3	4	5
Number of bacteria per cc	1500					

3. Suppose we want to know the expected bacteria count at 2 PM Thursday, 3.25 days after the initial count. Use the values in your table to estimate the number of bacteria. Explain your thinking.

4. To answer this question more precisely, it would be helpful if we can write a function for the bacteria count in terms of the number of days since Monday at 8 AM. To do this, it is helpful to look for a pattern. However, if you calculated the bacteria count for each number of days in the table, then the process you used to get the number of bacteria is probably camouflaged; this means it may be difficult to identify a pattern that can lead you to a generalized expression. Instead, consider writing the number of bacteria for each number of days in terms of 1500, the initial bacteria count. Ask yourself: If you began with a count of 1500 bacteria, how do you get the number of bacteria after 1 day? Do you see this is $1500 \cdot 2$? Then the number after 2 days is found by doubling the number of bacteria after 1 day so we now have $(1500 \cdot 2) \cdot 2$. Then the number after 3 days is found by doubling the number after 2 days so we have $(1500 \cdot 2 \cdot 2) \cdot 2$. Do you see a pattern? Do you see a pattern? As the number of days increases, what stays the same in the expressions for the number of bacteria? What is changing in the expressions for the number of bacteria?

5. Use the pattern from Problem 3 to write a function *P* that represents the number of bacteria per cc after *t* days. (Be sure your function gives you the same data you wrote in the table of Problem 2.)

6. How can you use the function to determine the number of bacteria present after 3.25 days?

7. Use graphing technology to graph the function. Explain how to use the graph to determine the bacteria count after 3.25 days.

8. If nothing is done and the bacteria continue to double, how long will it take for the count to reach 3 million bacteria? Write an equation to represent this situation. Find at least 2 different ways to solve the equation.

To solve the equation you wrote in Problem 8 algebraically, we need a strategy to isolate the exponent *t*. This strategy requires logarithms that are defined in the next task.

What is a Logarithm? (Spotlight Task)

Back to Task Table

****If you have been using the rapid recall quizzes in the appendix, this should really be a very very quick task for students. They will (by now) already know that LogArithm = Logical Arithmetic, and they will already have seen how it's just thinking backwards. In fact, with the Logical Arithmetic, this task will be quick AND will be a real, "AHA!" for them. So quick, in fact, that there is now an additional activity involving a card sort to reinforce the verbal→log→exponential→meaning to logarithms.*

Mathematical Goals

Understand the concept of a logarithm
Develop the ability to move flexibly from exponential form to logarithmic form and vice versa
Understand how logarithms can be used to solve problems
Use the calculator to evaluate common and natural logarithms

Georgia Standards of Excellence

MGSE9-12.F.IF.8b Use the properties of exponents to interpret expressions for exponential functions. *For example, identify percent rate of change in functions such as* $y = (1.02)^t$, $y = (0.97)^t$, $y = (1.01)^{(12t)}$, $y = (1.2)^{(t/10)}$, *and classify them as representing exponential growth and decay. (Limit to exponential and logarithmic functions.)*

MGSE9-12.F.BF.5 Understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents.

Standards for Mathematical Practice

1. Reason abstractly and quantitatively.
2. Construct viable arguments and critique the reasoning of others.
3. Model with mathematics.
4. Use appropriate tools strategically.
5. Attend to precision.
6. Look for and make use of structure.

Introduction

In this task students learn there is a direct connection between a logarithm and an exponent.

Materials

Calculator

As a society, we are accustomed to performing an action and then undoing or reversing that action. Identify the action that undoes each of those named.

Putting on a jacket *Removing a jacket*

Opening a door *Closing a door*

Walking forward *Walking backward*

Depositing money in a bank *Withdrawing money from a bank*

In mathematics we also find it useful to be able to undo certain actions.
What action undoes adding 5 to a number? *Subtracting 5 from the result*

What action undoes multiplying a number by 4? *Dividing the result by 4*

What action undoes squaring a number?

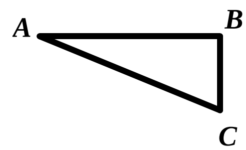
Taking the square root of the result

We say that addition and subtraction are inverse operations because one operation undoes the other. Multiplication and division are also inverse operations; squaring and taking the square root are inverse operations.

Inverse operations in mathematics help us solve equations. Consider the equation $2x + 3 = 35$. This equation implies some number (represented by x) has been multiplied by 2; then 3 has been added to the product for a result of 35. To determine the value of x, we subtract 3 from 35 to undo adding 3. This means that 2x must equal to 32. To undo multiplying the number by 2, we divide 32 by 2 and find the number represented by x is 16.

Explain how inverse operations are used in the solution of the following problems.

In right triangle ABC with right angle B, if BC is 8 cm and AC is 17 cm, determine the measure of angle A.



Since ABC is a right triangle, then
$sin\ A = \frac{8}{17}$ $so\ A = sin^{-1}\left(\frac{8}{17}\right)$.

If $\sqrt{x+8} = 10$, determine the value of x.

We've taken a number x, added 8 to it, then taken the square root of this sum and gotten 10. So we must reverse our steps using the inverse operations. Take the result 10 and first square it to get 100 and then subtract 8 from 100.

Solve $x^3 = 27$ for x.

If we have cubed a number x to get 27 so we must take the cube root of the result, 27, to determine the original number x.

Solve $2x = 10$ for x.

We have multiplied a number by 2 and gotten 10 so we must divide the result 10 by 2 to determine the original number x.

In problem 8 of Task 3, "Bacteria in the Swimming Pool," we obtained the equation $1500(2)^t = 3000000$ to solve for t. This equation is equivalent to equivalent to $2^t = 2000$. Why? While in Task 3 we had no algebraic way to solve this equation because we lacked a strategy to isolate the exponent t. Our goal in this current task is to continue our idea of "undoing" to solve an equation; specifically, we need to find an action that will undo raising 2 to a power. This action needs to report the exponent to which 2 has been raised in order to obtain 2000. In order to rewrite $2^t = 2000$ so t is isolated, we need to define logarithms. Logarithms allow us to rewrite an exponential equation so that the exponent is isolated. Specifically, if $a = b^c$, then "c is the logarithm with base b of a" and is written as $\log_b a = c$. (We read "$\log_b a = c$" as "log base b of a is c.")

Using logarithms we can write $2^t = 2000$ as $\log_2 2000 = t$. These two expressions are equivalent, and in the expression $\log_2 2000 = t$ we have t isolated. Although this is a good thing, we still need a way to evaluate the expression $\log_2 2000$. We know it equals the exponent to which 2 must be raised in order to obtain a value of 2000, but we still don't know how to calculate this value. Hang on…we will get there in the next task! First some preliminary work must be done!

Let's look at a few examples:
$10^2 = 100$ is equivalent to $\log_{10}100 = 2$. Notice that 10 is the *base* in both the exponential form and the logarithmic form. Also notice that the logarithm is the exponent to which 10 is raised to obtain 100.
Evaluate $\log_4 64$. This question asks for the exponent to which 4 is raised to obtain 64. In other words, 4 to what power equals 64? ____*3*____
Consider the following problem: $\log_2 n = 4$. This equation is equivalent to $2^4 = n$; thus, $n = 16$.
The relationship between exponents and logarithms must be understood clearly. The following practice problems will help you gain this understanding.

Rewrite each exponential equation as a logarithmic equation.

$6^2 = 36$ $\quad$ $log_6 36 = 2$

$10^3 = 1000$ $\quad$ $log_{10} 1000 = 3$

$25^{\frac{1}{2}} = 5$ $\quad$ $log_{25} 5 = \frac{1}{2}$

Rewrite each logarithmic equation as an exponential equation.

$\log_4 16 = 2$ $\quad$ $4^2 = 16$

$\log_6 1 = 0$ $\quad$ $6^0 = 1$

$\log_3 n = t$ $\quad$ $3^t = n$

Evaluate each of the following.

$\log_{10}(0.1)$ *We want the exponent that 10 must be raised in order to get 0.1 or* $\frac{1}{10}$. *That exponent is -1 so* $log_{10}(0.1) = -1$

$\log_3 81$ $\quad$ *4*

$\log_2 \frac{1}{16}$ $\quad$ *-4*

$\log_5 5$ $\quad$ *1*

Between what two whole numbers is the value of $\log_3 18$?

$log_3 18$ *is the exponent to which 3 must be raised to obtain 18.*
Since $3^2 = 9$ *and* $3^3 = 27$, *then the exponent to which 3 must be raised to obtain 18 is between 2 and 3.*

Between what two whole numbers is the value of $\log_2 50$?

Since $2^5 = 32$ *and* $2^6 = 64$, *then the exponent 2 must be raised to obtain 50 is between 5 and 6.*

Solve each logarithmic equation for x.

a) $\log_9 81 = x$
b) $\log_2 32 = x$
c) $\log_7 1 = x$
d) $\log_8 x = 3$
e) $\log_5(3x + 1) = 2$
f) $\log_6(4x - 7) = 0$

Solution: a. $x = 2$ *b.* $x = 5$ *c.* $x = 0$ *d.* $x = 512$ *e.* $x = 8$ *f.* $x = 2$

Hopefully you now have an understanding of the relationship between exponents and logarithms. In logarithms, just as with exponential expressions, any positive number can be a base except 1 (we will explore this fact later). Logarithms which use 10 for the base are called common logarithms and are expressed simply as *log x*. It is not necessary to write the base. Calculators are programmed to evaluate common logarithms.

Use your calculator to evaluate log 78. First think about what this expression means.

Solution: $x \approx 1.89$

Understanding logarithms can help solve more complex exponential equations. Consider solving the following equation for x:

$$10^x = 350$$

We know that $10^2 = 100$ and $10^3 = 1000$ so x should be between 2 and 3. Rewriting $10^x = 350$ as the logarithmic equation $x = \log 350$, we can use the calculator to determine the value of x to the nearest hundredth. *Solution:* $x = 2.54$

Solve each of the following for x using logarithms. Determine the value of x to the nearest hundredth.

1. $10^x = 15$ — $x \approx 1.18$
2. $10^x = 0.3458$ — $x \approx -0.46$
3. $3(10^x) = 2345$ — $x \approx 2.89$
4. $2(10^x) = -6538$ — $x \approx 3.51$

Logarithms that use the irrational number e as a base are of particular importance in many applications. Recall an irrational number is represented by a non-terminating, non-repeating decimal number. The value of e is $2.718281828\ldots$. The function $y = \log_e x$ is the natural logarithmic function and has a base of e. The shorthand for $y = \log_e x$ is $y = \ln x$. Calculators are also programmed to evaluate natural logarithms.

Consider $\ln 34$ which means the exponent to which the base e must be raised to obtain 34. The calculator evaluates $\ln 34$ as approximately 3.526. This value makes sense because $e^{3.526}$ is approximately 33.9877, a value very close to 34!

Evaluate $\ln 126$. Use an exponential expression to confirm your solution makes sense. *Approx 4.84 ; since* $e^{4.84} \approx 126.47$, *the solution makes sense.*

Evaluate $\ln e$. Explain why your answer makes sense. *ln e means the exponent to which e is raised to obtain e. Since* $e^1 = e$, *then this exponent is 1 so* $\ln e = 1$.

If $\ln x = 7$, determine the value of x to the nearest hundredth. HINT: Write the logarithmic equation in exponential form.

$\ln x = 7$ *is equivalent to* $e^7 = x$. *So x is approx 1096.63.*

If $e^x = 85$, determine the value of x to the nearest hundredth. HINT: Write the exponential equation in logarithmic form.

$e^x = 85$ *is equivalent to* $\ln 85 = x$ *so x is approx 4.44.*

STUDENT COPY-CARD SORT-SHOULD BE CUT APART BEFORE BEING PLACED INTO STUDENT HANDS

A **The logarithm, base two, of 16 is some number.**	£ **Two to the power of some number is 16.**	U $\log_2 16 = x$	Q **4**	S $2^x = 16$
P **The logarithm, base 16, of 2 is a number.**	B **Sixteen to the power of some number is 2.**	O $\log_{16} 2 = x$	W $\frac{1}{4}$	G $16^x = 2$
Ω **The logarithm with what base of 2 is 16?**	M Some number to the power of 16 is 2.	N $\log_x 2 = 16$	E $2^{\frac{1}{16}}$	V $x^{16} = 2$
L **The logarithm with what base of 16 is 2?**	F Some number to the power of 2 is 16.	T $\log_x 16 = 2$	I **4**	Z $x^2 = 16$
μ **The logarithm, base two, of some number is 16.**	C **Two to the power of 16 is some number.**	R $\log_2 x = 16$	K **65,536**	J $2^{16} = x$
H **The logarithm, base 16, of some number is 2.**	X **Sixteen to the power of 2 is some number.**	Y $\log_{16} x = 2.$	θ 256	Я $16^2 = x$

What is a Logarithm? (Spotlight Task)

As a society, we are accustomed to performing an action and then undoing or reversing that action. Identify the action that undoes each of those named.
Putting on a jacket

Opening a door

Walking forward

Depositing money in a bank

In mathematics we also find it useful to be able to undo certain actions.
What action undoes adding 5 to a number?

What action undoes multiplying a number by 4?

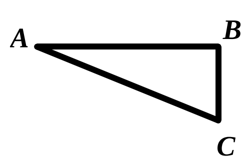
What action undoes squaring a number?

We say that addition and subtraction are inverse operations because one operation undoes the other. Multiplication and division are also inverse operations; squaring and taking the square root are inverse operations.

Inverse operations in mathematics help us solve equations. Consider the equation $2x + 3 = 35$. This equation implies some number (represented by x) has been multiplied by 2; then 3 has been added to the product for a result of 35. To determine the value of x, we subtract 3 from 35 to undo adding 3. This means that 2x must equal to 32. To undo multiplying the number by 2, we divide 32 by 2 and find the number represented by x is 16.

Explain how inverse operations are used in the solution of the following problems.

In right triangle ABC with right angle B, if BC is 8 cm and AC is 17 cm, determine the measure of angle A.



If $\sqrt{x+8} = 10$, determine the value of x.

Solve $x^3 = 27$ for x.

Solve $2x = 10$ for x.

In problem 8 of Task 3, "Bacteria in the Swimming Pool," we obtained the equation $1500(2)^t = 3000000$ to solve for *t*. This equation is equivalent to equivalent to $2^t = 2000$. Why? While in Task 3 we had no algebraic way to solve this equation because we lacked a strategy to isolate the exponent *t*. Our goal in this current task is to continue our idea of "undoing" to solve an equation; specifically, we need to find an action that will undo raising 2 to a power. This action needs to report the exponent to which 2 has been raised in order to obtain 2000. In order to rewrite $2^t = 2000$ so t is isolated, we need to define logarithms. Logarithms allow us to rewrite an exponential equation so that the exponent is isolated. Specifically, if $a = b^c$, then "c is the logarithm with base b of a" and is written as $\log_b a = c$. (We read "$\log_b a = c$" as "log base b of a is c.")

Using logarithms we can write $2^t = 2000$ as $\log_2 2000 = t$. These two expressions are equivalent, and in the expression $\log_2 2000 = t$ we have *t* isolated. Although this is a good thing, we still need a way to evaluate the expression $\log_2 2000$. We know it equals the exponent to which 2 must be raised in order to obtain a value of 2000, but we still don't know how to calculate this value. Hang on…we will get there in the next task! First some preliminary work must be done!

Let's look at a few examples:
***$10^2 = 100$* is equivalent to *$\log_{10}100 = 2$*.** Notice that 10 is the *base* in both the exponential form and the logarithmic form. Also notice that the logarithm is the exponent to which 10 is raised to obtain 100.

Evaluate *$\log_4 64$*. This question asks for the exponent to which 4 is raised to obtain 64. In other words, 4 to what power equals 64? ________
Consider the following problem: $\log_2 n = 4$. This equation is equivalent to $2^4 = n$; thus $n = 16$.

The relationship between exponents and logarithms must be understood clearly. The following practice problems will help you gain this understanding.

Rewrite each exponential equation as a logarithmic equation.
$6^2 = 36$
$10^3 = 1000$
$25^{\frac{1}{2}} = 5$

Rewrite each logarithmic equation as an exponential equation.
$\log_4 16 = 2$
$\log_6 1 = 0$
$\log_3 n = t$

Evaluate each of the following.

$\log_{10}(0.1)$ $\quad$ $\log_3 81$ $\quad$ $\log_2 \frac{1}{16}$ $\quad$ $\log_5 5$

Between what two whole numbers is the value of $\log_3 18$?

Between what two whole numbers is the value of $\log_2 50$?

Solve each logarithmic equation for x.

a) $\log_9 81 = x$
b) $\log_2 32 = x$
c) $\log_7 1 = x$
d) $\log_8 x = 3$
e) $\log_5(3x + 1) = 2$
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Hopefully you now have an understanding of the relationship between exponents and logarithms. In logarithms, just as with exponential expressions, any positive number can be a base except 1 (we will explore this fact later). Logarithms which use 10 for the base are called common logarithms and are expressed simply as *log x*. It is not necessary to write the base. Calculators are programmed to evaluate common logarithms.

Use your calculator to evaluate log 78. First think about what this expression means.

Understanding logarithms can help solve more complex exponential equations. Consider solving the following equation for x:

$$10^x = 350$$

We know that $10^2 = 100$ and $10^3 = 1000$ so *x* should be between 2 and 3. Rewriting $10^x = 350$ as the logarithmic equation $x = \log 350$, we can use the calculator to determine the value of x to the nearest hundredth.

Solve each of the following for *x* using logarithms. Determine the value of x to the nearest hundredth.

1. $10^x = 15$
2. $10^x = 0.3458$
3. $3(10^x) = 2345$
4. $2(10^x) = -6538$

Logarithms that use the irrational number e as a base are of particular importance in many applications. Recall an irrational number is represented by a non-terminating, non-repeating decimal number. The value of e is 2.718281828…. The function $y = \log_e x$ is the natural

logarithmic function and has a base of e. The shorthand for $y = \log_e x$ is $y = \ln x$. Calculators are also programmed to evaluate natural logarithms.
Consider $\ln 34$ which means the exponent to which the base e must be raised to obtain 34. The calculator evaluates $\ln 34$ as approximately 3.526. This value makes sense because $e^{3.526}$ is approximately 33.9877, a value very close to 34!

Evaluate $\ln 126$. Use an exponential expression to confirm your solution makes sense.

Evaluate $\ln e$. Explain why your answer makes sense.

If $\ln x = 7$, determine the value of x to the nearest hundredth. HINT: Write the logarithmic equation in exponential form.

If $e^x = 85$, determine the value of x to the nearest hundredth. HINT: Write the exponential equation in logarithmic form.

The cards you have been given are to be sorted. There will be six matches of five cards each. You will see a verbal description of the exponential function, a verbal description of the logarithmic function that means the same thing, the logarithmic equation written out, the exponential equation written out, and the solution to the equations. Make the matches, and then be prepared to tell:

a) Of the two equations that you saw, the exponential and the logarithmic, which one helped you find the solution the easiest?

b) How does the solution that you found work for both the logarithmic and the exponential equation?

c) Which ones could you have solved without any work at all except just using your calculator?

STUDENT COPY-CARD SORT-SHOULD BE CUT APART BEFORE BEING PLACED INTO STUDENT HANDS

A **The logarithm, base two, of 16 is some number.**	£ **Two to the power of some number is 16.**	U $\log_2 16 = x$	Q **4**	S $2^x = 16$
P **The logarithm, base 16, of 2 is a number.**	B **Sixteen to the power of some number is 2.**	O $\log_{16} 2 = x$	W $\frac{1}{4}$	G $16^x = 2$
Ω **The logarithm with what base of 2 is 16?**	M Some number to the power of 16 is 2.	N $\log_x 2 = 16$	E $2^{\frac{1}{16}}$	V $x^{16} = 2$
L **The logarithm with what base of 16 is 2?**	F Some number to the power of 2 is 16.	T $\log_x 16 = 2$	I **4**	Z $x^2 = 16$
μ **The logarithm, base two, of some number is 16.**	C **Two to the power of 16 is some number.**	R $\log_2 x = 16$	K **65,536**	J $2^{16} = x$
H **The logarithm, base 16, of some number is 2.**	X **Sixteen to the power of 2 is some number.**	Y $\log_{16} x = 2.$	θ 256	Я $16^2 = x$

Evaluating Logarithms that are not Common or Natural

Back to Task Table

Mathematical Goals
Develop a strategy for evaluating logarithms that have bases different from 10 or *e*
Use technology to evaluate logarithms with bases other than 10 or *e*

Georgia Standards of Excellence

MGSE9-12.A.SSE.3 Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression. *(Limit to exponential and logarithmic functions.)*

MGSE9-12.A.SSE.3c Use the properties of exponents to transform expressions for exponential functions. *For example, the expression* 1.15^t *, where t is in years, can be rewritten as* $[1.15^{(1/12)}]^{(12t)} \approx 1.012^{(12t)}$ *to reveal the approximate equivalent monthly interest rate if the annual rate is 15%.*

Standards for Mathematical Practice
1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Introduction
This task ultimately leads to the change of base formula with common logarithms. Knowing this formula helps students evaluate logarithms with bases other than 10 or e. However, the deeper understanding gained by deriving this formula is extremely important. You may want to work through this task as a whole class, posing questions along the way for students to discuss with their neighbors.

Materials
Calculator

What does the expression $\log 15$ mean? What is the value of $\log 15$? How can you show the value you obtained is correct?

$\log 15$ *means the exponent to which 10 is raised to yield 15;* $\log 15 \approx 1.176$; *Observe* $10^{1.176} \approx 14.996 \approx 15$

You can use your calculator to obtain the values of $\log 15$ because the calculator is programmed to evaluate common logarithms. The calculator can also evaluate natural logarithms such as $\ln 20$. A good question is …..how do I evaluate a logarithm such as $\log_2 45$ that has a base that is not 10 or e? This task is designed to help you evaluate such expressions. To make sense of the idea, you must read each part carefully and look for patterns that will help you reach a generalization.

First of all, explain what $\log_2 45$ means. Also, determine between what two whole numbers $\log_2 45$ is located and explain your thinking.

$log_2 45$ means the exponent to which 2 is raised to yield 45

Now let's see how to determine its value. Do you agree that any positive number can be written as a power of 10? Let's try it!

- How can you write 1 as a power of 10?
- How can you write 10 as a power of 10?
- What about 2? What about 45? What about 70?

Did you use guess and check to determine the exponent or did you use your understanding of logarithms to calculate the exponent? For example, what does $\log 2$ mean? What does $\log 45$ mean?

Since any positive number can be written as a power of 10, we can use this fact to help us evaluate an expression such as $\log_2 45$.

Now $\log_2 45$ equals some value. Let's call it x. (And we know x is the exponent to which __2_ must be raised in order to obtain _45).

So $\log_2 45 = x$. This means $2^x = 45$. Why?

Now any number can be written as a power of 10 so let r and s represent the exponents such that $45 = 10^r$ and $2 = 10^s$.

This means that $10^{sx} = 10^r$. Why?
And now we know that $sx = r$. Why?
Now we know that $x = \frac{r}{s}$. Why?

Remember that $\log_2 45 = x$ so this means that $\log_2 45 = \frac{r}{s}$
But $45 = 10^r$ and $2 = 10^s$ so we know that $r = \log 45$ and $s = \log 2$. Why?
Therefore, $\log_2 45 = \frac{\log 45}{\log 2}$. Notice that we've written $\log_2 45$ in terms of common logarithms.
Since we can evaluate $\frac{\log 45}{\log 2}$, we now know the value of $\log_2 45$.

Use the same strategy to evaluate $\log_6 132$.

In summary, we found $\log_2 45 = \frac{\log 45}{\log 2}$. Then you showed $\log_6 132 =$ ____$\frac{\log 132}{\log 6}$____.

What patterns are you observing?

Based on the patterns you've noticed, can you suggest an easy way using common logarithms to evaluate $\log_2 79$? $\quad \log_2 79 = \frac{\log 79}{\log 2}$

Now let's generalize; that is, given any expression $\log_b a$, what expression in terms of common logarithms is equivalent to $\log_b a$? $\quad \log_b a = \frac{\log a}{\log b}$

This is the <u>change of base formula</u> using common logarithms.

The generalization you've made is based on **inductive reasoning**. Now use **deductive reasoning** to prove your generalization.

Let $\log_b a = x$. This means that $b^x = a$. Let r and s be real numbers such that $a = 10^r$ and $b = 10^s$. Then $r = \log a$ and $s = \log b$. Since $b^x = a$ this means that $(10^s)^x = 10^r$. But this means that $sx = r$. So $x = \frac{r}{s}$ which means that $x = \frac{\log a}{\log b}$

Since the calculator can also evaluate natural logarithms, use the change of based formula outlined above to evaluate $\log_2 45$ using natural logarithms rather than common logarithms. Begin by writing 2 and 45 as powers of e. What do you observe?

When students work through this, they find that $\frac{\ln 45}{\ln 2} = \frac{\log 45}{\log 2}$; that is, the ratios of these logarithms are equal. In fact, the more general change of base formula says that $\log_b a = \frac{\log_c a}{\log_c b}$ for any positive real number c.

Remember the problem "Bacteria in the Swimming Pool"? In this problem the bacteria count in a heated swimming pool was 1500 per cubic centimeter on Monday morning at 8 AM, and the count doubled each day thereafter. We determined the function for the number of bacteria t days after the initial count was $P(t) = 1500(2)^t$. In the last question of this task, we wanted to know how long it would take for the count to reach 3 million bacteria. This meant we needed to solve the equation $1500(2)^t = 3000000$. At the time we had no way to solve the equation algebraically. Use what you have learned about logarithms to find algebraically the solution to $1500(2)^t = 3000000$.

Solution:

$1500(2)^t = 3000000$ $\quad$ *So* $(2)^t = 2000$ $\quad$ *This means that* $t = \log_2 2000$

So $t = \frac{\log 2000}{\log 2}$ *(about 11). Thus, the bacteria count reaches 3 million in approx 11 days.*

Evaluating Logarithms that are not Common or Natural

What does the expression $\log 15$ mean? What is the value of $\log 15$? How can you show the value you obtained is correct?

You can use your calculator to obtain the values of $\log 15$ because the calculator is programmed to evaluate common logarithms. The calculator can also evaluate natural logarithms such as $\ln 20$. A good question is …..how do I evaluate a logarithm such as $\log_2 45$ that has a base that is not 10 or e? This task is designed to help you evaluate such expressions. To make sense of the idea, you must read each part carefully and look for patterns that will help you reach a generalization.

First of all, explain what $\log_2 45$ means. Also, determine between what two whole numbers $\log_2 45$ is located and explain your thinking.

Now let's see how to determine its value. Do you agree that any positive number can be written as a power of 10? Let's try it!
How can you write 1 as a power of 10? How can you write 10 as a power of 10? What about 2? What about 45? What about 70?

Did you use guess and check to determine the exponent or did you use your understanding of logarithms to calculate the exponent? For example, what does $\log 2$ mean? What does $\log 45$ mean?

Since any positive number can be written as a power of 10, we can use this fact to help us evaluate an expression such as $\log_2 45$.

Now $\log_2 45$ equals some value. Let's call it x. (And we know x is the exponent to which ______ must be raised in order to obtain __________.)

So $\log_2 45 = x$. This means $2^x = 45$. Why?

Now any number can be written as a power of 10 so let r and s represent the exponents such that $45 = 10^r$ and $2 = 10^s$.
This means that $10^{sx} = 10^r$. Why?

And now we know that $sx = r$. Why?

Now we know that $x = \frac{r}{s}$. Why?

Remember that $\log_2 45 = x$ so this means that $\log_2 45 = \frac{r}{s}$ But $45 = 10^r$ and $2 = 10^s$ so we know that $r = \log 45$ and $s = \log 2$. Why?

Therefore, $\log_2 45 = \frac{\log 45}{\log 2}$. Notice that we've written $\log_2 45$ in terms of common logarithms.

Since we can evaluate $\frac{\log 45}{\log 2}$, we now know the value of $\log_2 45$.

We found in the previous problem that $\log_2 45 = \frac{\log 45}{\log 2}$.

Use the same strategy to evaluate $\log_6 132$.

In problem 4 we found $\log_2 45 = \frac{\log 45}{\log 2}$. In problem 5 you showed $\log_6 132 =$ ___________.
What patterns are you observing?

Based on the patterns you've noticed, can you suggest an easy way using common logarithms to evaluate $\log_2 79$?

Now let's generalize; that is, given any expression $\log_b a$, what expression in terms of common logarithms is equivalent to $\log_b a$?

The generalization you've made is based on **inductive reasoning**. Now use **deductive reasoning** to prove your generalization.

Since the calculator can also evaluate **natural logarithms**, use the strategy outlined above to evaluate $\log_2 45$ using **natural logarithms** rather than common logarithms.
Begin by writing 2 and 45 as powers of e. What do you observe?

Remember the problem “Bacteria in the Swimming Pool”? In this problem the bacteria count in a heated swimming pool was 1500 per cubic centimeter on Monday morning at 8 AM, and the count doubled each day thereafter.

We determined the function for the number of bacteria t days after the initial count was $P(t) = 1500(2)^t$.

In the last question of this task, we wanted to know how long it would take for the count to reach 3 million bacteria.

This meant we needed to solve the equation $1500(2)^t = 3000000$.

At the time we had no way to solve the equation algebraically.

Use what you have learned about logarithms to algebraically find the solution to $1500(2)^t = 3000000$.

The Logarithmic Function

Back to Task Table

Mathematical Goals
Graph logarithmic functions
Identify characteristics of logarithmic functions
Develop the ability to move flexibly from exponential form to logarithmic form and vice versa

Georgia Standards of Excellence

MGSE9-12.F.IF.7 Graph functions expressed algebraically and show key features of the graph both by hand and by using technology. *(Limit to exponential and logarithmic functions.)*

Standards for Mathematical Practice
1. Reason abstractly and quantitatively.
2. Construct viable arguments and critique the reasoning of others.
3. Model with mathematics.
4. Use appropriate tools strategically.
5. Look for and make use of structure.

Introduction
Students graph various logarithmic functions by hand and with technology to determine the characteristics of these graphs.

Materials
Graphing calculator or some other graphing utility

So far we have looked at logarithms as exponents, but in this task we will extend our study of logarithms by looking at the logarithmic function. The logarithmic function is defined as $y = \log_b x$ for $b > 0$ and $b \neq 1$. Associated with every logarithmic function is a base number. Given an input value, the logarithmic function returns the exponent to which the base number is raised to obtain this input; thus, the output of the logarithmic function is an exponent.

In Task 2 we saw there is a relationship between exponents and logarithms. The ability to go from an exponential expression to a logarithmic expression is powerful. Not surprisingly, there is a connection between an exponential function and a logarithmic function. In fact, $y = \log_b x$ implies $b^y = x$. Why? The connection between an exponential function and a logarithmic function will be explored more deeply in Unit 6.

For each function given, complete the table of values and then use these points to graph the function on graph paper.

$$y = \log x$$

x	$y = log\, x$
1	0
10	1
20	1.3
50	1.7
100	2

$$y = \log_2 x$$

x	$y = log_2\, x$
1	0
2	1
4	2
8	3
16	4

What common characteristics of these functions do you see? In particular, determine the domain and range of the functions and any intercepts. Also describe any characteristics of their graphs such as increasing/decreasing, asymptotes, end-behavior, etc.

Use graphing technology to graph $y = \log x$. Does your graph agree with your hand-drawn graph?

Use graphing technology to graph $y = \log_2 x$. (Remember you can write any logarithmic expression in terms of common logarithms—this will allow you to graph $y = \log_2 x$ with your technology.) Does your graph agree with your hand-drawn graph?

Use graphing technology to graph $y = \ln x$.

How does the graph of the logarithmic function change as the base b changes?

Use graphing technology to graph each function.
$y = -\log x$

$$y = -\log_2 x$$
$$y = -\ln x$$

How do these graphs compare to the graphs of $y = \log x$ and $y = \log_2 x$ and $y = \ln x$?
The graphs are reflections of each other over the x-axis.

Use what you know about transformations of functions to explain the relationship between $y = \log x$ and $y = -\log x$?

$y = -log\, x$ reflects $y = log\, x$ over the x-axis.

Does the same relationship hold for the graphs of $y = \log_2 x$ and $y = -\log_2 x$?
For $y = \ln x$ and $y = -\ln x$? yes; yes
In general, what is the relationship between $y = \log_b x$ and $y = -\log_b x$?

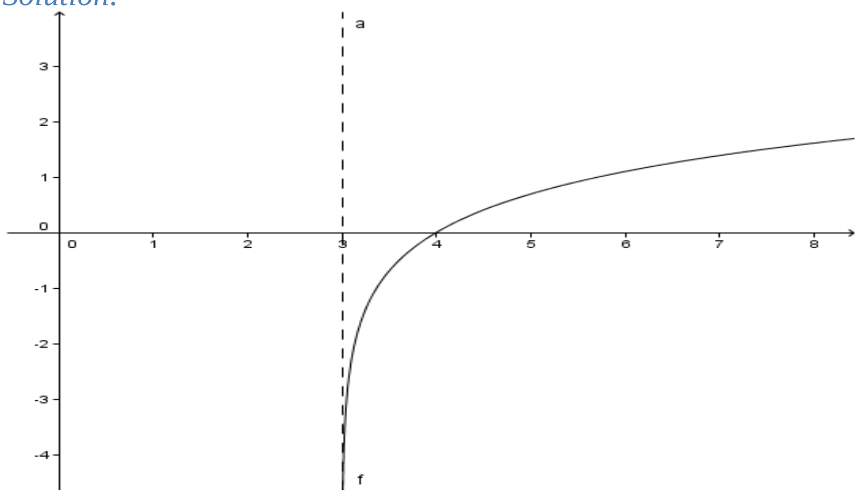
$y = -\,log_b\, x$ reflects the graph of $y = log_b\, x$ over the x-axis.

Graph $y = \log(x - 3)$. How does this graph compare to that of $y = \log x$?

Based on what you know about transformations of functions, describe in words how $y = \log(x - 3)$ transforms the parent function $y = \log x$.

$y = log(x - 3)$ translates $y = log\, x$ 3 units horizontally to the right.

Solution:



Solution: The transformation was a horizontal shift to the right 3 units so all the attributes of the graph of $y = log\, x$ are shifted to the right 3 units. The argument of the function is (x – 3). Since the logarithmic function is defined only for positive numbers, this means $x - 3 > 0$; solving this inequality for x, we find x > 3 which represents the domain of the function. The asymptote is x = 3. The new intercept is (4, 1).

Use technology to graph $y = \log(3x - 5)$. Key attributes such as domain, asymptote, and intercepts can often be determined algebraically. Consider the following questions to help you determine these attributes algebraically. Confirm your solutions match what you see on the graph.

To determine the domain, solve the inequality $3x - 5 > 0$. Explain why this makes sense.
To determine the asymptote, solve the equation $3x - 5 = 0$. Explain why this makes sense.
Use your understanding of the intercepts of any function to determine the intercepts of $y = \log(3x - 5)$. HINT: In general, how do you find the intercepts of any function?
.

Solution: The argument of the logarithmic function must be positive; therefore, the argument (3x - 5) must be positive and this happens when x > 5/3. Thus, the domain is x > 5/3.
The logarithmic function is undefined when the argument of the function is 0.
Asymptote is x = 5/3.

An x-intercept is where the graph intersects the x-axis, or where the y-coordinate is equal to 0. Setting y = 0, we get the following:

$0 = \log(3x - 5) \rightarrow 10^0 = 3x - 5 \rightarrow 1 = 3x - 5 \rightarrow x = 2$

So the x-intercept is (2, 0).

Use technology to graph $y = \log_5(-3x + 8)$. Determine its domain, asymptote, *x*-intercept, and *y*-intercept (if applicable) algebraically. Confirm your solutions agree with the graph.
$y = \log_5(-3x + 8)$

Domain: ____________ Asymptote: ______________

x-intercept: __________ *y*-intercept: ______________

Solution: Domain: x < 8/3 Asymptote: x = 8/3
x-intercept: (3, 0) y-intercept: (0, 1.292)

The Logarithmic Function

So far we have looked at logarithms as exponents, but in this task we will extend our study of logarithms by looking at the logarithmic function. The logarithmic function is defined as $y = \log_b x$ for $b > 0$ and $b \neq 1$. Associated with every logarithmic function is a base number. Given an input value, the logarithmic function returns the exponent to which the base number is raised to obtain this input; thus, the output of the logarithmic function is an exponent.

In Task 2 we saw there is a relationship between exponents and logarithms. The ability to go from an exponential expression to a logarithmic expression is powerful. Not surprisingly, there is a connection between an exponential function and a logarithmic function. In fact, $y = \log_b x$ implies $b^y = x$. Why? The connection between an exponential function and a logarithmic function will be explored more deeply in Unit 6.

For each function given, complete the table of values and then use these points to graph the function on graph paper.

$$y = \log x$$

x	y = log x
1	
10	
20	
50	
100	

$$y = \log_2 x$$

x	y = log_2 x
1	
2	
4	
8	
16	

What common characteristics of these functions do you see? In particular, determine the domain and range of the functions and any intercepts. Also describe any characteristics of their graphs such as increasing/decreasing, asymptotes, end-behavior, etc.

Use graphing technology to graph $y = \log x$. Does your graph agree with your hand-drawn graph?

Use graphing technology to graph $y = \log_2 x$. (Remember you can write any logarithmic expression in terms of common logarithms—this will allow you to graph $y = \log_2 x$ with your technology.) Does your graph agree with your hand-drawn graph?

Use graphing technology to graph $y = \ln x$.

How does the graph of the logarithmic function change as the base b changes?

Use graphing technology to graph each function.
$y = -\log x$

$$y = -\log_2 x$$
$$y = -\ln x$$

How do these graphs compare to the graphs of $y = \log x$ and $y = \log_2 x$ and $y = \ln x$?

Use what you know about transformations of functions to explain the relationship between $y = \log x$ and $y = -\log x$?

Does the same relationship hold for the graphs of $y = \log_2 x$ and $y = -\log_2 x$?
For $y = \ln x$ and $y = -\ln x$?
In general, what is the relationship between $y = \log_b x$ and $y = -\log_b x$?

Graph $y = \log(x - 3)$. How does this graph compare to that of $y = \log x$?

Based on what you know about transformations of functions, describe in words how $y = \log(x - 3)$ transforms the parent function $y = \log x$.

Use technology to graph $y = \log(3x - 5)$. Key attributes such as domain, asymptote, and intercepts can often be determined algebraically. Consider the following questions to help you determine these attributes algebraically. Confirm your solutions match what you see on the graph.
To determine the domain, solve the inequality $3x - 5 > 0$. Explain why this makes sense.
To determine the asymptote, solve the equation $3x - 5 = 0$. Explain why this makes sense.
Use your understanding of the intercepts of any function to determine the intercepts of $y = \log(3x - 5)$. HINT: In general, how do you find the intercepts of any function?
.

Use technology to graph $y = \log_5(-3x + 8)$. Determine its domain, asymptote, x-intercept, and y-intercept (if applicable) algebraically. Confirm your solutions agree with the graph.

$y = \log_5(-3x + 8)$

Domain: ____________ Asymptote: ______________

x-intercept: __________ y-intercept: ______________

How Long Does It Take?

[Back to Task Table](#)

Mathematical Goals

Represent a real-life situation with an exponential function
Solve exponential equations graphically and algebraically with logarithms

Georgia Standards of Excellence

MGSE9-12.A.SSE.3 Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression. *(Limit to exponential and logarithmic functions.)*

MGSE9-12.A.SSE.3c Use the properties of exponents to transform expressions for exponential functions. *For example, the expression* 1.15^t *, where t is in years, can be rewritten as* $[1.15^{(1/12)}]^{(12t)} \approx 1.012^{(12t)}$ *to reveal the approximate equivalent monthly interest rate if the annual rate is 15%.*

MGSE9-12.F.IF.8 Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function. *(Limit to exponential and logarithmic functions.)*

MGSE9-12.F.IF.8b Use the properties of exponents to interpret expressions for exponential functions. *For example, identify percent rate of change in functions such as* $y = (1.02)^t$, $y = (0.97)^t$, $y = (1.01)^{(12t)}$, $y = (1.2)^{(t/10)}$, *and classify them as representing exponential growth and decay. (Limit to exponential and logarithmic functions.)*

MGSE9-12.F.BF.5 Understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents.

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Introduction

In this task students represent a situation with an exponential function. The function is then used to answer related questions. Students will use both graphical and algebraic approaches to solve exponential equations.

Materials

Graphing calculator or some other graphing utility

1. **A Population Problem**: A new solar system was discovered far from the Milky Way in 1999. After much preparation, NASA decided to send a group of astronauts to explore Exponentia, one of the planets in the system. Upon landing on the planet, the astronauts discovered life on the planet. Scientists named the creatures Viètians (vee-et-ee-ans), after the French mathematician François Viète who led the way in developing our present system of notating exponents. After observing the species for a number of years, NASA biologists determined that the population was growing by 10% each year.

   a. The estimated number of Viètians was 1 million in 1999 and their population increases 10% a year. Complete the table to show the population for the next 4 years after 1999.

Years since 1999	0	1	2	3	4
Population in millions	*1*	*1 + (.1)1 1.1 1.1*	*1.1+(.1)(1.1) =1.1(1+.1) =1.1(1.1) =$(1.1)^2$ =1.21 million*	*$(1.1)^2$+.1$(1.1)^2$ =$(1.1)^2$(1+.1) =$(1.1)^2$(1.1) = $(1.1)^3$ =1.331 million*	*1.4641 million*

   b. Write an equation for the population of Exponentia, *P*, as a function of the number of years, *t*, since 1999. How can you express the population as an expression in the table rather than as a computed value to help you see patterns to create the function?

   Solution: $P(t) = 1.1^t$, where population is measured in millions

   c. What was the population in 2005? What will the population be in 2015 if the population growth rate remains the same?

   Solution:

2005 is 6 years after 1999. Thus, we find P(6) = 1.771561million = 1,771,561 Viètians.
2015 *is 16 years after 1999. We find $P(16) = 1.1^{16}$ = 4.594972986 million = 4,594,972* Viètians.

   d. Use technology to graph the function in part (b) .

Solution: The horizontal axis represents the number of years since 1999 and the vertical axis represents the population in millions.

      i. In the context of this problem about the population of Exponentia, what are the domain and range?

      Domain: _______________ Range: ________________

Solution: Domain: non-negative real numbers (the variable is the number of years since 1999 so $t \geq 0$. *Range:* $y > 1.1$

ii. What are some characteristics of the graph you can identify?

Solution: The y-intercept is 1 and this occurs at t = 0; this means there were 1 million people 0 years after 1999 (so in the year 1999). Also, the population is increasing. It is not a linear relationship. WARNING: We can't be sure the function will continue to model the population of Exponentia; it will continue to model the population only if the growth rate continues to be 10% per year.

e. Suppose you want to know when the population reached 2 million. Write an equation that could be solved to answer this question. Determine the answer graphically and algebraically.

Solution: $1.1^x = 2$. *Graphically, we could see where the line y = 2 intersects the graph in part (d). The point of intersection is (7.2725, 2) so it took approximately 7.3 years after 1999 for the population to reach 2 million. Algebraically, we have* $\log_{1.1} 2 = x$ *so* $\frac{\log 2}{\log 1.1}$ *or approx 7.3 years.*

2. Suppose there are 25 bacteria in a Petri dish, and the number of bacteria doubles every 4 hours.
   a. How many bacteria will there be in 4 hours? In 8 hours? 1 hour? 2 hours? Record your answers in the table. Explain how you came up with your answers. (You can return to your answers later to make any corrections if you find your strategy was incorrect.)

Time (hours)	0	1	2	4	8
Number of bacteria	25				

Solution:
(Explanations will vary. At this point, students may not have the answers for 1 hour and 2 hours correct. They will be able to go back and correct it.)

   b. Write a function for the number of bacteria present after *t* hours. What does your exponent need to represent? How can you determine this exponent if you know the number of hours that has passed?

Solution: $P(t) = 25(2)^{t/4}$

   c. Use the function to check your answers that you wrote in the table of part (a). Do you need to make any corrections? If so, make these corrections.

   d. Use your function to determine the number of bacteria after 24 hours.
      P(24) = 1600 bacteria

e. Determine how long it will take to have 5000 bacteria. Determine the answer graphically and algebraically.

In a little over 30 hours.

f. The bacteria double every 4 hours. Suppose we want to know the growth rate per hour. Use properties of exponents to rewrite the function you obtained in part (b) so the exponent is *t*, not $\frac{t}{4}$. How can you now determine the growth rate per hour?

$P(t) = 25(2)^{\frac{t}{4}} = 25\left(2^{\frac{1}{4}}\right)^{t}$ *Now* $2^{\frac{1}{4}} \approx 1.1892$ *so* $P(t) \approx 25(1.1892)^{t} = 25(1 + .1892)^{t}$

Since t is measured in hours this means that the population is 1.1892 times what it was the previous hour. Thus, it must have increased by a factor of .1892 or by 18.92%.

3. Suppose for a particular patient and dosing regimen a drug reaches its peak level of 300 mg in the bloodstream. The drug is then eliminated from the bloodstream at a rate of 20% per hour.

   a. How much of the drug remains in the bloodstream 2 hours after it reaches its peak level of 300 mg? How much is there 5 hours after the peak level? Make a table of values to record your answers. So that a pattern is more apparent, write the expressions used to obtain your answers.

Time (hours) since reaching peak level	0	1	2	3	4	5
Amount of drug (mg) in bloodstream	300	*240*	*192*	*153.6*	*122.88*	*98.304*

300(1 -.2) = 240; 240 * .8 = 192; 192 * .8 = 153.6; 153.6 * .8 = 122.88; 122.88 * .8 = 98.304*

   b. Using your work from part (a), write expressions for each computed value using the initial amount, 300 mg.

Solution:
Because 20% is eliminated every hour, 80% of the drug remains in the bloodstream every hour.
After 1 hour: $300 * (.8) = 240$*; after 2 hours:* $300 * (.8)^2 = 192$*;*
after 3 hours: $300 * (.8)^3 = 153.6$*; after 4 hours:* $300 * (.8)^4 = 122.88$*;*
after 5 hours: $300 * (.8)^5 = 98.304$

   c. Write a function *f* that gives the amount of the drug in the patient's bloodstream *t* hours after reaching its peak level.

Solution: $f(x) = 300(.8)^t$

   d. Use the function you wrote in part (c) to compute the amount of the drug after 1 hour, 2 hours, 3 hours, 4 hours, and 5 hours. Are these amounts the same as those you wrote in the table in part (a)?

Solution: $f(0) = 300(.8)^0 = 300$
$f(1) = 300(.8)^1 = 240$
$f(2) = 300(.8)^2 = 192$
$f(3) = 300(.8)^3 = 153.6$
$f(4) = 300(.8)^4 = 122.88$
$f(5) = 300(.8)^5 = 98.304$

e. Use technology to graph the function. Explain how to use the graph to determine how long it will take to have less than 10 mg of the drug in the bloodstream.

Solution: Using a graph, we can trace the graph to determine where the vitamin concentration dips below 10 mg. This occurs around 15.3 hours.

f. Write an equation that you could solve to determine when exactly 10 mg of the drug remains in the bloodstream. Solve the equation algebraically. Can you use the solution to this equation to answer the question in part (e)?

Solution: The equation is $300(.8)^t = 10.$ → $.8^t = 1/30.$ *The solution is approximately 15.24 hours or approximately 15 hours, 14 minutes, 24 seconds.*

4. Which of the problems in this section represent exponential growth? Which represent exponential decay?

Problems 1 & 2 are examples of exponential growth; problem 3 is exponential decay.

How Long Does It Take?

1. A Population Problem: A new solar system was discovered far from the Milky Way in 1999. After much preparation, NASA decided to send a group of astronauts to explore Exponentia, one of the planets in the system. Upon landing on the planet, the astronauts discovered life on the planet. Scientists named the creatures Viètians (vee-et-ee-ans), after the French mathematician François Viète who led the way in developing our present system of notating exponents. After observing the species for a number of years, NASA biologists determined that the population was growing by 10% each year.

   a. The estimated number of Viètians was 1 million in 1999 and their population increases 10% a year. Complete the table to show the population for the next 4 years after 1999.

Years since 1999	0	1	2	3	4
Population in millions	1				

   b. Write an equation for the population of Exponentia, *P*, as a function of the number of years, *t*, since 1999. How can you express the population as an expression in the table rather than as a computed value to help you see patterns to create the function?

   c. What was the population in 2005? What will the population be in 2015 if the population growth rate remains the same?

   d. Use technology to graph the function in part (b) .

      i. In the context of this problem about the population of Exponentia, what are the domain and range?

      Domain: ________________ Range: _________________

      ii. What are some characteristics of the graph you can identify?

   e. Suppose you want to know when the population reached 2 million. Write an equation that could be solved to answer this question. Determine the answer graphically and algebraically.

2. Suppose there are 25 bacteria in a Petri dish, and the number of bacteria doubles every 4 hours.
   a. How many bacteria will there be in 4 hours? In 8 hours? 1 hour? 2 hours? Record your answers in the table. Explain how you came up with your answers. (You can return to your answers later to make any corrections if you find your strategy was incorrect.)

Time (hours)	0	1	2	4	8
Number of bacteria	25				

   b. Write a function for the number of bacteria present after *t* hours. What does your exponent need to represent? How can you determine this exponent if you know the number of hours that has passed?

   c. Use the function to check your answers that you wrote in the table of part (a). Do you need to make any corrections? If so, make these corrections.

   d. Use your function to determine the number of bacteria after 24 hours.

   e. Determine how long it will take to have 5000 bacteria. Determine the answer graphically and algebraically.

   f. The bacteria double every 4 hours. Suppose we want to know the growth rate per hour. Use properties of exponents to rewrite the function you obtained in part (b) so the exponent is *t*, not $\frac{t}{4}$. How can you now determine the growth rate per hour?

3. Suppose for a particular patient and dosing regimen a drug reaches its peak level of 300 mg in the bloodstream. The drug is then eliminated from the bloodstream at a rate of 20% per hour.
   a. How much of the drug remains in the bloodstream 2 hours after it reaches its peak level of 300 mg? How much is there 5 hours after the peak level? Make a table of values to record your answers. So that a pattern is more apparent, write the expressions used to obtain your answers.

Time (hours) since reaching peak level	0	1	2	3	4	5
Amount of drug (mg) in bloodstream	300					

b. Using your work from part (a), write expressions for each computed value using the initial amount, 300 mg.

c. Write a function *f* that gives the amount of the drug in the patient's bloodstream *t* hours after reaching its peak level.

d. Use the function you wrote in part (c) to compute the amount of the drug after 1 hour, 2 hours, 3 hours, 4 hours, and 5 hours. Are these amounts the same as those you wrote in the table in part (a)?

e. Use technology to graph the function. Explain how to use the graph to determine how long it will take to have less than 10 mg of the drug in the bloodstream.

f. Write an equation that you could solve to determine when exactly 10 mg of the drug remains in the bloodstream. Solve the equation algebraically. Can you use the solution to this equation to answer the question in part (e)?

4. Which of the problems in this section represent exponential growth? Which represent exponential decay?

Zombies Revisited – Can You Model Zombie Growth? (Spotlight Task)

Back to Task Table

Mathematical Goals

Represent a supposedly real-life situation with an exponential function
Solve exponential equations graphically and algebraically with logarithms

Georgia Standards of Excellence

MGSE9-12.A.SSE.3 Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression. *(Limit to exponential and logarithmic functions.)*

MGSE9-12.A.SSE.3c Use the properties of exponents to transform expressions for exponential functions. *For example, the expression* 1.15^t *, where t is in years, can be rewritten as* $[1.15^{(1/12)}]^{(12t)} \approx 1.012^{(12t)}$ *to reveal the approximate equivalent monthly interest rate if the annual rate is 15%.*

MGSE9-12.F.IF.8 Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function. *(Limit to exponential and logarithmic functions.)*

MGSE9-12.F.IF.8b Use the properties of exponents to interpret expressions for exponential functions. *For example, identify percent rate of change in functions such as* $y = (1.02)^t$, $y = (0.97)^t$, $y = (1.01)^{(12t)}$, $y = (1.2)^{(t/10)}$, *and classify them as representing exponential growth and decay. (Limit to exponential and logarithmic functions.)*

MGSE9-12.F.BF.5 Understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents.

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Introduction

This task comes after "How Long Does it Take?" because that task shows them how to create an equation for an exponential equation given certain characteristics. This task really zones in on true understanding of not only exponential growth, but also how that would affect their graphing *choices. The numbers get so big that students will have to make decisions about the best way to represent the phenomena in a way that makes sense. With an AC or Honors class, an exploration into logarithmic graph paper and how it scales values from small to very large could be interesting for students to see their logarithmic graph suddenly become linear!*

As you probably know, The Walking Dead is filmed in Georgia. That makes Georgia PRIME location for a Zombie Apocalypse.
If each Georgia Zombie is able to bite three people every 7 days, and the entire state of Georgia has an estimated 9,919,945 as of 2013.*,write the equation that models Zombie Growth in the State of Georgia.

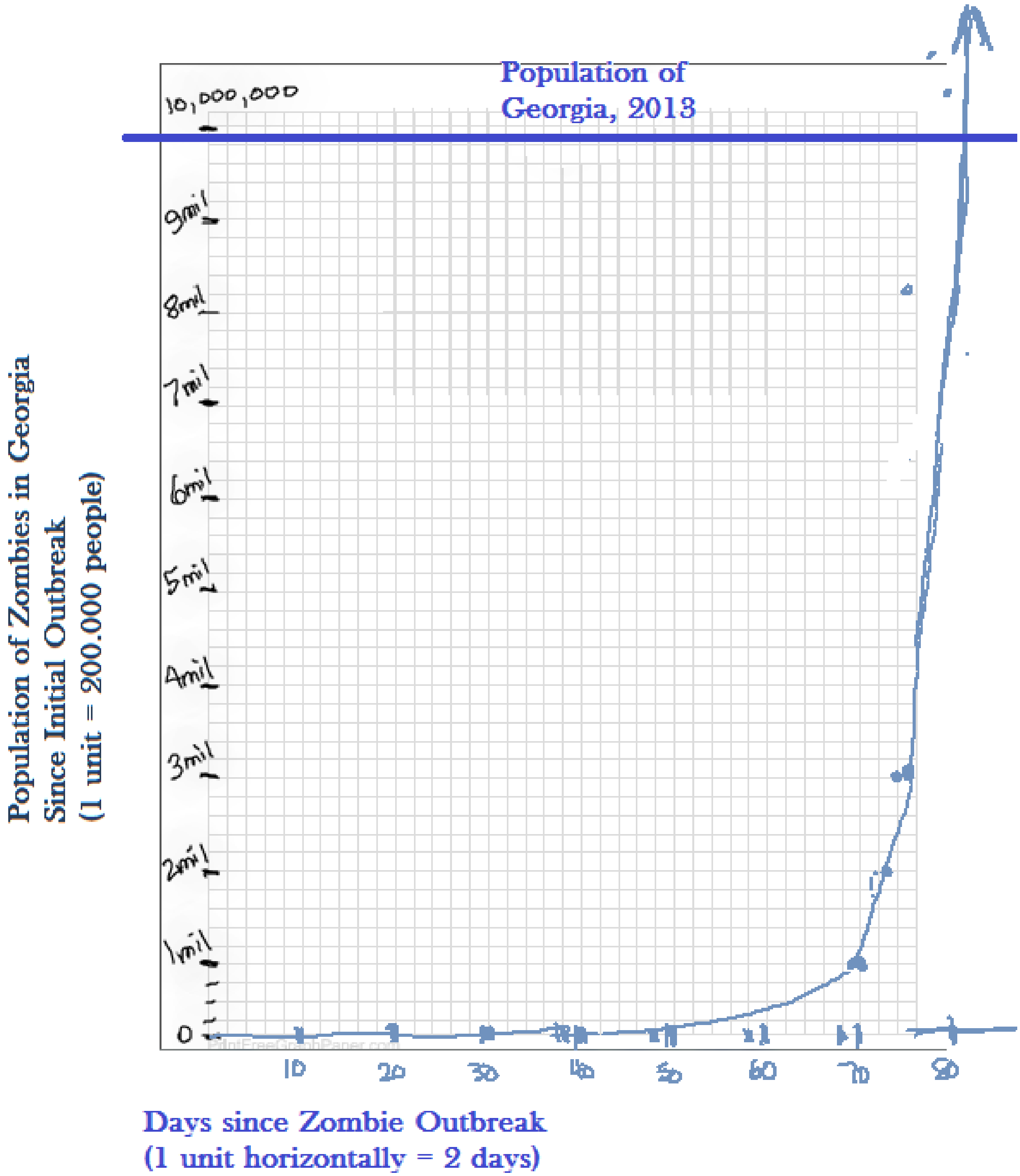
$$z(x) = 1 * (4)^{\frac{t}{7}}$$

Create a chart showing the growth.

day	ZOMBIE TOTAL	day	ZOMBIE TOTAL	day	ZOMBIE TOTAL	day	ZOMBIE TOTAL	day	ZOMBIE TOTAL	day	ZOMBIE TOTAL	day	ZOMBIE TOTAL
0	1	11	8.8	22	78	33	689.1	44	6086.6	55	53761.5	77	4194304
1	1.2	12	10.8	23	95.1	34	840	45	7419.7	56	65536	78	5112913.8
2	1.5	13	13.1	24	115.9	35	1024	46	9044.7	57	79889.3	79	6232711.8
3	1.8	14	16	25	141.3	36	1248.3	47	11025.6	58	97386.1	80	7597760.8
4	2.2	15	19.5	26	172.3	37	1521.7	48	13440.4	59	118715	81	9261774.1
5	2.7	16	23.8	27	210	38	1854.9	49	16384	60	144715.2	82	11290229.1
6	3.3	17	29	28	256	39	2261.2	50	19972.3	61	176409.8	83	13762943.5
7	4	18	35.3	29	312.1	40	2756.4	51	24346.5	62	215046	84	16777216
8	4.9	19	43.1	30	380.4	41	3360.1	52	29678.8	63	262144	85	20451655.4
9	5.9	20	52.5	31	463.7	42	4096	53	36178.8	64	319557.1	86	24930847.2
10	7.2	21	64	32	565.3	43	4993.1	54	44102.5	65	389544.5	87	30391043.1

Graph the equation on graph paper that models the growth and that demonstrates the point at which all citizens of Georgia would (theoretically) be zombies.

Notice the graph…..the number of zombies present for several days is almost imperceptible from this distance. Bu then something very interesting happens. It shoots up very quickly. It is advised, by the way, that you NOT give too much help on this graph, and do not to give students pre-defined axe that you have already set up appropriately for them, although it IS a good idea to require that the graph be on graph paper and (somewhat) proportional to the actual situation. The reason for this is that students should grapple with graphing in terms of how big to make the axes, what incremental units to use, and how big to make each unit. The notion that they need to go at least as high as the phenomenon they want to see (2013 population in the state of Georgia) and that they need to start at the beginning isn't automatic. The idea that it's a good idea to test values in the equation to get a handle on how to create a graph also is a skill they can only learn if we do not give too cooperative a set of axes. But do have plenty of graph paper on hand. This type of grappling with data does take extra paper ***(See Next Page)***



When will Georgia be overrun with zombies? Calculate it down to the <u>second</u>.

Sometime halfway in between 81 and 82 days, the population of zombies exceeds the population of Georgia.

To Solve

$$\ln\left(1(4)^{x/7}\right) = \ln(9919945)$$

$$(7)\ \frac{x}{7}\ln(4) = \ln(9919945)\ (7)$$

$$\frac{x\,\cancel{\ln 4}}{\cancel{\ln 4}} = \frac{7\ln(9919945)}{\ln 4}$$

$$x = \frac{7\ln(9919945)}{\ln 4}$$

$$x \approx 81.34665242$$

.

To Convert All the Way Down to Seconds

≈ 81.34665242 days

$= 81$ DAYS

$0.34665242 * 24$ hours/day

≈ 8.319658053 hours

$+\ 8$ HOURS

0.319658053 hours $* 60$ minutes/hour

≈ 19.17948315

$+\ 19$ MINUTES

$0.17948315 * 60$ seconds/minute

≈ 10.76898916

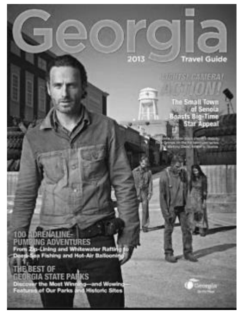
$+\ 11$ SECONDS

:

81 days, 8 hours, 19 minutes, and 11 seconds.

*http://www.worldpopulationstatistics.com/georgia-population-2013/.

Zombies Revisited – Can You Model Zombie Growth? (Spotlight Task)



As you probably know, The Walking Dead is filmed in Georgia. That makes Georgia PRIME location for a Zombie Apocalypse.
If each Georgia Zombie is able to bite three people every 7 days, and the entire state of Georgia has an estimated 9,919,945 as of 2013*, write the equation that models Zombie Growth in the State of Georgia.

Create a chart showing the growth.

Graph the equation on graph paper that models the growth and that demonstrates the point at which all citizens of Georgia would (theoretically) be zombies.

When will Georgia be overrun with zombies? Calculate it down to the second.

**http://www.worldpopulationstatistics.com/georgia-population-2013/*.

Half-Life

[Back to Task Table]

Mathematical Goals
Understand the concept of half-life
Use an exponential function to represent a half-life situation
Solve exponential equations graphically and algebraically using logarithms

Georgia Standards of Excellence

MGSE9-12.A.SSE.3 Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression. *(Limit to exponential and logarithmic functions.)*

MGSE9-12.A.SSE.3c Use the properties of exponents to transform expressions for exponential functions. *For example, the expression* 1.15^t *, where t is in years, can be rewritten as* $[1.15^{(1/12)}]^{(12t)} \approx 1.012^{(12t)}$ *to reveal the approximate equivalent monthly interest rate if the annual rate is 15%.*

MGSE9-12.F.IF.8 Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function. *(Limit to exponential and logarithmic functions.)*

MGSE9-12.F.IF.8b Use the properties of exponents to interpret expressions for exponential functions. *For example, identify percent rate of change in functions such as* $y = (1.02)^t$, $y = (0.97)^t$, $y = (1.01)^{(12t)}$, $y = (1.2)^{(t/10)}$, *and classify them as representing exponential growth and decay. (Limit to exponential and logarithmic functions.)*

MGSE9-12.F.BF.5 Understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents.

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Introduction
This task includes several problems about half-life.

Materials
Graphing calculator or some other graphing utility

Problem 1: A Caffeine Problem The half-life of caffeine is 5 hours; this means that approximately ½ of the caffeine in the bloodstream is eliminated every 5 hours. Suppose you drink a can of Instant Energy, a 16-ounce energy drink that contains 80 mg of caffeine. Suppose the caffeine in your bloodstream peaks at 80 mg.

How much caffeine will remain in your bloodstream after 5 hours? 10 hours? 1 hour? 2 hours? Record your answers in the table. Explain how you came up with your answers. (You can return to your answers later to make any corrections if you find your strategy was incorrect.)

Time (hours) since peak level reached	0	1	2	5	10
Caffeine in bloodstream (mg)	80	*69.644*	*60.629*	*40*	*20*

(Explanations will vary. At this point, students may not have the answers for 1 hour and 2 hours correct. They will be able to go back and correct it.)

Write an exponential function *f* to model the amount of caffeine remaining in the blood stream *t* hours after the peak level. What does your exponent need to represent? How can you determine this exponent if you know the number of hours that has passed?

Solution: $f(t) = 80(.5)^{t/5}$

Use the function you wrote in question (2) *to* check your answers for the table in question (1). Make any necessary corrections. (Be careful when entering fractional exponents in the calculator. Use parentheses.)

Solution: $f(0) = 80(.5)^{0/5} = 80$; $f(1) = 80(.5)^{1/5} = 69.644$; $f(2) = 80(.5)^{2/5} = 60.629$; $f(5) = 80(.5)^{1} = 40$; $f(10) = 80(.5)^{2} = 20$

Determine the amount of caffeine remaining in the bloodstream 3 hours after the peak level. What about 8 hours after peak level? 20 hours?

Solution: $f(3) = 80(.5)^{3/5} = 52.78$; $f(8) = 80(.5)^{8/5} = 26.39$; $f(20) = 80(.5)^{4} = 5$

The half-life of caffeine varies among individuals. For example, some medications extend the half-life to 8 hours. This means that ½ of the caffeine is eliminated from the bloodstream every 8 hours.

Write a function for this new half-life time (assuming a peak level of 80 mg of caffeine).

Solution: $f(t) = 80(.5)^{t/8}$

Determine the amount of caffeine in the bloodstream after 1 hour, 2 hours, 5 hours, 10 hours, and 20 hours. (Be sure to consider how many 8-hour time intervals are used in each time value.)
Solution: after 1 hour approx 73.4 mg; after 2 hrs approx 67.3 mg ;
after 5 hrs approx 51.9 mg ; after 10 hrs approx 33.6 mg; after 20 hrs approx 14.1 mg

Which half-life time results in the caffeine being eliminated faster? Explain why this makes sense.
Solution: The caffeine is eliminated faster with a half-life of 5 hours than with a half-life of 8 hours. To make sense of this, consider an 8- hour period of time. With a half-life of 8 hours, there is still ½ of the caffeine after this 8-hour period of time. However, with a half-life of 5 hours ½ of the caffeine was eliminated in 5 hours so for the next 3 hours of the 8-hour time period more of the caffeine is being eliminated. Thus, a half-life of 5 hours is eliminating the caffeine faster.

Consider again the function in question (2) resulting from a half-life of caffeine of 5 hours. Use the properties of exponents to rewrite the function so the exponent is just t, not $\frac{t}{5}$. Can you now determine the percent of caffeine that remains in the bloodstream *each hour*? Explain how.

About 87% of the amount in the preceding hour

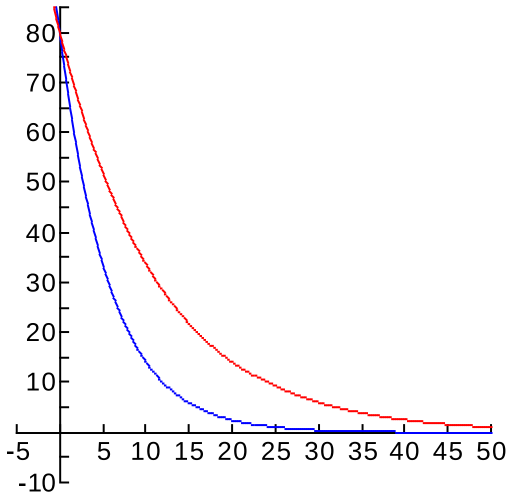
The function in question (5) referred to a half-life of 8 hours. Use properties of exponents to help you determine the percent of caffeine that remains in the bloodstream *each hour*?

About 92%

Graph the functions from questions (2) and (5) on the same coordinate plane. Compare the graphs of the two functions. How are the graphs similar? Different?

Do the graphs intersect? Where?

The graphs intersect only at (0, 80). In both situations, the beginning amount of caffeine is 80 mg but the caffeine is eliminated more quickly with a half-life of 5 years than with 8 years so the graph of $f(t) = 80(.5)^{t/5}$ (blue graph) decreases faster than the graph of $f(t) = 80(.5)^{t/8}$ (red graph).



Problem 2: Carbon-Dating Scientists use carbon-dating to determine the ages of once-living things. The radioactive isotope carbon-14 (C14) is widely used in radiocarbon dating. This form of carbon is formed when plants absorb atmospheric carbon dioxide into their organic material during photosynthesis. Animals eat the plants which introduces carbon into their bodies. After the organism dies, no more C14 is formed and the C14 in the material decays without being replaced. The half-life of carbon-14 is approximately 5730 years; this means that it takes 5730 years for half of the amount of carbon-14 in the material to decay.

An archaeologist found a piece of human bone fragment. How long will it take for there to be only 50% of the amount of C14 in the bone fragment?
5730 years

Examine the way the table below has been set up. Explain why inputs of time are represented as 5730, 2(5730), and 3(5730).

Time since death (yrs)	0	5730	2(5730)	3(5730)
Proportion of C14 remaining	1	$1(.5) = .5^1$	$1(.5)(.5) = .5^2 = .25$	$1(.5)(.5)(.5) = .5^3 = .125$

Complete the table above. So that you can more easily determine the function, write the EXPRESSIONS that lead to the computed proportions. Do these help you see a pattern?

The exponent is the number of half-lives so you must divided by 5730 to determine the number of half-lives for a given number of years.

Find a function *f* that represents the proportion of the initial C14 remaining in the fragment *t* years after the death of the person to which it belonged.

$f(t) = (.5)^{\frac{t}{5730}}$

Based on the values in the table, can you estimate the percent of C14 remaining in the fragment 2000 years after the person's death? Explain your thinking.

$f(2000) = (.5)^{2000} \approx .785$ *Thus, about 78.5% remains 2000 years after death.*

Graph the function. Graphically determine the percent of C14 remaining in the fragment 2000 years after the person's death. How do you determine the answer algebraically?

It is determined that the fragment contains 64% of the amount of C14 that is normally found in the bone of a living person. Approximately how long ago did the person die?

$$.64 = (.5)^{\frac{t}{5730}}$$

$log_{.5}(.64) = \frac{t}{5730}$ *so* $t = 5730\, log_{.5}(.64)$ *which is about 3689 years.*

Half-Life

Problem 1: A Caffeine Problem: The half-life of caffeine is 5 hours; this means that approximately ½ of the caffeine in the bloodstream is eliminated every 5 hours. Suppose you drink a can of Instant Energy, a 16-ounce energy drink that contains 80 mg of caffeine. Suppose the caffeine in your bloodstream peaks at 80 mg.

How much caffeine will remain in your bloodstream after 5 hours? 10 hours? 1 hour? 2 hours? Record your answers in the table. Explain how you came up with your answers. (You can return to your answers later to make any corrections if you find your strategy was incorrect.)

Time (hours) since peak level reached	0	1	2	5	10
Caffeine in bloodstream (mg)	80				

Write an exponential function *f* to model the amount of caffeine remaining in the blood stream t hours after the peak level. What does your exponent need to represent? How can you determine this exponent if you know the number of hours that has passed?

Use the function you wrote in part (b) to check your answers for the table in part (a). Make any necessary corrections. (Be careful when entering fractional exponents in the calculator. Use parentheses.)

Determine the amount of caffeine remaining in the bloodstream 3 hours after the peak level. What about 8 hours after peak level? 20 hours?

The half-life of caffeine varies among individuals. For example, some medications extend the half-life to 8 hours. This means that ½ of the caffeine is eliminated from the bloodstream every 8 hours.
Write a function for this new half-life time (assuming a peak level of 80 mg of caffeine).

Determine the amount of caffeine in the bloodstream after 1 hour, 2 hours, 5 hours, 10 hours, and 20 hours. (Be sure to consider how many 8-hour time intervals are used in each time value.)

Which half-life time results in the caffeine being eliminated faster? Explain why this makes sense.

Consider again the function in question (2) *resulting* from a half-life of caffeine of 5 hours. Use the laws of exponents to rewrite the function so the exponent is just t, not $\frac{t}{5}$. Can you now determine the percent of caffeine that remains in the bloodstream *each hour*? Explain how.

The function in question (5) referred to a half-life of 8 hours. Use properties of exponents to help you determine the percent of caffeine that remains in the bloodstream *each hour*?

Graph the functions from questions (2) and (5) on the same coordinate plane. Compare the graphs of the two functions. How are the graphs similar? Different?

Do the graphs intersect? Where?

Problem 2: Carbon-Dating: Scientists use carbon-dating to determine the ages of once-living things. The radioactive isotope carbon-14 (C14) is widely used in radiocarbon dating. This form of carbon is formed when plants absorb atmospheric carbon dioxide into their organic material during photosynthesis. Animals eat the plants which introduces carbon into their bodies. After the organism dies, no more C14 is formed and the C14 in the material decays without being replaced. The half-life of carbon-14 is approximately 5730 years; this means that it takes 5730 years for half of the amount of carbon-14 in the material to decay.

An archaeologist found a piece of human bone fragment. How long will it take for there to be only 50% of the amount of C14 in the bone fragment?

Examine the way the table below has been set up. Explain why inputs of time are represented as 5730, 2(5730), and 3(5730).

Time since death (yrs)	0	5730	2(5730)	3(5730)
Proportion of C14 remaining	1			

Complete the table above. So that you can more easily determine the function, write the EXPRESSIONS that lead to the computed proportions. Do these help you see a pattern?

Find a function f that represents the proportion of the initial C14 remaining in the fragment t years after the death of the person to which it belonged.

Based on the values in the table, can you estimate the percent of C14 remaining in the fragment 2000 years after the person's death? Explain your thinking.

Graph the function. Graphically determine the percent of C14 remaining in the fragment 2000 years after the person's death. How do you determine the answer algebraically?

It is determined that the fragment contains 64% of the amount of C14 that is normally found in the bone of a living person. Approximately how long ago did the person die?

How Does Your Money Grow?

Back to Task Table

Mathematical Goals

Meaningfully develop the formulas for compounded and continuously compounded interest
Recognize compounded interest as a special application of an exponential function
Solve problems involving compounded interest

Georgia Standards of Excellence

MGSE9-12.F.IF.8 Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function. *(Limit to exponential and logarithmic functions.)*

MGSE9-12.F.IF.8b Use the properties of exponents to interpret expressions for exponential functions. *For example, identify percent rate of change in functions such as* $y = (1.02)^t$, $y = (0.97)^t$, $y = (1.01)^{(12t)}$, $y = (1.2)^{(t/10)}$, *and classify them as representing exponential growth and decay. (Limit to exponential and logarithmic functions.)*

MGSE9-12.F.BF.5 Understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents.

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Introduction

In this task the formulas for compounded interest are developed in a meaningful way.

Materials

Graphing calculator or some other graphing utility

Be sure to explain what compounded interest means—that you are earning interest on interest.

1. Suppose you invest \$1000 in a savings account that earns 3% interest compounded per year. This amount that you invest is called your *principal*.
    a. How much money will you have at the end of year 1? After 2 years? After 6 years? Organize your work in the table.

Time in Years	0	1	2	3	4	5
Total Amount o f Money	*1000*	*1030*	*1061*	*1093*	*1126*	*1159*

b. Write a function to express the total amount of money *A* in terms of the number of years *t*.

Each year you have the previous amount together with 3% of the previous amount. So, each year's amount is 1.03 times the previous year's amount.
$A(t) = 1000(1.03)^t$

2. Suppose you invest $1000 at 3% per year but instead of earning the interest yearly, the account earns the interest compounded semi-annually. Thus, each quarter you earn ½ of the 3% interest or $\frac{3}{2}$ % interest per quarter .
   a. Why? (Do you see that $\frac{3}{2}$ % is equivalent to $\frac{.03}{2}$?)
   b. Complete the table to show the amount of money you have.

Time in quarters	0	1	2 (1 yr)	3	4 (2 yrs)	5
Total amt of money	*1000*	*1015*	*1030*	*1046*	*1061*	*1077*

c. After 1 year, how does the amount of money in problem 2 compare to that in problem 1? What caused the difference in the amounts?

The interest at the end of 1 year and at the end of 2 years when the interest is compounded semiannually is slightly larger than when it is compounded yearly. The increase in the number of compounding periods per year caused the increase.

d. The function you wrote in part (b) of question (1) now becomes
$A(t) = 1000\left(1 + \frac{.03}{2}\right)^{2t}$. Explain why.

In the next sequence of questions, the goal is to write the function you wrote in part (b) of question (1)in a more general way. Consider the sequence of questions carefully, paying close attention to patterns.

e. How would your function in part (d) of question (2) change if your interest was compounded quarterly? How would your function in part (d) of question (2) change if your interest was compounded monthly?

Quarterly: $A(t) = 1000\left(1 + \frac{.03}{4}\right)^{4t}$

Monthly: $A(t) = 1000\left(1 + \frac{.03}{12}\right)^{12t}$

f. Look for patterns. How would your function in part (d) of question (2) change if your interest was compounded *n* times per year? Write this function.

$$A(t) = 1000\left(1 + \frac{.03}{n}\right)^{nt}$$

g. Now take the function you wrote in part (f) of question (2). How would this function change if the annual interest rate was 5.2%?

$$A(t) = 1000\left(1 + \frac{.052}{n}\right)^{nt}$$

h. Look for patterns. How would the function in part (g) of question (2) change if the annual interest rate was *r*? Write this function.

$$A(t) = 1000\left(1 + \frac{r}{n}\right)^{nt}$$

i. Finally, if the amount of money invested was \$850, how would the function you wrote in part (h) of question (2) change?

$$A(t) = 850\left(1 + \frac{r}{n}\right)^{nt}$$

j. Look for patterns. How would the function in part (i) of question (2) change if the amount invested were represented by P? Write this function.

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

Congratulations!! If all has gone well with the above questions in parts (e - j) of question (2), you have just written a generalization for the total amount of money A a person would have if he invests an initial amount P at an annual rate r compounded n times per year for t years!!!

You should have gotten $(t) = P\left(1+\frac{r}{n}\right)^{nt}$. If you did not get this, go back to find your error. This formula is referred to as the compounded interest formula. The formula is more commonly expressed as $A = P\left(1+\frac{r}{n}\right)^{nt}$.

Often money is compounded continuously rather than compounded 4 times a year or 12 times a year, or some other finite number of times per year. When money is compounded continuously, we use the continuously compounded interest formula $A = Pe^{rt}$ where P is the initial amount of money (or principal) invested at a yearly rate r compounded continuously for t years. This formula uses the irrational number e which equals 2.718281828… Since e is irrational, it neither terminates nor repeats. But where does the formula $A = Pe^{rt}$ come from and what does it mean to say that "interest is compounded continuously"? Let's make sense of this.
The continuously compounded interest formula comes from $A = P\left(1+\frac{r}{n}\right)^{nt}$. To get a sense of what is happening, let's simplify the situation to consider \$1 invested at 100% per year for 1 year. If you substitute this information into $A = P\left(1+\frac{r}{n}\right)^{nt}$, you get $A = 1\left(1+\frac{1}{n}\right)^{n}$ which is equivalent to $A = \left(1+\frac{1}{n}\right)^{n}$. Why?

When money is compounded continuously, you can imagine that the number of times it is compounded per year gets infinitely large—that is, n gets infinitely large. Use technology to investigate what happens to the expression $\left(1+\frac{1}{n}\right)^{n}$ as n increases in value. Record the value of $\left(1+\frac{1}{n}\right)^{n}$ for each value of n. (Some values of n are given in the table.)

Frequency of Compounding	Number of times compounded in a year (n)	$\left(1+\frac{1}{n}\right)^{n}$
Annually	1	*2*
Semiannually	2	*2.25*
Quarterly	*4*	*2.44140625*
Monthly	*12*	*2.61303529022*
Weekly	*52*	*2.69255969544*
Daily	*365*	*2.71456748202*
Hourly	8760	*2.71812669063*
Every Minute	*525600*	*2.7182792154*
Every Second	31536000	*2.71828247254*

Examine your data in the table. As the number of compounding times gets infinitely large, what happens to the value of $\left(1+\frac{1}{n}\right)^{n}$? What do you think would happen if you continued to use larger and larger values of n? (Try it!)

Solution: As the value of n increases, $\left(1+\frac{1}{n}\right)^n$ gets closer to the value of e. If we continued to use larger and larger values of n, we would get closer and closer to the value of e.

Let's see what happens if we invest \$1 at rates different from 100% for 1 year; this time we will use r for our annual rate of interest so that we can let r vary. Again, we want the number of times the interest is compounded per year to increase. The formula $A = P\left(1+\frac{r}{n}\right)^{nt}$ now becomes $A = 1\left(1+\frac{r}{n}\right)^n$ which is equivalent to $A = \left(1+\frac{r}{n}\right)^n$. Why?

With technology, investigate what happens to the expression $\left(1+\frac{r}{n}\right)^n$ as *n* gets infinitely large. You will investigate this expression for different values of *r*. Begin by letting *r* = 90% or .9. Then you are looking at $\left(1+\frac{.9}{n}\right)^n$ as *n* increases in value. You can then consider *r* = 80% or .8 so you are investigating $\left(1+\frac{.8}{n}\right)^n$. The following table can help you organize your work. In the final column you can choose your own value of *r* to create the expression $\left(1+\frac{r}{n}\right)^n$. Remember there is no limit on the value of *n*; instead, *n* continues to get larger and larger!

Frequency of Compounding	Number of times compounded in a year (*n*)	$\left(1+\frac{.9}{n}\right)^n$	$\left(1+\frac{.8}{n}\right)^n$	
Annual	1	*1.9*	*1.8*	
Semiannual	2	*2.1025*	*1.96*	
Quarterly	*4*	*2.2519*	*2.0736*	
Monthly	*12*	*2.3818*	*2.1694*	
Weekly	*52*	*2.4407*	*2.212*	
Daily	*365*	*2.4569*	*2.2236*	
Hourly	8760	*2.4595*	*2.2255*	
Every Minute	*525600*	*2.4596*	*2.2255*	
Every Second	31536000	*2.4596*	*2.2255*	

Examine the data in the table. As the number of compounding times gets infinitely large, what happens to the value of $\left(1+\frac{r}{n}\right)^n$? Do you see for each value of *r* the value of $\left(1+\frac{r}{n}\right)^n$ appears to be getting closer to a particular number? Let's see if we can determine what that number is. As a suggestion, use your calculator to evaluate $e^{.9}$. Now evaluate $e^{.8}$. Conjecture? What number do you think $\left(1+\frac{r}{n}\right)^n$ gets closer to as *n* gets larger and larger? Test your conjecture for different values of *r* and LARGE values of *n*!

Did you see that as n increases in value, the expression $\left(1+\frac{r}{n}\right)^n$ gets closer and closer to the value of e^r for any value r? Therefore, if we are compounding interest continuously, the compound interest formula $A = P\left(1+\frac{r}{n}\right)^{nt}$ becomes $A = Pe^{rt}$ because $\left(1+\frac{r}{n}\right)^n$ can be replaced with the value e^r for infinitely large values of n. Show how.

$$A = P\left(1+\frac{r}{n}\right)^{nt} = P\left[\left(1+\frac{r}{n}\right)^{n}\right]^{t} = P\,[e^r]^t = Pe^{rt}$$

Assume your friend, Natalie, has $10,000 to invest. Complete the following chart to show how much she would earn if her money was invested in each of the specified accounts for 10 years.

Frequency of compounding	Annual interest rate	Formula with values substituted in	Amount after 10 years
Quarterly	3.65%	*A = 10,000(1 + .0365/4)^40*	*$14,381.32*
Monthly	3.65%	*A = 10,000(1 + .0365/12)^(120)*	*$14,397.16*
Continuously	3.6%	*A = 10,000e^(.036*10)*	*$14,333.29*

Which account would you suggest for Natalie? ________________

Solution: Natalie would make the most money with the account in which the interest is compounded monthly at 3.65% per year.

Natalie is particularly interested in how long it will take her money to double if she invests her $10,000 at 3.65% compounded monthly .
The equation that will help us determine how long it will take Natalie's money to double is

$$10000\left(1+\frac{.0365}{12}\right)^{12t} = 20000 \; .$$

Explain why this equation is correct.

We've used the compounded interest formula for investing $1000 at 3.65% compounded 12 times per year but we don't know how many years so we leave the variable t; we want the amount to be $20,000 so these are set equal to each other.

The above equation is equivalent to $\left(1+\frac{.0365}{12}\right)^{12t}=2$. Explain why.

Divided both sides by 10000.

Use logarithms to solve the equation to find out how long it will take for her money to double.

$$\left(1+\frac{.0365}{12}\right)^{12t}=2$$

$$(1.003041667)^{12t}=2$$

$$log_{1.003041667}2=12t$$

$$\frac{log\,2}{log\,1.003041667}=12t$$

$$228.2304027=12t$$
$$t\approx 19\ years$$

Natalie is also interested in how long it will take her $10,000 to double if she invested it at 3.7% compounded continuously.
The equation that can be solved to answer this question is

$$10000e^{.037t}=20000.$$

Explain why this equation is correct.

The above equation is equivalent to $e^{.037t}=2$. Explain why.

If we rewrite this equation as a logarithmic equation we get $\log_e 2=.037t$ which is equivalent to $\ln 2=.037t$. Solve this equation algebraically to find how long it takes Natalie's money to double.

Solution: It will take approximately 18.7 years for the money to double at 3.7% compounded continuously.

Would your answer be different if Natalie invested $50,000 at 3.7% compounded continuously? What if she invested $100? What about $1? Explain. (Try some of the examples if you need to.)

The results of this problem may surprise students!

Solution: No, the answer would not be different. If we are looking at when the money doubles, each equation would reduce to the same equation, namely, $2 = e^{.037t}$.

How Does Your Money Grow?

1. Suppose you invest \$1000 in a savings account that earns 3% interest compounded per year. This amount that you invest is called your *principal*.
   a. How much money will you have at the end of year 1? After 2 years? After 6 years? Organize your work in the table.

Time in Years	0	1	2	3	4	5

   b. Write a function to express the total amount of money *A* in terms of the number of years *t*.

2. Suppose you invest \$1000 at 3% per year but instead of earning the interest yearly, the account earns the interest compounded semi-annually. Thus, each quarter you earn ½ of the 3% interest or $\frac{3}{2}$ % interest per quarter .
   a. Why? (Do you see that $\frac{3}{2}$ % is equivalent to $\frac{.03}{2}$?)
   b. Complete the table to show the amount of money you have.

Time in quarters	0	1	2 (1 yr)	3	4 (2 yrs)	5
Total amt of money						

   c. After 1 year, how does the amount of money in problem 2 compare to that in problem 1? What caused the difference in the amounts?

   d. The function you wrote in part (b) of question (1) now becomes
   $A(t) = 1000\left(1 + \frac{.03}{2}\right)^{2t}$. Explain why.

In the next sequence of questions, the goal is to write the function you wrote in part (b) in a more general way. Consider the sequence of questions carefully, paying close attention to patterns.

   e. How would your function in part (d) change if your interest was compounded quarterly? How would your function in part (d) of question (2) change if your interest was compounded monthly?

f. Look for patterns. How would your function in part (d) change if your interest was compounded *n* times per year? Write this function.

g. Now take the function you wrote in part (f). How would this function change if the annual interest rate was 5.2%?

h. Look for patterns. How would the function in part (ii) change if the annual interest rate was *r*? Write this function.

i. Finally, if the amount of money invested was $850, how would the function you wrote in part (iv) change?

j. Look for patterns. How would the function in part (i) change if the amount invested were represented by P? Write this function.

Congratulations!! If all has gone well with the above questions in part (e), you have just written a generalization for the total amount of money A a person would have if he invests an initial amount P at an annual rate r compounded n times per year for *t* years!!! You should have gotten $(t) = P\left(1+\frac{r}{n}\right)^{nt}$. If you did not get this, go back to find your error. This formula is referred to as the compounded interest formula. The formula is more commonly expressed as $A = P\left(1+\frac{r}{n}\right)^{nt}$.

Often money is compounded continuously rather than compounded 4 times a year or 12 times a year, or some other finite number of times per year. When money is compounded continuously, we use the continuously compounded interest formula $A = Pe^{rt}$ where *P* is the initial amount of money (or principal) invested at a yearly rate *r* compounded continuously for *t* years. This formula uses the irrational number *e* which equals 2.718281828… Since *e* is irrational, it neither terminates nor repeats. But where does the formula $A = Pe^{rt}$ come from and what does it mean to say that "interest is compounded continuously"? Let's make sense of this.
The continuously compounded interest formula comes from $A = P\left(1+\frac{r}{n}\right)^{nt}$. To get a sense of what is happening, let's simplify the situation to consider $1 invested at 100% per year for 1

year. If you substitute this information into $A = P\left(1+\frac{r}{n}\right)^{nt}$, you get $A = 1\left(1+\frac{1}{n}\right)^{n}$ which is equivalent to $A = \left(1+\frac{1}{n}\right)^{n}$. Why?

When money is compounded continuously, you can imagine that the number of times it is compounded per year gets infinitely large—that is, *n* gets infinitely large. Use technology to investigate what happens to the expression $\left(1+\frac{1}{n}\right)^{n}$ as *n* increases in value. Record the value of $\left(1+\frac{1}{n}\right)^{n}$ for each value of *n*. (Some values of *n* are given in the table.)

Frequency of Compounding	Number of times compounded in a year (n)	$\left(1+\frac{1}{n}\right)^{n}$
Annually	1	
Semiannually	2	
Quarterly		
Monthly		
Weekly		
Daily		
Hourly	8760	
Every Minute		
Every Second	31536000	

Examine your data in the table. As the number of compounding times gets infinitely large, what happens to the value of $\left(1+\frac{1}{n}\right)^{n}$? What do you think would happen if you continued to use larger and larger values of *n*? (Try it!)

Let's see what happens if we invest $1 at rates different from 100% for 1 year; this time we will use r for our annual rate of interest so that we can let r vary. Again, we want the number of times the interest is compounded per year to increase. The formula $A = P\left(1+\frac{r}{n}\right)^{nt}$ now becomes $A = 1\left(1+\frac{r}{n}\right)^{n}$ which is equivalent to $A = \left(1+\frac{r}{n}\right)^{n}$. Why?

With technology, investigate what happens to the expression $\left(1+\frac{r}{n}\right)^{n}$ as *n* gets infinitely large. You will investigate this expression for different values of *r*. Begin by letting *r* = 90% or .9. Then you are looking at $\left(1+\frac{.9}{n}\right)^{n}$ as *n* increases in value. You can then consider *r* = 80% or .8 so you are investigating $\left(1+\frac{.8}{n}\right)^{n}$. The following table can help you organize your work. In the final column you can choose your own value of *r* to create the expression $\left(1+\frac{r}{n}\right)^{n}$. Remember there is no limit on the value of *n*; instead, *n* continues to get larger and larger!

Frequency of Compounding	Number of times compounded in a year (n)	$\left(1+\frac{.9}{n}\right)^n$	$\left(1+\frac{.8}{n}\right)^n$	
Annual	1			
Semiannual	2			
Quarterly				
Monthly				
Weekly				
Daily				
Hourly	8760			
Every Minute				
Every Second	31536000			

Examine the data in the table. As the number of compounding times gets infinitely large, what happens to the value of $\left(1+\frac{r}{n}\right)^n$? Do you see for each value of *r* the value of $\left(1+\frac{r}{n}\right)^n$ appears to be getting closer to a particular number? Let's see if we can determine what that number is. As a suggestion, use your calculator to evaluate $e^{.9}$. Now evaluate $e^{.8}$. Conjecture? What number do you think $\left(1+\frac{r}{n}\right)^n$ gets closer to as *n* gets larger and larger? Test your conjecture for different values of *r* and LARGE values of *n*!

Did you see that as *n* increases in value, the expression $\left(1+\frac{r}{n}\right)^n$ gets closer and closer to the value of e^r for any value r? Therefore, if we are compounding interest continuously, the compound interest formula $A = P\left(1+\frac{r}{n}\right)^{nt}$ becomes $A = Pe^{rt}$ because $\left(1+\frac{r}{n}\right)^n$ can be replaced with the value e^r for infinitely large values of *n*. Show how.

Assume your friend, Natalie, has $10,000 to invest. Complete the following chart to show how much she would earn if her money was invested in each of the specified accounts for 10 years.

Frequency of compounding	Annual interest rate	Formula with values substituted in	Amount after 10 years
Quarterly	3.65%		
Monthly	3.65%		
Continuously	3.6%		

Which account would you suggest for Natalie? ________________

Natalie is particularly interested in how long it will take her money to double if she invests her $10,000 at 3.65% compounded monthly.
The equation that will help us determine how long it will take Natalie's money to double is

$$10000\left(1+\frac{.0365}{12}\right)^{12t}=20000 .$$

Explain why this equation is correct.

The above equation is equivalent to $\left(1+\frac{.0365}{12}\right)^{12t}=2$. Explain why.

Use logarithms to solve the equation to find out how long it will take for her money to double.

Natalie is also interested in how long it will take her $10,000 to double if she invested it at 3.7% compounded continuously.
The equation that can be solved to answer this question is

$$10000e^{.037t}=20000.$$

Explain why this equation is correct.

The above equation is equivalent to $e^{.037t}=2$. Explain why.

If we rewrite this equation as a logarithmic equation we get $\log_e 2=.037t$ which is equivalent to $\ln 2=.037t$. Solve this equation algebraically to find how long it takes Natalie's money to double.

Would your answer be different if Natalie invested $50,000 at 3.7% compounded continuously? What if she invested $100? What about $1? Explain. (Try some of the examples if you need to.)

Applications of Logarithmic Functions

Back to Task Table

Mathematical Goals

Work with several applications of logarithmic functions
Use exponential and logarithmic functions to solve problems

Georgia Standards of Excellence

MGSE9-12.F.IF.8 Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function. *(Limit to exponential and logarithmic functions.)*

MGSE9-12.F.IF.8b Use the properties of exponents to interpret expressions for exponential functions. *For example, identify percent rate of change in functions such as* $y = (1.02)^t$, $y = (0.97)^t$, $y = (1.01)^{(12t)}$, $y = (1.2)^{(t/10)}$, *and classify them as representing exponential growth and decay. (Limit to exponential and logarithmic functions.)*

MGSE9-12.F.BF.5 Understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents.

MGSE9-12.F.LE.4 For exponential models, express as a logarithm the solution to $ab^{(ct)} = d$ where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology.

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Introduction

Students explore several applications of logarithmic functions including pH levels and the magnitude of earthquakes.

Materials

Graphing calculator or some other graphing utility

Logarithmic functions are used to model various situations in many different fields. In this task, you will investigate some of these situations.

Problem 1: Acidity of Solutions

NASA had to ensure the astronauts would have sufficient and safe drinking water while traveling to Exponentia. The scientists had to test the pH of the water that was to be stored on the space shuttle to be sure it was within a safe range.

"pH" stands for "potential" of "hydrogen." The "H" in pH is capitalized because the symbol for hydrogen is H. The concentration of hydrogen ions, $[H^+]$, in a substance determines whether the substance is more acidic or alkaline. $[H^+]$, however, is usually a very large or very small number so we use logarithms to convert $[H^+]$ to pH. pH provides a more convenient way to express how acidic or alkaline a substance is.

The pH scale ranges from 0 to 14. A solution with a pH of 7 means it is a neutral solution. Pure water has a pH of 7. A pH less than 7 means the solution is acidic. A pH greater than 7 means the solution is alkaline (basic). The smaller the pH, the more acidic is the solution; the greater the pH, the more alkaline is the solution.

pH is often measured for soil, water, blood, urine, and many chemical reactions; pH is an important value that has significance and consequences. For example, the pH of normal human blood and tissues is about 7.4; if this pH is changed by 0.2 or more, either up or down, it is a life-threatening situation. The ideal range for the pH of water in a swimming pool is 7.2 to 7.8. When the water in a swimming pool falls below 7.2, humans experience eye and skin irritation and pool equipment corrodes. Levels above 7.8 inhibit chlorine's ability to neutralize viruses, bacteria and other health risks in the water, and also cause eye irritation.

pH is the negative logarithm of the concentration of free hydrogen ions, measured in moles per liter (moles/L). The formula to convert the concentration of hydrogen ions to pH is
$pH = -\log [H^+]$.

1. Consider the general common logarithmic function, $f(x) = \log x$. How will the graph of the pH conversion function, $g(x) = -\log x$, differ from the graph of $f(x)$? Specifically, how, if at all, are the domain, range, intercepts, asymptotes, increasing/decreasing changed? What kind of graphical transformation is this?

Solution: The conversion function is a vertical reflection of f(x) over the x-axis. The domain, range, intercept, and asymptote will remain the same. However, the graph of g(x) is decreasing.

2. If a water sample has a pH of 5, use the conversion formula to determine the concentration of hydrogen ions in the sample.

Solution: $-5 = \log [H^+] \rightarrow 10^{-5} = [H^+]$***. So the concentration is*** 10^{-5} ***moles/L.***

3. Suppose another water sample has a pH of 7. How does the concentration of hydrogen ions in this sample compare to the concentration of hydrogen ions in the sample with pH of 5?

Solution: If the pH is 7, the concentration of hydrogen ions is 10^{-7} moles/L. Water with a pH of 5 has a hydrogen ion concentration that is 10^{-5} moles/L. Thus, the concentration of hydrogen ions in the water sample with pH 5 is 100 times that of the water sample with pH of 7.

4. The $[H^+]$ in drinking water should range between approximately 3.16 x 10^{-9} and 10^{-6}. Determine the approximate range for the pH of drinking water.

Solution: The approximate pH range is between 6.0 and 8.5.

5. The concentration of hydrogen ions of a solution is measured and found to be 10^{-4}. Is this solution more or less acidic than drinking water? Explain.

Solution: The pH of this solution is 4. The lower the pH, the more acidic it is. Therefore, this substance is more acidic than our drinking water.

Problem 2: Intensity of Sound[1]

The loudness of sound, *D*, measured in decibels (dB) is given by the formula

$$D = 10\log\frac{I}{10^{-16}}$$

where *I* is the intensity measured in watts per square cm (w/cm^2). The denominator, 10^{-16}, is the approximate intensity of the least sound audible to the human ear.[2]

1. If a normal conversation is held at an intensity of 3.16 x 10^{-10} w/cm^2, how many decibels is this? Simplify as much as possible before using the calculator. (You should use some properties of exponents.)

Solution: $D = 10 \log (I / 10^{-16}) = 10 \log (3.16 \times 10^{-10} / 10^{-16}) = 10 \log (3.16 \times 10^{6}) \approx 65$ dB

2. Suppose the whisper of the ventilation system in the space shuttle had an intensity of 10^{-15} w/cm^2. How many decibels is this? Do not use a calculator. Explain how you determined the answer.

Solution: $D = 10 \log (I / 10^{-16}) = 10 \log (10) = 10$ dB. We know that log (10) means the exponent, a, that the base 10 is raised to in order for $10^a = 10$. So a = 1. Therefore, the equation is now 10 (1) = 10 dB.

[2] Adapted from *Discovering Advanced Algebra: An Investigative Approach* from Key Curriculum Press, 2004.

3. The loudest a rock concert may be held is 120 dB. This is also how loud a space shuttle launch is from a viewing area for non-essential NASA personnel. What is the intensity of the launch from this site? Leave your answer in exponential notation. (You will need to use that logarithms and exponential functions are inverses.)[3]

Solution: $D = 10 \log (I / 10^{-16}) \rightarrow 12 = \log (I/10^{-16}) \rightarrow 10^{12} = I/10^{-16} \rightarrow 10^{-4}\ w/cm^2 = I$

Problem 3: Magnitude of Earthquakes

Most earthquakes are so small enough hardly to be noticed; however, some can be very powerful causing widespread death and destruction and can even trigger tsunamis. The Richter magnitude scale was developed in 1935 by Charles F. Richter of the California Institute of Technology. The magnitude is a number that characterizes the relative size of an earthquake. Magnitude is based on a measurement of ground motion recorded by a seismograph. The Richter scale is a base-10 logarithm scale; each increase of 1 magnitude means 10 times the ground motion. For example, an earthquake with magnitude 6.3 has 10 times the ground motion of an earthquake of magnitude 5.3; thus, the 6.3 earthquake is 10 times the size of the 5.3. An earthquake of magnitude 7 has 10^2 or 100 times the ground motion of an earthquake of magnitude 5 so the magnitude 7 earthquake is 100 times the size of the magnitude 5 earthquake.

1. One earthquake measured 2 on the Richter scale. A second earthquake measured 8 on the Richter scale. Compare the sizes of the two earthquakes.

2. In 2002, an earthquake of magnitude 7.9, one of the largest on U.S. land, occurred in the Denali National Park in Alaska. On April 29, 2003, an earthquake in Fort Payne, Alabama was felt by many residents of northern Georgia. The magnitude was 4.6. How does the size of the Alabama earthquake compare with the size of the Denali earthquake?

Solution: The Denali earthquake was $10^{3.3}$ times the size of the Alabama earthquake.

3. Rather than discuss relative size of an earthquake, we often prefer to discuss the amount of energy released by an earthquake. A formula that relates the number on a Richter scale to the energy of an earthquake is $r = 0.67 \log E - 7.6$, where r is the number on the Richter scale and E is the energy in ergs.

   a. What is the Richter number of an earthquake that releases 3.9×10^{15} ergs of energy? (Be careful when inputting this into the calculator.)

[3] Information obtained from the Space Shuttle Recording Project.

Solution: r = 2.846

b. How much energy was released by the 2002 Denali earthquake? By the 2003 Alabama earthquake?

Solution:

Denali earthquake: $r = 0.67 \log E - 7.6$
$7.9 = 0.67 \log E - 7.6$
$\log E = 15.5/.67 \rightarrow 10^{23.12} = E \rightarrow 1.36 \times 10^{23}$ ***ergs of energy***

Alabama earthquake: $r = 0.67 \log E - 7.6$
$4.6 = 0.67 \log E - 7.6$
$\log E = 12.2/.67 \rightarrow 10^{18.21} = E \rightarrow 1.62 \times 10^{18}$ ***ergs of energy***

Applications of Logarithmic Functions

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    a. What is the Richter number of an earthquake that releases 3.9×10^{15} ergs of energy? (Be careful when inputting this into the calculator.)

    b. How much energy was released by the 2002 Denali earthquake? By the 2003 Alabama earthquake?

Newton's Law of Cooling-Coffee, Donuts, and (later) Corpses. (Spotlight Task)

Back to Task Table

(Three Parts-Coffee, Donuts, Death)

Mathematical Goals
Utilizing real-world situations students will apply the concepts of exponential growth and decay to real-world problems.

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to Precision
7. Look for and Make Use of Structure
8. Look for and Express Regularity in Repeated Reasoning

Georgia Standards of Excellence

MGSE9-12.A.SSE.3 Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression. *(Limit to exponential and logarithmic functions.)*

MGSE9-12.A.SSE.3c Use the properties of exponents to transform expressions for exponential functions. *For example, the expression 1.15^t, where t is in years, can be rewritten as $[1.15^{(1/12)}]^{(12t)} \approx 1.012^{(12t)}$ to reveal the approximate equivalent monthly interest rate if the annual rate is 15%.*

Analyze functions using different representations

MGSE9-12.F.IF.7 Graph functions expressed algebraically and show key features of the graph both by hand and by using technology. *(Limit to exponential and logarithmic functions.)*

MGSE9-12.F.IF.7e Graph exponential and logarithmic functions, showing intercepts and end behavior, ~~and trigonometric functions, showing period, midline, and amplitude.~~

MGSE9-12.F.IF.8 Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function. *(Limit to exponential and logarithmic functions.)*

MGSE9-12.F.IF.8b Use the properties of exponents to interpret expressions for exponential functions. *For example, identify percent rate of change in functions such as* $y = (1.02)^t$, $y = (0.97)^t$, $y = (1.01)^{(12t)}$, $y = (1.2)^{(t/10)}$, *and classify them as representing exponential growth and decay. (Limit to exponential and logarithmic functions.)*

Build new functions from existing functions

MGSE9-12.F.BF.5 Understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents.

Construct and compare linear, quadratic, and exponential models and solve problems

MGSE9-12.F.LE.4 For exponential models, express as a logarithm the solution to $ab^{(ct)} = d$ where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology.

Newton's Law of Cooling-Coffee, Donuts, and (later) Corpses.

(Three Parts-Coffee, Donuts, Death)

PART 1: Coffee and Newton's Law of Cooling
Modified from www.haverford.edu

Sir Isaac Newton found that the temperature of something heated will cool down at different rates, depending on the rate of the environment in which it is cooling. The "Newton's Law of Cooling" equation was derived based on this function:

$$T(t) = T_e + (T_0 - T_e)\, e^{-kt},$$

where $T(t)$ is the temperature of the object at time t, T_e is the constant temperature of the environment, T_0 is the initial temperature of the object, and k is a constant that depends on the material properties of the object.

1. Look at the statement about k. It is saying that k is a constant that depends on the material. Can you think of two liquids that would cool at different rates? What physical properties of the two liquids make that happen? *The density of two different materials often determines how fast it cools off-for liquids, generally thick liquids take longer.*

NEWTON'S LAW OF COOLING (student notes)Vocabulary and Variables:

(T_0) pronounced: "T-sub-zero"	*"Sub-Zero" almost always means, "at the beginning." This mea at the first measurement.*
(T_e) pronounced: "T-sub-e"	*This means temp of the environment.*
k	*The constant of cooling we discussed.*
t	*The time elapsed.*
T(t)	*The Temperature at any given time.*

You discuss Newton's Law of Cooling, give formula and variables here….this is where student takes notes.

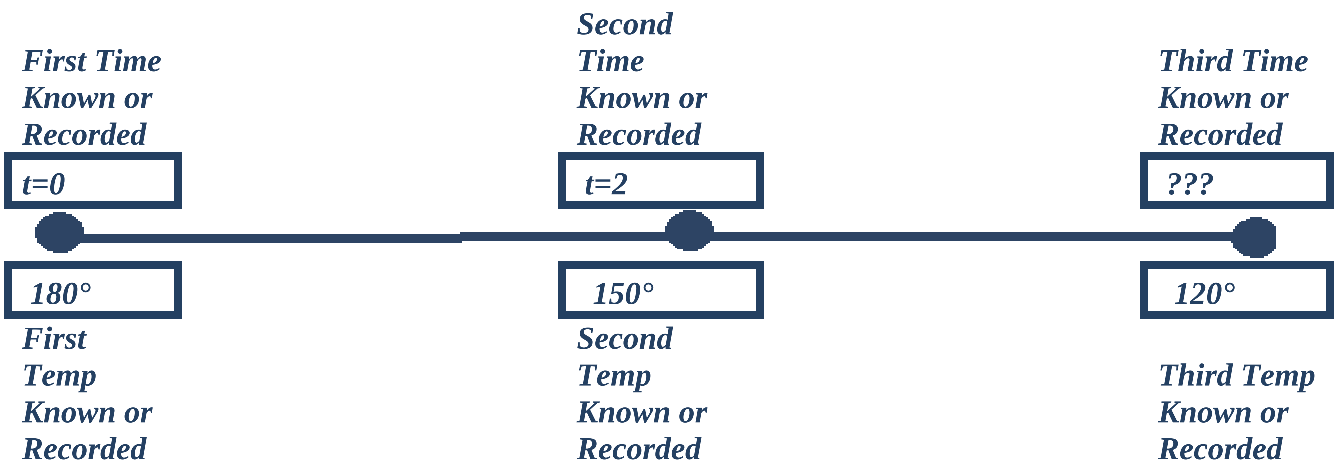
SAMPLE PROBLEM: In a 72° room, my 180° coffee will be 150° after two minutes. I like my coffee at 120°. How long should I wait?

See next page for explanation.

PART 1: COFFEE:

SAMPLE PROBLEM: In a 72° room, my 180° coffee will be 150° after two minutes. I like my coffee at 120°. How long should I wait? Use the info about how long it takes for my coffee to get to find *k*

It might be helpful to imagine the diagram below…three "checkpoints" with a cloud to represent the temp of the environment. You need to have two completed checkpoints to be able to find "k." You can see bellow that we know t=0 sand t=2. There may be times when you have the first and third, second and third, etc. The cloud below represents the "environmental temperature."



Since you have both the first and second "checkpoints," these are the two that you use to find the constant. T(t), the resultant temp, is the end point of those two data checkpoints.

$$T(t) = T_e + (T_0 - T_e)\,e^{-kt}$$

$$150° = 72° + (180° - 72°)\,e^{-k*2}$$

$$150-72=(108)e^{-2k}$$

$$\frac{78}{108} = e^{-2k}$$

$$\ln\left(\frac{78}{108}\right) = \ln(e^{-2k})$$

$$\ln\left(\frac{78}{108}\right) = -2k$$

$$\frac{\ln\left(\frac{78}{108}\right)}{-2} = k$$

$$k \approx 0.1627112002^{**}$$

*****TIP: If students hit "STO→" and then the variable "K" on their calculators, they can keep from having to type in the long decimals!!!! (Store's Usually Beside the Four)-they can now type "K" and the calculator remembers the K for that problem. The variable "k" must be re-figured for each problem though!!!**

PART 1: COFFEE, *continued….*

$$T(t) = T_e + (T_0 - T_e)\, e^{-kt}$$

$$120° = 72° + (180° - 72°)e^{-0.1627\,t}$$

$$120° = 72° + (180° - 72°\ e^{-0.1627\,t})$$

$$48° = (108)\ e^{-0.1627\,t}$$

$$\frac{48}{108} = e^{-0.1627\,t}$$

$$\ln\left(\frac{48}{108}\right) = \ln\left(e^{-0.1627\,t}\right)$$

$$\ln\left(\frac{48}{108}\right) = -0.1627t$$

$$\frac{\ln\left(\frac{48}{108}\right)}{-0.1627} = t$$

$$t \approx 4.98 \text{ minutes}$$

(It will take about five minutes for the coffee to cool down to 120°

Challenges

1. In a 72° room, my 180° coffee will be 150° after two minutes. How long will it take to get 75°?

   About 22 minutes (setup here was identical to first example…didn't have to re-figure k).

2. What is the temperature after 30 minutes**?** ***72.8°F…if students get 1.366° they failed to observe order of operation, and add/subtracted 72 + 180-72 straight across rather than multiplying (180-72) times the result of $e^{-0.1627*30}$!!! Bad Bad!!!***
3. Boiling water (212° at sea level) is left in a 70° and after 5 minutes it is 180°. What is the constant of cooling? ***k≈0.5107***
4. Using this info from the previous question, how long will it take to have it cool to 98°? ***≈32 minutes***
5. Heating is cooling in reverse. Use the same constant *k* as in #3. If an ice cube is placed in the same room. How long will it take to become 50°? (Presume the ice is 32° when frozen). ***≈12.5 minutes***

PART 2: Donuts and Coffee Lab

Suppose you visited Krispy Kreme. You ordered hot coffee and hot donuts (YUM!). You are handed your hot coffee, but informed that you will have to wait 15 minutes for your hot donuts to finish traveling their conveyor belt trip through the waterfall of sweet glaze. You like to add creamer to your coffee, but you still want the coffee to be as hot as possible after those fifteen minutes so that you can enjoy both the hot donuts and the hottest possible coffee.

The question is: should you add the creamer as soon as you get it (2 minutes after it is brewed) or should you add the creamer in about 12 minutes?

Objectives:

1. To determine whether we should add the room temperature creamer after 2 minutes or after 12 minutes if we wish to drink the coffee as hot as possible about 15 minutes after it is poured. (If the coffee is allowed to sit for half an hour, it will likely be room temperature, so either case would give the same final temperature after half an hour.)
2. To determine the rate of cooling of the two cases.
3. To explore logarithms in a chemistry setting.
4. To create graphs of both situations, and to analyze the graphs

MY HYPOTHESIS ABOUT WHAT IS GOING TO HAPPEN, AND REASONS WHY I KNOW I AM RIGHT:

Answers Vary-as long as student can explain why they think what is going to happen.

Materials (one of each per group-pairs is best).
Per group of 2-3 students:

- 1 hot plate per group*
- 2 250 ml beakers per group *
- 2 150 ml beakers per group *
- 2 thermometers per group*
- water
- milk
- instant coffee
- graduated cylinder *
- timer (cell phones have these)*
- paper towel
- graph paper
- pencil and paper to record observations

Chemistry teachers may have all of the lab items in this lab, and may allow you to borrow them or maybe even switch rooms for the day so that you can run this lab! However the TI/Vernier LabPro works quite well, and as long as you have two of the SAME amounts of coffee for this lab, lab glassware isn't necessarily needed. Equal volume containers and thermometers will work. It may be advised to have a bucket for throwaway coffee as well. If you have access to brewed coffee, you might choose that, but this lab heats water up to 80 rather than allowing the coffee to cool down to 80 so that you don't have to wait for cooling. Hot coffee is actually much hotter than 80°F.

Procedure:
You should have already written down your hypothesis of what you think will happen, now that we have discussed Newton's Law on the previous page.

One partner will do Part A (add milk early) below, and the other partner will do Part B (wait to add the milk). Measure exactly as directed, and carefully follow each instruction to a "T." Do Part A and Part B at the same time.

Part A. Add the milk early:	**Part B. Add the milk later:**
1. Measure 150 ml of water with instant coffee into a 250 ml beaker and heat it on a hot plate until it the temperature reaches 80 degrees.	**1. Measure 150 ml of water with instant coffee into a 250 ml beaker and heat it on a hot plate until the temperature is 80 degrees.**
2. Put 40 ml of milk in a 150 ml beaker.	**2. Put 40 ml of milk in a 150 ml beaker.**
3. When the coffee has reached 80 degrees, carefully remove it from the hot plate using a paper towel as a pot holder.	**3. When the coffee has reached 80 degrees, carefully remove it from the hot plate using a paper towel as a pot holder.**
4. The temperature of the water may continue to increase a few degrees after it is removed from the heat. When the temperature returns to 80 degrees start taking the temperature every 30 seconds.	**4. The temperature of the water may continue to increase a few degrees after it is removed from the heat. When the temperature returns to 80 degrees start taking the temperature every 30 seconds.**
HERE (AT 2 MINUTES) IS WHERE PART A AND PART B DIVERGE!	
5. At the two minute (2 minutes) mark, add the milk and record the temperature immediately. Then continue taking the temperatures every 30 seconds.	**5. At the 12 minute mark (12 seconds), add the milk and take the temperature immediately, then continue taking the temperature every 30 seconds.**
6. Using Newton's Law of Cooling, identify	**6. Using Newton's Law of Cooling, identify**

which of your measurements would be T_0, which of your measurements would be T_e, what is "t," and what is k?	**which of your measurements would be T_0, which of your measurements would be T_e, what is "t," and what is k?**
7. Record the temperatures in a chart and keep taking temperatures each 30 seconds until the time reaches 15 minutes.	**7. Record the temperatures in a chart and keep taking temperatures every 30 seconds until the time had reached 15 minutes.**
Graph both on the same piece of graph paper so that they can easily be compared. (Or else take turns graphing on both so that both members can have a copy of both graphs-check with teacher)	
8. Make a graph of temperature versus time, temperature on the y axis and time on the x axis.	**8. Make a graph of temperature versus time, temperature on the y axis and times on the x axis.**
9. Compare your graph with your partner. Which slope is steeper? Who has the higher temperature after fifteen minutes? What does this tell you?	**9. Compare your graph with your partner. Which slope is steeper? Who has the higher temperature after fifteen minutes? What does this tell you?**

Data and Results

10. Record the data in the table: Note that the last three entries are at extended time periods so that you can start graphing while you are waiting to finish. ***Sample data below. Room was roughlty73° (results are more dramatic if you blast the A/C all day), and coffee was the correct 80° at time t=0.***

Time (Minutes)	Temperature A	Temperature B	Time (Minutes)	Temperature A	Temperature B	Time (Minutes)	Temperature A	Temperature B	Time (Minutes)	Temperature A	Temperature B	Time (Minutes)	Temperature A	Temperature B
0	*80*	*80*	**3.5**	*77.4*	*79.2*	**7**	*76.8*	*78.4*	**10.5**	*76.3*	*78.2*	**14**	*75.8*	*77.2*
0.5	*78.8*	*79.8*	**4**	*77.4*	*79.0*	**7.5**	*76.8*	*78.4*	**11**	*76.2*	*78.1*	**14..5**	*75.8*	*77.2*
1	*78.4*	*79.6*	**4.5**	*77.2*	*78.9*	**8**	*76.6*	*78.4*	**11.5**	*76.2*	*78.0*	**15**	*75.8*	*77.0*
1.5	*78.1*	*79.6*	**5**	*77.2*	*78.9*	**8.5**	*76.6*	*78.4*	**12**	*76.0*	*78.0*	**15.5**	*75.7*	*76.9*
2	*77.8*	*79.4*	**5.5**	*77.0*	*78.8*	**9**	*76.5*	*78.3*	**12.5**	*76.0*	*77.7*	**20**	*75.4*	*76.0*
2.5	*77.7*	*79.4*	**6**	*77.0*	*78.7*	**9.5**	*76.5*	*78.2*	**13**	*76.0*	*77.6*	**25**	*75.2*	*75.2*
3	*77.6*	*79.2*	**6.5**	*76.9*	*78.7*	**10**	*76.4*	*78.2*	**13.5**	*75.9*	*77.5*	**30**	*75.2*	*74.7*

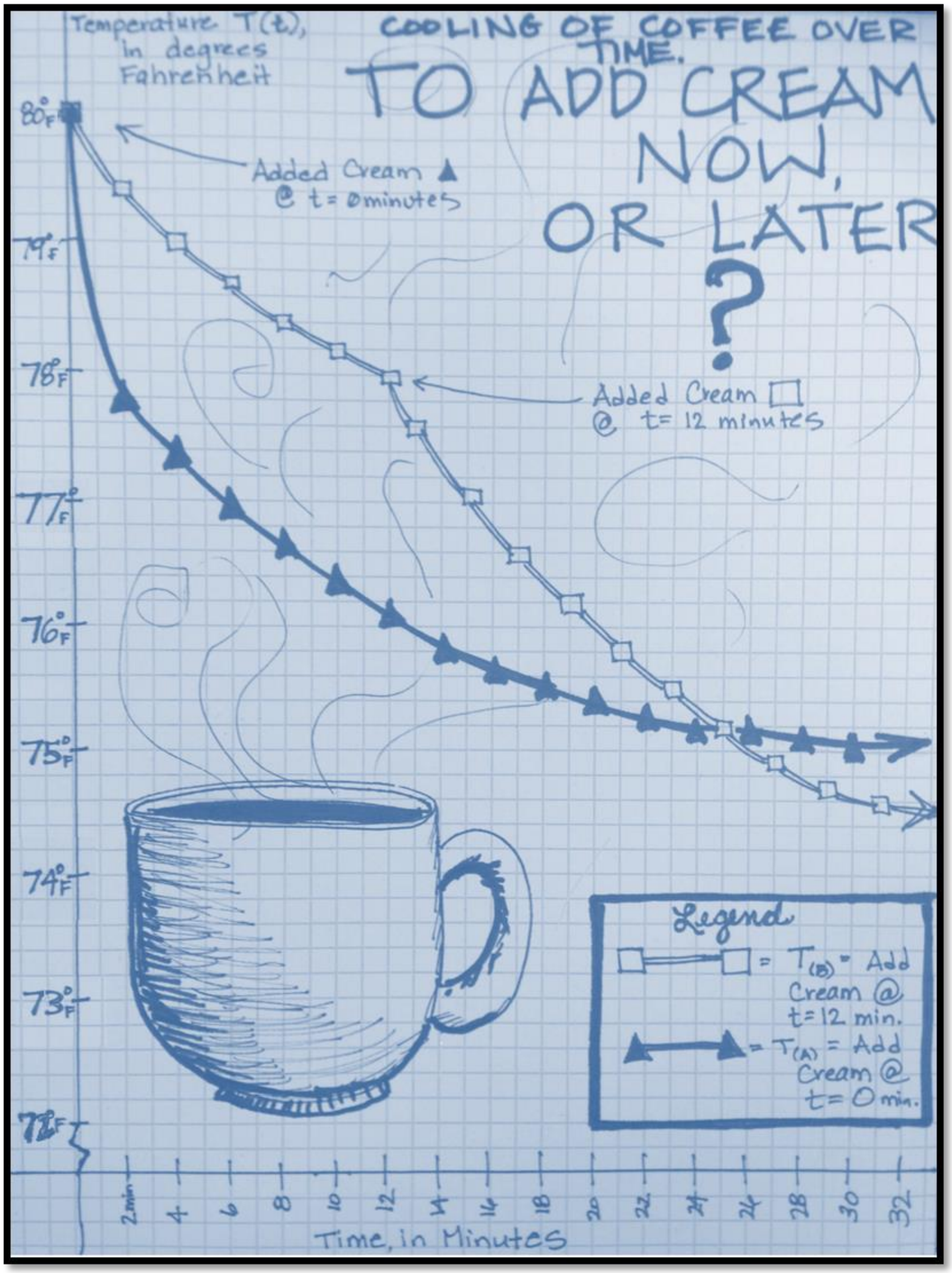
11. GRAPH the data on the same sheet of graph paper, same set of axes (or else take turns graphing on both so that both members can have a copy of both graphs) Only graph points for even numbers including zero, since your thermometer probably doesn't measure to the hundredth, so that you can actually see the change

See graph, next page. The axes do not need to start at zero-you aren't in a freezing cold classroom, hopefully. For the sample range of data shown, the temperature never go below 75°. Creating a graph that is large enough to be seen, but without a lot of empty space on graph paper is a skill they should practice, so it is better to give them empty graph paper and tell them to graph the situation than it is to give them axes that are more "friendly."

It is also a good idea to have them graph only every 2 minutes rather than every one of the 30 seconds. This is because most high school lab thermometers do not register temperatures with greater specificity than a tenth of a degree. If students graph every

single half-minute point, their graphs will look more like stair steps than smooth graphs. But there is a good discussion, even there. "Do you think the coffee is still cooling? Why does it look like it's the same from 4 to 4.5 minutes?

Sample Graph from Data in Chart, Previous Page



12. Discussion – What were the results, and was my hypothesis correct, or not?

The results should turn out as follows: The final temperature of the coffee with cream added early (Part A) is a few degrees above the coffee with cream added later (Part B). If the room temperature is 24 or 25 C, there will be little difference. The experiment works best if the temperature of the room is about 20 C, as in the wintertime (OR crank down the A/C). Also, the graph for Part A after the cream is added is more horizontal, or has a lower slope, than the graph for Part B until the cream is added. That shows that if the temperature between the liquid (coffee) and the room is greater, the cooling is faster and if the difference in temperatures is less, the cooling is slower. Or, the rate is greater is the temperature difference is greater. The time to cool is proportional to the difference between the substance and the ambient temperature. Pretty Cool!.

13. Discussion – What did you learn?

Answers Vary-hopefully they learned that (because the difference between the creamer and the coffee was greater at the beginning) it is better to wait until the temperature difference is smaller, because the greater the difference, the more quickly the coffee will cool.

14. Did anything surprise you about your results?

Answers Vary

15. ExtensionA: What if (instead of coffee) it was Hot Chocolate or Broccoli and Cheese Soup that was cooling? Do you think these would cool faster, or more slowly?

Denser substances typically cool more slowly.

16. ExtensionB: What sources of error might there have been in your data?

Measurement errors will typically be the sources of error, along with thermometer reading errors, especially if the thermometer is an older mercury one. Timing would be a source of error-even a few seconds difference between pouring the two cups of coffee makes a big difference because the cooling starts to take place immediately.

Assessment-Turn in the Following: This Hypothesis/Conclusion/Discussion Page, Your Data, and your Graphs

Have the students turn in their hypothesis, data charts, graphs, and conclusion, and possible sources of error (reading the thermometer, adding cream at correct time, graphing, etc).

Extensions:

The most obvious extension to this task is to ask students to compare their actual temperature measurements with theoretical temperature measurements For "Part A,", students could derive the equation using data from Step 5 (the 2 minute mark) to Step 7 (the 15 minute mark), and then test to see if the in-between actual data points match the equation's theoretical data points. "Part B," could do the same. An equation could be derived from Step 1 to Step 5 (the 2 minute mark to the 12 minute mark) and test times in between to see if the real data lines up with the theoretical data. Then the same could occur from Steps 5 to 7.

The result can be applied to discussions involving climate and the cooling or earth and water - cooling of ground in dessert areas, cooling of water in lakes and oceans, and warming of oceans for swimming. In a liquid mixture, the hot liquid is an energy source and the cold liquid is an energy receiver. Heat energy is transferred from the hot to the cold substance.

A class should observe Newton's Law of Cooling directly several ways; any of these experiments would demonstrate that hotter water cools faster than cooler water.

- ***They could just heat water, remove it from the hot plate and then record the temperature every minute as it cooled.***
- ***A shorter experiment would be for each group in the class to have the same quantity of water at different temperatures. As a class, set time zero and record the original temperature ant then record the temperature at some set time later, say three minutes. The class could compare the average rate of temperature drop in three minutes for different initial temperatures.***
- ***You could also say that you recorded cooling data and have them graph the data. Here is some data and a computer generated graph.***

The question we are asking here that needs to be considered here is how fast does something cool off, or how fast does the temperature change? What are the crucial factors that affect the rate of cooling? Newton's Law of Cooling addresses these questions. Newton's Law says that

the time a substance takes to cool off depends on the temperature difference between the substance and the surroundings.

PART 3: Corpses

Newton's Law of Cooling and CSI: REAL APPLICATION

Crime Scene

A detective is called to the scene of a crime in a college science lab where the dead body of an unnamed chemistry student has just been found in a closet. It is clear the body was there for some time-possibly even while students were working in the lab the previous night. The detective arrives on the scene at 5:41 am and begins her investigation. Immediately, the temperature of the body is taken and is found to be 78.0^{o}F. The detective checks the programmable thermostat and finds that the lab has been kept at a constant 71^{o} F for several days.

After evidence from the crime scene is collected, the temperature of the body is taken once more and found to be 76.6^{o} F. This last temperature reading was taken exactly one hour after the first one.

Based on key-card entry records, it is clear that there were only four students in the lab the night before:

- The dead chemistry student arrived at the lab at 7pm the previous night and never left the lab.
- Edgar got into the science lab at 6pm, but he left at 10pm.
- Franny got into the science lab at midnight and worked until 2am.
- Geoffrey got into the science lab at 10 pm and worked until midnight.

The next day the detective is asked by another investigator, "What time did our victim die?" Assuming that the victim's body temperature was normal (98.6^{o} F) prior to death, what is her answer to this question?

Newton's Law is how detectives determine time of death! Solve the crime. Find the dead student's constant of cooling, her time of death, and name the murderer.

How Long Ago? (Answer will be negative…)	*First Time Known or Recorded*	*Second Time Known or Recorded*
????	*t=0*	*t=60*
98.6°	*78.0°*	*76.6°*
First Temp Known or Recorded	*Second Temp Known or Recorded*	*Third Temp Known or Recorded*

Using the two known time/temp pairs, we can fill out the Newton's Law of Cooling Formula to Find k.

$$T(t) = T_e + (T_0 - T_e)\, e^{-kt}$$

$$76.6° = 71° + (78° - 71°)\, e^{-k*60}$$

$$76.6°-71°=(7)e^{-2k}$$

$$\frac{5.6}{7} = e^{60k}$$

$$\ln\left(\frac{5.6}{7}\right) = \ln(e^{-60k})$$

$$\ln\left(\frac{5.6}{7}\right) = -60k$$

$$\frac{\ln\left(\frac{5.6}{7}\right)}{-60} = k$$

$$k \approx 0.0037190592^{**}$$

$$\mathrm{T}(t) = \mathrm{T_e} + (\mathrm{T_0} - \mathrm{T_e})\, e^{-kt}$$

$$98.6°=71° + (7°)e^{-0.0037\,t}$$

$$98.6° = 71° + (7°)e^{-0.0037\,t}$$

$$27.6°=(7)\, e^{-0.0037\,t}$$

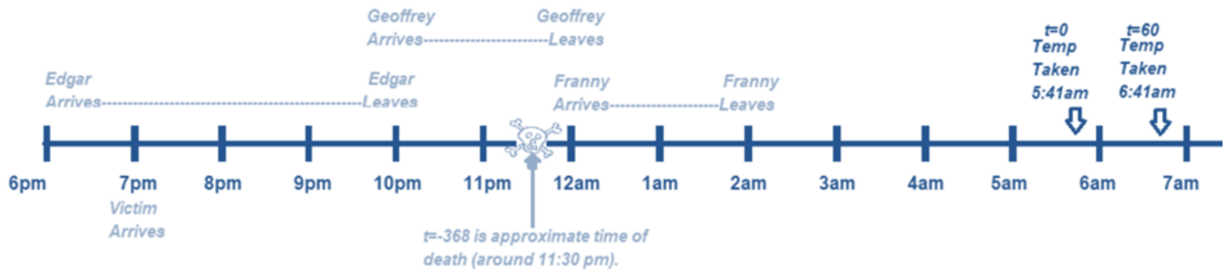
$$\frac{27.6}{7} = e^{-0.0037\,t}$$

$$\ln\left(\frac{27.6}{7}\right) = \ln\left(e^{-0.0037\,t}\right)$$

$$\ln\left(\frac{27.6}{7}\right) = -0.0037t$$

$$\frac{\ln\left(\frac{27.6}{7}\right)}{-0.0037} = t$$

t ≈ -368.89 minutes ago….roughly 6 hours, 5 minutes ago.



Looks like Geoffrey has some explaining to do.

Newton's Law of Cooling-Coffee, Donuts, and (later) Corpses. (Spotlight Task)

(Three Parts-Coffee, Donuts, Death)

Mathematical Goals
Utilizing real-world situations students will apply the concepts of exponential growth and decay to real-world problems.

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to Precision
7. Look for and Make Use of Structure
8. Look for and Express Regularity in Repeated Reasoning

Georgia Standards of Excellence

MGSE9-12.A.SSE.3 Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression. *(Limit to exponential and logarithmic functions.)*

MGSE9-12.A.SSE.3c Use the properties of exponents to transform expressions for exponential functions. *For example, the expression* 1.15^t *, where t is in years, can be rewritten as* $[1.15^{(1/12)}]^{(12t)} \approx 1.012^{(12t)}$ *to reveal the approximate equivalent monthly interest rate if the annual rate is 15%.*

<u>Analyze functions using different representations</u>

MGSE9-12.F.IF.7 Graph functions expressed algebraically and show key features of the graph both by hand and by using technology. *(Limit to exponential and logarithmic functions.)*

MGSE9-12.F.IF.7e Graph exponential and logarithmic functions, showing intercepts and end behavior, ~~and trigonometric functions, showing period, midline, and amplitude.~~

MGSE9-12.F.IF.8 Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function. *(Limit to exponential and logarithmic functions.)*

MGSE9-12.F.IF.8b Use the properties of exponents to interpret expressions for exponential functions. *For example, identify percent rate of change in functions such as $y = (1.02)^t$, $y = (0.97)^t$, $y = (1.01)^{(12t)}$, $y = (1.2)^{(t/10)}$, and classify them as representing exponential growth and decay. (Limit to exponential and logarithmic functions.)*

<u>Build new functions from existing functions</u>

MGSE9-12.F.BF.5 Understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents.

<u>Construct and compare linear, quadratic, and exponential models and solve problems</u>

MGSE9-12.F.LE.4 For exponential models, express as a logarithm the solution to $ab^{(ct)} = d$ where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology.

NEWTON'S LAW OF COOLING-Two Cups of Coffee
Modified from www.haverford.edu

Sir Isaac Newton found that the temperature of something heated will cool down at different rates, depending on the rate of the environment in which it is cooling. The "Newton's Law of Cooling" equation was derived based on this function:

$$T\,(t) = T_e + (T_0 - T_e\,)\,e^{-kt},$$

where $T\,(t)$ is the temperature of the object at time t, T_e is the constant temperature of the environment, T_0 is the initial temperature of the object, and k is a constant that depends on the material properties of the object.

1. Look at the statement about k. It is saying that k is a constant that depends on the material. Can you think of two liquids that would cool at different rates? What physical properties of the two liquids make that happen?

NEWTON'S LAW OF COOLING (student notes)Vocabulary and Variables:

(T_0) pronounced: "T-sub-zero"	
(T_e) pronounced "T-sub-e"	
k	
t	
T(t)	

PART 1: COFFEE:SAMPLE PROBLEM: In a 72° room, my 180° coffee will be 150° after two minutes. I like my coffee at 120°. How long should I wait? Use the info about how long it takes for my coffee to get to find *k*

Challenges

1. In a 72° room, my 180° coffee will be 150° after two minutes. How long will it take to get 75°?
2. What is the temperature after 30 minutes?
3. Boiling water (212° at sea level) is left in a 70° and after 5 minutes it is 180° What is the constant of cooling?
4. Using this info from the previous question, how long will it take to have it cool to 98°?

5. Heating is cooling in reverse. Use the same constant k as in #3. If an ice cube is placed in the same room. How long will it take to become 50°? (Presume the ice is 32° when frozen).

PART 2: DONUTS AND COFFEE:

Introduction:

Suppose you visited Krispy Kreme. You ordered hot coffee and hot donuts (YUM!). You are handed your hot coffee, but informed that you will have to wait 15 minutes for your hot donuts to finish traveling their conveyor belt trip through the waterfall of sweet glaze. You like to add creamer to your coffee, but you still want the coffee to be as hot as possible after those fifteen minutes so that you can enjoy both the hot donuts and the hottest possible coffee.

The question is: should you add the creamer as soon as you get it (2 minutes after it is brewed) or should you add the creamer in about 12 minutes?

Objectives:

1. To determine whether we should add the room temperature creamer after 2 minutes or after 12 minutes if we wish to drink the coffee as hot as possible about 15 minutes after it is poured. (If the coffee is allowed to sit for half an hour, it will likely be room temperature, so either case would give the same final temperature after half an hour.)
2. To determine the rate of cooling of the two cases.
3. To explore logarithms in a chemistry setting.
4. To create graphs of both situations, and to analyze the graphs

MY HYPOTHESIS ABOUT WHAT IS GOING TO HAPPEN, AND REASONS WHY I KNOW I AM RIGHT:

Materials (one of each per group-pairs is best).

per group of 2-3 students:

- 1 hot plate per class*
- 2 250 ml beakers *
- 2 150 ml beakers *
- 2 thermometers*
- water
- milk
- instant coffee
- graduated cylinder *
- timer *
- paper towel
- graph paper
- pencil and paper to record observations

Procedure:
You should have already written down your hypothesis of what you think will happen, now that we have discussed Newton's Law on the previous page.

One partner will do Part A (add milk early) below, and the other partner will do Part B (wait to add the milk). Measure exactly as directed, and carefully follow each instruction to a "T." Do Part A and Part B at the same time.

Part A. Add the milk early	**Part B. Add the milk later:**
1. Measure 150 ml of water with instant coffee into a 250 ml beaker and heat it on a hot plate until it the temperature reaches 80 degrees.	**1. Measure 150 ml of water with instant coffee into a 250 ml beaker and heat it on a hot plate until the temperature is 80 degrees.**
2. Put 40 ml of milk in a 150 ml beaker.	**2. Put 40 ml of milk in a 150 ml beaker.**
3. When the coffee has reached 80 degrees, carefully remove it from the hot plate using a paper towel as a pot holder.	**3. When the coffee has reached 80 degrees, carefully remove it from the hot plate using a paper towel as a pot holder.**
4. The temperature of the water may continue to increase a few degrees after it is removed from the heat. When the temperature returns to 80 degrees start taking the temperature every 30 seconds.	**4. The temperature of the water may continue to increase a few degrees after it is removed from the heat. When the temperature returns to 80 degrees start taking the temperature every 30 seconds.**
HERE (AT 2 MINUTES) IS WHERE PART A AND PART B DIVERGE!	
5. At the two minute (2 minutes) mark, add the milk and record the temperature immediately. Then continue taking the temperatures every 30 seconds.	**5. At the 12 minute mark (12 seconds), add the milk and take the temperature immediately, then continue taking the temperature every 30 seconds.**
6. Using Newton's Law of Cooling, identify which of your measurements would be T0, which of your measurements would be Te, what is "t," and what is k?	**6. Using Newton's Law of Cooling, identify which of your measurements would be T0, which of your measurements would be Te, what is "t," and what is k?**
7. Record the temperatures in a chart and keep taking temperatures each 30 seconds until the time reaches 15 minutes.	**7. Record the temperatures in a chart and keep taking temperatures every 30 seconds until the time had reached 15 minutes.**

Graph both on the same piece of graph paper so that they can easily be compared. (Or else take turns graphing on both so that both members can have a copy of both graphs-check with teacher)	
8. Make a graph of temperature versus time, temperature on the y axis and time on the x axis.	**8. Make a graph of temperature versus time, temperature on the y axis and times on the x axis.**
9. Compare your graph with your partner. Which slope is steeper? Who has the higher temperature after fifteen minutes? What does this tell you?	**9. Compare your graph with your partner. Which slope is steeper? Who has the higher temperature after fifteen minutes? What does this tell you?**

Data and Results

10. Record the data in the table: Note that the last three entries are at extended time periods so that you can start graphing while you are waiting to finish.

Time (Minutes)	Temperature A	Temperature B	Time (Minutes)	Temperature A	Temperature B	Time (Minutes)	Temperature A	Temperature B	Time (Minutes)	Temperature A	Temperature B	Time (Minutes)	Temperature A	Temperature B
0			3.5			7			10.5			14		
0.5			4			7.5			11			14..5		
1			4.5			8			11.5			15		
1.5			5			8.5			12			15.5		
2			5.5			9			12.5			20		
2.5			6			9.5			13			25		
3			6.5			10			13.5			30		

11. GRAPH the data on the same sheet of graph paper, same set of axes (or else take turns graphing on both so that both members can have a copy of both graphs) Only graph points for even numbers including zero, since your thermometer probably doesn't measure to the hundredth, so that you can actually see the change

12. Discussion – What were the results, and was my hypothesis correct, or not?

13. Discussion – What did you learn?

14. Did anything surprise you about your results?

15. Extension A: What if (instead of coffee) it was Hot Chocolate or Broccoli and Cheese Soup that was cooling? Do you think these would cool faster, or more slowly?

16. Extension B: What sources of error might there have been in your data?

Assessment-Turn in the Following: This Hypothesis/Conclusion/Discussion Page, and your Graphs

PART 3: CORPSES
Newton's Law of Cooling and CSI: REAL APPLICATION

Crime Scene

A detective is called to the scene of a crime in a college science lab where the dead body of an unnamed chemistry student has just been found in a closet. It is clear the body was there for some time-possibly even while students were working in the lab the previous night. The detective arrives on the scene at 5:41 am and begins her investigation. Immediately, the temperature of the body is taken and is found to be 78.0^{o}F. The detective checks the programmable thermostat and finds that the lab has been kept at a constant 71^o F for several days.

After evidence from the crime scene is collected, the temperature of the body is taken once more and found to be 76.6^o F. This last temperature reading was taken exactly one hour after the first one.

Based on key-card entry records, it is clear that there were only four students in the lab the night before:

- The dead chemistry student arrived at the lab at 7pm the previous night and never left the lab.
- Edgar got into the science lab at 6pm, but he left at 10pm.
- Franny got into the science lab at midnight and worked until 2am.
- Geoffrey got into the science lab at 10 pm and worked until midnight.

The next day the detective is asked by another investigator, "What time did our victim die?" Assuming that the victim's body temperature was normal (98.6^o F) prior to death, what is her answer to this question?

Newton's Law is how detectives determine time of death! Solve the crime. Find the dead student's constant of cooling, her time of death, and name the murderer.

Formative Assessment Lesson: Graphing Logarithmic and Exponential Functions

Back to Task Table

Source: Georgia Mathematics Design Collaborative

This lesson is intended to help you assess how well students are able to:

- Graph exponential and logarithmic functions
- Determine x and y intercepts and asymptotes of exponential and logarithmic functions
- Describe the domain, range and end behaviors of exponential and logarithmic functions
- Understand that exponential and logarithmic functions are inverses of each other

STANDARDS ADDRESSED IN THIS TASK:

MGSE9-12.F.IF.7e Graph exponential and logarithmic functions, showing intercepts and end behavior, ~~and trigonometric functions, showing period, midline, and amplitude.~~

MGSE9-12.F.BF.5 Understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents.

STANDARDS FOR MATHEMATICAL PRACTICE:

This lesson uses all of the practices with emphasis on:

1. Make sense of problems and persevere in solving them
2. Reason abstractly and quantitatively
3. Construct viable arguments and critique the reasoning of others
6. Attend to precision

TASK DESCRIPTION, DEVELOPMENT AND DISCUSSION:

Tasks and lessons from the Georgia Mathematics Design Collaborative are specifically designed to help teachers effectively formatively assess their students. The way the tasks and lessons are designed gives the teacher a clear understanding of what the students are able to do and not do. Within the lesson, teachers will find suggestions and question prompts that will help guide students towards understanding.

The task, *Graphing Logarithmic and Exponential Functions*, is a Formative Assessment Lesson (FAL) that can be found on the Georgia Mathematics online professional learning community, edWeb.net. Once logged in, the task can be found directly at:
https://www.edweb.net/?14@@.5ad26830

John and Leonhard at the Café Mathematica

Back to Task Table

Mathematical Goals

- Students will use prior knowledge of exponential functions and logarithms to model and solve practical scenarios.

Georgia Standards of Excellence

Analyze functions using different representations.

MGSE9-12.F.IF.7e Graph exponential and logarithmic functions, showing intercepts and end behavior, ~~and trigonometric functions, showing period, midline, and amplitude.~~

MGSE9-12.F.IF.8b Use the properties of exponents to interpret expressions for exponential functions. *For example, identify percent rate of change in functions such as* $y = (1.02)^t$, $y = (0.97)^t$, $y = (1.01)^{(12t)}$, $y = (1.2)^{(t/10)}$, *and classify them as representing exponential growth and decay.*

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Introduction

This is a relatively straightforward task that focuses on using what students already know about exponential functions and the uses of logarithms to create models and, more often here, to use those models to solve problems. Like other times during this unit, this is in no way a comprehensive treatment of exponential functions, logarithms, or logarithmic functions, but instead is here to give students a brief exposure to the different contexts where these important tools can be used.

Materials

- Pencil
- Handout
- Calculator

John and Leonhard at the Café Mathematica

John Napier is running late in meeting with his friend Leonhard Euler at a local coffee shop, The Café Mathematica. John is behind schedule because he has spent all morning making a discovery, namely that:

$$f(x) = log_b x \quad \textit{has some relation to} \quad x = b^{f(x)}$$

While Leonhard is waiting for his hopelessly tardy friend, he begins scribbling out the solutions to some exponential problems that have been posed by a few of his professor friends at the local university on some spare napkins at the table.

---

1. The population of a town increases according to the model

$$P(t) = 2500e^{0.0293t}$$

where t is the time in years, with $t = 0$ corresponding to 2010.

(a) Find the projected population of the town in 2012, 2015, and 2018.

$$\mathbf{2012 \rightarrow P(2) = 2500e^{0.0293(2)} \approx 2651}$$

$$\mathbf{2015 \rightarrow P(5) = 2500e^{0.0293(5)} \approx 2895}$$

$$\mathbf{2018 \rightarrow P(8) = 2500e^{0.0293(8)} \approx 3161}$$

(b) Use a graphing calculator to graph the function for the years 2010 through 2030.

Students produce a graph on the calculator to answer (c) below.

(c) Use a graphing calculator to approximate the population in 2025 and 2030.

$$\mathbf{P(25) \approx 5201}$$

$$\mathbf{P(30) \approx 6021}$$

(d) Verify your answers in part (c) algebraically.

$$\boldsymbol{P(25) = 2500e^{0.0293(25)} \approx 5201}$$

$$\boldsymbol{P(30) = 2500e^{0.0293(30)} \approx 6021}$$

2. A certain population increases according to the model $P(t) = 250e^{0.47t}$. Use the model to determine the population when $t = 5$. Round your answer to the nearest integer.

A. 400 B. 2621 C. 1998

D. 1597 E. None of these

The answer is B because $\boldsymbol{P(5) = 250e^{0.47(5)} \approx 2621}$

3. You go to work for a company that pays \$0.01 the first day, \$0.02 the second day, \$0.04 the third day, and so on. If the daily wage keeps doubling, what will your total income be after working 15 days?

A. \$.15 B. \$2.02 C. \$32

D. \$327.67 E. \$32,767

The model for this scenario is $\boldsymbol{a_n = 0.01(2)^{n-1}}$***, which students should recognize as having come from the formula for the nth term of a geometric sequence, which of course is an exponential function. However, it is very easy to overlook the objective of the question, which is not to find what you're paid on the 15th day, but to find your TOTAL income after working for 15 days. Therefore, the sum of a geometric series is necessary***

$$\boldsymbol{S_{15} = 0.01\left(\frac{1-2^{15}}{1-2}\right) = 327.67}$$

Therefore, the answer is D.

4. You bought a guitar 6 years ago for \$400. If its value decreases by about 13% per year, how much is your guitar worth now?

A. \$173.45 B. \$226.55 C. \$322

D. \$351.23 E. \$832.78

It is important for students to recognize an appropriate exponential base in building the associated function. Many students will be tempted to make the base 0.13 instead of reasoning out that, since this is decay, the guitar isn't worth 13% of the previous year's value every year, but instead 13% <u>less</u> than the previous year's value every year. Therefore,

$$\boldsymbol{guitar\ value = 400(1-0.13)^n = 400(0.87)^6 \approx 173.45}$$

Therefore, the answer is A.

5. The amount of a certain radioactive substance remaining after t years decreases according to the function $N = N_0 e^{-0.0315t}$ where N_0 is the initial amount of the substance and t = time in years. How much of a 25 gram sample will remain after 20 years?

A. 13.31 grams B. 46.94 grams C. 0.53 grams D. 1.88 grams

$\boldsymbol{25e^{-0.0315(20)} \approx 13.31}$ ***so the answer is A***

6. Let Q (in grams) represent the mass of a quantity of carbon-14, which has a half-life of 5730 years. The quantity present after t years is

$$Q = 10\left(\frac{1}{2}\right)^{\frac{t}{5730}}$$

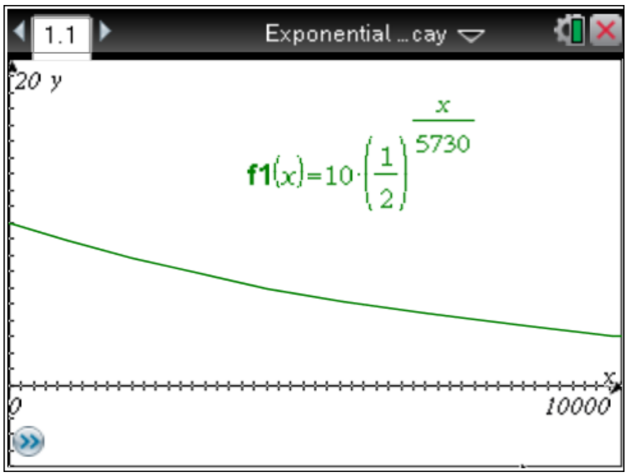
(a) Determine the initial quantity (when $t = 0$).

$$\boldsymbol{10\left(\frac{1}{2}\right)^{\frac{0}{5730}} = 10\ grams}$$

(b) Determine the quantity present after 2000 years.

$$10\left(\frac{1}{2}\right)^{\frac{2000}{5730}} \approx 7.851 \; grams$$

(c) Sketch the graph of the function $Q(x)$ over the interval $t = 0$ to $t = 10{,}000$



Now that John has shown up at The Café Mathematica, he wishes to share his new knowledge with his buddy Leonhard. John offers Leonhard the following problems to work out and discuss, so Leonhard immediately asks for new napkins on which to scribble profusely. Up until John's discovery, the two couldn't figure these problems out!

---

1. The value of a snowmobile can be modeled by the equation $y = 4500(0.93)^t$ where t is the number of years since the car was purchased. After how many years will the value of the snowmobile be about $2500?

A. 7 years B. 8 years C. 9 years D. 10 years

$$2500 = 4500(0.93)^t$$

$$log\, 0.556 = t\, log\, 0.93$$

$$t \approx 8 \text{ years so the answer is } B$$

2. The amount of a certain radioactive substance remaining after t years decreases according to the function $N = N_0e^{-0.0315t}$ where N_0 is the initial amount of the substance and t = time in years. Approximately how many years will it take for a 30 gram sample to decay to 15 grams?

A. -22 years B. 22 years C. 18.70 years D. 5.83 years

$$15 = 30e^{-0.0315t}$$

$$ln\frac{1}{2} = -0.0315t$$

$$t \approx 22 \text{ years so the answer is } B$$

3. The formula for finding the number of bacteria present is given by $P = P_0(2)^{2t}$ where P is the final population, P_0 is the initial population and t is the time measured in hours. If the population contained 275 bacteria at $t = 0$, approximately how long will it take for 15,000 bacteria to be present?

A. 2.25 hours B. -2.88 hours C. -2.25 hours D. 2.88 hours

$$15000 = 275(2)^{2t}$$

$$log\, 54.545 = 2t\, log\, 2$$

$$t \approx 2.88 \text{ hours so the answer is } D$$

After John shares his excitement over his newly-discovered logarithms, Leonhard decides to change the subject (apparently Euler was a very jealous mathematician) to an investment opportunity.

Leonhard has an investment opportunity for John that will pay John 8.73% interest compounded annually if John makes an initial investment of $50,000.

(a) How long will it take for John to double his money?

$$100,000 = 50,000\left(1+\frac{0.0873}{1}\right)^{1t} \text{ so } t \approx 8.282 \text{ years}$$

(b) How long will it take for John's investment to have a value of $68,000?

$$68,000 = 50,000\left(1+\frac{0.0873}{1}\right)^{1t} \text{ so } t \approx 3.674 \text{ years}$$

(c) Leonhard is being dishonest with John. The investment actually pays <u>8.73% interest compounded continually</u>. Leonhard, who is extremely jealous that John has discovered logarithms, plans on keeping the extra interest for himself when John is paid 8.73% interest compounded annually. After five years, how much money would Leonhard make off of cheating John?

$$A = 50,000\left(1+\frac{0.0873}{1}\right)^{1(5)} \approx 75983$$

$$A = 50,000e^{0.0873(5)} \approx 77364$$

So Leonhard would keep 77364 – 75983 = $1381 of John's money.

(d) Luckily, John doesn't fall for Leonhard's investment trick. John tells Leonhard that he would only invest $50,000 in an investment that would double in five years. If interest was being compounded continuously, what interest rate would John need to do this?

$$100,000 = 50,000e^{5r}$$

$$\ln 2 = 5r$$

$$r \approx 0.139$$

So a 13.9% interest rate would be required for John's money to double in five years.

Seized with guilt, Leonhard breaks into tears and confesses his scheme to John. John hugs Leonhard and tells him that he forgives him. The two walk out of The Café Mathematica together, and the other customers couldn't be happier. They were afraid of the two crazy men at the table scribbling things on napkins and arguing about lumber. And they didn't even order anything.

John and Leonhard at the Café Mathematica

John Napier is running late in meeting with his friend Leonhard Euler at a local coffee shop, The Café Mathematica. John is behind schedule because he has spent all morning making a discovery, namely that:

$$f(x) = log_b x \quad \textit{has some relation to} \quad x = b^{f(x)}$$

While Leonhard is waiting for his hopelessly tardy friend, he begins scribbling out the solutions to some exponential problems that have been posed by a few of his professor friends at the local university on some spare napkins at the table.

---

1. The population of a town increases according to the model

$$P(t) = 2500e^{0.0293t}$$

where t is the time in years, with $t = 0$ corresponding to 2010.

(a) Find the projected population of the town in 2012, 2015, and 2018.

(b) Use a graphing calculator to graph the function for the years 2010 through 2030.

(c) Use a graphing calculator to approximate the population in 2025 and 2030.

(d) Verify your answers in part (c) algebraically.

2. A certain population increases according to the model $P(t) = 250e^{0.47t}$. Use the model to determine the population when $t = 5$. Round your answer to the nearest integer.

A. 400 B. 2621 C. 1998

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(a) Determine the initial quantity (when $t = 0$).

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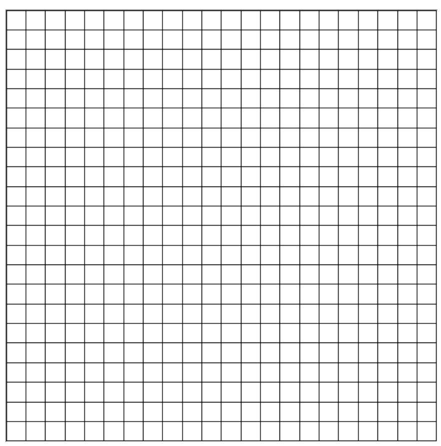
(c) Sketch the graph of the function $Q(x)$ over the interval $t = 0$ to $t = 10{,}000$



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Culminating Task: Jason's Graduation Presents

[Back to Task Table]

Mathematical Goals
To write expressions in logarithmic form to exponential form and vice versa
Solve exponential and logarithmic equations
Understand the relationship between logarithms and exponents

Georgia Standards of Excellence

MGSE9-12.F.IF.8 Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function. *(Limit to exponential and logarithmic functions.)*

MGSE9-12.F.IF.8b Use the properties of exponents to interpret expressions for exponential functions. *For example, identify percent rate of change in functions such as* $y = (1.02)^t$, $y = (0.97)^t$, $y = (1.01)^{(12t)}$, $y = (1.2)^{(t/10)}$, *and classify them as representing exponential growth and decay. (Limit to exponential and logarithmic functions.)*

MGSE9-12.F.BF.5 Understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents.

MGSE9-12.F.LE.4 For exponential models, express as a logarithm the solution to $ab^{(ct)} = d$ where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology.

Standards for Mathematical Practice
1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

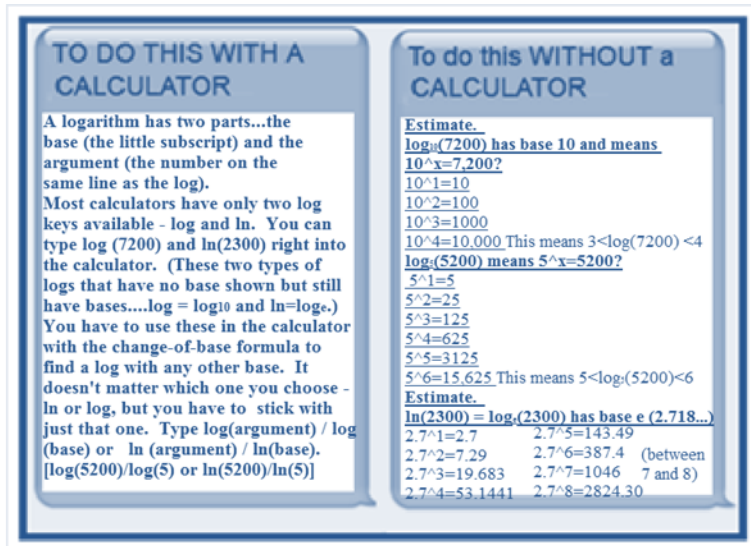
Introduction
This task provides another opportunity for students to use exponential and logarithmic functions in context.

Materials
Graphing calculator or some other graphing utility

Jason's Graduation Present

1. You are at a baseball game when you get a text from your friend Jason. See to the left. Jason's uncle, an eccentric mathematics teacher, has decided to make Jason's graduation gift a challenge. You have to text him back the instructions for how to evaluate and order those logarithms with a calculator and without a calculator. Finish the texts below and help Jason.

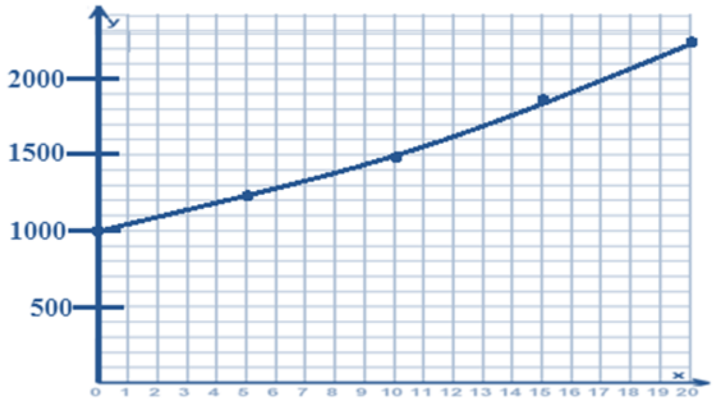
$\log 72000 \approx \$4.86$ $\quad$ $\log_5 5200 \approx \$5.32$ $\quad$ $\ln 2300 \approx \$7.74$



2. Jason received $1000 in graduation gifts. He found a savings plan that will pay him 4% interest compounded continuously. Use the continuously compounded interest formula to write the amount of money *A* Jason will have as a function of the time in *t* years.

$$A(t) = 1000e^{.04t}$$

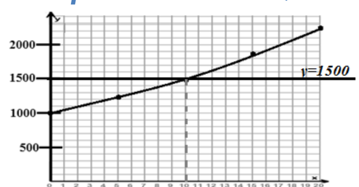
3. Use technology to graph this function and graph a rough sketch below showing Jason's investment's growth until he turns 38 (the next 20 years).



4. Jason wants to know how long it will take for him to have $1500 in his account. How long will it take?

Explain how to find the answer from the graph.

Also graph the line y = 1500 and find the point of intersection. Pt of intersection is (10.1366, 1500) so it will take a little over 10 years for him to have $1500.



Find/explain the answer algebraically.

$$1500 = 1000e^{.04t}$$
$$1.5 = e^{.04t}$$
$$\ln 1.5 = .04t$$
$$t \approx 10 \text{ years}$$

5. One of Jason's graduation gifts is a trip on Amtrak from his home in Atlanta to New York to visit a cousin. Jason is afraid he may experience motion sickness so he has decided to take 50 mg of dimenhydrinate (which will help prevent motion sickness) before boarding Amtrak. Suppose that 85% of this medication remains in the bloodstream after 1 hour. Represent the amount of dimenhydrinate in Jason's bloodstream for the first 4 hours of taking the drug. Let *t* be the number of hours after reaching its peak level of 50 mg. Explain how you determined the amount of dimenhydrinate in Jason's bloodstream.

Time in hours (t)	*0*	*1*	*2*	*3*	*4*
Amt of med. In system (f(t)	*50*	*50(0.85) ≈42.5mg*	*50(0.85)(0.85) ≈36.1mg*	*50(0.85)(0.85) (0.85) ≈30.7mg*	*50(0.85)(0.85) (0.85) (0.85) ≈26.1mg*

About 26mg of the drug remains in his system. Each hour you take 85% (multiply by 0.85) to find out how much is left.

$$f(t) = 50(0.85)^t$$

(Find the amount of the drug (f(t), in mg) in Jason's system at time t=4 hours)

$$f(4) = 50(0.85)^4$$
$$f(4) = 26 \text{ mg}$$

6. Write an exponential function for the amount of dimenhydrinate in Jason's bloodstream t hours after reaching its peak level of 50 mg. How long does it take for Jason to have only half the amount of dimenhydrinate in his bloodstream? Determine the solution in at least two different ways and explain your thinking.

Time in hours (t)	*0*	*1*	*2*	*3*	*4*	*5*
Amt of med. In system (f(t)	*50*	*≈42.5mg*	*≈36.1mg*	*≈30.7mg*	*≈26.1mg*	*≈22.2mg*

Sometime between 4 and 5 hours is when the drug was at 25mg…shortly after 4 hours, the drug was half gone.

The equation to solve is

$$25 = 50(0.85)^t$$
$$\frac{25}{50} = (0.85)^t$$
$$log\left(\frac{25}{50}\right) = log\ (0.85)^t$$
$$log\left(\frac{25}{50}\right) = t * log(0.85)$$
$$\frac{log\left(\frac{25}{50}\right)}{log(0.85)} = t$$
$$t \approx 4.265$$

It will take approx 4.3 hours for him to have only 25 mg in his bloodstream.

7. At what level might you consider Jason's bloodstream cleared of dimenhydrinate? Why did you choose this level?

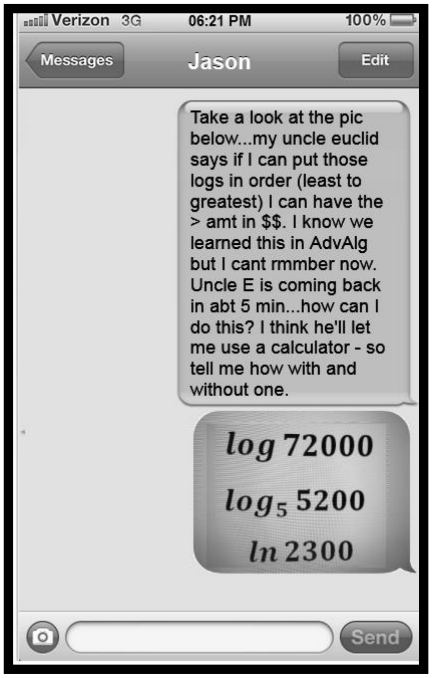
Some students may choose a level of 1 mg as being virtually cleared of dimenhydrinate since 1 mg is a small amount. Theoretically, however, the bloodstream is never cleared since only 15% is eliminated every hour.

8. Using your established criteria, find how long it takes for the dimenhydrinate to clear Jason's bloodstream. Explain how you found your solution.

   It will take a little over 24 hours.

Jason's Graduation Present

1. You are at a baseball game when you get a text from your friend Jason. See to the left.



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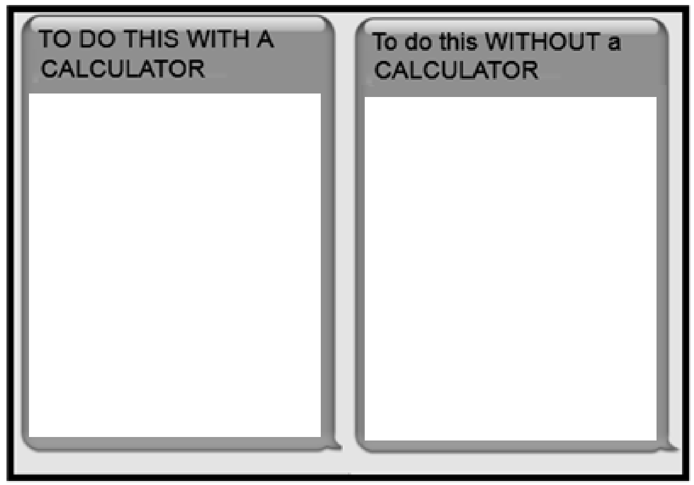
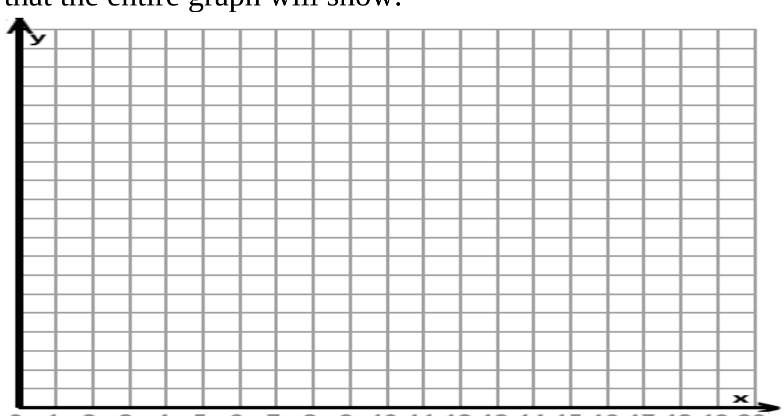


2. Jason received $1000 in graduation gifts. He found a savings plan that will pay him 4% interest compounded continuously. Use the continuously compounded interest formula to write the amount of money *A* Jason will have as a function of the time in *t* years.

3. Use technology to graph this function and graph a rough sketch below showing Jason's investment's growth until he turns 38 (the next 20 years). Label the y-axis with units that are appropriate so that the entire graph will show.



4. Jason wants to know how long it will take for him to have \$1500 in his account. How long will it take?	
Explain how to find the answer from the graph.	Find/explain the answer algebraically.

5. One of Jason's graduation gifts is a trip on Amtrak from his home in Atlanta to New York to visit a cousin. Jason is afraid he may experience motion sickness so he has decided to take 50 mg of dimenhydrinate (which will help prevent motion sickness) before boarding Amtrak. Suppose that 85% of this medication remains in the bloodstream after 1 hour. Represent the amount of dimenhydrinate in Jason's bloodstream for the first 4 hours of taking the drug. Let t be the number of hours after reaching its peak level of 50 mg. Explain how you determined the amount of dimenhydrinate in Jason's bloodstream.

6. Write an exponential function for the amount of dimenhydrinate in Jason's bloodstream t hours after reaching its peak level of 50 mg. How long does it take for Jason to have only half the amount of dimenhydrinate in his bloodstream? Determine the solution in at least two different ways and explain your thinking.

7. At what level might you consider Jason's bloodstream cleared of dimenhydrinate? Why did you choose this level?

8. Using your established criteria, find how long it takes for the dimenhydrinate to clear Jason's bloodstream. Explain how you found your solution.

APPENDIX-SUGGESTED MINI ASSESSMENTS THROUGHOUT UNIT

The following pages contain an exponent chart and quick quizzes to use as warm ups, ticket out the door, quick practice during class, or in any way teachers feel is appropriate for their students.

Using these items will focus attention to facts and connections used within the framework for this Unit.

The first exponent chart is filled in for students and several cells are "X-ed" out. Students will find those not "X-ed" out very useful to increase efficiency in solving problems involving exponents.

The second (blank) chart is provided for teachers to use as quick assessments or to provide to students for the opportunity for self-evaluation.

The next pages are copy-ready versions of "Quick Recall" activities.

Exponent Chart: You will find this unit much easier if you know these.											
Name____**Date**________________											
Bases -1-10, Non-"X'ed" Values											
$(-1)^0$	0^0	1^0	2^0	3^0	4^0	5^0	6^0	7^0	8^0	9^0	10^0
1	**undefined**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
$(-1)^1$	0^1	1^1	2^1	3^1	4^1	5^1	6^1	7^1	8^1	9^1	10^1
-1	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
$(-1)^2$	0^2	1^2	2^2	3^2	4^2	5^2	6^2	7^2	8^2	9^2	10^2
1	**0**	**1**	**4**	**9**	**16**	**25**	**36**	**49**	**64**	**81**	**100**
$(-1)^3$	0^3	1^3	2^3	3^3	4^3	5^3	6^3	7^3	8^3	9^3	10^3
-1	**0**	**1**	**8**	**27**	**64**	**125**	**216**	**343**	**512**	**729**	**1000**
$(-1)^4$	0^4	1^4	2^4	3^4	4^4	5^4	6^4	7^4	8^4	9^4	10^4
1	**0**	**1**	**16**	**81**	**256**	**625**	~~1296~~	~~2401~~	~~4096~~	~~6561~~	**10000**
$(-1)^5$	0^5	1^5	2^5	3^5	4^5	5^5	6^5	7^5	8^5	9^5	10^5
-1	**0**	**1**	**32**	~~243~~	~~1024~~	~~3125~~	~~7776~~	~~16807~~	~~32768~~	~~59049~~	**100000**
$(-1)^6$	0^6	1^6	2^6	3^6	4^6	5^6	6^6	7^6	8^6	9^6	10^6
1	**0**	**1**	**64**	~~729~~	~~4096~~	~~15625~~	~~46656~~	~~117649~~	~~262144~~	~~531441~~	**1000000**
$(-1)^7$	0^7	1^7	2^7	3^7	4^7	5^7	6^7	7^7	8^7	9^7	10^7
-1	**0**	**1**	**128**	~~2187~~	~~16384~~	~~78125~~	~~279936~~	~~823543~~	~~2097152~~	~~4782969~~	**10000000**
$(-1)^8$	0^8	1^8	2^8	3^8	4^8	5^8	6^8	7^8	8^8	9^8	10^8
1	**0**	**1**	**256**	~~6561~~	~~65536~~	~~390625~~	~~1679616~~	~~5764801~~	~~16777216~~	~~43046721~~	**100000000**
$(-1)^9$	0^9	1^9	2^9	3^9	4^9	5^9	6^9	7^9	8^9	9^9	10^9
-1	**0**	**1**	**512**	~~19683~~	~~262144~~	~~1953125~~	~~10077696~~	~~40353607~~	~~134217728~~	~~387420489~~	**1000000000**
$(-1)^{10}$	0^{10}	1^{10}	2^{10}	3^{10}	4^{10}	5^{10}	6^{10}	7^{10}	8^{10}	9^{10}	10^{10}
1	**0**	**1**	**1024**	~~59049~~	~~1048576~~	~~9765625~~	~~60466176~~	~~282475249~~	~~1073741824~~	~~3486784401~~	**10000000000**

Exponent Chart
Fill-in-the-Blanks

Name____**Date**________________

Bases 1-10, Non-“X’ed” Values											
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$(-1)^2$	0^2	1^2	2^2	3^2	4^2	5^2	6^2	7^2	8^2	9^2	10^2
$(-1)^3$	0^3	1^3	2^3	3^3	4^3	5^3	6^3	7^3	8^3	9^3	10^3
$(-1)^4$	0^4	1^4	2^4	3^4	4^4	5^4	6^4	7^4	8^4	9^4	10^4
$(-1)^5$	0^5	1^5	2^5	3^5	4^5	5^5	6^5	7^5	8^5	9^5	10^5
$(-1)^6$	0^6	1^6	2^6	3^6	4^6	5^6	6^6	7^6	8^6	9^6	10^6
$(-1)^7$	0^7	1^7	2^7	3^7	4^7	5^7	6^7	7^7	8^7	9^7	10^7
$(-1)^8$	0^8	1^8	2^8	3^8	4^8	5^8	6^8	7^8	8^8	9^8	10^8
$(-1)^9$	0^9	1^9	2^9	3^9	4^9	5^9	6^9	7^9	8^9	9^9	10^9
$(-1)^{10}$	0^{10}	1^{10}	2^{10}	3^{10}	4^{10}	5^{10}	6^{10}	7^{10}	8^{10}	9^{10}	10^{10}

QUICK RECALL 1

1 **QUICK RECALL EXPONENTS,**	**1** **QUICK RECALL EXPONENTS,**	**1** **QUICK RECALL EXPONENTS,**	**1** **QUICK RECALL EXPONENTS,**

MIXED	**MIXED**	**MIXED**	**MIXED**
$(-1)^4$ =__________	$(-1)^4$ =__________	$(-1)^4$ =__________	$(-1)^4$ =__________
$(-1)^{10}$ =__________	$(-1)^{10}$ =__________	$(-1)^{10}$ =__________	$(-1)^{10}$ =__________
$(0)^6$ =__________	$(0)^6$ =__________	$(0)^6$ =__________	$(0)^6$ =__________
1^{10} =__________	1^{10} =__________	1^{10} =__________	1^{10} =__________
2^7 =__________	2^7 =__________	2^7 =__________	2^7 =__________
2^9 =__________	2^9 =__________	2^9 =__________	2^9 =__________
2^{10} =__________	2^{10} =__________	2^{10} =__________	2^{10} =__________
3^0 =__________	3^0 =__________	3^0 =__________	3^0 =__________
3^1 =__________	3^1 =__________	3^1 =__________	3^1 =__________
4^2 =__________	4^2 =__________	4^2 =__________	4^2 =__________
5^0 =__________	5^0 =__________	5^0 =__________	5^0 =__________
5^2 =__________	5^2 =__________	5^2 =__________	5^2 =__________
7^0 =__________	7^0 =__________	7^0 =__________	7^0 =__________
9^0 =__________	9^0 =__________	9^0 =__________	9^0 =__________
10^{10} =__________	10^{10} =__________	10^{10} =__________	10^{10} =__________
12^2 =__________	12^2 =__________	12^2 =__________	12^2 =__________
15^2 =__________	15^2 =__________	15^2 =__________	15^2 =__________
16^2 =__________	16^2 =__________	16^2 =__________	16^2 =__________
17^2 =__________	17^2 =__________	17^2 =__________	17^2 =__________
19^2 =__________	19^2 =__________	19^2 =__________	19^2 =__________

QUICK RECALL 2

2 **QUICK RECALL EXPONENTS, MIXED**	**2** **QUICK RECALL EXPONENTS, MIXED**	**2** **QUICK RECALL EXPONENTS, MIXED**	**2** **QUICK RECALL EXPONENTS, MIXED**
18^2 =__________	18^2 =__________	18^2 =__________	18^2 =__________
2^9 =__________	2^9 =__________	2^9 =__________	2^9 =__________
12^2 =__________	12^2 =__________	12^2 =__________	12^2 =__________
9^1 =__________	9^1 =__________	9^1 =__________	9^1 =__________
4^3 =__________	4^3 =__________	4^3 =__________	4^3 =__________
$(-1)^1$ =__________	$(-1)^1$ =__________	$(-1)^1$ =__________	$(-1)^1$ =__________
6^3 =__________	6^3 =__________	6^3 =__________	6^3 =__________
$(-1)^7$ =__________	$(-1)^7$ =__________	$(-1)^7$ =__________	$(-1)^7$ =__________
5^4 =__________	5^4 =__________	5^4 =__________	5^4 =__________
7^2 =__________	7^2 =__________	7^2 =__________	7^2 =__________
14^2 =__________	14^2 =__________	14^2 =__________	14^2 =__________
4^4 =__________	4^4 =__________	4^4 =__________	4^4 =__________
10^9 =__________	10^9 =__________	10^9 =__________	10^9 =__________
1^6 =__________	1^6 =__________	1^6 =__________	1^6 =__________
5^2 =__________	5^2 =__________	5^2 =__________	5^2 =__________
2^3 =__________	2^3 =__________	2^3 =__________	2^3 =__________
8^0 =__________	8^0 =__________	8^0 =__________	8^0 =__________
2^6 =__________	2^6 =__________	2^6 =__________	2^6 =__________
10^3 =__________	10^3 =__________	10^3 =__________	10^3 =__________
9^3 =__________	9^3 =__________	9^3 =__________	9^3 =__________

QUICK RECALL #3

3 **QUICK RECALL EXPONENTS, MIXED**	**3** **QUICK RECALL EXPONENTS, MIXED**	**3** **QUICK RECALL EXPONENTS, MIXED**	**3** **QUICK RECALL EXPONENTS, MIXED**
25^2 =__________	25^2 =__________	25^2 =__________	25^2 =__________
$(0)^5$ =__________	$(0)^5$ =__________	$(0)^5$ =__________	$(0)^5$ =__________
4^3 =__________	4^3 =__________	4^3 =__________	4^3 =__________
16^2 =__________	16^2 =__________	16^2 =__________	16^2 =__________
1^{10} =__________	1^{10} =__________	1^{10} =__________	1^{10} =__________
7^3 =__________	7^3 =__________	7^3 =__________	7^3 =__________
19^2 =__________	19^2 =__________	19^2 =__________	19^2 =__________
8^2 =__________	8^2 =__________	8^2 =__________	8^2 =__________
2^6 =__________	2^6 =__________	2^6 =__________	2^6 =__________
$(-1)^1$ =__________	$(-1)^1$ =__________	$(-1)^1$ =__________	$(-1)^1$ =__________
18^2 =__________	18^2 =__________	18^2 =__________	18^2 =__________
17^2 =__________	17^2 =__________	17^2 =__________	17^2 =__________
5^3 =__________	5^3 =__________	5^3 =__________	5^3 =__________
3^2 =__________	3^2 =__________	3^2 =__________	3^2 =__________
0^0 =__________	0^0 =__________	0^0 =__________	0^0 =__________
$(-1)^0$ =__________	$(-1)^0$ =__________	$(-1)^0$ =__________	$(-1)^0$ =__________
2^2 =__________	2^2 =__________	2^2 =__________	2^2 =__________
10^8 =__________	10^8 =__________	10^8 =__________	10^8 =__________
9^2 =__________	9^2 =__________	9^2 =__________	9^2 =__________
20^2 =__________	20^2 =__________	20^2 =__________	20^2 =__________

QUICK RECALL #4

4 **QUICK RECALL EXPONENTS,**	**4** **QUICK RECALL EXPONENTS,**	**4** **QUICK RECALL EXPONENTS,**	**4** **QUICK RECALL EXPONENTS,**
6^3 =__________	6^3 =__________	6^3 =__________	6^3 =__________
19^2 =__________	19^2 =__________	19^2 =__________	19^2 =__________
$(0)^6$ =__________	$(0)^6$ =__________	$(0)^6$ =__________	$(0)^6$ =__________
4^4 =__________	4^4 =__________	4^4 =__________	4^4 =__________
6^2 =__________	6^2 =__________	6^2 =__________	6^2 =__________
1^3 =__________	1^3 =__________	1^3 =__________	1^3 =__________
2^4 =__________	2^4 =__________	2^4 =__________	2^4 =__________
13^2 =__________	13^2 =__________	13^2 =__________	13^2 =__________
5^4 =__________	5^4 =__________	5^4 =__________	5^4 =__________
14^2 =__________	14^2 =__________	14^2 =__________	14^2 =__________
3^4 =__________	3^4 =__________	3^4 =__________	3^4 =__________
2^3 =__________	2^3 =__________	2^3 =__________	2^3 =__________
2^8 =__________	2^8 =__________	2^8 =__________	2^8 =__________
$(-1)^7$ =__________	$(-1)^7$ =__________	$(-1)^7$ =__________	$(-1)^7$ =__________
4^3 =__________	4^3 =__________	4^3 =__________	4^3 =__________
10^0 =__________	10^0 =__________	10^0 =__________	10^0 =__________
8^3 =__________	8^3 =__________	8^3 =__________	8^3 =__________
5^3 =__________	5^3 =__________	5^3 =__________	5^3 =__________
$(-1)^6$ =__________	$(-1)^6$ =__________	$(-1)^6$ =__________	$(-1)^6$ =__________
6^3 =__________	6^3 =__________	6^3 =__________	6^3 =__________

QUICK RECALL #5

5 **QUICK RECALL EXPONENTS, MIXED**	**5** **QUICK RECALL EXPONENTS, MIXED**
$(-1)^7$ =__________	**$(-1)^7$ =__________**
$(0)^4$ =__________	**$(0)^4$ =__________**
1^0 =__________	**1^0 =__________**
2^3 =__________	**2^3 =__________**
2^8 =__________	**2^8 =__________**
5^4 =__________	**5^4 =__________**
9^2 =__________	**9^2 =__________**
4^4 =__________	**4^4 =__________**
5^3 =__________	**5^3 =__________**
6^3 =__________	**6^3 =__________**
Logical Arithmetic: **If 6^x =____1_____,** **x=___________**	**Logical Arithmetic:** **If 6^x =____1_____,** **x=___________**
Logical Arithmetic **If 9^x =___81______** **x=___________**	**Logical Arithmetic** **If 9^x =___81______,** **x=__________**
Logical Arithmetic **If 4^x =___64______,** **x=___________**	**Logical Arithmetic** **If 4^x =___64______,** **x=___________**
Logical Arithmetic **If 3^x =____9_____,** **x=___________**	**Logical Arithmetic** **If 3^x =____9_____,** **x=___________**
Logical Arithmetic **If 6^x =___216____,** **x=___________**	**Logical Arithmetic** **If 6^x =___216____,** **x=___________**

QUICK RECALL #6

6 **QUICK RECALL EXPONENTS, MIXED**	**6** **QUICK RECALL EXPONENTS, MIXED**
$(-1)^8$ =_______	**$(-1)^8$ =_______**
1^7 =_________	**1^7 =_________**
2^9 =_________	**2^9 =_________**
4^4 =_________	**4^4 =_________**
7^2 =_________	**7^2 =_________**
7^3 =_________	**7^3 =_________**
10^{10} =_________	**10^{10} =_________**
18^2 =_________	**18^2 =_________**
Logical Arithmetic: **If 5^x =625** **x=__________**	**Logical Arithmetic:** **If 5^x =____625** **x=__________**
Logical Arithmetic **If 3^x =___81______** **x=__________**	**Logical Arithmetic** **If 3^x =___81______** **x=__________**
Logical Arithmetic **If 10^x =____10____,** **x=__________**	**Logical Arithmetic** **If 10^x =____10____,** **x=__________**
Logical Arithmetic **If 9^x =____729__,** **x=__________**	**Logical Arithmetic** **If 9^x =____729__,** **x=__________**
Logical Arithmetic **If 11^x =__121____,** **x=__________**	**Logical Arithmetic** **If 11^x =__121____,** **x=____________**
Logical Arithmetic **If 14^x =___196____,** **x=__________**	**Logical Arithmetic** **If 14^x =___196____,** **x=__________**
Logical Arithmetic **If 4^x =____1____,** **x=__________**	**Logical Arithmetic** **If 4^x =____1____,** **x=__________**
Logical Arithmetic **If 15^x =___125,** **x=__________**	**Logical Arithmetic** **If 15^x =___125,** **x=__________**

QUICK RECALL #7

8 **QUICK RECALL EXPONENTS, MIXED**	**8** **QUICK RECALL EXPONENTS, MIXED**
$(-1)^5$ =__________	**$(-1)^5$ =__________**
$(0)^5$ =__________	**$(0)^5$ =__________**
2^5 =__________	**2^5 =__________**
2^8 =__________	**2^8 =__________**
4^3 =__________	**4^3 =__________**
6^3 =__________	**6^3 =__________**
7^0 =__________	**7^0 =__________**
8^2 =__________	**8^2 =__________**
10^5 =__________	**10^5 =__________**
11^2 =__________	**11^2 =__________**
EXPLANATION: **LogArithm*: base 2,(8)*** ***IS ASKING…*** ***With Base 2, what exponent gives you 8?*** **=___3____** ***(because 2^3=8)***	***EXPLANATION:*** **LogArithm*: base 2,(8)*** ***IS ASKING…*** ***With Base 2, what exponent gives you 8?*** **=___3____** ***(because 2^3=8)***
LogArithm: base 6,(1) =__________	**LogArithm: base 6,(1) =__________**
LogArithm: base 2,(2) =__________	**LogArithm: base 2,(2) =__________**
LogArithm:base16,(256) =__________	**LogArithm:base16,(256) =__________ __**
LogArithm:base16,(256) =__________	**LogArithm:base16,(256) =__________**
LogArithm: base 4,(64) =__________	**LogArithm: base 4,(64) =__________**

QUICK RECALL #8

$(-1)^5$ =__________
$(0)^5$ =__________
2^5 =__________
2^8 =__________
4^3 =__________
6^3 =__________
7^0 =__________
8^2 =__________
10^5 =__________
11^2 =__________

LogArithm base 2 , (2)=__________

LogArithm base 6 , (1)=__________

LogArithm base 4 , (64)=__________

LogArithm base 6 , (36)=__________

LogArithm base 7 , (7)=__________

LogArithm base 8 , (64)=__________

LogArithm base 10 , (1)=__________

LogArithm base 11 , (121)=__________

LogArithm base 16 , (256)=__________

LogArithm base 17 , (289)=__________

QUICK RECALL #8

$(-1)^5$ =__________
$(0)^5$ =__________
2^5 =__________
2^8 =__________
4^3 =__________
6^3 =__________
7^0 =__________
8^2 =__________
10^5 =__________
11^2 =__________

LogArithm base 2 , (2)=__________

LogArithm base 6 , (1)=__________

LogArithm base 4 , (64)=__________

LogArithm base 6 , (36)=__________

LogArithm base 7 , (7)=__________

LogArithm base 8 , (64)=__________

LogArithm base 10 , (1)=__________

LogArithm base 11 , (121)=__________

LogArithm base 16 , (256)=__________

LogArithm base 17 , (289)=__________

QUICK RECALL #9

$(-1)^5$ =__________

$(0)^5$ =__________

2^5 =__________

2^8 =__________

4^3 =__________

6^3 =__________

7^0 =__________

8^2 =__________

10^5 =__________

11^2 =__________

LogArithm base 2 , (2)=__________

LogArithm base 6 , (1)=__________

LogArithm base 4 , (64)=__________

LogArithm base 6 , (36)=__________

LogArithm base 7 , (7)=__________

LogArithm base 8 , (64)=__________

LogArithm base 10 , (1)=__________

LogArithm base 11 , (121)=__________

LogArithm base 16 , (256)=__________

LogArithm base 17 , (289)=__________

QUICK RECALL #9

$(-1)^5$ =__________

$(0)^5$ =__________

2^5 =__________

2^8 =__________

4^3 =__________

6^3 =__________

7^0 =__________

8^2 =__________

10^5 =__________

11^2 =__________

LogArithm base 2 , (2)=__________

LogArithm base 6 , (1)=__________

LogArithm base 4 , (64)=__________

LogArithm base 6 , (36)=__________

LogArithm base 7 , (7)=__________

LogArithm base 8 , (64)=__________

LogArithm base 10 , (1)=__________

LogArithm base 11 , (121)=__________

LogArithm base 16 , (256)=__________

LogArithm base 17 , (289)=__________

APPENDIX-PENTAGON ZOMBIE PLAN ARTICLE

Exclusive: The Pentagon Has a Plan to Stop a Zombie Apocalypse. Seriously.
Retrieved from www.foreignpolicy.com

By Gordon Lubold -a national security reporter for Foreign Policy.
May 14, 2014

The U.S. military has always been the one place in government with a plan, forever in preparation mode and ready to yank a blueprint off the shelf for almost any contingency. Need a response for a Russian nuclear missile launch? Check. Have to rescue a U.S. ambassador kidnapped by drug lords? Yup, check, got that covered. How about a detailed strategy for surviving a zombie apocalypse? As it turns out, check.

Incredibly, the Defense Department has a response if zombies attacked and the armed forces had to eradicate flesh-eating walkers in order to "preserve the sanctity of human life" among all the "non-zombie humans."

Buried on the military's secret computer network is an unclassified document, obtained by Foreign Policy, called "CONOP 8888." It's a zombie survival plan, a how-to guide for military planners trying to isolate the threat from a menu of the undead -- from chicken zombies to vegetarian zombies and even "evil magic zombies" -- and destroy them.

"This plan fulfills fictional contingency planning guidance tasking for U.S. Strategic Command to develop a comprehensive [plan] to undertake military operations to preserve 'non-zombie' humans from the threats posed by a zombie horde," CONOP 8888's plan summary reads. "Because zombies pose a threat to all non-zombie human life, [Strategic Command] will be prepared to preserve the sanctity of human life and conduct operations in support of any human population -- including traditional adversaries."

CONOP 8888, otherwise known as "Counter-Zombie Dominance" and dated April 30, 2011, is no laughing matter, and yet of course it is. As its authors note in the document's "disclaimer section," "this plan was not actually designed as a joke."

Military planners assigned to the U.S. Strategic Command in Omaha, Nebraska during 2009 and 2010 looked for a creative way to devise a planning document to protect citizens in the event of an attack of any kind. The officers used zombies as their muse. "Planners ... realized that training examples for plans must accommodate the political fallout that occurs if the general *public mistakenly believes that a fictional training scenario is actually a real plan," the authors wrote, adding: "Rather than risk such an outcome by teaching our augmentees using the fictional 'Tunisia' or 'Nigeria' scenarios used at [Joint Combined Warfighting School], we elected to use a completely-impossible scenario that could never be mistaken for a real plan."*

Navy Capt. Pamela Kunze, a spokeswoman for Strategic Command, acknowledged the document exists on a "secure Internet site" but took pains to explain that the zombie survival guide is only a creative endeavor for training purposes. "The document is identified as a training tool used in an in-house training exercise where students learn about the basic concepts of military plans and order development through a fictional training scenario," she wrote in an email. "This document is not a U.S. Strategic Command plan."

This isn't the first time zombies have been used to inspire trainers or the American public. The Centers for Disease Control (CDC) built an entire public awareness campaign for emergency preparedness around zombies. "Get a kit, make a plan, be prepared," one CDC poster warns as a dead-eyed woman peeks over a blanket.

But the military appears to have come up with the idea first. And of course, should there be a zombie apocalypse, the military indeed has a plan.

CONOP 8888 is designed to "establish and maintain a vigilant defensive condition aimed at protecting humankind from zombies," according to the plan's purpose, and, "if necessary, conduct operations that will, if directed, eradicate zombie threats to human safety." Finally, the plan provides guidance to "aid civil authorities in maintaining law and order and restoring basic services during and after a zombie attack."

The "worst case threat scenario," according to the plan, suggests a rather dark situation: a zombie attack in which there would be high "transmissibility," lots of zombies eating lots of people, zombies infecting humans at a rapid rate, and little or no immunity and few effective countermeasures.

Under "Zombie Threat Summary," the plan highlights the different kinds of zombie adversaries one might find in such an attack. They include not only vegetarian zombies ("zombie life forms originating from any cause but pose no direct threat to humans because they only eat plant life"); evil magic zombies ("EMZs are zombie life forms created via some form of occult experimentation in what might otherwise be referred to as 'evil magic'"); and also chicken zombies.

"Although it sounds ridiculous, this is actually the only proven class of zombie that actually exists," the plan states. So-called "CZs" occur when old hens that can no longer lay eggs are euthanized by farmers with carbon monoxide, buried, and then claw their way back to the surface. "CZs are simply terrifying to behold and are likely only to make people become vegetarians in protest to animal cruelty," CONOP 8888 notes.

The catalog of the walking dead also includes zombies that come from outer space; those deliberately created by Frankensteinian bio-engineers; and humans that have been invaded by a pathogen that turns them into zombies.

The catalog of the walking dead also includes zombies that come from outer space; those deliberately created by Frankensteinian bio-engineers; and humans that have been invaded by a pathogen that turns them into zombies.

The plan reviews, extensively, the various phases of saving the world from zombie rule and reads not unlike the phases of a counterinsurgency campaign: from "shape" to "deter" to "seize initiative" to "dominate" to "stabilize" and, finally, in the final, confidence-building phase, *"restore civil authority." That final phase includes the directive to "prepare to redeploy the forces to attack surviving zombie holdouts."*

Finally, "[a]s directed by POTUS and SECDEF," using military-ese for the president of the United States and the defense secretary, "provide support to federal, state and tribal agencies' efforts to restore basic services in zombie-related disaster areas."

If the military's mantra is to "be prepared," then writing a zombie survival guide -- even if it is just for an imaginative exercise -- makes sense. "I hope we've invested a similar level of intellectual rigor against dragon egg hatching contingencies," one defense official quipped.

CONOP 8888