Patterns and Functions

Students investigate properties of perimeter, area, and volume related to various geometric two- and three-dimensions shapes. They conjecture, test, discuss, verbalize, and generalize patterns. Through this process they *discover* the salient features of the pattern, *construct* understandings of concepts and relationships, develop a language to talk about the pattern, *integrate*, and *discriminate* between the pattern and other patterns. When relationships between quantities in a pattern are studied, knowledge about important mathematical relationships and functions emerges.

Learning Objectives

Students will:

- compute perimeter, area, and volume of various geometric figures
- compute maximum and minimum area of geometric figures, given linear dimensions restrictions

Materials

Square tiles
A piece of grid paper for each student
Tables at a Birthday Party Activity Sheet

Instructional Plan

Investigation: Perimeter and Area

Pose the following problem to the students:

> Tanya Teen started her own summer business - putting on birthday parties for small children. Her neighbors agreed to loan her square card tables to seat the children for refreshments. However, when some of the neighbors were away on vacation, Tanya couldn't use their tables, and she really hated hauling the tables back and forth. Therefore, using as few tables as possible was important to her. Because all the children wanted to sit together, she had to place the card tables together into rectangles. Only one child could sit on each side of a card table. Her first party had eighteen children. How many tables did Tanya need to borrow?

This problem is also found on the Tables at a Birthday Party activity sheet.

Tables at a Birthday Party Activity Sheet

Students may use tiles to display a 1 × 2 banquet table on the overhead projector.

Ask, *How many people can be seated?* [6]

Alternatively, the problem can be posed as, *What is the area of a rectangle with a perimeter of 18 units?*

Allow small groups of students to explore the problem with the square tiles.

Ask, *Which rectangles used the fewest square tiles?* [1 × 9 or 9 × 1]
The most square tiles? [4 × 5 or 5 × 4]

How many tables did Tanya need?

The language of tables and people sitting at tables is an appropriate story to link the world of students to the world of mathematics. In this example, tables represent area, and people sitting at tables represent perimeter. Translate the language of unit tables and people into area and perimeter. In describing the dimensions of a rectangle, use bottom and side edge in place of length and width. This allows students to include rectangles 4 × 5 and 5 × 4 as different rectangles. Later the terms *width* and *length* can be introduced. Arrange all the rectangles with a perimeter of 18 in a table, such as the one shown below.

Length (units)	**Width (units)**	**Perimeter (units)**	**Area (square units)**
1	8	18	8
2	7	18	14
3	6	18	18
4	5	18	20
5	4	18	20
6	3	18	18
7	2	18	14
8	1	18	8

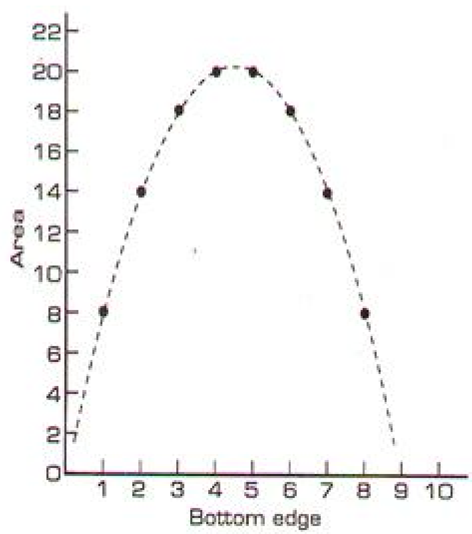
A blank table is found on the Tables at a Birthday Party activity sheet.

Ask the students to *observe patterns in the chart*. As the length increases, the width decreases, and as the length increases, the area increases to a certain point and then starts to decrease. The later pattern is a verbal description of the graph called a parabola, as shown below. This question builds an understanding of the effect of changing one variable, say length, on another variable, such as width or area.

If we allow lengths to be real numbers, is there a rectangle with a perimeter of 18 that has a larger area? Students will usually try a rectangle with a side of length 4.5 units. The area is 20.25 square units. Trying sides with lengths larger and smaller than 4.5, say 4.4 or 4.6 or 4.55, will yield areas that are smaller than 20.25. The shape of the rectangle with largest area for a perimeter of 18 is the square rectangle that is closest in shape to a square, 4 × 5 (if we restrict our lengths to whole numbers).

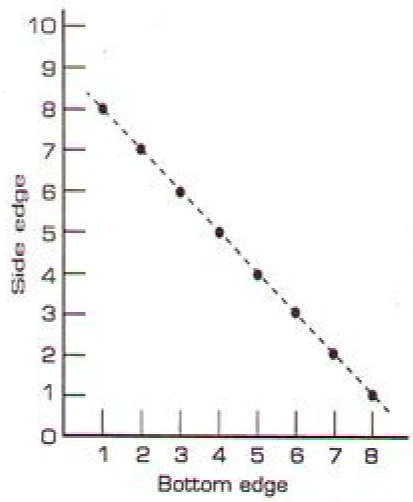
Several graphs can be drawn to represent the data in the table above. The graph of the area versus the length (or width) is a *parabola,* which is the graph of a *quadratic* function.

Area vs. Length



The graph of the length versus the width is a *linear* graph.

Length vs. Width



Using the graph of the parabola, ask students, *What happens between the points (4, 5) and (5, 4)?*
This is where the maximum point will occur if we extend our dimensions to include real numbers.

Ask, *What are the dimensions of the rectangle whose area is 16 square units? [6.5 × 2.5]*

What is the area of the rectangle whose length is 5.5 units? [19.25]
The graph visually describes the effects of increasing the length on the area.

Can a length be 9? [No, because the width would then be 0.]

Ask, *What happens if the graph is allowed to cross the horizontal axis?*

Investigation: Predicting the Maximum Area

Have students predict which rectangle has the maximum area for a fixed perimeter.

Discussion. Ask the class to predict when the area seems to "turn around." It reaches its maximum area when the shape of the rectangle is as close to a square as possible. If we have real numbers, then the rectangle with largest (maximum) area is a square. Try out this conjecture with perimeters of 20 units and 24 units. Have students guess when the area turns around and then find all the rectangles with integer lengths that have the given perimeter.

For a perimeter of 20 units, students may need to list all pairs of whole numbers whose sum is 20/2, or 10, and note that the pair (5,5) is the turn around. Thus the square with sides of length 5 and area of 25 is the rectangle with the largest area for a fixed perimeter of 20 units.

For a perimeter of 24 units, the rectangle with the largest area is also a square, with sides of length 6 units and area of 36 square units.

For a perimeter of 34, which rectangle has the maximum area? If the dimensions are whole numbers, then the rectangle with dimensions 8 × 9 has the maximum area. If the dimensions are real numbers, then the square whose sides have length 8.5 has the maximum area.

Investigation: Generalizing the Process for Finding the Maximum Area

Have students generalize the pattern for finding the area of a rectangle given a fixed perimeter.

Discussion. Ask the students to describe how they found all the rectangles. Allow the students time to talk about the process and then to translate the process into symbols. Find all pairs of whole numbers whose sum is half the perimeter. Use these pairs of numbers as the lengths and widths of the rectangles and then compute the areas. Or find the pair of numbers that are almost equal; this rectangle has the maximum area. For real numbers, since we know that the rectangle with the maximum area for a fixed perimeter is a square, use the number that is one-fourth the perimeter for the length of the side of the square. The following sets of equations could also be used:

Basic Equations

2(length + width) = Perimeter, or
length + width = Perimeter ÷ 2

length × width = area

Advanced Equations

Perimeter = 18
2(length + width) = Perimeter, or

$$l \times w = 9$$
$$w = 9 - l$$
$$l \times w = \text{Area}$$
$$l(9 - l) = \text{Area},$$

Or if P is any perimeter and A is the corresponding area, then $l(P/2 - l) = A$

Looking toward Algebra: The equation that describes the area of a rectangle with fixed perimeter is a quadratic equation, which is studied in more detail in algebra. Quadratic equations always have the shape of a parabola (as shown previously), except some open down and have a maximum point and some open up and have a minimum point. The maximum or minimum pint can also be found algebraically.

Have students observe the symmetry of the parabola and how this relates to the pairs of rectangles; for example, the rectangles 4 × 5 and 5 × 4 have the same area. They lie on the same horizontal line. The perpendicular line drawn from the maximum point to the horizontal axis is the line of symmetry. This problem provides intuition and understanding for some very important concepts of algebra and calculus. Maximum and minimum points of more general functions are studied in calculus. Geometrically, the maximum or minimum points occur when the tangent line to the graph is a horizontal line or has slope zero. As the students generalize the patterns they have observed and discussed, they are "sneaking" up on the notion of variable in a very natural way.

Investigation: Factor Pairs

Have students find all the factor pairs for 36.

Discussion. Ask students to describe all the rectangles that have an area of 36. Let them use square tiles or grid paper. Some students will recognize that this task is equivalent to finding all the factor pairs of 36. To find the rectangle with the least perimeter, students need to find the rectangle with a shape closest to a square or the factor pair where the factor pairs begin to repeat. For 36, these are the factor pairs:

1 × 36

2 × 18

3 × 12

4 × 9

6 × 6

9 × 4

12 × 3

18 × 2

36 × 1

The factor pairs repeat after the pair of factors (6, 6). The factors in this pair are equal, thus 36 is a square.

Let students explore an area of 12, 28, or 64.
Ask them to conjecture about when the factor pairs will begin to repeat.
Ask, If you were going to find all the factor pairs of a number, how many numbers would you have to check? [1 through the number that when squared is closest to the number; for 12 it is 3; for 28 it is 4; for 64 it is 8.]
Ask students if they can predict which numbers are perfect squares (square of a whole number). The number is a square number if it has a factor pair with both factors equal or with an odd number of factors.

Assessment Options

1. To evaluate students' understanding of the concept of "maximum area," the teacher can ask them to write about the process of dinging the rectangles with a greatest area, given a fixed perimeter. This question asks the students to reflect on the class exploration and discussion and helps the student (and teacher) assess the level of understanding.
2. Find the perimeter of all rectangles with whole-number dimensions whose area is 72 square units. Make a table and graph your data. Use the table or graph to answer the following questions:

    Which rectangle has the least perimeter? The greatest perimeter?
    If we allow the dimensions to be rational numbers (fractions), which rectangle has the least perimeter? The greatest perimeter?

Extensions

1. **Fixed Area**

    How many people can be seated if we have 24 unit tables?
    Or, what is the perimeter of a rectangular table whose area is 24 square units?

    Discussion. Allow the students to use the square tiles to form tables and to list all rectangles with an area of 24 square units in a chart. Stress that we are trying to find all rectangles with a given length. After the students have explored this problem, organize the data.

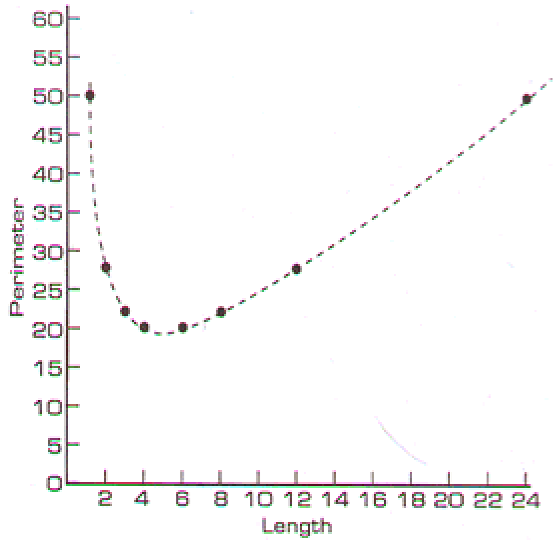
Length (units)	Width (units)	Area (square units)	Perimeter (units)
1	24	24	50
2	12	24	58
3	8	24	22
4	6	24	20
6	4	24	20
8	3	24	22
12	2	24	28
24	1	24	50

Ask students to observe the patterns.
Are they the same as the problem with fixed perimeter?

As the length increases, the width decreases. But the perimeter decreases to a certain point and then starts to increase, but the decrease and increase are not as symmetric as in the pattern with fixed perimeter. Graphing the problem displays this pattern visually. The maximum perimeter is 50 units for the long, skinny 1 × 24 rectangle. The minimum perimeter is 20 units for the fat, "almost square" 4 × 6 rectangle.

Ask, Could we have a larger perimeter? Students might suggest cutting the tables in half to form a rectangle with dimensions of 1/2 × 48. Other students will claim that if we allow rational (or real) numbers, we could go on forever.

What about the minimum perimeter? Could we get a smaller perimeter? [Yes, if we try lengths between 4 and 6] Try 5 for one side, then the other side will have to be 24/5 (the area is fixed at 24 square units). The perimeter is 19.6 units, which is less than 20 units. This process can go on until we reach a number that when squared is 24. Students can use a calculator to compute the square root of 24, or they can estimate the square root of 24 as 4.9: 4.92 = 24.01, or approximately 24.



Call attention to the fact that this graph is not a parabola (its shape is called a hyperbola).

Ask the students to read information from the graph: If the perimeter is 30 units, what are the dimensions of the rectangle?
If the length is 7 units, what is the perimeter of the rectangle?
If the length is 24 units, what is the perimeter?

For older students, this process can be generalized as

$P = 2(l + w)$ and $A = lw$, or $w = A/l$. Thus $P = 2(l + A/l)$.

2. **Fixed Area for Plane Figures**

   For a fixed perimeter, which plane figures will have the greatest area?

   Discussion. If we extend our investigations to other figures in the plane, such as polygons and circles, students can discover that the circle is the plane figure that has the most area for a fixed perimeter.

NCTM Standards and Expectations

Algebra 6-8

1. Use symbolic algebra to represent situations and to solve problems, especially those that involve linear relationships.
2. Identify functions as linear or nonlinear and contrast their properties from tables, graphs, or equations.
3. Model and solve contextualized problems using various representations, such as graphs, tables, and equations.

Geometry 6-8

1. Create and critique inductive and deductive arguments concerning geometric

ideas and relationships, such as congruence, similarity, and the Pythagorean relationship.

Number & Operations 6-8

1. Understand the meaning and effects of arithmetic operations with fractions, decimals, and integers.

References

- Phillips, Elizabeth, et al. *Patterns and Functions*, 1, 41 - 46. Reston, VA: NCTM, 1991.