This column is devoted to mathematics for fun. What better purpose is there for mathematics? To appear here, a theorem or problem or remark does not need to be profound (but it is allowed to be); it may not be directed only at specialists; it must attract and fascinate.

We welcome, encourage, and frequently publish contributions from readers—either new notes, or replies to past columns.

Please send all submissions to the Mathematical Entertainments Editor, **Alexander Shen,** Institute for Problems of Information Transmission, Ermolovoi 19, K-51 Moscow GSP-4, 101447 Russia; e-mail:shen@landau.ac.ru

Lights Out

The board for this game is an $m \times n$ rectangular array of lamps. Each lamp may be on or off. Each lamp works as a button changing the state (on/off) of the lamp and all its neighbors. Thus the maximal number of lamps affected by one button is five, the minimal number is three (for the corner button). Initially all lamps are on; the goal is to switch all the lamps off by a succession of button-pushes.

I heard about this game about ten years ago from Michael Sipser (MIT), who told me that it is always solvable and there is a very nice proof of this using linear algebra. Recently Prof. Óscar Martín-Sánches and Cristóbal Pareja-Flores wrote an article about this puzzle (to appear; see also their site http://dalila.sip.ucm.es/miembros/cpareja/lo), where they provide a detailed proof for the 5×5-game. (By the way, they have found this puzzle in toy stores!)

Here is the solution using linear algebra. First of all, we may forget about the rectangle; let V be the set of vertices of an arbitrary undirected graph. Each vertex has a lamp and a button that changes the state of this lamp and all its neighbors. The set of all configurations of lamps forms a linear space over $\mathbb{Z}/2\mathbb{Z}$. Each vector is a function of type $V \to \{0,1\}$. Here 1/0 means on/off, and vector addition is performed modulo 2. The dimension of this space is the number of lamps, i.e., $|V|$. For each vertex v we consider a function f_v that equals 1 in the neighborhood of v and 0 elsewhere. We need to prove that the function u that is equal to 1 everywhere can be represented as a linear combination of functions f_v.

It is enough to show that any linear functional α that maps all f_v to zero equals zero on u. Any linear functional $\alpha:\{0,1\}^V \to \{0,1\}$ can be represented as $\alpha(f) = \sum\{f(v)|v \in A\}$ for some $A \subseteq V$ (the sum is computed modulo 2). Therefore the statement can be reformulated as follows: if A has even-sized intersection with the neighborhood of any vertex v, then $|A|$ is even. To see that this inference holds, consider the restriction of our graph to A. Each vertex $a \in A$ has odd degree in the restricted graph, but the sum of the degrees of graph A is of course even; therefore the number of vertices of the restricted graph, $|A|$, is even.

We get also the criterion saying whether the state $c \in \{0,1\}^V$ is solvable. Here it is: $\sum\{c(v)|v \in A\} = 0$ for any subset $A \subseteq V$ having even-size intersection with the neighborhood of any vertex. (Such a set A can be called "neutral": if we press all buttons in A, all lamps return to their initial state.) One may ask for an "elementary" solution; indeed Sipser reports,

> . . . An epilog to the lamp problem. A generalization (which may make the problem easier) appeared in the problem section of *American Mathematical Monthly* [see problem 10197, vol. 99, no. 2, February 1992, p. 162 and vol. 100, no. 8, Oct. 93, pp. 806–807]. A very sharp new student here (named Marcos Kiwi) found a nice solution to it. Say that the lamps are nodes of a given undirected graph. The button associated with a lamp switches both its state and the state

of all its neighbors. Then we prove that there is a way to switch all states as follows.

Use induction on n (the number of nodes of the graph). First say n is even. For each lamp, remove it, and take the inductively given solution on the smaller graph. Replace the lamp and see whether the solution switches it. If yes, then we are done. If no for every lamp, then take the linear sum of all the above solutions given for all the lamps. Every lamp is switched an odd number of times $(n-1)$, so we are done.

If n is odd, then there must be a node of even degree. Do the above procedure for only the nodes in the neighborhood of this node, including itself. In addition press the button of this lamp. This also switches all lamps an odd number of times.

Last year this problem appeared on the All-Russia Math Olympiad. One of the participants, Ilia Meszirov, rediscovered Kiwi's argument. He also gave an elementary proof (not using linear algebra) for the statement mentioned above (a state having even-sized intersection with any neutral set is solvable).