The mathematics of stopping your car

Eric Wood

National Institute of Education, Singapore

<ewood@nie.edu.sg>

In addition to looking for applications that can be profitably examined algebraically, numerically and geometrically it is also helpful to use ideas that typical students might find interesting. Most secondary students are eager to obtain their driving license and part of becoming a good driver is understanding how long it takes to stop your vehicle and being able to use this information to make safe driving decisions; but what is the relationship between speed and stopping distance? This article presents some of the basic mathematics nested in a discussion of stopping distances and illustrates how you can make use of this information to engage students in an exploration of both linear and quadratic functions.

Basic principles

There are two components to stopping distance: the distance the car travels after the brakes are applied (braking distance) and the distance the car travels between the time the driver realises the need to stop and the brakes are applied (reaction distance). Furthermore, reaction times vary depending on whether the driver is anticipating something happening or whether something unexpected occurs. These are referred to as alert reaction time (ART) and unexpected reaction time (URT). Typical values are 0.7 seconds for ART and 1.5 seconds for URT which combines the time it takes to move your foot from the accelerator to the brake pedal and the extra time required to process this unexpected information and to decide what to do in a surprise situation.

The braking distance is frequently given in road test reports for vehicles. Although such reports only give a single speed and distance, a few basics physics formulae can be used to generate table data which is suitable for students to analyse. The data in the Table 1 has been assembled from a variety of sources and standardised with respect to units and speed range, accurate to 1 decimal place. A range of vehicles are listed in Table 1 to provide teachers with sufficient data in the event that they wish to use this information as

the basis of student explorations which require them to make comparisons of various kinds.

Table 1. Alert reaction distance (ARD), unexpected reaction distance (URD)
and braking distances for five types of car at different initial speeds.

Speed (km/h)	Speed (m/s)	ARD (m)	URD (m)	Braking distance (m)				
				Ferrari 550 Maranello	Lexus LS400	Mercedes C36	Nissan 200SX	Toyota Corolla
0	0	0	0	0	0	0	0	0
10	2.8	1.9	4.2	0.4	0.6	0.4	0.5	0.7
20	5.6	3.9	8.3	1.7	2.2	1.8	1.9	2.8
30	8.3	5.8	12.5	3.7	5.0	4.0	4.3	6.2
40	11.1	7.8	16.7	6.6	8.9	7.1	7.6	11.0
50	13.9	9.7	20.8	10.4	14.0	11.1	11.9	17.2
60	16.7	11.7	25.0	14.9	20.1	16.0	17.2	24.8
70	19.4	13.6	29.2	20.3	27.4	21.8	23.4	33.8
80	22.2	15.6	33.3	26.5	35.8	28.4	30.6	44.1
90	25.0	17.5	37.5	33.6	45.3	36.0	38.7	55.8
100	27.8	19.4	41.7	41.5	55.9	44.4	47.8	68.9
110	30.6	21.4	45.8	50.2	67.7	53.8	57.8	83.4
120	33.3	23.3	50.0	59.7	80.5	64.0	68.8	99.2

Interpreting the data

Have the students plot the URD, the ARD and the braking distance for the Ferrari on the same axes and join the points with either a straight line or a smooth curve, whichever seems appropriate (see Figure 1).[1] This may be the first time students have seen a non-linear graph and helps give the idea that such things are not in any way 'strange' but occur all the time. Typical questions might include:
- Which reaction distance should be larger for a given speed? Why?
- Find the gradient of the two lines. What do you notice? Can you explain?
- What is more significant at 8.3 m/s (30 km/h), reaction distance or braking distance? Explain.
- Is the braking distance at 22.2 m/s (80 km/h) double the braking distance at 11.1 m/s (40 km/h)? Explain.
- Is the alert reaction distance at 22.2 m/s (80 km/h) double the braking distance at 11.1 m/s (40 km/h)? Explain.

1. For consistency of units the speeds are plotted in m/s; but it is useful for students to get a sense of how fast these speeds are in terms they understand and this is the reason that the speed in km/h is also given in the table.
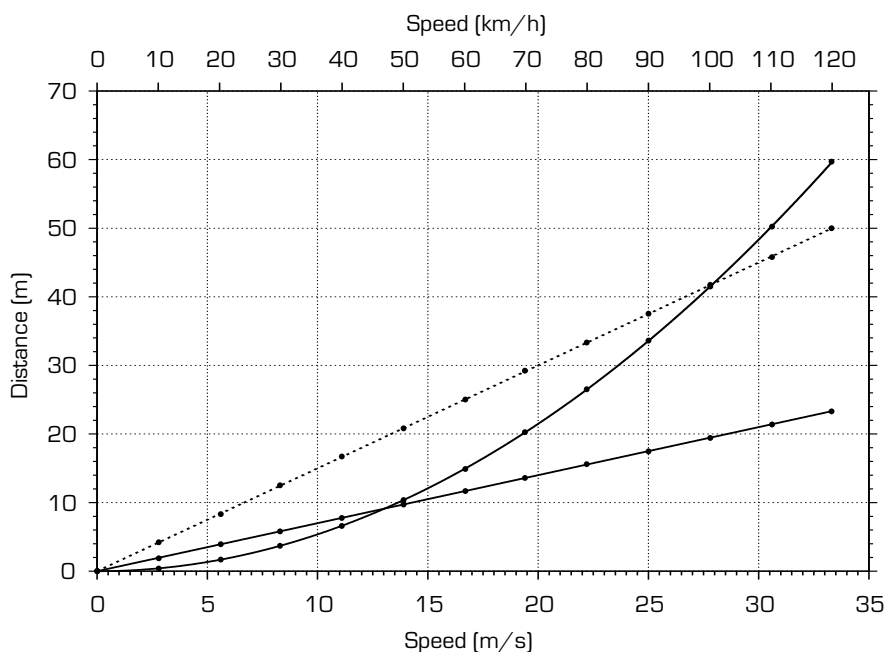
Figure 1. Graph of the unexpected reaction distance (dotted line), the alert reaction distance (solid line), and the braking distance (solid curve), against time for a Ferrari.

The gradients of the straight lines will give the two reaction times — a fact that can be confirmed by noting that the gradient is a ratio of quantities measured in m and m/s which results in seconds for the units of the gradient. The steeper the line, the longer the reaction time and therefore the greater the reaction distance. Furthermore, because the relationship between speed and reaction distance is linear, doubling one will cause the other to double and so on. This leads naturally to the equation

$$reaction\ distance = reaction\ time \times speed$$
or
$$y = mx.$$

In this case the equations are $y = 0.7x$ and $y = 1.5x$ for the lines representing ARD and URD respectively. The relationship between braking distance and speed does not show this same pattern, however, and so it is not surprising that the graph is not linear. In fact, doubling the speed results in a quadrupling of the braking distance. So how are the two quantities related? A spreadsheet is a useful way of examining this data numerically to see if we can establish the mathematical relationship.

One way of looking at the data numerically is to calculate the differences between successive data values. Students often use this technique to establish missing terms in sequences. For example in the sequence 3, 7, 11, 15… it is easy to see that the difference between successive terms is 4 and we can see that the functional rule which generates this sequence $x \rightarrow 4x - 1$ is which is indeed linear. With the sequence 1, 4, 9, 16, 25… the differences are 3, 5, 7, 9 and so on. Clearly the relationship is not linear but if we find the differences of the differences (second differences) we get 2, 2, 2, 2… In this case we can see that the functional rule to generate the sequence is $x \rightarrow x^2$ and this

suggests that when the second differences are constant the relationship is quadratic. This same technique can be used on data sets where the *x*-values (in this case speed) are evenly spaced. If a spreadsheet is used to calculate the first differences for the ARD data we can see (subject to rounding error) that they are reasonably constant (~1.9) and this confirms the linear relationship we found by graphing. For the braking distance data, the first differences are clearly not constant but the second differences are reasonably constant (~1) and this *suggests* that the relationship between braking distance and speed *may* be quadratic.

Now that a numerical approach has given us a clue about the relationship between speed and braking distance we can go back to geometric and algebraic techniques to establish what the actual relationship is. Have students draw a graph similar to the previous one but this time graph distance versus the square of the speed (see Figure 2). The points should be collinear, and if a straight line is drawn though them students should find the gradient to be approximately 0.054. So the straight line has equation $Y = 0.054X$ and this tells us that the relationship between braking distance and speed is, in fact, $y = 0.54x^2$ since in this case $Y \to y$ and $X \to x^2$. The value 0.054 is related to the braking efficiency of a car; smaller values will mean that a car takes less distance to stop.[2] Students could verify that this equation is correct by graphing the data and the newly found quadratic function with either *Graphmatica* or a graphing calculator to see if it passes through the points. One advantage of this method is that it requires more analysis on the part of the students than merely using a 'blackbox' curve fitting strategy; and, it provides for numerical and geometrical confirmation of the algebraic form of the equation.
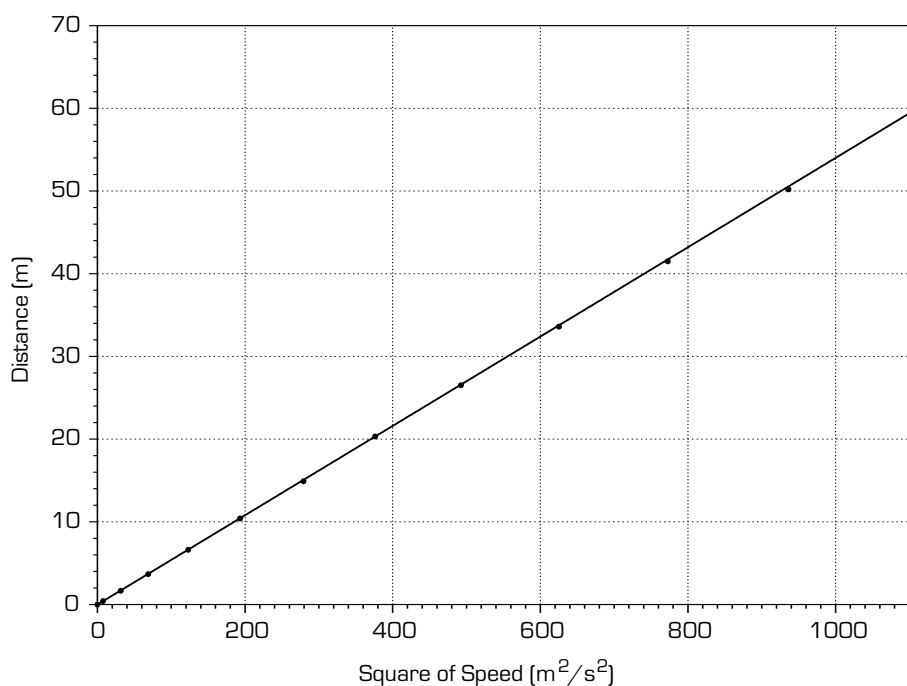

Figure 2. Braking distance against the square of speed.

2. Actually it is proportional to the reciprocal of the deceleration so a large deceleration means a small value.

Putting it all together — stopping distance

As a final exercise students could be asked to now consider the idea of stopping distance which is *braking distance + reaction distance*. Adding together the alert reaction distances and the braking distances and then plotting this data versus speed along with the equation $y = 0.054x^2 + 0.7x$ with *Graphmatica* or a graphing calculator will allow them to verify that the equation does fit (see Figure 3).
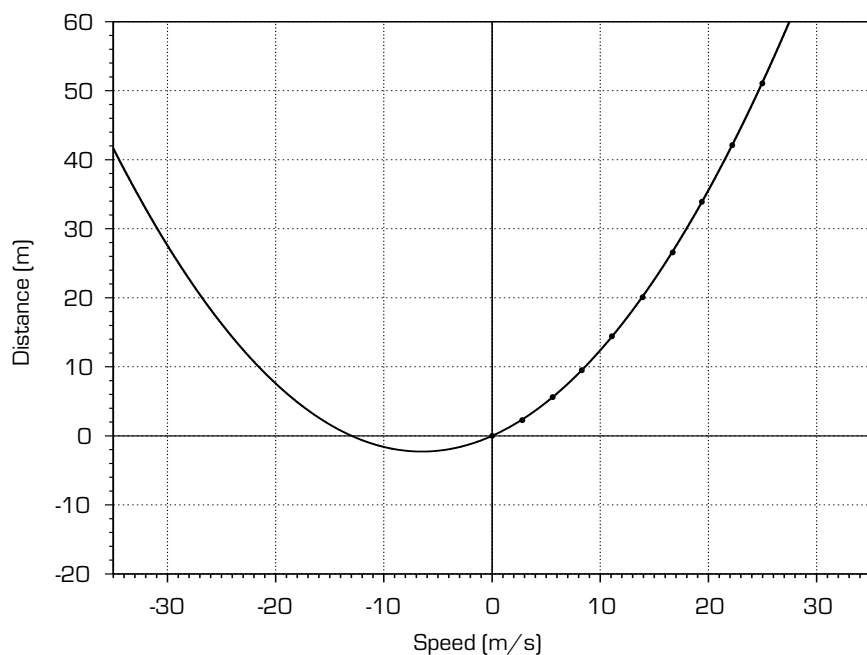

Figure 3. Stopping distance against speed.

In this graph a traditional *x*-axis and *y*-axis have be added to show the more traditional view of the parabola. Of course it is important to remember that whenever we create a mathematical model for something such as stopping distance we must realise that although the mathematical function may be defined for negative values, it may not have any value as a model for such values.

A challenge

After this discussion students could be challenged to answer the following question:

A police officer is investigating a traffic violation. The car, a Ferrari 550 Maranello, has left skid marks 37 metres long on the roadway in its attempt to stop at a red light but has ended up in the intersection when spotted by the police officer. The officer charges the driver with running a red light but is also concerned about how fast the car was travelling. The driver claims he was only going about 70 km/h when he applied the brakes.

(a) Is he telling the truth? Justify your answer.

(b) If you think he is lying, how fast would you estimate he was travelling at the time his brakes were applied?

Since skid marks are often made when brakes are applied, we could use braking distance data to get a rough idea of speed[3]. Either the graph or the formula can be used to get a speed estimate of somewhere between 90 and 100 km/h.

---

3. The police do use a slightly more complicated formula than this simplified version although the principle is exactly the same. Anti-lock brakes do still leave skid marks but they are not as obvious as those left by cars without them.