# Georgia
# Standards of Excellence
# Curriculum Frameworks

# Mathematics

## GSE Geometry

## Unit 4: Circles and Volume

Georgia Department of Education

Richard Woods, Georgia's School Superintendent

*"Educating Georgia's Future"*

# Unit 4
# Circles and Volume
## Table of Contents

## OVERVIEW

In this unit students will:

- Understand and Apply theorems about circles
- Find Arc Length and Area of Sectors of circles
- Explain Volume Formulas and Use them to solve problems

Although the units in this instructional framework emphasize key standards and big ideas at specific times of the year, routine topics such as estimation, mental computation, and basic computation facts should be addressed on an ongoing basis. Ideas related to the eight practice standards should be addressed constantly as well. This unit provides much needed content information and excellent learning activities. However, the intent of the framework is not to provide a comprehensive resource for the implementation of all standards in the unit. A variety of resources should be utilized to supplement this unit. The tasks in this unit framework illustrate the types of learning activities that should be utilized from a variety of sources. To assure that this unit is taught with the appropriate emphasis, depth, and rigor, it is important that the **"Strategies for Teaching and Learning"** in the Comprehensive Course Overview and the tasks listed under **"Evidence of Learning"** be reviewed early in the planning process.

## STANDARDS ADDRESSED IN THIS UNIT

Mathematical standards are interwoven and should be addressed throughout the year in as many different units and activities as possible in order to emphasize the natural connections that exist among mathematical topics.

## KEY STANDARDS

**Understand and apply theorems about circles**

**MGSE9-12.G.C.1** Understand that all circles are similar.

**MGSE9-12.G.C.2** Identify and describe relationships among inscribed angles, radii, chords, tangents, and secants. Include the relationship between central, inscribed, and circumscribed angles; inscribed angles on a diameter are right angles; the radius of a circle is perpendicular to the tangent where the radius intersects the circle.

**MGSE9-12.G.C.3** Construct the inscribed and circumscribed circles of a triangle, and prove properties of angles for a quadrilateral inscribed in a circle.

**MGSE9-12.G.C.4** Construct a tangent line from a point outside a given circle to the circle.

**Find arc lengths and areas of sectors of circles**

**MGSE9-12.G.C.5** Derive using similarity the fact that the length of the arc intercepted by an angle is proportional to the radius, and define the radian measure of the angle as the constant of proportionality; derive the formula for the area of a sector.

**Explain volume formulas and use them to solve problems**

**MGSE9-12.G.GMD.1** Give informal arguments for geometric formulas.

a. Give informal arguments for the formulas of the circumference of a circle and area of a circle using dissection arguments and informal limit arguments.
b. Give informal arguments for the formula of the volume of a cylinder, pyramid, and cone using Cavalieri's principle.

**MGSE9-12.G.GMD.2** Give an informal argument using Cavalieri's principle for the formulas for the volume of a sphere and other solid figures.

**MGSE9-12.G.GMD.3** Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems.

**Visualize relationships between two-dimensional and three-dimensional objects**

**MGSE9-12.G.GMD.4** Identify the shapes of two-dimensional cross-sections of three-dimensional objects, and identify three-dimensional objects generated by rotations of two-dimensional objects.

## STANDARDS FOR MATHEMATICAL PRACTICE

Refer to the Comprehensive Course Overview for more detailed information about the Standards for Mathematical Practice.

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

*SMP = Standards for Mathematical Practice*

## ENDURING UNDERSTANDINGS

- Understand and Apply Theorems about Circles
- Find Arc Lengths and Areas of Sectors of Circles
- Explain Volume Formulas and Use them to solve problems
- Extend the study of identifying cross sections of three dimensional shapes to identifying solids of revolution

## ESSENTIAL QUESTIONS

- How are Congruent Chords related?
- How does the GBI use the fact that the perpendicular bisectors of the sides of a triangle are concurrent to solve crimes?
- How does the location of the vertex of an angle effect the formula for finding the angle measure?
- When two lines intersect they create an angle. How does the location of the vertex of that angle effect the formula that I will use to find the lengths of those line segments that get created?
- What are some real life examples of lines intersecting circles and will I be able to solve them?
- Where is Arc Length on a cookie?
- Where is a Sector on a cookie?
- How did ancient mathematicians construct perfect tangents to a circle, without the software or measuring tools that we have today?
- How does Cavalieri's Principle apply to finding the Volume of a cylinder, even if it is oblique or not standing straight up?
- How can you identify the resulting solid of revolution when the graph of a simple function is rotated around an axis?

## CONCEPTS/SKILLS TO MAINTAIN

The introduction to all of the parts of a circle and the relationships of all of those parts to each other will be new to students this year. The concepts of Area, Surface Area, and Volume of triangles, special quadrilaterals, and right rectangular prisms were introduced in the 6$^{th}$ Grade Unit 5. This knowledge was built on in the 7$^{th}$ Grade Unit 5 and expanded to include the slicing of right rectangular pyramids. The Volumes of Cones, Cylinders, and Spheres were previously covered in the 8$^{th}$ Grade Unit 3. The purpose of re-visiting these formulas here in Analytic Geometry is to formalize the students understanding of the development of these formulas; to take them from a memorization and use of the formulas to an understanding and application level.

## SELECTED TERMS AND SYMBOLS

- **Arc**: an unbroken part of a circle; minor arcs have a measure less than $180^0$; semi-circles are arcs that measure exactly $180^0$; major arcs have a measure greater than $180^0$
- **Arc Length**: a portion of the circumference of the circle
- **Arc Measure**: The angle that an arc makes at the center of the circle of which it is a part.
- **Cavalieri's Principle**: A method, with formula given below, of finding the volume of any solid for which cross-sections by parallel planes have equal areas. This includes, but is not limited to, cylinders and prisms.

  Formula: Volume = $Bh$, where $B$ is the area of a cross-section and $h$ is the height of the solid.
- **Central Angle**: an angle whose vertex is at the center of a circle
- **Chord**: a segment whose endpoints are on a circle
- **Circumcenter**: The point of intersection of the perpendicular bisectors of the sides of a given triangle; the center of the circle circumscribed about a given triangle.
- **Circumscribed Circle**: a circle containing an inscribed polygon; for this unit the polygon will be a triangle and so the center of the circle will be the circumcenter of the triangle.
- **Composite Figures**: If a figure is made from two or more geometric figures, then it is called a Composite Figure.
- **Inscribed**: an inscribed planar shape or solid is one that is enclosed by and "fits snugly" inside another geometric shape or solid.
- **Inscribed Angle**: an angle whose vertex is on the circle and whose sides contain chords of a circle
- **Inscribed Circle**: a circle enclosed in a polygon, where every side of the polygon is a tangent to the circle; specifically for this unit the polygon will be a triangle and so the center of the Inscribed Circle is the incenter of the triangle
- **Inscribed Polygon**: a polygon whose vertices all lie on a circle
- **Lateral Area**: The sum of the areas of the lateral (vertical) faces of a cylinder, cone, frustum or the like.
- **Major and Minor Arcs**: Given two points on a circle, the minor arc is the shortest arc linking them. The major arc is the longest.
- **Point of Tangency**: the point where a tangent line touches a circle.
- **Secant Line**: a line in the plane of a circle that intersects a circle at exactly two points
- **Secant Segment**: a segment that contains a chord of a circle and has exactly one endpoint outside of the circle
- **Sector**: the region bounded by two radii of the circle and their intercepted arc
- **Slant Height**: The diagonal distance from the apex of a right circular cone or a right regular pyramid to the base.
- **Tangent Line**: a line in the plane of a circle that intersects a circle at only one point, the point of tangency

## EVIDENCE OF LEARNING

By the conclusion of this unit, students should be able to demonstrate the following competencies:

- Select an appropriate theorem or formula to use to solve a variety of situations involving circles, their segments, and the angles created, as well as volumes of such solids as the cylinder, cone, pyramid, and sphere.
- Construct Inscribed and Circumscribed Circles of triangles
- Complete a Formal Proof of the opposite angles of an Inscribed Quadrilateral being supplementary.
- Find the Arc Length and Area of any sector of a circle
- Use Cavalieri's Principle to show that the Volume of an Oblique Solid can be found using Right Solids.

## FORMATIVE ASSESSMENT LESSONS (FAL)

**Formative Assessment Lessons** are intended to support teachers in formative assessment. They reveal and develop students' understanding of key mathematical ideas and applications. These lessons enable teachers and students to monitor in more detail their progress towards the targets of the standards. They assess students' understanding of important concepts and problem solving performance, and help teachers and their students to work effectively together to move each student's mathematical reasoning forward.

More information on Formative Assessment Lessons may be found in the Comprehensive Course Overview.

## SPOTLIGHT TASKS

A Spotlight Task has been added to each GSE mathematics unit in the Georgia resources for middle and high school. The Spotlight Tasks serve as exemplars for the use of the Standards for Mathematical Practice, appropriate unit-level Georgia Standards of Excellence, and research-based pedagogical strategies for instruction and engagement. Each task includes teacher commentary and support for classroom implementation. Some of the Spotlight Tasks are revisions of existing Georgia tasks and some are newly created. Additionally, some of the Spotlight Tasks are 3-Act Tasks based on 3-Act Problems from Dan Meyer and Problem-Based Learning from Robert Kaplinsky.

## TASKS

The following tasks represent the level of depth, rigor, and complexity expected of all Analytic Geometry students. These tasks, or tasks of similar depth and rigor, should be used to demonstrate evidence of learning. It is important that all elements of a task be addressed throughout the learning process so that students understand what is expected of them. While some tasks are identified as a performance task, they may also be used for teaching and learning (learning/scaffolding task).

| Task Name | Task Type *Grouping Strategy* | Content Addressed | SMPs Addressed |
|---|---|---|---|
| Lifting the Rope (Spotlight Task) | Discovery Task *Partner or Individual* | Introductory task to unit, revisited later. | 2, 3, 4, 5 |
| Circles and their Relationships among Central Angles, Arcs, and Chords | Learning and Applying Task *Partner Task* | Prove all circles are similar; relationships among inscribed angles, radii, and chords; perform geometric constructions of inscribed and circumscribed circles of a triangle; properties of angles for a quadrilateral inscribed in a circle. | 1, 3, 5 |
| Arcs, Strings, and Radii (Spotlight Task) | Discovery Task *Individual and Partner* | Connecting arc lengths with radians. | 2, 3, 5 |
| Return to Lifting the Rope (Spotlight Task) | Learning and Applying Task *Partner Task* | Similarity of circles, circumference | 2, 3, 4, 5 |
| Investigating Angle Relationships in Circles | Discovery Task *Partner or Individual with technology Task* | Relationships among inscribed angles, radii, and chords. | 1, 3, 5 |
| Chords, Secants, and Tangents | Learning and Performance Task *Whole Group/Partner Task* | Relationships among inscribed angles, radii, and chords; perform geometric constructions of inscribed and circumscribed circles of a triangle; properties of angles for a quadrilateral inscribed in a circle; perform the geometric construction of a tangent line from a point outside a given circle to the circle. | 1, 3, 5 |
| Inscribing and Circumscribing Right Triangles (FAL) | Formative Assessment Lesson | Inscribing and circumscribing right triangles. | 1, 2, 3, 7 |

| | | | |
|---|---|---|---|
| Geometry Problems: Circles and Triangles (FAL) | Formative Assessment Lesson | Solve problems by determining the lengths of the sides in right triangles and find the measurements of shapes by decomposing complex shapes into simpler ones. | 1 – 3, 6, 7 |
| Circles in Triangles | Short Cycle Task | Proving geometric theorems and understanding and applying theorems about circles. | 3, 7 |
| Grain Storage | Achieve CCSS-CTE Classroom Tasks | Covers a variety of related content from previous tasks in the unit. | 1 – 8 |
| Arc Length and Area of a Sector | Discovery Task *Partner Task* | Use similarity to define and find arc length; define the radian measure of the angle as the constant of proportionality; derive the formula for the area of a sector; provide informal arguments for area and volume formulas. | 2 – 7 |
| Sectors of Circles (FAL) | Formative Assessment Lesson | Computing perimeters, areas, and arc lengths of sectors using formulas and finding the relationships between arc lengths and areas of sectors after scaling. | 1, 7 |
| Volumes of Cylinders, Cones, Pyramids, and Spheres | Performance Task *Partner or Whole Group Task* | Provide informal arguments for area and volume formulas; provide informal arguments using Cavalieri's principle for the volume of a sphere and other solid figures; use formulas for cylinders, pyramids, cones, and spheres to solve problems. | 1 – 5 |
| Evaluating Statements about Enlargements (FAL) | Formative Assessment Lesson | Computing perimeters, areas and volumes using formulas and finding the relationships between perimeters, areas, and volumes of shapes after scaling. | 3, 7 |
| Calculating Volume of Compound Objects (FAL) | Formative Assessment Lesson | Computing measurements using formulas; decomposing compound shapes into simpler ones; and using right triangles and their properties to solve real-world problems. | 1, 6 |
| A Golden Crown | Short Cycle Task | Analyzing proportional relationships and using them to solve real world mathematical problems. | 1 – 8 |
| Bestsize Cans | Short Cycle Task | Explaining volume formulas and using them to solve problems. | 1 – 8 |

| | | | |
|---|---|---|---|
| Funsize Cans | Short Cycle Task | Explaining volume formulas and using them to solve problems. | 3, 7 |
| Propane Tanks | Short Cycle Task | Explaining volume formulas and using them to solve problems, and visualizing relationships between two-dimensional and three-dimensional objects. | 1 – 8 |
| Tennis Balls in a Can | Discovery Task | Three Dimensional Shapes and solids of Revolution | 1-8 |
| 2D Representations of 3D Objects (FAL) | Formative Assessment Lesson | Visualize relationships between 2D and 3D objects. | 3,5,6,7 |
| Rate of Change | Culminating Task *Partner/Small Group Task* | Provide informal arguments for area and volume formulas; provide informal arguments using Cavalieri's principle for the volume of a sphere and other solid figures; use formulas for cylinders, pyramids, cones, and spheres to solve problems. | 1 – 5 |

## Lifting the Rope (Spotlight Task)

[Back to Task Table]

***The following task is meant to serve as an introduction to the unit. No formal outcomes are expected in this task. Rather, the task is used to set a problem that will be returned to after future tasks. During this introductory activity, students should draw on their intuition and provide whatever justification or thoughts they have.***

**Standards Addressed in this Task**
Although this task is only meant as an opening task to the unit with no formal outcomes, the task does set the basis for investigating the standards:

**MGSE9-12.G.C.1** Understand that all circles are similar.

**MGSE9-12.G.C.5** Derive using similarity the fact that the length of the arc intercepted by an angle is proportional to the radius, and define the radian measure of the angle as the constant of proportionality; derive the formula for the area of a sector.

**MGSE9-12.G.GMD.1** Give informal arguments for geometric formulas.

a. Give informal arguments for the formulas of the circumference of a circle and area of a circle using dissection arguments and informal limit arguments.
b. Give informal arguments for the formula of the volume of a cylinder, pyramid, and cone using Cavalieri's principle.

**Standards for Mathematical Practice**
Although this task is only meant as an opening task to the unit with no formal outcomes, the task does set the basis for engaging in the following practices:

**2.** Reason abstractly and quantitatively.
**3.** Construct viable arguments and critique the reasoning of others.
**4.** Model with mathematics.
**5.** Use appropriate tools strategically.

***COMMENTS***
***A teacher should bring different sized balls and string into the class to aid this investigation. They can vary the task to have students actually engage in wrapping string around the balls, or they can choose to save that activity for the second implementation of this task (after Arcs, Strings, and Radii). The task can also be modified to a 3-ACTs task by shooting a video of balls with taut string and then string pulled one-foot off of the balls. A picture of the earth can be used.***

Imagine wrapping the circumference of a tennis ball and basketball with pieces of string so that the pieces of string wrap perfectly around the balls. Now, imagine lifting the entire pieces of string 1 foot off of each ball. Which ball requires more string to "close" the string circle that is 1 foot off of the ball?

Imagine wrapping the circumference of the Moon and Earth with pieces of string so that the pieces of string wrap perfectly around the surfaces (we'll imagine the surfaces are smooth). Now, imagine lifting the entire pieces of string 1 foot off of each surface. Which surface requires more string to "close" the string circle? How do these compare to the amount of string needed to "close" the string circles off of the tennis and basketball?

***COMMENTS***

***Students should not be given a significant amount of time on this task. The task is intended to generate questions and a problem to solve for the Unit. So, the teacher can hold this as a whole class discussion, or give the students a few minutes in small groups/individually to determine initial thoughts and reasoning for those thoughts. As the students describe their thoughts, it is a good idea to ask the students what quantities they might want to measure and compare to answer these questions.***

***Intuitively, the students might say that the larger the object, the more string it will take to "close" the string circle.***

## Lifting the Rope (Spotlight Task)

Imagine wrapping the circumference of a tennis ball and basketball with pieces of string so that the pieces of string wrap perfectly around the balls. Now, imagine lifting the entire pieces of string 1 foot off of each ball. Which ball requires more string to "close" string circle that is 1 foot off of the ball?

Imagine wrapping the circumference of the Moon and Earth with pieces of string so that the pieces of string wrap perfectly around the surfaces (we'll imaging the surfaces are smooth). Now, imagine lifting the entire pieces of string 1 foot off of each surface. Which surface requires more string to "close" the string circle? How do these compare to the amount of string needed to "close" the string circles off of the tennis and basketball?

## Circles and their Relationships among Central Angles, Arcs, and Chords

**Back to Task Table**

**Standards Addressed in this Task**
**MGSE9-12.G.C.1** Understand that all circles are similar.

**MGSE9-12.G.C.2** Identify and describe relationships among inscribed angles, radii, chords, tangents, and secants. Include the relationship between central, inscribed, and circumscribed angles; inscribed angles on a diameter are right angles; the radius of a circle is perpendicular to the tangent where the radius intersects the circle.

**MGSE9-12.G.C.3** Construct the inscribed and circumscribed circles of a triangle, and prove properties of angles for a quadrilateral inscribed in a circle.

**Standards for Mathematical Practice**
**1. Make sense of problems and persevere in solving them** by requiring students to interpret and make meaning of a problem and find a logical starting point, and to monitor their progress and change their approach to solving the problem, if necessary.

**2. Reason abstractly and quantitatively** by requiring students to make sense of quantities and their relationships to one another in problem situations.

**3. Construct viable arguments and critique the reasoning of others** by engaging students on discussion of why they agree or disagree with responses, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

**5. Use appropriate tools strategically** by expecting students to consider available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a compass, a calculator, software, etc.

**Common Student Misconceptions**
1. Students sometimes confuse inscribed angles and central angles. For example they will assume that the inscribed angle is equal to the arc like a central angle.
2. Students may confuse the segment theorems. For example, they will assume that the inscribed angle is equal to the arc like a central angle.
3. Students will conclude that because lengths are longer on larger circles, circles are not similar.

***COMMENTS***
***If a teacher chooses to implement the Circle Book or list suggestion, it should be kept in mind that the purpose of the assignment is for the students to have a place to find all of the theorems, without having to look them up in a separate resource. If the learning object is collected before an assessment, it should be returned to the students to use to prepare for the assessment. If it is collected as an assessment, it should be returned to the students to use to prepare for any cumulative assessment that they may take.***

***Open with the following question as a class discussion.***

1. What is a circle?

***SOLUTION***
***The set of all points equidistant from a given point with the distance being defined as the radius of the circle.***

***COMMENTS***
***Have the students work on the following tasks as individuals first, then comparing with other students when prompted in the activity in order to open up the task to student inquiry. The students will need a compass, string, and ruler. Wikki Stix (waxed string) are preferred (http://www.wikkistix.com/).***

2. Using a compass and piece of string, and as an individual:
   a. Construct a circle with a radius equal in length to your string. Be sure to mark the center of your circle.
   b. Construct a chord on your circle that is approximately 0.25 string lengths.
   c. Construct another chord on your circle that is approximately 1.5 string lengths.
   d. Connect the center of the circle to the endpoints of the chords, forming two central angles.

3. As a group:
   a. Compare your constructions. What might this tell you about your circles? Test your conjecture with a different chord length than above.
   b. Draw three new circles of different sizes. Construct arcs on each circle that are approximately 0.5 string lengths, 1 string length, and 2 string lengths. Connect the center of each circle to the endpoints of the arcs, forming central angles.
   c. Compare the constructed angles across your circles. What do you notice? What might this tell you about your circles?
   d. Using a ruler, estimate the following measures in inches or centimeters (use consistent units throughout).

| Quantity | Circle 1 | Circle 2 | Circle 3 |
|---|---|---|---|
| Arc length (0. 5 string arc) | | | |
| Arc Length (1 string arc) | | | |
| Arc Length (2 string arc) | | | |
| Chord Length (0. 5 string arc) | | | |
| Chord Length (1 string arc) | | | |
| Chord Length (2 string arc) | | | |
| Radius | | | |
| Diameter | | | |
| Circumference | | | |

4. A fellow student asks, "Are all circles similar?" What is your answer to this question? Be sure to provide support for your answer.

***COMMENTS***
***You could adapt a commonly used pi-day activity here of having students measure with a cloth tape measure, or string and ruler, a variety of circular objects and calculate the Circumference/ Diameter Ratio for each to see that all circles have the same linear ratio of approximately 3.14 and write a paragraph proof.***

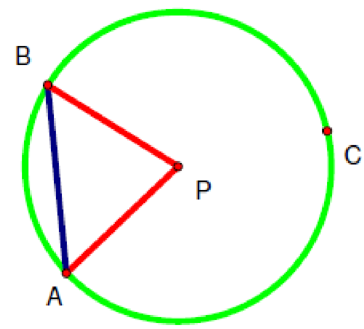
***SOLUTION***
***Since similar objects have linear measurements that are proportional to each other and since all circles have the linear measurement ratio of circumference to diameter of pi, then all circles must be similar to each other. Students should use their responses to 2-3 to exhibit proportionality (e.g., if the radius is scaled by some factor, the arc length subtended by an angle is scaled by the same factor; or, a ratio between two of the quantities stays constant across circles). When drawn to a whole class discussion, groups can compare their table results, thus providing a variety of circles and support for their arguments. They should also connect back to the arguments for similarity given in Unit 1.***

***During the activity, students should also be noting that using the same string arc measure gives the same chord length and angle openness regardless of circle size. Likewise, the students should note that constructing a chord length of a particular string size yields an angle of the same openness regardless of the circle size. Emphasize that when comparing measures, it is necessary to identify that the similarity basis is relative to angle measures. So,***

***corresponding chords have the same central angle, as do corresponding arcs. In the case of the circumference, the angle is 360 degrees.***

***An informal argument for similarity can involve a "perspective" argument. For instance, if we have some circle, we can imagine this circle being mapped onto a larger or smaller circle by moving the circle closer or further from us.***

Now we will introduce some notation and terminology needed to study circles.
Consider the figure at right.
$m\angle APB = 75^{\circ}$




Circles are identified by the notation $\circ P$, where $P$ represents the point that is the center of the circle.

A **central angle** of a circle is an angle whose vertex is at the center of the circle.
$\angle APB$ is a central angle of $\circ P$.

A portion of a circle's circumference is called an **arc**. An arc is defined by two endpoints and the points on the circle between those two endpoints. If a circle is divided into two unequal arcs, the shorter arc is called the **minor arc** and the longer arc is called the **major arc**. If a circle is divided into two equal arcs, each arc is called a **semicircle**.

In our figure, we call the portion of the circle between and including points $A$ and $B$, arc $AB$ notated by $\widehat{AB}$. We call the remaining portion of the circle arc $ACB$, *or* $\widehat{ACB}$. Note that major arcs are usually named using three letters in order to distinguish it from the minor arc.

We say that the central angle $\angle APB$ *intercepts* or has $\widehat{AB}$ . We also say that $\widehat{AB}$ *subtends* or has the central angle $\angle APB$. Note that when we refer to the arc of a central angle, we usually mean the minor arc unless otherwise stated.

Arcs are measured in two different ways - using degree measure and using linear measure. Usually when we refer to the **measure** of an arc, we are referring to the degree measure. The **measure** of a minor arc is defined to be the measure of the central angle that intercepts the arc. The measure of a major arc is 360° minus the measure of the minor arc with the same endpoints. In the figure above, the measure of $\widehat{AB}$ is 75° because that is the measure of its central angle. The measure of $\widehat{ACB}$ is 360° – 75° or 285°.

The **length** of an arc is different from its measure. The length is given in linear units (e.g., inches, centimeters, and feet). We will investigate the length of an arc in a later task. **Congruent arcs** have equal degree measures and equal lengths. **Equivalent arcs** have equal degree measures. For example, the arcs of 0.25 string lengths on Circles 1, 2, and 3 are equivalent because they each have the same degree measure. But, they are not congruent because they have different lengths. It follows that **congruent circles** have the same circumference measure and length.

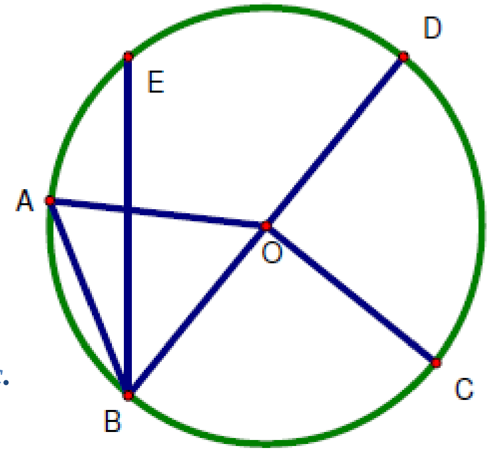
A **chord** is a *segment* whose endpoints lie on the circle. In the above figure, segment $\overline{AB}$ is a chord of $\circ P$.

5. How many chords can be in a circle?

***SOLUTION***
***Infinitely many***

What is the longest chord in a circle?
Explain how you know?

***SOLUTION***
***The Diameter is the longest chord because it passes through the center of the circle and has a length of 2r.***




6. Refer to the figure at the right.
   Identify and name each of the following.
   Be sure to use the correct notation.

***SOLUTION***

a. Two different central angles ***$\angle DOC$, $\angle COB$, etc.***
b. A minor arc ***arc AB***
c. A major arc ***arc DBC***
d. A semicircle ***arc DCB***
e. Two different chords ***segment BC, segment AB***
f. The central angle subtended by $\widehat{AD}$ $\angle AOD$

Use your protractor to help you find the following measures:

g. The measure of $\widehat{AC}$ ***approximately 150 degrees***
h. The measure of $\widehat{DEC}$ ***approximately 270 degrees***

7. Now it is time to use some of the terminology you have learned. Consider the following two theorems:

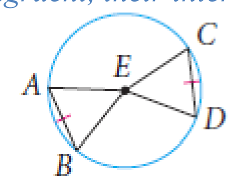
*In the same circle or congruent circles, if two chords are congruent, their intercepted arcs are congruent.*

*In the same circle or congruent circles, if two arcs are congruent, then their chords are congruent.*

   a. Prove that each of the theorems is true.

***SOLUTION***

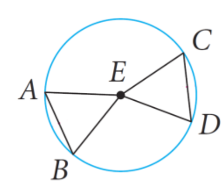
*"In the same circle or congruent circles, if two chords are congruent, their intercepted arcs are congruent" is proven below in a two column proof format.*




***Given: Chord AB and Chord CD are Congruent.***
***Prove: arc AB is congruent to arc CD***

| ***Statements*** | ***Reasons*** |
|---|---|
| $\overline{AB} \cong \overline{CD}$ | ***Given*** |
| $\overline{EA} \cong \overline{EB} \cong \overline{EC} \cong \overline{ED}$ | ***All radii are congruent*** |
| $\Delta AEB \cong \Delta CED$ | ***SSS*** |
| $\angle AEB \cong \angle CED$ | ***Corresponding Parts of Congruent Triangles are Congruent*** |
| $\widehat{AB} \cong \widehat{CD}$ | ***congruent Central Angles create congruent arc measures*** |

*"In the same circle or congruent circles, if two arcs are congruent, then their chords are congruent" is proven below in a flow chart proof format.*




$\widehat{AB} \cong \widehat{CD} \rightarrow \angle AEB \cong \angle CED \rightarrow$

***Given*** ***Congruent Arcs imply congruent central angle measures***

$\overline{EA}$ @$\overline{EB}$ @$\overline{EC}$ @$\overline{ED}$ →

***All radii are congruent***

$\rightarrow \Delta AEB \cong \Delta CED \rightarrow \overline{AB} \cong \overline{CD}$

***SAS*** ***CPCTC***
***To be proven***

b. Write the two theorems as one biconditional statement.

***SOLUTION***
***Two chords of the same circle or congruent circles are congruent if and only if their intercepted arcs are congruent.***

8. Use a compass to construct a circle on an unlined sheet of paper. Label the center of your circle.

    a. Draw any chord, other than a diameter, on your circle.
    Use your compass and a straightedge to construct a segment that represents the distance from the center of your circle to the chord. What is the relationship between the chord and the segment representing this distance?

***SOLUTION***
***The chord and this distance are perpendicular meaning that this is the distance the chord is from the center of the circle.***

    b. Mary made the following conjecture: "If two chords of a circle are the same distance from the center of the circle, the chords are congruent." Mary is correct. Use what you learned in *Item 5a* to help convince Mary that her conjecture is correct.

***COMMENTS***
***Teachers may choose to prove this theorem to the students, have the students prove it, or have the students discuss convincing arguments, recognizing that they are not a formal proof.***

    c. State the converse of Mary's conjecture.

***SOLUTION***
***"In a circle, if two chords are congruent, then they are the same distance from the center."***

    d. Write Mary's conjecture and its converse as a biconditional statement.

***SOLUTION***
***In a circle two chords are congruent if and only if they are the same distance from the center.***

    e. When a conjecture has been proven, it can be stated as a theorem. Write and illustrate this theorem in your Circle Book.

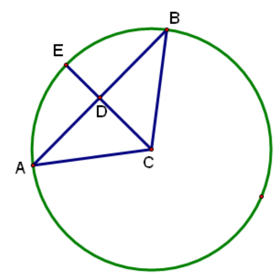
***COMMENTS***
***Teachers may want to have students try a few example problems of this theorem before moving onto the next part.***

9. Ralph made the following conjecture: “A radius perpendicular to a chord bisects the chord.”
   a. Use your construction from *item 8a* to help convince yourself that Ralph’s conjecture and the converse are true.

***COMMENTS***
***Teachers may decide to have students construct the perpendicular bisector of their chord and see that the center of the circle is on the perpendicular bisector, therefore proving by construction that the radius is perpendicular to the chord and bisects the chord at the point of intersection. By having multiple students complete the construction with a variety of chords, or by showing this construction on a dynamic geometry software program, such as Geometer’s Sketchpad or Geogebra, or the http://www.geogebra.org/cms/ website, you will be addressing the topic in #7 as well.***

   b. Write Ralph’s conjecture and its converse as a biconditional statement and illustrate it in your *Circle Book*.

   c. Ralph also believes that a radius perpendicular to a chord bisects the arc intercepted by the chord.
   Is this true? How do you know?




***SOLUTION***

| ***Statements*** | ***Reasons*** |
|---|---|
| $\overline{AD} \cong \overline{DB}$ **and** $\overline{AB} \perp \overline{EC}$ | ***Given*** |
| $\overline{CA} \cong \overline{CB}$ | ***All radii are congruent*** |
| $\overline{CD} \cong \overline{CD}$ | ***A segment is congruent to itself*** |
| $\Delta CAD \cong \Delta CBD$ | ***SSS*** |
| $\angle DCA \cong \angle DCB$ | ***Corresponding Parts of Congruent Triangles are Congruent*** |
| $\widehat{AE} \cong \widehat{EB}$ | ***Congruent central angles intercept congruent arcs*** |

10. Tevante examined his construction and his partner’s construction. He believes that *any* line that is a perpendicular bisector of a chord of a circle must also contain the center of the circle. Is he right? How do you know?

11. An investigator working for the Georgia Bureau of Investigation's crime lab has uncovered a jagged piece of a circular glass plate believed to have been used as a murder weapon. She needs to know the diameter of the plate. How might you use the information you learned in problem 10 to help determine the diameter of the circular plate?
    Use a compass, a straightedge, and a ruler to illustrate your answer.

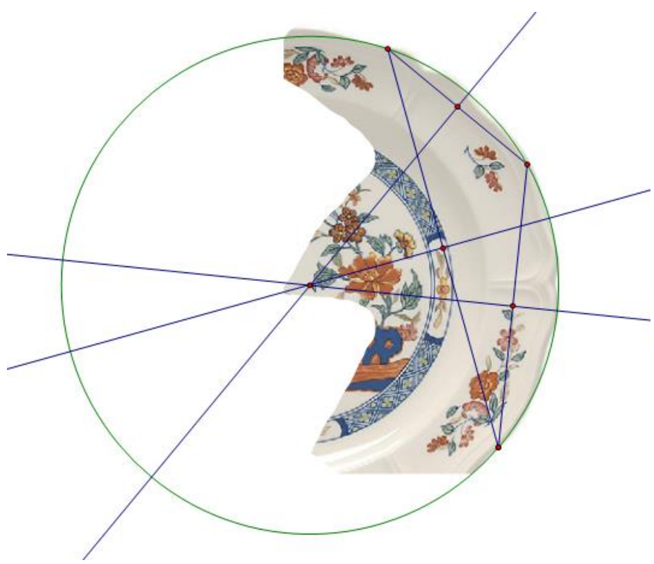
***<u>COMMENTS</u>***
***Students who have not been previously exposed to points of concurrency in triangles will need more support and direct teaching for this part of the unit.***

***An illustration using an interactive software such as Geometers Sketchpad or Geogebra may be useful. One is included below.***

***An entire lab day could be created here by providing students with pieces of broken plates and asking them to find the diameter.***

***<u>SOLUTION</u>***
***By identifying three points on the jagged piece, the investigator could construct a triangle and its three perpendicular bisectors that would meet that the Circumcenter of the triangle. Knowing that the circumcenter is equidistant to the vertices of the triangle and that the circle constructed with these vertices and center is the circumscribed circle will lead the investigator to have the circular plate reconstructed.***

***The distance from the circumcenter to any one of the triangles vertices will be the radius, which can be doubled to find the diameter of the plate.***



## Circles and their Relationships among Central Angles, Arcs, and Chords

1. What is a circle?

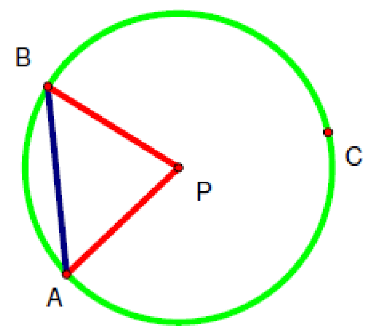
2. Using a compass and piece of string, and as an individual:
    a. Construct a circle with a radius equal in length to your string. Be sure to mark the center of your circle.
    b. Construct a chord on your circle that is approximately 0.25 string lengths.
    c. Construct another chord on your circle that is approxiately 1.5 string lengths.
    d. Connect the center of the circle to the endpoints of the chords, forming central angles.

3. As a group:
    a. Compare your constructions and estimations. What might this tell you about your circles? Test your conjecture with a different chord length than above.
    b. Draw three new circles of different sizes. Construct arcs on each circle that are approximately 0.25 string lengths, 1 string length, and 2 string lengths. Connect the center of each circle to the endpoints of the arcs, forming central angles.
    c. Compare the constructed angles across your circles. What do you notice? What might this tell you about your circles?
    d. Using a ruler, estimate the following measures in inches or centimeters (be consistent throughout).

| Quantity | Circle 1 | Circle 2 | Circle 3 |
|---|---|---|---|
| Arc length (0.25 string arc) | | | |
| Arc Length (1 string arc) | | | |
| Arc Length (2 string arc) | | | |
| Chord Length (0.25 string arc) | | | |
| Chord Length (1 string arc) | | | |
| Chord Length (2 string arc) | | | |
| Radius | | | |
| Diameter | | | |
| Circumference | | | |

4. A fellow student asks, "Are all circles similar?" What is your answer to this question? Be sure to provide support for your answer.

Now we will introduce some notation and terminology needed to study circles.
Consider the figure at right.
$m\angle APB = 75^{\circ}$




Circles are identified by the notation $\odot P$, where $P$ represents the point that is the center of the circle.

A **central angle** of a circle is an angle whose vertex is at the center of the circle.
$\angle APB$ is a central angle of $\odot P$.

A portion of a circle's circumference is called an **arc**. An arc is defined by two endpoints and the points on the circle between those two endpoints. If a circle is divided into two unequal arcs, the shorter arc is called the **minor arc** and the longer arc is called the **major arc**. If a circle is divided into two equal arcs, each arc is called a **semicircle**.

In our figure, we call the portion of the circle between and including points $A$ and $B$, arc $AB$ notated by $\widehat{AB}$. We call the remaining portion of the circle arc $ACB$, or $\widehat{ACB}$. Note that major arcs are usually named using three letters in order to distinguish it from the minor arc.

We say that the central angle $\angle APB$ *intercepts* or has $\widehat{AB}$. We also say that $\widehat{AB}$ *subtends* or has the central angle $\angle APB$. Note that when we refer to the arc of a central angle, we usually mean the minor arc unless otherwise stated.

Arcs are measured in two different ways - using degree measure and using linear measure. Usually when we refer to the **measure** of an arc, we are referring to the degree measure. The **measure** of a minor arc is defined to be the measure of the central angle that intercepts the arc. The measure of a major arc is 360° minus the measure of the minor arc with the same endpoints.

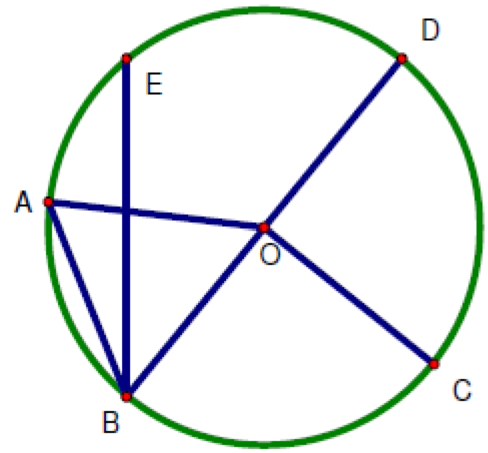
In the figure above, the measure of $\widehat{AB}$ is 75° because that is the measure of its central angle. The measure of $\widehat{ACB}$ is 360° – 75° or 285°.

The **length** of an arc is different from its measure. The length is given in linear units (e.g., inches, centimeters, and feet). We will investigate the length of an arc in a later task. **Congruent arcs** have equal degree measures and equal lengths. **Equivalent arcs** have equal degree measures. For example, the arcs of 0.25 string lengths on Circles 1, 2, and 3 are equivalent because they each have the same degree measure. But, they are not congruent because they have different lengths. It follows that **congruent circles** have the same circumference measure and length.

A **chord** is a *segment* whose endpoints lie on the circle. In the above figure, segment $\overline{AB}$ is a chord of $\odot P$.

5. How many chords can be in a circle?

What is the longest chord in a circle?
Explain how you know?




6. Refer to the figure at the right.
   Identify and name each of the following.
   Be sure to use the correct notation.

   a. Two different central angles
   b. A minor arc
   c. A major arc
   d. A semicircle
   e. Two different chords
   f. The central angle subtended by $\widehat{AD}$

Use your protractor to help you find the following measures:

   g. The measure of $\widehat{AC}$
   h. The measure of $\widehat{DEC}$

7. Now it is time to use some of the terminology you have learned. Consider the following two theorems:

*In the same circle or congruent circles, if two chords are congruent, their intercepted arcs are congruent.*

*In the same circle or congruent circles, if two arcs are congruent, then their chords are congruent.*

   a. Prove that each of the theorems is true.

   b. Write the two theorems as one biconditional statement.

8. Use a compass to construct a circle on an unlined sheet of paper. Label the center of your circle.

   a. Draw any chord, other than a diameter, on your circle.
   Use your compass and a straightedge to construct a segment that represents the distance from the center of your circle to the chord. What is the relationship between the chord and the segment representing this distance?

   b. Mary made the following conjecture: "If two chords of a circle are the same distance from the center of the circle, the chords are congruent." Mary is correct. Use what you learned in *Item 5a* to help convince Mary that her conjecture is correct.

   c. State the converse of Mary's conjecture.

   d. Write Mary's conjecture and its converse as a biconditional statement.

   e. When a conjecture has been proven, it can be stated as a theorem. Write and illustrate this theorem in your Circle Book.

9. Ralph made the following conjecture: "A radius perpendicular to a chord bisects the chord."
   a. Use your construction from *item 8a* to help convince yourself that Ralph's conjecture and the converse are true.

   b. Write Ralph's conjecture and its converse as a biconditional statement and illustrate it in your *Circle Book*.

   c. Ralph also believes that a radius perpendicular to a chord bisects the arc intercepted by the chord.
   Is this true? How do you know?

10. Tevante examined his construction and his partner's construction. He believes that *any* line that is a perpendicular bisector of a chord of a circle must also contain the center of the circle. Is he right? How do you know?

11. An investigator working for the Georgia Bureau of Investigation's crime lab has uncovered a jagged piece of a circular glass plate believed to have been used as a murder weapon. She needs to know the diameter of the plate. How might you use the information you learned in problem 10 to help determine the diameter of the circular plate?
    Use a compass, a straightedge, and a ruler to illustrate your answer.

## Arcs, Strings, and Radii (Spotlight Task)

[Back to Task Table](#)

**Standards Addressed in this Task**
**MGSE9-12.G.C.5** Derive using similarity the fact that the length of the arc intercepted by an angle is proportional to the radius, and define the radian measure of the angle as the constant of proportionality; derive the formula for the area of a sector.

**Standards for Mathematical Practice**

**2. Reason abstractly and quantitatively** by requiring students to make sense of quantities and their relationships to one another in problem situations.

**3. Construct viable arguments and critique the reasoning of others** by engaging students on discussion of why they agree or disagree with responses, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

**5. Use appropriate tools strategically** by expecting students to consider available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a compass, a calculator, software, etc.

***COMMENTS***
***Have the students work on the following tasks as individuals first, then comparing with other students when prompted in the activity. The students will need a compass, string, and a ruler. But, do not provide them a ruler until noted. Wikki Stix (waxed string) are preferred (http://www.wikkistix.com/). Each student who will be working in a group should have a string length differing from her or his group members.***

1. Let's make some more circles. As an individual:
   a. Construct a circle with a radius equal in length to your string.
   b. Using the center of the circle as a vertex, construct angles that subtend arc measures of 0.5, 1, 1.5, and 5 string lengths.
   c. Approximately how many string lengths will rotate around your circle's circumference?

***COMMENTS***
***For 1c, students should conclude that between 6 and 6.5 string lengths rotate around the circle's circumference. After they compare their activity as a group (see 2.), the teacher can draw their attention back to the circumference formula in order to have them connect this outcome to that formula (e.g., the circumference is 2pi, or approximately 6.28, times as large as the radius) and the implied similarity/proportionality (e.g., double the radius, double the circumference length).***

2. As a group:
   a. Compare your constructions and estimations. What do you notice about your angles?

b. Draw a new angle on a sheet of paper. Using the vertex of the angle as the center for each circle, draw circles with radii equivalent to the lengths of your strings. Estimate the string measure of the arc that subtends this angle on each of your circles.
c. Based on your observation, what can you conjecture about the relationship between the string arc measure and the radius of a circle for an angle of constant openness? Test your conjecture with another angle.
d. Using a ruler, your strings, and each of your circles, estimate the following measures in inches or centimeters (be consistent throughout) and determine the posed calculations.

| **Quantity** | **Circle 1** | **Circle 2** | **Circle 3** |
|---|---|---|---|
| Arc length (0.5 string arc) | | | |
| Arc Length (1 string arc) | | | |
| Arc Length (1.5 string arc) | | | |
| Arc Length (5 string arc) | | | |
| Radius | | | |
| Diameter | | | |
| Circumference | | | |
| Arc length (0.5 string arc)/Radius | | | |
| Arc length (1 string arc)/Radius | | | |
| Arc length (1.5 string arc)/Radius | | | |
| Arc length (5 string arc)/Radius | | | |
| Circumference/Radius | | | |
| Circumference/Diameter | | | |

3. Without constructing a circle and making measurements, fill out the following table. Justify the calculations you make to fill out the table entries and draw diagrams to illustrate each measurement.

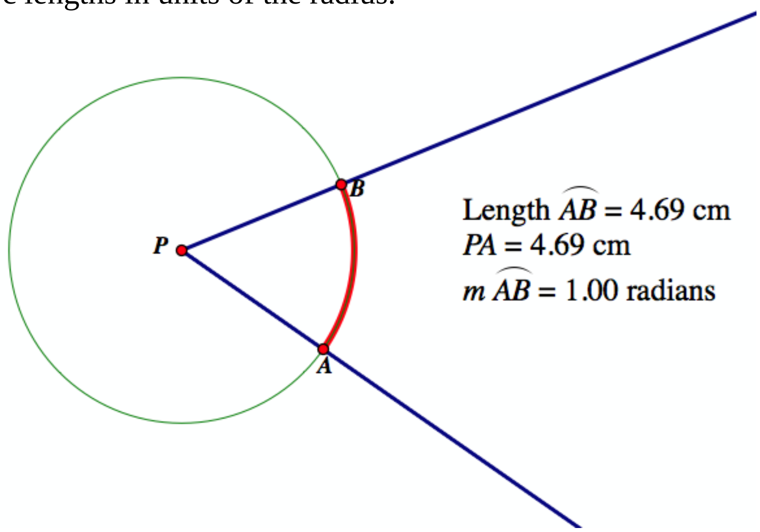
| Quantity | Circle 1 | Circle 2 | Circle 3 |
|---|---|---|---|
| Arc length (0.5 string arc) | ***.75*** | ***1.5*** | ***3*** |
| Arc Length (1 string arc) | ***1.5*** | ***3*** | ***6*** |
| Arc Length (1.5 string arc) | ***2.25*** | ***4.5*** | ***9*** |
| Arc Length (5 string arc) | ***7.5*** | ***15*** | ***30*** |
| Radius | 1.5 | 3 | 6 |
| Diameter | ***3*** | ***6*** | ***12*** |
| Circumference | $3\pi$ | $6\pi$ | $12\pi$ |
| Arc length (0.5 string arc)/Radius | ***0.5*** | ***0.5*** | ***0.5*** |
| Arc length (1 string arc)/Radius | ***1*** | ***1*** | ***1*** |
| Arc length (1.5 string arc)/Radius | ***1.5*** | ***1.5*** | ***1.5*** |
| Arc length (5 string arc)/Radius | ***5*** | ***5*** | ***5*** |
| Circumference/Radius | $2\pi$ | $2\pi$ | $2\pi$ |
| Circumference/Diameter | $\pi$ | $\pi$ | $\pi$ |

***SOLUTION***
***Building on the previous activity, students should continue observing that for an angle of constant openness, the same string arc measure is obtained. They should also extend the similarity conclusions from the previous lesson to justify why various table entries have equal values (e.g., the ratio of arc length to the radius must stay equal for an angle of constant openness by similarity).***

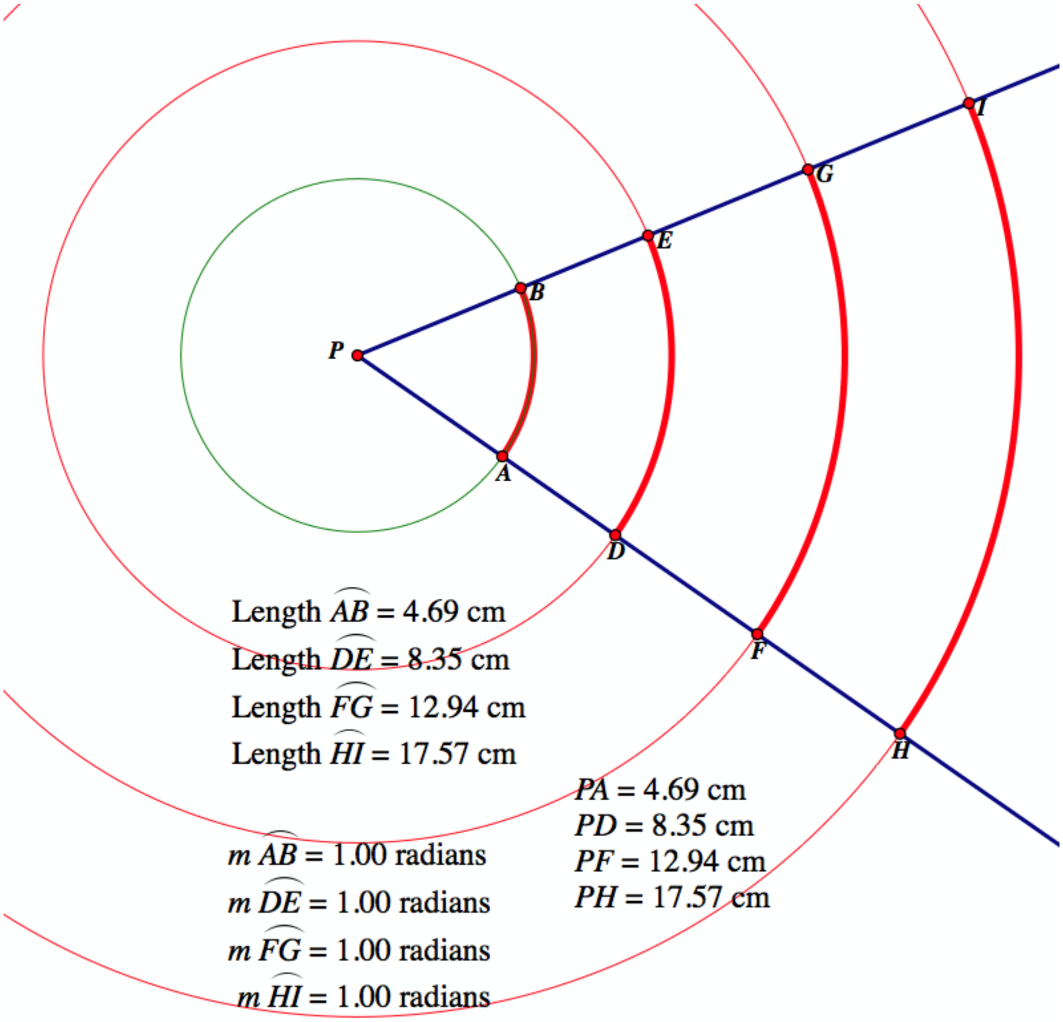
***On the second table, they should explain how they determined the values using the given radius and also how the values relate to each other from circle to circle. For example, with respect to the arc length entries, they should explain that since the string length is equivalent to the radius, the 0.5 string length arc should be 0.5 times as large as the corresponding radius. As another example, they can argue that if the radius is double (circle 1 to circle 2), then the arc length must also double by similarity. For the Arc length/Radius entries, they should be prompted to describe the meanings of these calculations (e.g., the arc length is so***

***many times as large as the radius) to determine that these values are the same as the string arc values because the string arc values give how many radii each arc length is.***

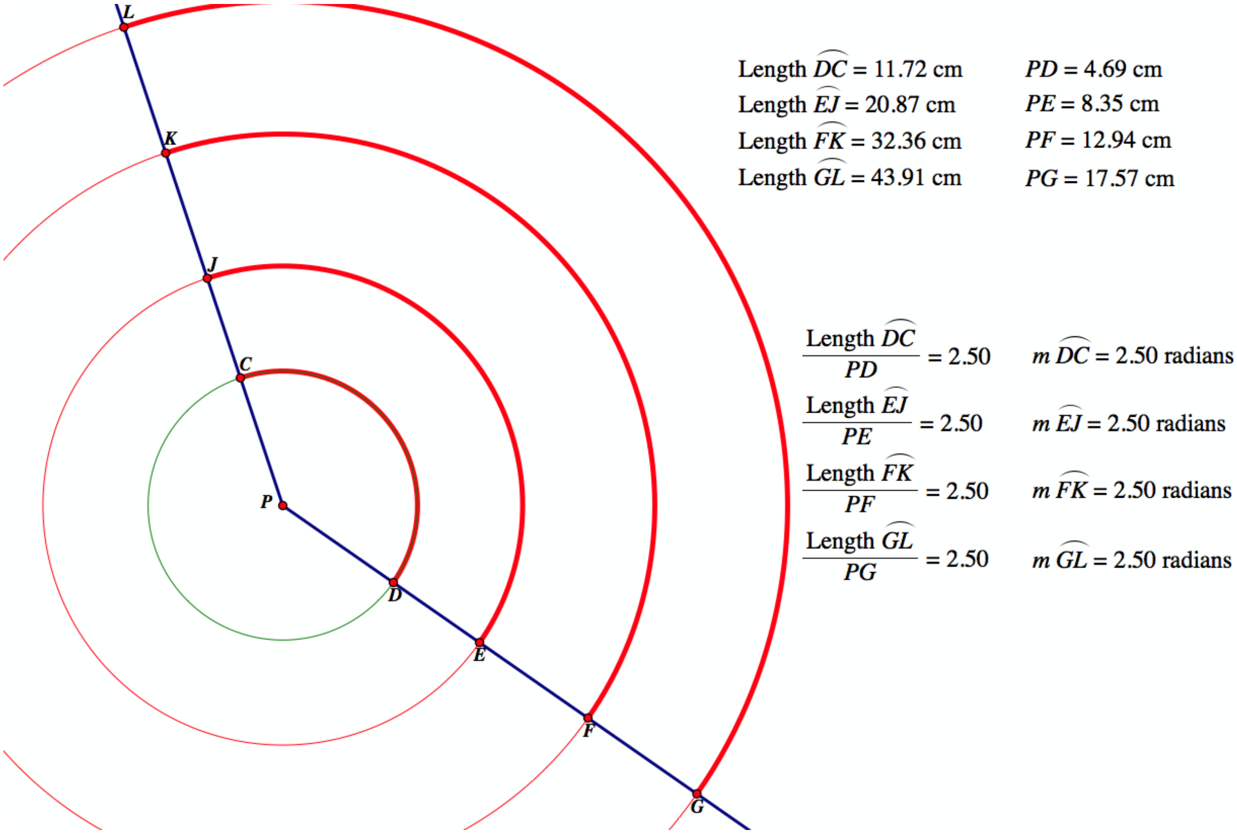
In the previous activity we introduced notation and terminology for arcs and described that degrees are a unit for measuring arcs. Another unit for measuring arcs is the **radian**, which is based in measuring arc lengths in units of the radius.




In the figure above, the measure of $\widehat{AB}$ is 1 radian because the arc length is equivalent to the radius. Also, as with degree measures, the arc intercepted by the central angle $\angle APB$ has the same measure regardless of size of the circle used to measure the arc. In the figure below, the same angle intercepts several arcs, each of which has a measure of 1 radian.




Radian measures for arcs are useful because they give the constant of proportionality between the arc lengths intercepted by an angle and the corresponding radius lengths. For the central angle $\angle APB$, the ratio between the arc length intercepted by the angle and the radius of the corresponding circle is always 1, meaning that the length of the arc is 1 times as large as the radius regardless of the circle used.




The angle $\angle CPD$ intercepts an arc that is 2.5 radians. The ratio between the arc length intercepted by the angle and the radius of the corresponding circle is always 2.5, meaning that the length of the intercept arc is 2.5 times as large as the radius regardless of the circle used.

4. Using the figure above with the central angle $\angle CPD$, answer the following questions.
   a. What is the arc length intercepted by angle $\angle CPD$ on a circle with a radius of 6.5 centimeters?

   b. What is the radius of the circle such that angle $\angle CPD$ intercepts an arc length of 13 centimeters?

   c. What is the arc length, $s$ centimeters, intercepted by angle $\angle CPD$ on a circle with a radius of $r$ centimeters?

***SOLUTION***
***16.25 inches***
***5.2 inches***
***s=2.5r***

5. Using the same figure, draw a new angle that intercepts an arc of approximately 3 radians.

    a. What is the arc length intercepted by your angle on a circle with a radius of 6.5 centimeters?

    b. What is the radius of the circle such that your angle intercepts an arc length of 13 centimeters?

    c. What is the arc length, *s* centimeters, intercepted by your angle on a circle with a radius of *r* centimeters?

***SOLUTION***
***19.5 inches***
***4.333 inches***
***s=3r***

6. A fellow student tells you that they drew an angle that intercepts an arc length of 7 centimeters on a circle with a radius of 2 centimeters. What is the arc measure in radians?

***SOLUTION***
***7/2=3.5 radians***

***COMMENTS***
***Throughout the final sequence of prompts students should be asked to be explicit about the quantities of the situation when describing their calculations. For instance, they should describe that dividing an arc length by the radius gives the number of radii in that arc, thus providing the radian measure and constant of proportionality. Teachers can conclude this activity by having students determine the formula for any given radian measure $\theta$, arc length s, and radius r. $\theta$=s/r as s/r represents the radii measure of the arc length.***

## Arcs, Strings, and Radii (Spotlight Task)

1. Let's make some more circles. As an individual:
   a. Construct a circle with a radius equal in length to your string.
   b. Using the center of the circle as a vertex, construct angles that subtend arc measures of 0.5, 1, 1.5, and 5 string lengths.
   c. Approximately how many string lengths will rotate around your circle's circumference?

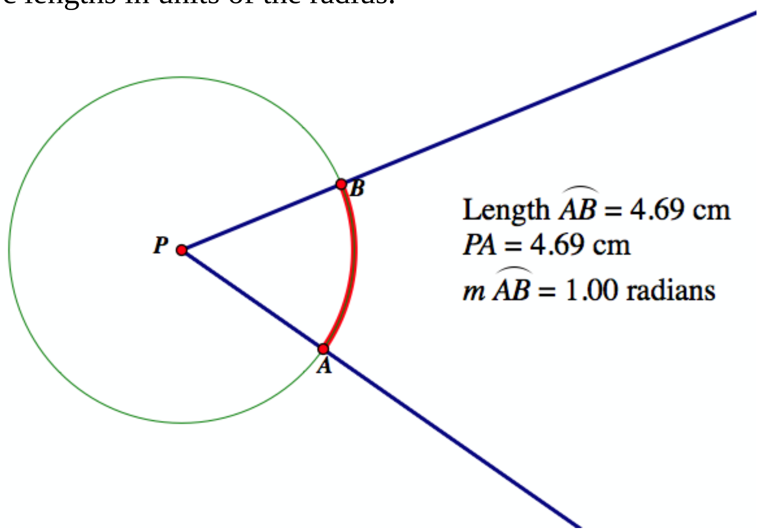
2. As a group:
   a. Compare your constructions and estimations. What do you notice about your angles?
   b. Draw a new angle on a sheet of paper. Using the vertex of the angle as the center for each circle, draw circles with radii equivalent to the lengths of your strings. Estimate the string measure of the arc that subtends this angle on each of your circles.
   c. Based on your observation, what can you conjecture about the relationship between the string arc measure and the radius of a circle for an angle of constant openness? Test your conjecture with another angle.
   d. Using a ruler, your strings, and each of your circles, estimate the following measures in inches or centimeters (be consistent throughout) and determine the posed calculations.

| Quantity | Circle 1 | Circle 2 | Circle 3 |
|---|---|---|---|
| Arc length (0.5 string arc) | | | |
| Arc Length (1 string arc) | | | |
| Arc Length (1.5 string arc) | | | |
| Arc Length (5 string arc) | | | |
| Radius | | | |
| Diameter | | | |
| Circumference | | | |
| Arc length (0.5 string arc)/Radius | | | |
| Arc length (1 string arc)/Radius | | | |
| Arc length (1.5 string arc)/Radius | | | |
| Arc length (5 string arc)/Radius | | | |
| Circumference/Radius | | | |
| Circumference/Diameter | | | |

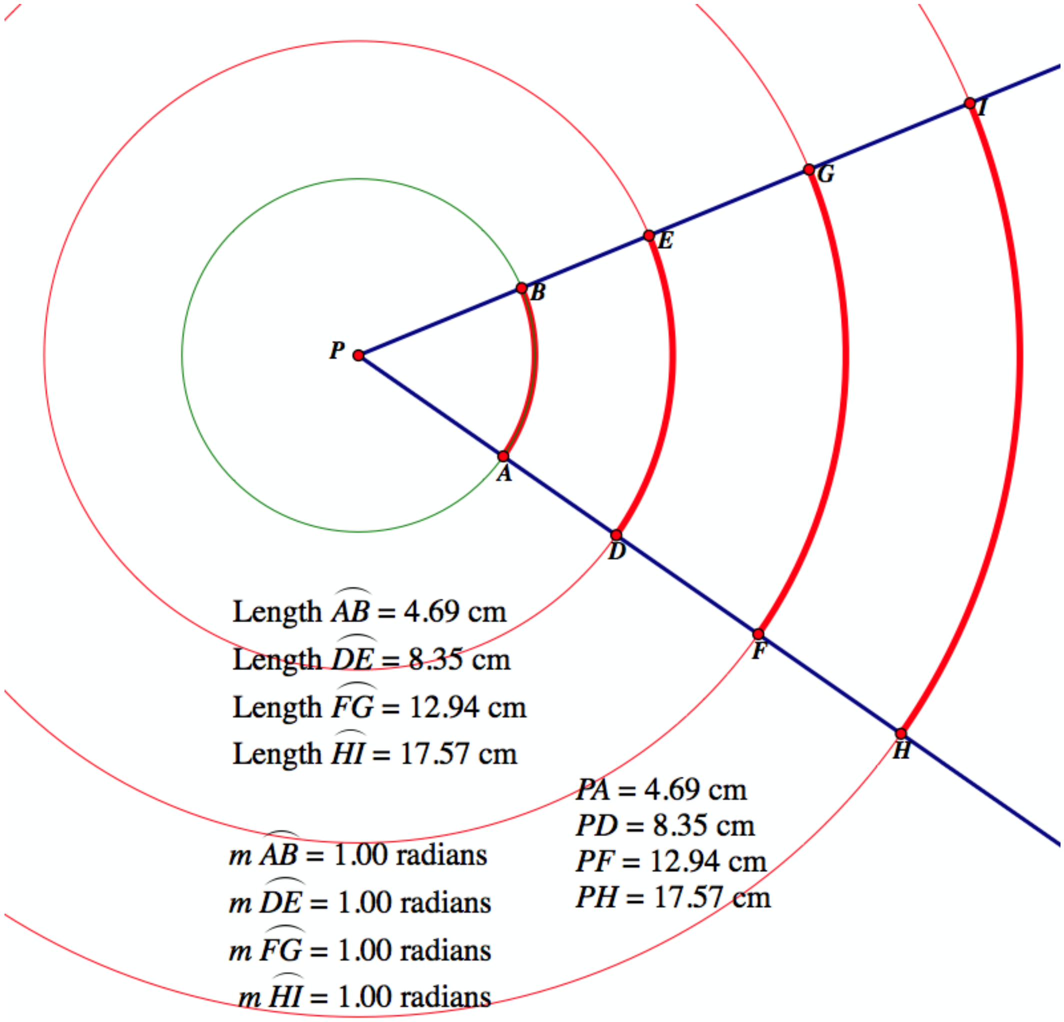
3. Without constructing a circle and making measurements, fill out the following table. Justify the calculations you make to fill out the table entries and draw diagrams to illustrate each measurement.

| **Quantity** | **Circle 1** | **Circle 2** | **Circle 3** |
|---|---|---|---|
| Arc length (0.5 string arc) | | | |
| Arc Length (1 string arc) | | | |
| Arc Length (1.5 string arc) | | | |
| Arc Length (5 string arc) | | | |
| Radius | 1.5 | 3 | 6 |
| Diameter | | | |
| Circumference | | | |
| Arc length (0.5 string arc)/Radius | | | |
| Arc length (1 string arc)/Radius | | | |
| Arc length (1.5 string arc)/Radius | | | |
| Arc length (5 string arc)/Radius | | | |
| Circumference/Radius | | | |
| Circumference/Diameter | | | |

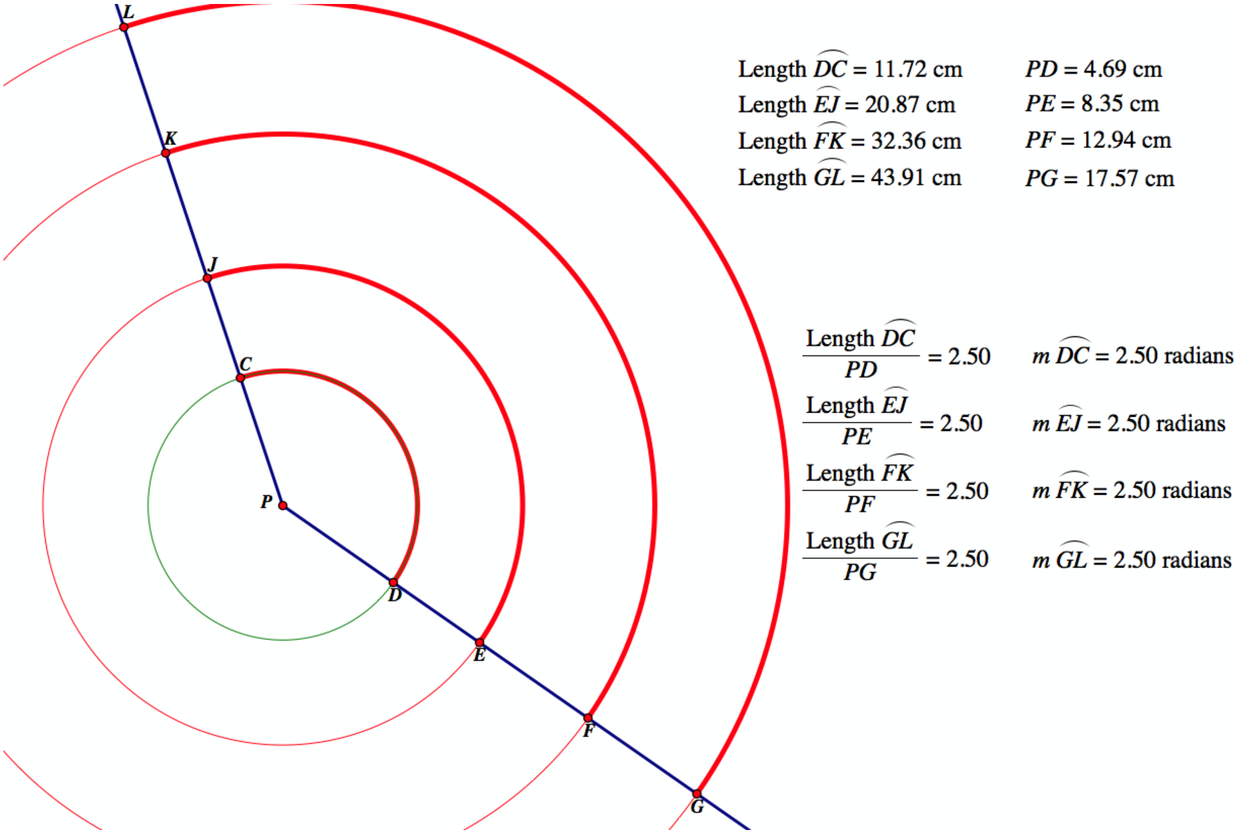
In the previous activity we introduced notation and terminology for arcs and described that degrees are a unit for measuring arcs. Another unit for measuring arcs is the **radian**, which is based in measuring arc lengths in units of the radius.




In the figure above, the measure of $\widehat{AB}$ is 1 radian because the arc length is equivalent to the radius. Also, as with degree measures, the arc intercepted by the central angle $\angle APB$ has the same measure regardless of size of the circle used to measure the arc. In the figure below, the same angle intercepts several arcs, each of which has a measure of 1 radian.




Radian measures for arcs are useful because they give the constant of proportionality between the arc lengths intercepted by an angle and the corresponding radius lengths. For the central angle $\angle APB$, the ratio between the arc length intercepted by the angle and the radius of the corresponding circle is always 1, meaning that the length of the arc is 1 times as large as the radius regardless of the circle used.




The angle $\angle CPD$ intercepts an arc that is 2.5 radians. The ratio between the arc length intercepted by the angle and the radius of the corresponding circle is always 2.5, meaning that the length of the intercept arc is 2.5 times as large as the radius regardless of the circle used.

4. Using the figure above with the central angle $\angle CPD$, answer the following questions.
   a. What is the arc length intercepted by angle $\angle CPD$ on a circle with a radius of 6.5 centimeters?

   b. What is the radius of the circle such that angle $\angle CPD$ intercepts an arc length of 13 centimeters?

   c. What is the arc length, $s$ centimeters, intercepted by angle $\angle CPD$ on a circle with a radius of $r$ centimeters?

5. Using the same figure, draw a new angle that intercepts an arc of approximately 3 radians.

a. What is the arc length intercepted by your angle on a circle with a radius of 6.5 centimeters?

b. What is the radius of the circle such that your angle intercepts an arc length of 13 centimeters?

c. What is the arc length, $s$ centimeters, intercepted by your angle on a circle with a radius of $r$ centimeters?

6. A fellow student tells you that they drew an angle that intercepts an arc length of 7 centimeters on a circle with a radius of 2 centimeters. What is the arc measure in radians?

## Return to Lifting the Rope (Spotlight Task)

**Back to Task Table**

***The following task returns to the Introductory Task from the beginning of the unit. The students should now reach a formal conclusion using those ideas from the previous activities.***

**Standards Addressed in this Task**

**MGSE9-12.G.C.1** Understand that all circles are similar.

**MGSE9-12.G.C.5** Derive using similarity the fact that the length of the arc intercepted by an angle is proportional to the radius, and define the radian measure of the angle as the constant of proportionality; derive the formula for the area of a sector.

**MGSE9-12.G.GMD.1** Give informal arguments for geometric formulas.

a. Give informal arguments for the formulas of the circumference of a circle and area of a circle using dissection arguments and informal limit arguments.
b. Give informal arguments for the formula of the volume of a cylinder, pyramid, and cone using Cavalieri's principle.

**Standards for Mathematical Practice**
Although this task is only meant as an opening task to the unit with no formal outcomes, the task does set the basis for engaging in the following practices:

**2.** Reason abstractly and quantitatively.
**3.** Construct viable arguments and critique the reasoning of others.
**4.** Model with mathematics.

***COMMENTS***
***A teacher should bring different sized balls and string into the class to aid this investigation. They can vary the task to have students actually engage in wrapping string around the balls. The task can also be modified to a 3-ACTs task by shooting a video of balls with taut string and then string pulled one-foot off of the balls. A picture of the earth can be used.***

Imagine wrapping the circumference of a tennis ball and basketball with pieces of string so that the pieces of string wrap perfectly around the balls. Now, imagine lifting the entire pieces of string 1 foot off of each ball. Which ball requires more string to "close" string circle that is 1 foot off of the ball?


Imagine wrapping the circumference of the Moon and Earth with pieces of string so that the pieces of string wrap perfectly around the surfaces (we'll imagine the surfaces are smooth). Now, imagine lifting the entire pieces of string 1 foot off of each surface. Which surface requires more string to "close" the string circle? How do these compare to the amount of string needed to "close" the string circles off of the tennis and basketball?

***COMMENTS***

***Students can draw on different arguments to conclude that the same amount of string must be used in each case. They could use the formula (e.g., c=2pi*(r+1)=2pi*r+2pi) to conclude this. Or, they could use that since each circle has a circumference of 2pi radii, increasing the radius by 1 unit increases the circumference by 2pi units (e.g., 6 increases of 1 unit and 1 increase of approximately .28 units). Or, the students can reason that the 2pi in the formula c=2pi*r is a rate of change such that for a 1 unit change in the radius, the circumference changes by 2pi units. Have the students present their different methods and make connections between these methods. Also, an extension is connecting to arc measure. In each case, the same arc length is needed to close the circle. But, as the circle increases in size, the arc measure needed to close the circle decreases.***

## Return to Lifting the Rope (Spotlight Task)

Imagine wrapping the circumference of a tennis ball and basketball with pieces of string so that the pieces of string wrap perfectly around the balls. Now, imagine lifting the entire pieces of string 1 foot off of each ball. Which ball requires more string to "close" string circle that is 1 foot off of the ball?

Imagine wrapping the circumference of the Moon and Earth with pieces of string so that the pieces of string wrap perfectly around the surfaces (we'll imaging the surfaces are smooth). Now, imagine lifting the entire pieces of string 1 foot off of each surface. Which surface requires more string to "close" the string circle? How do these compare to the amount of string needed to "close" the string circles off of the tennis and basketball?

## Investigating Angle Relationships in Circles

[Back to Task Table]

**Standards Addressed in this Task**

**MGSE9-12.G.C.2** Identify and describe relationships among inscribed angles, radii, chords, tangents, and secants. Include the relationship between central, inscribed, and circumscribed angles; inscribed angles on a diameter are right angles; the radius of a circle is perpendicular to the tangent where the radius intersects the circle.

**Standards for Mathematical Practice**

**1. Make sense of problems and persevere in solving them** by requiring students to interpret and make meaning of a problem and find a logical starting point, and to monitor their progress and change their approach to solving the problem, if necessary.

**3. Construct viable arguments and critique the reasoning of others** by engaging students on discussion of why they agree or disagree with responses, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

**5. Use appropriate tools strategically** by expecting students to consider available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a compass, a calculator, software, etc.

**Common Student Misconceptions**

1. Students sometimes confuse inscribed angles and central angles. For example they will assume that the inscribed angle is equal to the arc like a central angle.
2. Students may think they can tell by inspection whether a line intersects a circle in exactly one point. It may be beneficial to formally define a tangent line as the line perpendicular to a radius at the point where the radius intersects the circle.
3. Students may confuse the segment theorems. For example, they will assume that the inscribed angle is equal to the arc like a central angle.

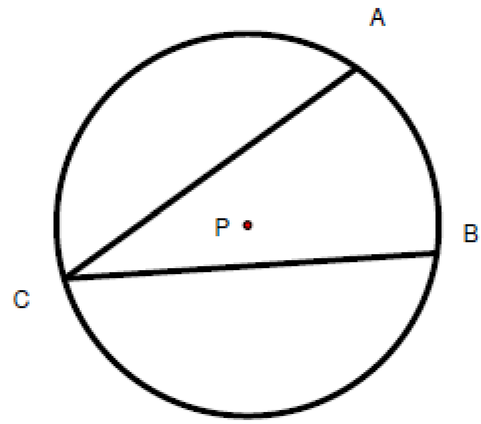
In this task, you will be investigating, discovering, and proving two theorems that involve circles and their inscribed angles. Afterwards, you will be expected to know and apply these theorems, and several others that you will be shown, to solve problems.

***COMMENTS***

***Discovery Learning is a great method of learning for students. It allows them to be the creator of a theorem and feel ownership of the material. The circle theorems covered in this unit are a great place in a geometry course to spend some time allowing and encouraging students to investigate their conjectures. The Inscribed Angle Theorem is chosen in this task as one of the theorems to investigate. It is up to the teacher to choose a method of investigation depending on the teacher's available resources. Please consider using any and all technology available such as the Cabri Jr software on most TI-84 plus calculators, Geometer's Sketchpad software, a free download from the website http://www.geogebra.org/cms/, or a free trial from www.explorelearning.com website. A group discovery can be accomplished as well using compass, straightedge, and protractor. The standards emphasize the necessity of a formal proof for the relationship of the angles of an inscribed quadrilateral. This has been included***

***in part 2 of this task. The standards also emphasize the need to apply the use of these theorems, so the teacher needs to provide adequate assignments involving problem solving in order to ensure student mastery.***

**Part 1: Inscribed Angles**




Definition: an **inscribed angle** is an angle whose vertex lies on the circle and whose sides are chords of the circle.

In ○P, ∠ACB is an inscribed angle.

1. Sketch another inscribed angle in ○P.

***SOLUTION***
***Drawings will vary.***

2. Now, you need to investigate the measure of an inscribed angle and its intercepted arc by following your teacher's instructions.

***COMMENTS***
***As stated above, the teacher will need to provide directions here based on the chosen method of discovery. Please be aware that most resources have student labs or student exploration guides prepared with step by step directions for students to follow.***

3. Write your conjecture here:

***SOLUTION***
***Students may answer that "the angle is half the measure of the arc" or they may choose to state their conjecture as "the arc measure is twice that of the angle measure". It will be beneficial for students to recognize and use the theorem as:***
***Inscribed Angles = ½ their intercepted arc***

Remember that a conjecture is not a theorem until it has been proved.

**Part 2: Quadrilaterals Inscribed in a Circle**

4. Define quadrilateral.

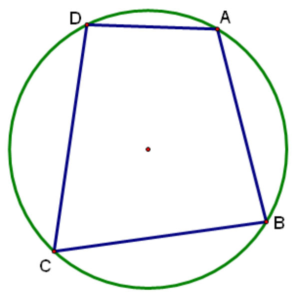
***SOLUTION***

***A quadrilateral is a polygon with four sides.***

A polygon is **inscribed** in a circle when every vertex of the polygon is on the circle.

5. Sketch a picture of a circle P with an inscribed quadrilateral ABCD.

***SOLUTION***
***It is important to remind students that polygons are named in consecutive order of their vertices.***




6. Now, you will investigate the relationships among the angles of the quadrilateral inscribed in a circle.

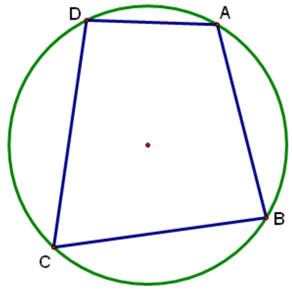
***COMMENTS***
***As stated above, the teacher will need to provide directions here based on the chosen method of discovery. Please be aware that most resources have student labs or student exploration guides prepared with step by step directions for students to follow.***

7. Write your conjecture here:

***SOLUTION***
***"The opposite angles are supplementary"***

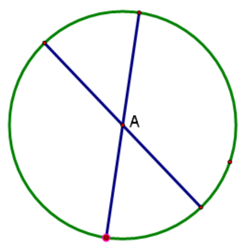
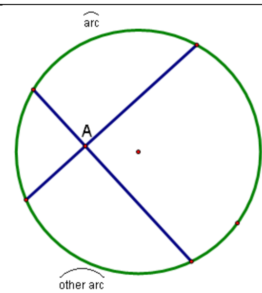
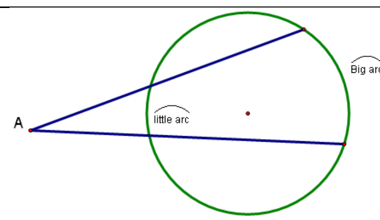
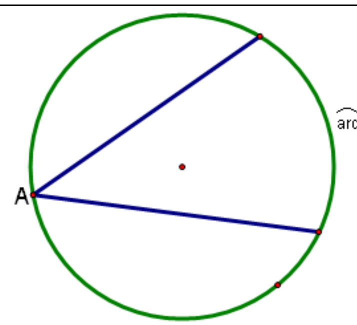
8. Write a proof of the theorem using your sketch from above.

**SOLUTION**



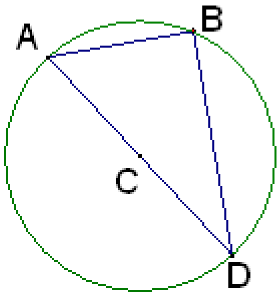

| Statements | Reasons |
|---|---|
| $\angle BCD, \angle BAD$ are inscribed angles | given |
| $m\angle BCD = \frac{1}{2} m\widehat{BAD}$<br>$m\angle BAD = \frac{1}{2} m\widehat{BCD}$ | Inscribed Angle Thm |
| $m\angle BCD + m\angle BAD = \frac{1}{2}(m\widehat{BAD} + m\widehat{BCD})$ | Addition Property of Equality |
| $m\angle BCD + m\angle BAD = \frac{1}{2}(360^\circ)$ | Arc Addition Postulate |
| $m\angle BCD + m\angle BAD = 180^\circ$ | multiplication |
| $\angle BCD, \angle BAD$ are supplementary | Definition of Supplementary Angles |

**Part 3: Graphic Organizer for Angle Theorems**

| Location of the Vertex | Picture | Theorem |
|---|---|---|
| **Inside the circle** <br> **At the Center** |   | $m\angle A = arc$ |
| **Not at the center** |   | $m\angle A = \frac{1}{2}(arc + other\ arc)$ |
| **Outside of the circle** |   | $m\angle A = \frac{1}{2}(Big\ arc - Little\ arc)$ |
| **On the circle** |   | $m\angle A = \frac{1}{2}(arc)$ |

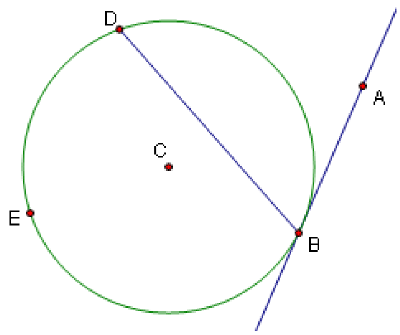
**Part 4: Apply these theorems to solve these special cases of inscribed angles.**

1. Find the m∠ABD, the inscribed angle of ○C.



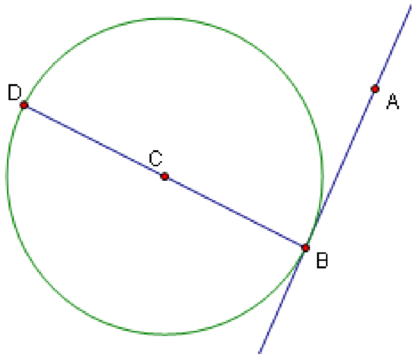

***SOLUTION***
***Since the angle intercepts a semi-circle it is 90º***

2. Find the m∠ABD, the inscribed angle of ○C, if $m\widehat{BED} = 300°$




***SOLUTION***
***Since the angle is an inscribed angle its measure will be half of its intercepted arc of 60ºand so it will measure 30º.***

3. Find the m∠ABD, the inscribed angle of ○C.

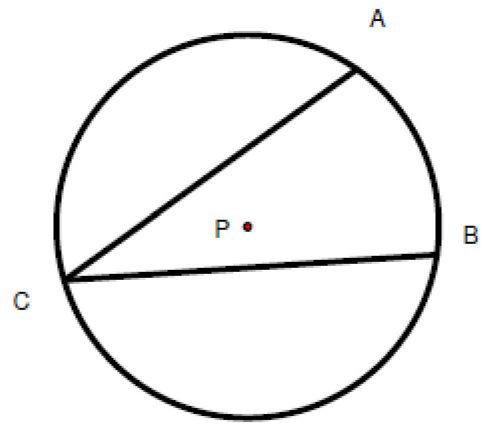



***SOLUTION***
***Since the inscribed angle intercepts a semi-circle it is 90ºimplying that any tangent is perpendicular to a radius at its point of tangency.***

# Investigating Angle Relationships in Circles

In this task, you will be investigating, discovering, and proving two theorems that involve circles and their inscribed angles. Afterwards, you will be expected to memorize and apply these theorems, and several others that you will be shown, to solve problems.

## Part 1: Inscribed Angles




Definition: an **inscribed angle** is an angle whose vertex lies on the circle and whose sides are chords of the circle.

In ○P, ∠ACB is an inscribed angle.

1. Sketch another inscribed angle in ○P.

2. Now, you need to investigate the measure of an inscribed angle and its intercepted arc by following your teacher's instructions.

3. Write your conjecture here:

Remember that a conjecture is not a theorem until it has been proved.

## Part 2: Quadrilaterals Inscribed in a Circle

4. Define quadrilateral.

A polygon is **inscribed** in a circle when every vertex of the polygon is on the circle.

5. Sketch a picture of a circle P with an inscribed quadrilateral ABCD.

6. Now, you will investigate the relationships among the angles of the quadrilateral inscribed in a circle.

7. Write your conjecture here:

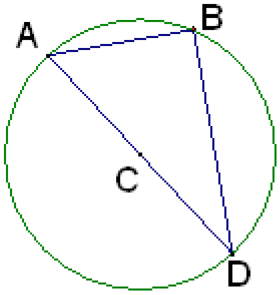
8. Write a proof of the theorem using your sketch from above.

**Part 3: Graphic Organizer for Angle Theorems**

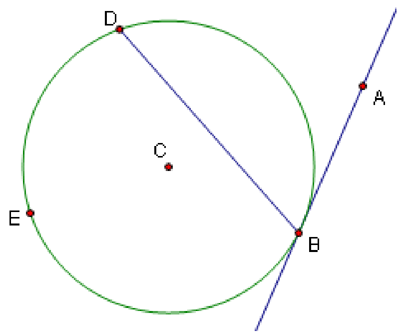
| Location of the Vertex | Picture | Theorem |
|---|---|---|
| **Inside the circle**<br>**At the Center** | | |
| **Not at the center** | | |
| **Outside of the circle** | | |
| **On the circle** | | |

**Part 4: Apply these theorems to solve these special cases of inscribed angles.**

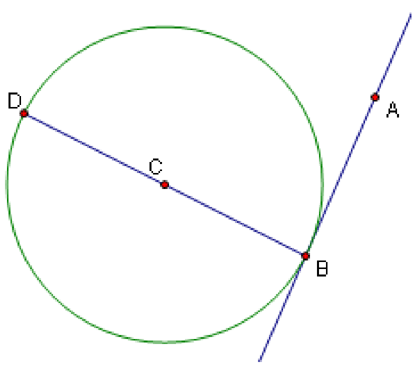
1. Find the m∠ABD, the inscribed angle of ○C.




2. Find the m∠ABD, the inscribed angle of ○C, if $m\widehat{BED} = 300°$.




3. Find the m∠ABD, the inscribed angle of ○C.



## Chords, Secants, and Tangents

[Back to Task Table](#)

**Standards Addressed in this Task**

**MGSE9-12.G.C.2** Identify and describe relationships among inscribed angles, radii, chords, tangents, and secants. Include the relationship between central, inscribed, and circumscribed angles; inscribed angles on a diameter are right angles; the radius of a circle is perpendicular to the tangent where the radius intersects the circle.

**MGSE9-12.G.C.3** Construct the inscribed and circumscribed circles of a triangle, and prove properties of angles for a quadrilateral inscribed in a circle.

**MGSE9-12.G.C.4** Construct a tangent line from a point outside a given circle to the circle.

**Standards for Mathematical Practice**

**1. Make sense of problems and persevere in solving them** by requiring students to interpret and make meaning of a problem and find a logical starting point, and to monitor their progress and change their approach to solving the problem, if necessary.

**3. Construct viable arguments and critique the reasoning of others** by engaging students on discussion of why they agree or disagree with responses, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

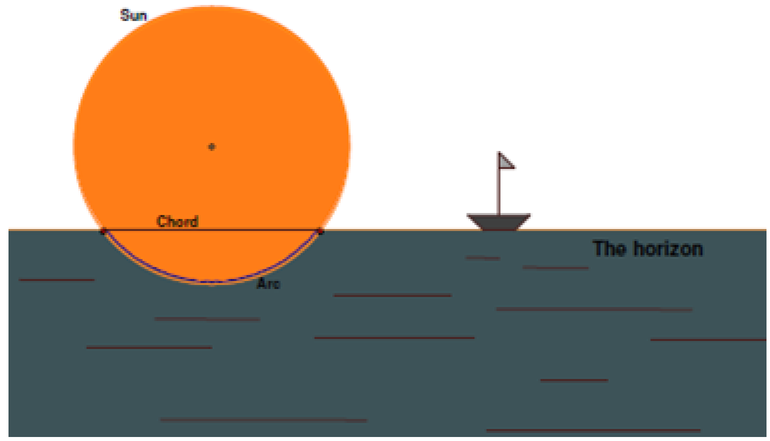
**5. Use appropriate tools strategically** by expecting students to consider available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a compass, a calculator, software, etc.

**Common Student Misconceptions**

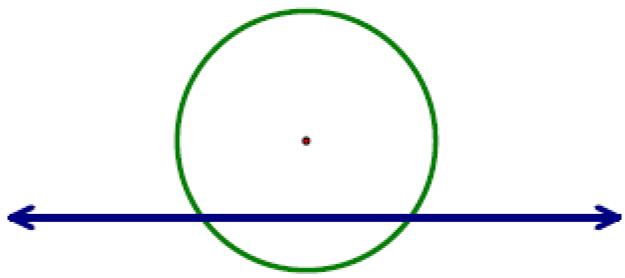
1. Students sometimes confuse inscribed angles and central angles. For example they will assume that the inscribed angle is equal to the arc like a central angle.
2. Students may think they can tell by inspection whether a line intersects a circle in exactly one point. It may be beneficial to formally define a tangent line as the line perpendicular to a radius at the point where the radius intersects the circle.
3. Students may confuse the segment theorems. For example, they will assume that the inscribed angle is equal to the arc like a central angle.

## Part 1: Sunrise on the First Day of the New Year

It is customary for people in Asia to visit the seashores on the eastern sides of their countries on the first day of the year. While watching the sun rise over the ocean, visitors wish for good luck in the New Year.




As the sun rises, the horizon cuts the sun at different positions. Although a circle is not a perfect representation of the sun, we can simplify this scene by using a circle to represent the sun and a line to represent the horizon.



1. Using the simplified diagram above, sketch and describe the different types of intersections the sun and the horizon may have.

***SOLUTION***
***Students should have at least a tangent line, another secant line, and a secant that creates a diameter sketched on their papers.***

2. Definitions:
A **tangent line** is a line that intersects a circle in exactly one point.
A **secant line** intersects a circle in two points.

Do any of your sketches in contain tangent or secant lines?
If so, label them.

***SOLUTION***
***Be sure the diameter is labeled as a secant line, as well.***

Is it possible for a line to intersect a circle in 3 points? 4 points?
Explain why or why not.

***SOLUTION***

***No, a line can only intersect a circle at one point as in the case of the tangent, or two points, as in the case of a secant. The Center of a circle is not a point on a circle and so therefore the diameter is not intersecting at three points.***

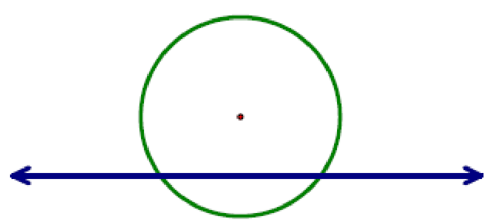
3. When a secant line intersects a circle in two points, it creates a chord. As you have already learned, a **chord** is a segment whose endpoints lie on the circle. How does a chord differ from a secant line?

***SOLUTION***
***A chord is a segment not a line.***

4. Look again at our representation of the sun and the horizon.

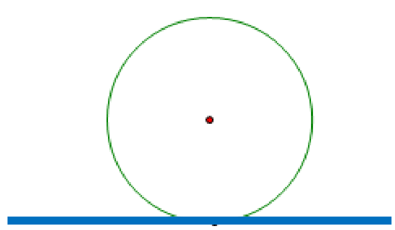
Let $d$ represent the distance between the center of a circle and a line $l$. Let $r$ represent the length of a radius of the circle.

a. Draw and describe the relationship between $d$ and $r$ when $l$ is a secant line



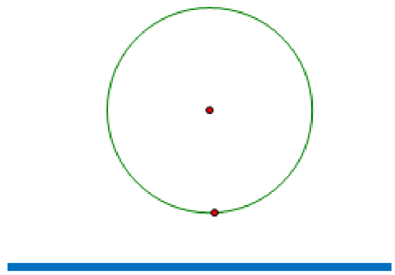
***SOLUTION***
$d < r$

b. Draw and describe the relationship between $d$ and $r$ when $l$ is a tangent line



***SOLUTION***
$d = r$

c. Draw and describe the relationship between *d* and *r* when *l* does not intersect the circle



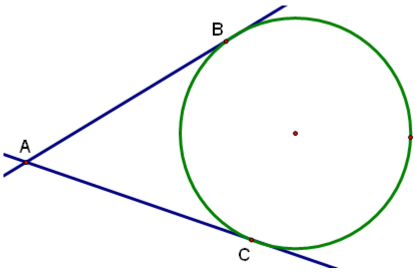
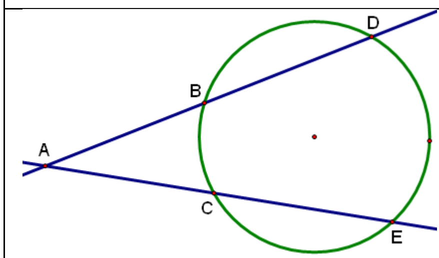
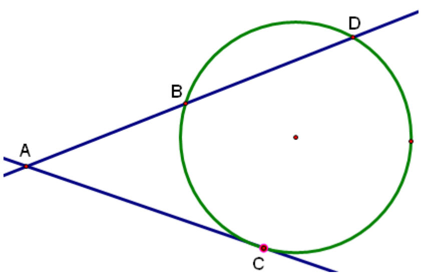
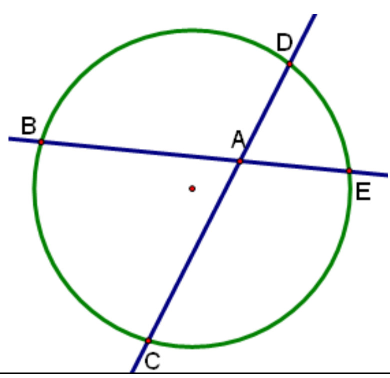
***SOLUTION***
***d > r***

5. You just compared the length of a radius of a circle to the distance from the center of the circle to a tangent line. What does this comparison tell you about the relationship of a tangent line to a radius at the point of tangency. Explain your thinking. Relate this to the problem you solved in the task *Investigating Angle Relationships in Circles* (Part 4, question #3).

***SOLUTION***
***The tangent line is perpendicular to the radius at the point of tangency.***

**Part 2: The Segment Theorems Graphic Organizer**

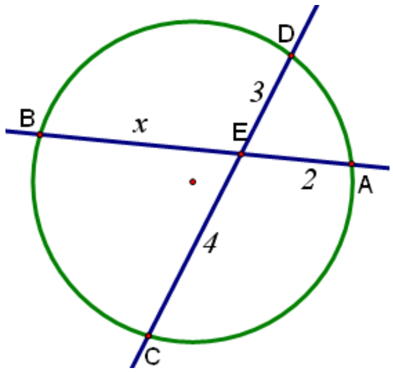
In the remaining items of this task, we will work with the relationships between the lengths of the segments created when these lines intersect.

| Picture | Type | Theorem |
|---|---|---|
|  | **2 tangents**<br><br>**vertex outside** | ***The tangent segments are congruent***<br><br>***AB = AC***<br><br>***whole² = whole²*** |
|  | **2 secants**<br><br>**vertex outside** | $AB \bullet AD = AC \bullet AE$<br><br>***Outside x Whole = Outside x Whole***<br><br>***"outside times whole bless my soul"*** |
|  | **Secant and tangent**<br><br>**vertex outside** | $AB \bullet AD = AC \bullet AC$<br>$AB \bullet AD = AC^2$<br><br>***Outside x Whole = Outside x Whole***<br>***Outside x Whole = Whole x Whole***<br>***Outside x Whole = Whole²*** |
|  | **2 secants**<br><br>**VERTEX INSIDE** | $BA \bullet AE = DA \bullet AC$<br><br>***part x other part = part x other part***<br><br>***"part times part cross my heart"*** |

**Part 3: Apply the Theorems to Solve Problems**
Sketch a picture for each problem, choose a theorem, set up an equation, and then solve.

1. Chords $\overline{AB}$ and $\overline{CD}$ intersect inside a circle at point E. $AE= 2$, $CE =4$ , and $ED =3$ .
   Find $EB$.

***SOLUTION***

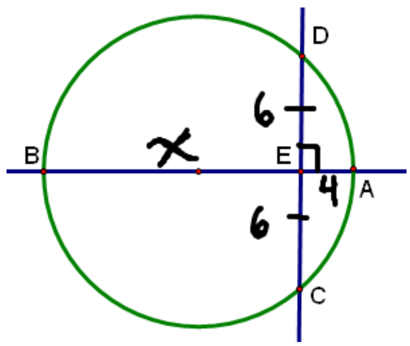



$AE \bullet EB = CE \bullet ED$
$2 \bullet x = 4 \bullet 3$
$2x = 12$
$x = 6$

***EB = 6***

2. A diameter of a circle is perpendicular to a chord whose length is 12 inches. If the length of the shorter segment of the diameter is 4 inches, what is the length of the longer segment of the diameter?

***SOLUTION***

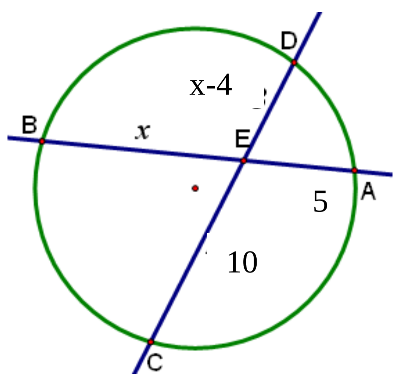



$AE \bullet EB = CE \bullet ED$
$4 \bullet x = 6 \bullet 6$
$4x = 36$
$x = 9$

***The longer segment, EB = 9***

3. Chords $\overline{AB}$ and $\overline{CD}$ intersect inside a circle at point E. $AE= 5$, CE $=10$, $EB = x$, and $ED = x\text{-}4$.
   Find $EB$ and $ED$.

***SOLUTION***

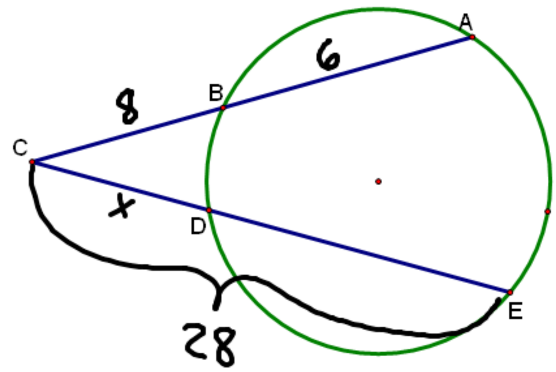



$AE \bullet EB = CE \bullet ED$

$$5(x) = 10(x-4)$$
$$5x = 10x - 40$$
$$-5x = -40$$
$$x = 8$$

***EB = 8 and ED = 4***

4. Two secant segments are drawn to a circle from a point outside the circle. The external segment of the first secant segment is 8 centimeters and its internal segment is 6 centimeters. If the entire length of the second secant segment is 28 centimeters, what is the length of its external segment?




**SOLUTION**

$CB \bullet CA = CD \bullet CE$

$8 \bullet 14 = x \bullet 28$

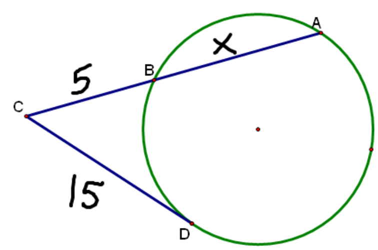
$112 = 28x$

$4 = x$

***The external segment is 4cm***

5. A tangent segment and a secant segment are drawn to a circle from a point outside the circle. The length of the tangent segment is 15 inches. The external segment of the secant segment measures 5 inches. What is the measure of the internal secant segment?

**SOLUTION**




$CB \bullet CA = CD^2$

$5 \bullet (x+5) = 15^2$

$5x + 25 = 225$

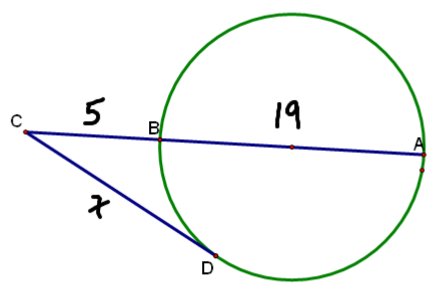
$5x = 200$

$x = 40$

***The internal secant segment is 40in.***

6. The diameter of a circle is 19 inches. If the diameter is extended 5 inches beyond the circle to point *C*, how long is the tangent segment from point *C* to the circle?

**SOLUTION**




$CB \bullet CA = CD^2$

$5 \bullet 24 = x^2$

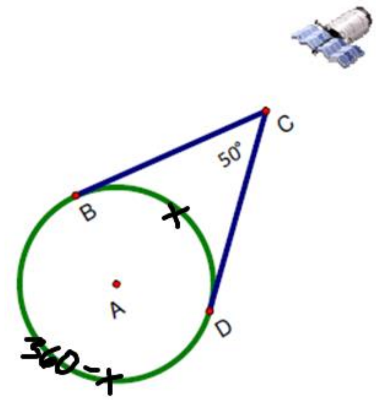
$120 = x^2$

$\pm\sqrt{120} = x$

$x = \pm 2\sqrt{30} \approx 10.954$

***The tangent segment is approximately 10.954in.***

7. A satellite orbits the earth so that it remains at the same point above the Earth's surface as the Earth turns. If the satellite has a 50° view of the equator, what percent of the equator can be seen from the satellite?




***SOLUTION***

$$m<C=\frac{1}{2}(Big\,arc-Little\,arc)$$

$$m<C=\frac{1}{2}(360-x-x)$$

$$m<C=\frac{1}{2}(360-2x)$$

$$m<C=180-x$$

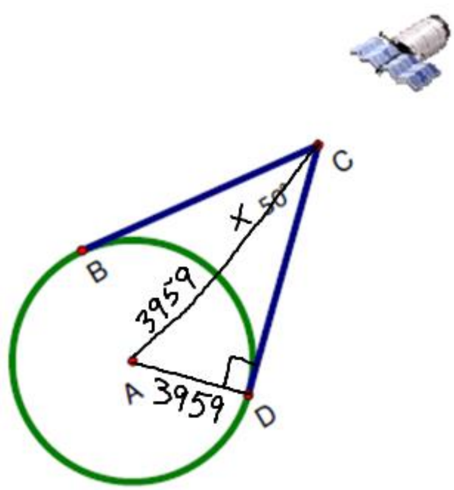
$$50=180-x$$

$$x=130$$

***Since the earth is 360°the percent viewable is***

$$\frac{130}{360}=.361\approx 36.1\%$$

8. The average radius of the Earth is approximately 3959 miles.
a. How far above the Earth's surface is the satellite described in *Problem 7?*




***SOLUTION***

$$\sin(25^{o})=\frac{3959}{3959+x}$$

$$.4226=\frac{3959}{3959+x}$$

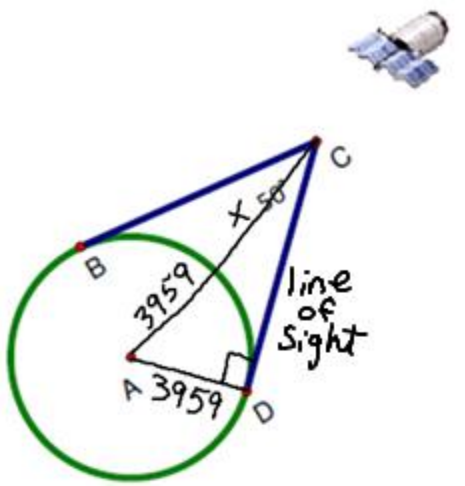
$$1673.07+.4226x=3959$$

$$.4226x=2285.9266$$

$$x=5409.1968$$

***The satellite is approximately 5,409 miles above the earth.***

b. What is the length of the longest line of sight from the satellite to the Earth's surface? Identify this line of sight using the diagram.




***SOLUTION***

$$3959^2 + l^2 = (3959 + 5409)^2$$

$$15673681 + l^2 = 87759424$$

$$l^2 = 72085743$$

$$l = 8490.33$$

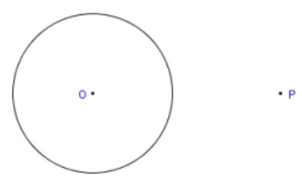
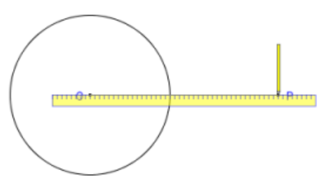
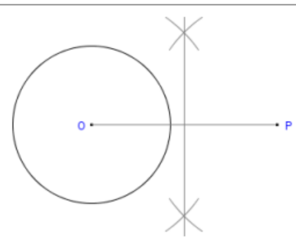
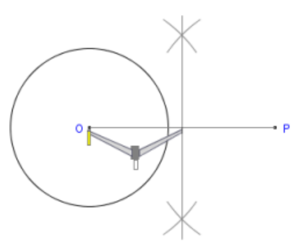
## Part 4: Constructions in Euclidean Geometry

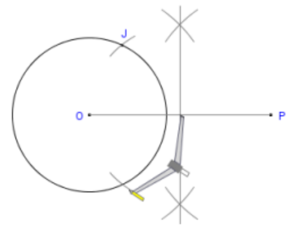
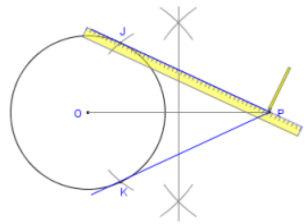
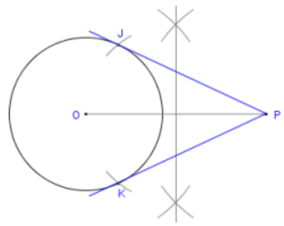
1. Using a compass and straight edge, construct a tangent line to a circle from a given exterior point.

***SOLUTION***
***Step by step instructions are provided here from:***
***http://www.mathopenref.com/printtangents.html***

**Tangents to a circle from a point** Back to main page Print
Printable instructions worksheet.

This construction assumes you are already familiar with Constructing the Perpendicular Bisector of a Line Segment.

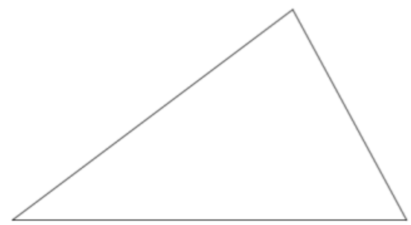
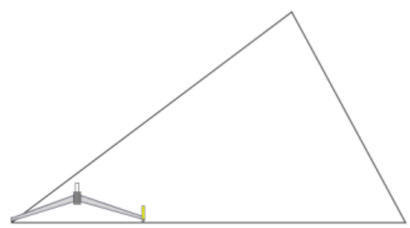
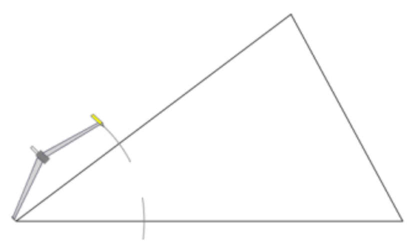
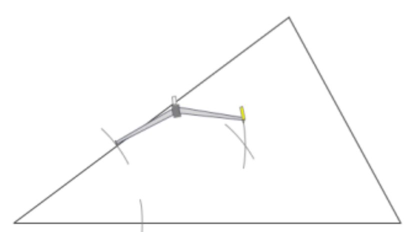
| After doing this | Your work should look like this |
|---|---|
| We start with a given circle with center O, and a point P outside the circle. |  |
| **1.** Draw a straight line between the center O of the given circle and the given point P. |  |
| **2.** Find the midpoint of this line by constructing the line's perpendicular bisector.<br><br>(See Constructing the Perpendicular Bisector of a Line Segment. |  |
| **3.** Place the compass on the midpoint just constructed, and set it's width to the center O of the circle. |  |

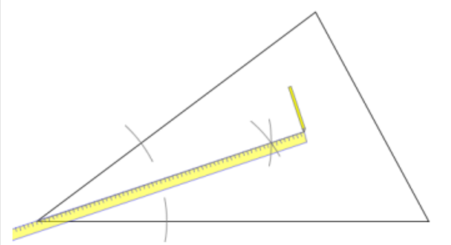
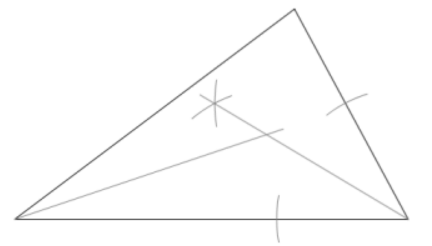
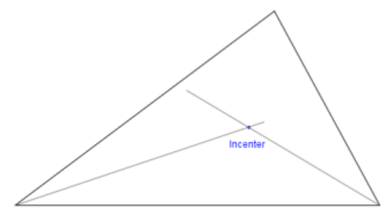
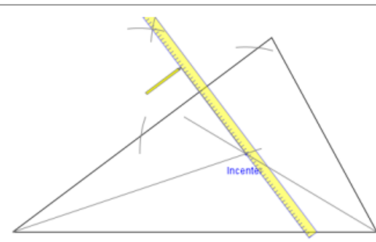
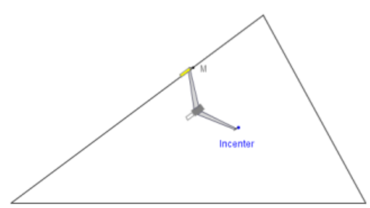
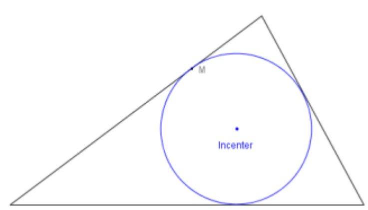
| | |
|---|---|
| **4.** Without changing the width, draw an arc across the circle in the two possible places. These are the contact points J, K for the tangents. |   |
| **5.** Draw the two tangent lines from P through J and K. |   |
| **6.** Done. The two lines just drawn are tangential to the given circle and pass through P. |   |

2. Using a compass and a straight edge, **construct** the inscribed circle for the given triangle.

**Incircle of a triangle**
Printable instructions worksheet.

Back to main page Print

| After doing this | Your work should look like this |
|---|---|
| We start with the given triangle. |  |
| The steps 1-6 establish the incenter and are identical to those in Constructing the Incenter of a Triangle | |
| **1.** Place the compass point on any of the triangle's vertices. Adjust the compass to a medium width setting. The exact width is not important. |  |
| **2.** Without changing the compass width, strike an arc across each adjacent side. |  |
| **3.** Change the compass width if desired, then from the point where each arc crosses the side, draw two arcs inside the triangle so that they cross each other, using the same compass width for each. |  |

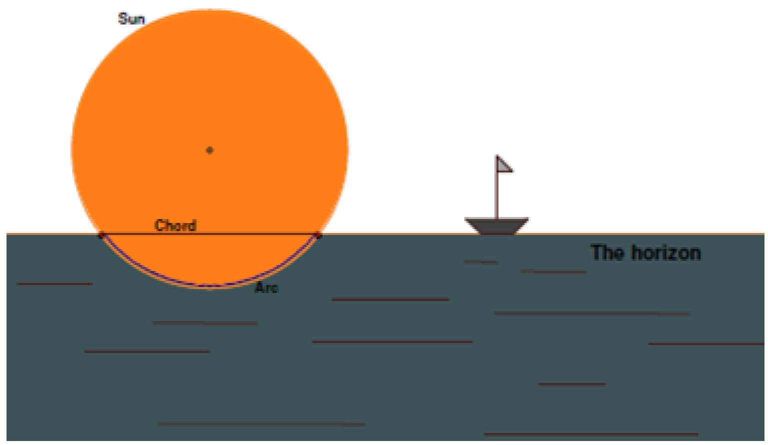
| | |
|---|---|
| **4.** Using the straightedge, draw a line from the vertex of the triangle to where the last two arcs cross. |  |
| **5.** Repeat all of the above at any other vertex of the triangle. You will now have two new lines drawn. |  |
| **6.** Where the two new lines intersect, mark a point as the incenter of the triangle. |  |
| **Optional Step** Repeat steps 1-4 for the third vertex. This will convince you that the three angle bisectors do, in fact, always intersect at a single point. But two are enough to find that point. | |
| **7.** Draw the perpendicular from the incenter to a side of the triangle. Label the point where it meets the side M.<br><br>See Constructing a Perpendicular from a Point for this procedure. |  |
| **8.** Place the compass on the incenter and set the width to point M. This is the radius of the incircle, sometimes called the inradius of the triangle. |  |
| **9.** Draw a full circle. |  |

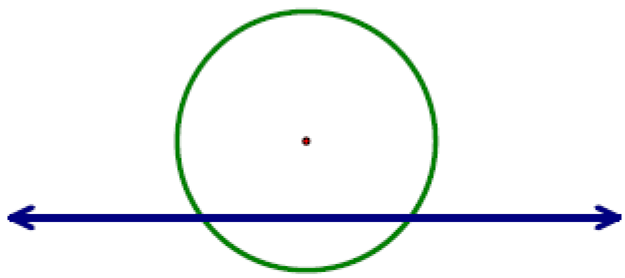
# Chords, Secants, and Tangents

## Part 1: Sunrise on the First Day of the New Year

It is customary for people in Asia to visit the seashores on the eastern sides of their countries on the first day of the year. While watching the sun rise over the ocean, visitors wish for good luck in the New Year.

As the sun rises, the horizon cuts the sun at different positions. Although a circle is not a perfect representation of the sun, we can simplify this scene by using a circle to represent the sun and a line to represent the horizon.



1. Using the simplified diagram above, sketch and describe the different types of intersections the sun and the horizon may have.

2. Definitions:
A **tangent line** is a line that intersects a circle in exactly one point.
A **secant line** intersects a circle in two points.

Do any of your sketches contain tangent or secant lines?
If so, label them.

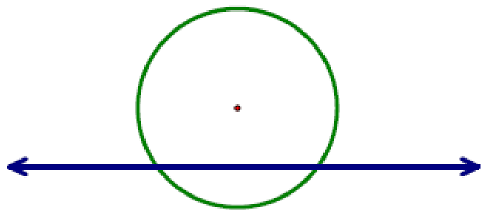
Is it possible for a line to intersect a circle in 3 points? 4 points?
Explain why or why not.

3. When a secant line intersects a circle in two points, it creates a chord. As you have already learned, a **chord** is a segment whose endpoints lie on the circle. How does a chord differ from a secant line?

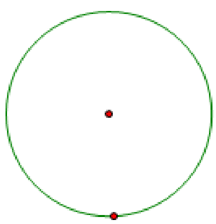
4. Look again at our representation of the sun and the horizon.

Let *d* represent the distance between the center of a circle and a line *l*. Let *r* represent the length of a radius of the circle.

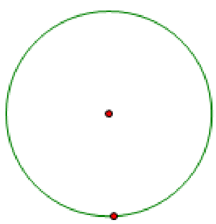
a. Draw and describe the relationship between *d* and *r* when *l* is a secant line



b. Draw and describe the relationship between *d* and *r* when *l* is a tangent line



c. Draw and describe the relationship between *d* and *r* when *l* does not intersect the circle



5. You just compared the length of a radius of a circle to the distance from the center of the circle to a tangent line. What does this comparison tell you about the relationship of a tangent line to a radius at the point of tangency?

Explain your thinking. Relate this to the problem you solved in the task *Investigating Angle Relationships in Circles* (Part 4, question #3).

**Part 2: The Segment Theorems Graphic Organizer**
In the remaining items of this task, we will work with the relationships between the lengths of the segments created when these lines intersect.

| Picture | Type | Theorem |
|---|---|---|
| | **2 tangents**<br><br>**vertex outside** | |
| | **2 secants**<br><br>**vertex outside** | |
| | **Secant and tangent**<br><br>**vertex outside** | |
| | **2 secants**<br><br>**VERTEX INSIDE** | |

**Part 3: Apply the Theorems to Solve Problems**
Sketch a picture for each problem, choose a theorem, set up an equation, and then solve.

1. Chords $\overline{AB}$ and $\overline{CD}$ intersect inside a circle at point E. $AE = 2$, $CE = 4$, and $ED = 3$. Find $EB$.

2. A diameter of a circle is perpendicular to a chord whose length is 12 inches. If the length of the shorter segment of the diameter is 4 inches, what is the length of the longer segment of the diameter?

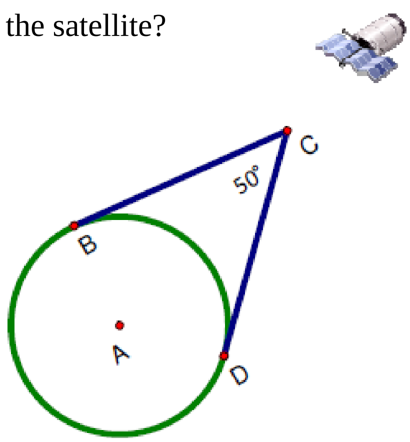
3. Chords $\overline{AB}$ and $\overline{CD}$ intersect inside a circle at point E. $AE = 5$, CE $= 10$, $EB = x$, and $ED = x-4$. Find $EB$ and $ED$.

4. Two secant segments are drawn to a circle from a point outside the circle. The external segment of the first secant segment is 8 centimeters and its internal segment is 6 centimeters. If the entire length of the second secant segment is 28 centimeters, what is the length of its external segment?

5. A tangent segment and a secant segment are drawn to a circle from a point outside the circle. The length of the tangent segment is 15 inches. The external segment of the secant segment measures 5 inches. What is the measure of the internal secant segment?

6. The diameter of a circle is 19 inches. If the diameter is extended 5 inches beyond the circle to point *C*, how long is the tangent segment from point *C* to the circle?

7. A satellite orbits the earth so that it remains at the same point above the Earth's surface as the Earth turns. If the satellite has a 50° view of the equator, what percent of the equator can be seen from the satellite?

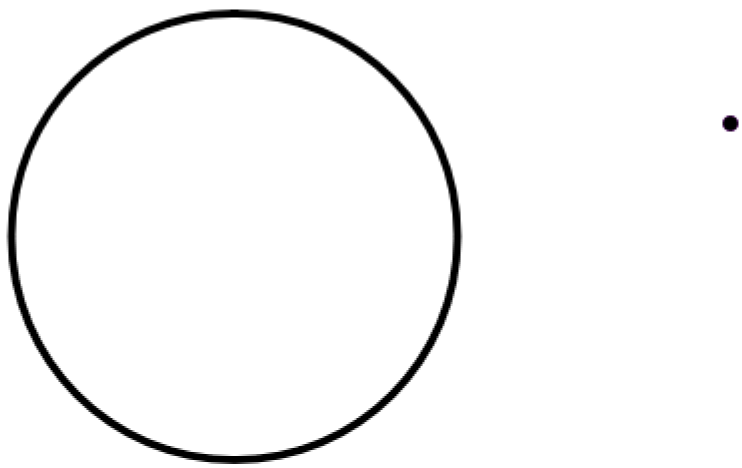



8. The average radius of the Earth is approximately 3959 miles.
a. How far above the Earth's surface is the satellite described in *Problem 7?*

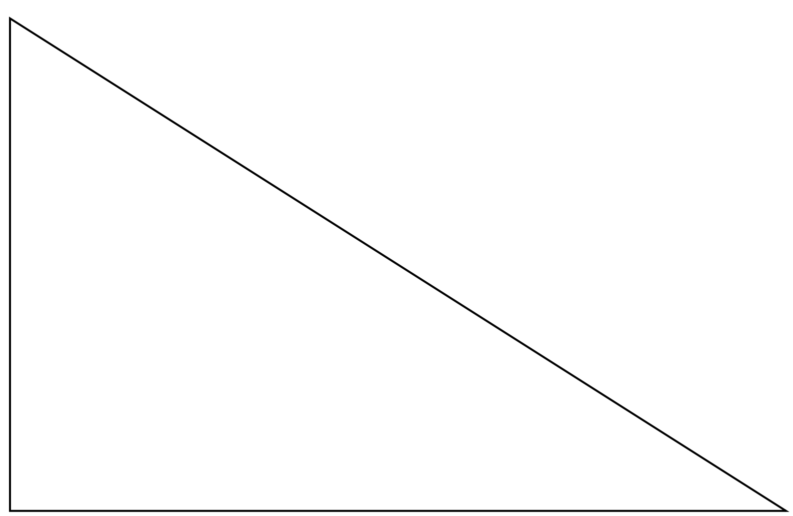
b. What is the length of the longest line of sight from the satellite to the Earth's surface? Identify this line of sight using the diagram.

**Part 4: Constructions in Euclidean Geometry**

1. Using a Compass and Straight Edge, Construct a Tangent Line to a circle from a given exterior point.



2. Using a compass and a straight edge, **Construct** the Inscribed Circle for the given triangle.



## Formative Assessment Lesson: Inscribing and Circumscribing Right Triangles

[**Back to Task Table**]

*Source: Formative Assessment Lesson Materials from Mathematics Assessment Project*
http://map.mathshell.org/materials/download.php?fileid=1194

### ESSENTIAL QUESTIONS:

- How do you decompose complex shapes into simpler ones in order to solve a problem?
- How do you find the relationship between radii of inscribed and circumscribed circles of right triangles?

### TASK COMMENTS:

Tasks and lessons from the Mathematics Assessment Project are specifically designed to help teachers effectively formatively assess their students. The way the tasks and lessons are designed gives the teacher a clear understanding of what the students are able to do and not do. Within the lesson, teachers will find suggestions and question prompts that will help guide students towards understanding. For more information access the MAP website:
http://www.map.mathshell.org/materials/background.php?subpage=formative

The task, *Inscribing and Circumscribing Right Triangles*, is a Formative Assessment Lesson (FAL) that can be found at the website:
http://map.mathshell.org/materials/lessons.php?taskid=403&subpage=problem

The FAL document provides a clear lesson design, from the opening of the lesson to the closing of the lesson.

The PDF version of the task can be found at the link below:
http://map.mathshell.org/materials/download.php?fileid=1194

### STANDARDS ADDRESSED IN THIS TASK:

#### Understand and apply theorems about circles

**MGSE9-12.G.C.1** Understand that all circles are similar.

**MGSE9-12.G.C.2** Identify and describe relationships among inscribed angles, radii, chords, tangents, and secants. Include the relationship between central, inscribed, and circumscribed angles; inscribed angles on a diameter are right angles; the radius of a circle is perpendicular to the tangent where the radius intersects the circle.

**MGSE9-12.G.C.3** Construct the inscribed and circumscribed circles of a triangle, and prove properties of angles for a quadrilateral inscribed in a circle.

**MGSE9-12.G.C.4** Construct a tangent line from a point outside a given circle to the circle.

**Standards for Mathematical Practice**
This lesson uses all of the practices with emphasis on:

**1. Make sense of problems and persevere in solving them** by requiring students to interpret and make meaning of a problem and find a logical starting point, and to monitor their progress and change their approach to solving the problem, if necessary.

**2. Reason abstractly and quantitatively** by requiring students to make sense of quantities and their relationships to one another in problem situations.

**3. Construct viable arguments and critique the reasoning of others** by engaging students on discussion of why they agree or disagree with responses, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

**7. Look for and make use of structure** by expecting students to apply rules, look for patterns and analyze structure.

# Formative Assessment Lesson: Geometry Problems: Circles & Triangles

**Back to Task Table**

*Source: Formative Assessment Lesson Materials from Mathematics Assessment Project*
http://map.mathshell.org/materials/download.php?fileid=696

## ESSENTIAL QUESTIONS:

- How do you solve problems by determining the lengths of the sides in right triangles?
- How do you find the measurements of shapes by decomposing complex shapes into simpler ones?

## TASK COMMENTS:

Tasks and lessons from the Mathematics Assessment Project are specifically designed to help teachers effectively formatively assess their students. The way the tasks and lessons are designed gives the teacher a clear understanding of what the students are able to do and not do. Within the lesson, teachers will find suggestions and question prompts that will help guide students towards understanding. For more information access the MAP website:
http://www.map.mathshell.org/materials/background.php?subpage=formative

The task, *Geometry Problems: Circles & Triangles*, is a Formative Assessment Lesson (FAL) that can be found at the website:
http://map.mathshell.org/materials/lessons.php?taskid=222&subpage=problem

The FAL document provides a clear lesson design, from the opening of the lesson to the closing of the lesson.

The PDF version of the task can be found at the link below:
http://map.mathshell.org/materials/download.php?fileid=696

## STANDARDS ADDRESSED IN THIS TASK:

### Understand and apply theorems about circles

**MGSE9-12.G.C.1** Understand that all circles are similar.

**MGSE9-12.G.C.2** Identify and describe relationships among inscribed angles, radii, chords, tangents, and secants. Include the relationship between central, inscribed, and circumscribed angles; inscribed angles on a diameter are right angles; the radius of a circle is perpendicular to the tangent where the radius intersects the circle.

**MGSE9-12.G.C.3** Construct the inscribed and circumscribed circles of a triangle, and prove properties of angles for a quadrilateral inscribed in a circle.

**MGSE9-12.G.C.4** Construct a tangent line from a point outside a given circle to the circle.

**Standards for Mathematical Practice**
This lesson uses all of the practices with emphasis on:
**1. Make sense of problems and persevere in solving them** by requiring students to interpret and make meaning of a problem and find a logical starting point, and to monitor their progress and change their approach to solving the problem, if necessary.

**2. Reason abstractly and quantitatively** by requiring students to make sense of quantities and their relationships to one another in problem situations.

**3. Construct viable arguments and critique the reasoning of others** by engaging students on discussion of why they agree or disagree with responses, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

**6. Attend to precision** by requiring students to calculate efficiently and accurately; and to communicate precisely with others by using clear mathematical language to discuss their reasoning.

**7. Look for and make use of structure** by expecting students to apply rules, look for patterns and analyze structure.

## Circles in Triangles (Short Cycle Task)

**Back to Task Table**

*Source: Balanced Assessment Materials from Mathematics Assessment Project*
http://www.map.mathshell.org/materials/download.php?fileid=764

**ESSENTIAL QUESTIONS:**

- How do you prove geometric theorems?
- How do you understand and apply theorems about circles?

**TASK COMMENTS:**

Tasks and lessons from the Mathematics Assessment Project are specifically designed to help teachers effectively formatively assess their students. The way the tasks and lessons are designed gives the teacher a clear understanding of what the students are able to do and not do. Within the lesson, teachers will find suggestions and question prompts that will help guide students towards understanding. For more information access the MAP website:
http://www.map.mathshell.org/materials/background.php?subpage=summative

The task, *Circles in Triangles*, is a Mathematics Assessment Project Assessment Task that can be found at the website:
http://www.map.mathshell.org/materials/tasks.php?taskid=256&subpage=apprentice

The PDF version of the task can be found at the link below:
http://www.map.mathshell.org/materials/download.php?fileid=764

The scoring rubric can be found at the following link:
http://www.map.mathshell.org/materials/download.php?fileid=765

**STANDARDS ADDRESSED IN THIS TASK:**

**Understand and apply theorems about circles**
**MGSE9-12.G.C.1** Understand that all circles are similar.

**MGSE9-12.G.C.2** Identify and describe relationships among inscribed angles, radii, chords, tangents, and secants. Include the relationship between central, inscribed, and circumscribed angles; inscribed angles on a diameter are right angles; the radius of a circle is perpendicular to the tangent where the radius intersects the circle.

**MGSE9-12.G.C.3** Construct the inscribed and circumscribed circles of a triangle, and prove properties of angles for a quadrilateral inscribed in a circle.

**MGSE9-12.G.C.4** Construct a tangent line from a point outside a given circle to the circle.

**Standards for Mathematical Practice**

This task uses all of the practices with emphasis on:

**3. Construct viable arguments and critique the reasoning of others** by engaging students on discussion of why they agree or disagree with responses, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

**7. Look for and make use of structure** by expecting students to apply rules, look for patterns and analyze structure.

## GRAIN STORAGE (Career and Technical Education Task)

**Back to Task Table**

*Source: National Association of State Directors of Career Technical Education Consortium*
PDF: http://www.achieve.org/files/CCSS-CTE-Grain-Storage-Capacity-FINAL.pdf
Word: http://www.achieve.org/files/CCSS-CTE-Grain-Storage-Capacity-FINAL.docx

**Introduction**
Students are to determine the storage capacity of a wheat storage facility and design a new one that will hold more.

**Standard Addressed in this Task**
**MGSE9-12.G.C.2** Identify and describe relationships among inscribed angles, radii, chords, tangents, and secants. Include the relationship between central, inscribed, and circumscribed angles; inscribed angles on a diameter are right angles; the radius of a circle is perpendicular to the tangent where the radius intersects the circle.

**Standards for Mathematical Practice**
**1. Make sense of problems and persevere in solving them** by requiring students to make sense of the problem and determine an approach.

**2. Reason abstractly and quantitatively** by requiring students to reason about quantities and what they mean within the context of the problem.

**4. Model with mathematics** by asking students to use mathematics to model a physical situation.

**6. Attend to precision** by expecting students to attend to units as they perform calculations. Rounding and estimation are a key part.

**7. Look for and make use of structure** by requiring students to use the structure of expressions.

**Common Student Misconceptions**
1. Students may confuse the segment and angle theorems.

## Arc Length and Area of a Sector

[Back to Task Table]

**Standards Addressed in the Task**

**MGSE9-12.G.C.5** Derive using similarity the fact that the length of the arc intercepted by an angle is proportional to the radius, and define the radian measure of the angle as the constant of proportionality; derive the formula for the area of a sector.

**MGSE9-12.G.GMD.1** Give informal arguments for geometric formulas.

a. Give informal arguments for the formulas of the circumference of a circle and area of a circle using dissection arguments and informal limit arguments.
b. Give informal arguments for the formula of the volume of a cylinder, pyramid, and cone using Cavalieri's principle.

**Standards for Mathematical Practice**

**2. Reason abstractly and quantitatively** by requiring students to make sense of quantities and their relationships to one another in problem situations.

**3. Construct viable arguments and critique the reasoning of others** by engaging students on discussion of why they agree or disagree with responses, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

**4. Model with mathematics** by expecting students to apply the mathematics concepts they know in order to solve problems arising in everyday situations, and reflect on whether the results are sensible for the given scenario.

**5. Use appropriate tools strategically** by expecting students to consider available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a compass, a calculator, software, etc.

**6. Attend to precision** by requiring students to calculate efficiently and accurately; and to communicate precisely with others by using clear mathematical language to discuss their reasoning.

**7. Look for and make use of structure** by expecting students to apply rules, look for patterns and analyze structure.

**Common Student Misconceptions**

1. Sectors and segments are often used interchangeably in everyday conversation. Care should be taken to distinguish these two geometric concepts.
2. The formulas for converting radians to degrees and vice versa are easily confused. Knowing that the degree measure of given angle is always a number larger than the radian measure can help students use the correct unit.

3. An informal survey of students from elementary school through college showed the number pi to be the mathematical idea about which more students were curious than any other. There are at

least three facets to this curiosity: the symbol π itself, the number 3.14159..., and the formula for the area of a circle. All of these facets can be addressed here, at least briefly.
4. Many students want to think of infinity as a number. Avoid this by talking about a quantity that becomes larger and larger with no upper bound.
5. The inclusion of the coefficient 1/3 in the formulas for the volume of a pyramid or cone and 4/3 in the formula for the volume of a sphere remains a mystery for many students. In high school, students should attain a conceptual understanding of where these coefficients come from.
6. Concrete demonstrations, such as pouring water from one shape into another should be followed by more formal reasoning.

Define Arc Length.

***SOLUTION***
***Arc length is the distance making up the curved line around an arc, or portion, of a circle.***

Define Sector of a Circle.

***SOLUTION***
***Sector of a circle is the area of a portion of a circle formed by two radii and the arc intercepted by the central angle created from the two radii. "a pizza or pie slice"***

**Part 1: Hands on Activity**

***COMMENTS***
***This lab works well as an introduction or as a review activity. It works well with partners with each student having his own cookie and the partnership only using 1 cookie and setting aside the spare cookie in case of a mis-cut, so that the back-up cookie can be used to start over again, if necessary.***

***Sugar Cookies work the best since chocolate chips can inhibit a straight cut. Cookies can be bought from the bakery of your local grocery store. If you fear they will be crumbly, stale, or hard to cut place a slice or two of cheap loaf bread in the container with them over night and the cookies will pull the moisture out of the bread, creating a soft and cut-able cookie.***

**COOKIE LAB**

Materials: Large Soft Cookie, String, Protractor, Ruler, Knife, Paper Towel

1. Find the circumference of the cookie in cm using the string and the ruler.

   Circumference = _______cm

2. Find the measure of the diameter in cm.

   Diameter = ________cm

3. What is the ratio of the Circumference to the Diameter?

$$\frac{C}{d} = \_\_\_\_\_\_$$

4. The formula for Area of a circle is $\pi r^2$; where r = radius of circle

5. Find the Area of the cookie. ____________$cm^2$

*Cut the cookie in half on the diameter. Then cut each half of the cookie into two **unequal** sectors. You will have 4 different pieces of cookie.*

6. Using the protractor, find the Angle Measure of each sector's central angle.

Angle 1 = _______° Angle 2 = _______°

Angle 3 = _______° Angle 4 = _______°

*You may now eat your cookie!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!*

7. The formula for the **length of an arc** in a circle is $Arc\ Length = \frac{2\pi r\theta}{360}$

where r = radius of circle and $\theta$= central angle

8. Using the Arc Length formula, find the measure of each sector's arc length.

Arc Length 1 = ________cm

Arc Length 2 = ________cm

Arc Length 3 = ________cm

Arc Length 4 = ________cm

9. What is the total length of the 4 arcs? _________cm

10. How does it compare to the circumference of the cookie?

11. The formula for the **Area of a Sector** is $Area\ of\ Sector = \frac{\pi r^2 \theta}{360}$
where r = radius of circle and $\theta$ = central angle

12. Find the Area of each sector.

Area of sector 1 = _________cm²

Area of sector 2 = _________ cm²

Area of sector 3 = _________ cm²

Area of sector 4 = _________ cm²

13. What is the total area of the four sectors? _________ cm²

14. How does it compare to the area of the original cookie?

15. Explain why the 4 arc lengths should add to the circumference of your circle. If they did not add to the circumference of your cookie, explain why they did not.

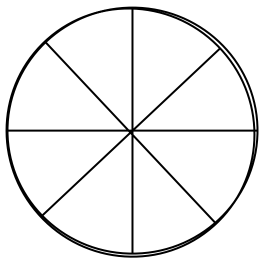
16. Explain why the 4 sector areas should add to the area of your cookie. If they did not sum to equal the area, explain why.

**Part 2: Understanding the Formulas**

## Investigating the Area of a Circle

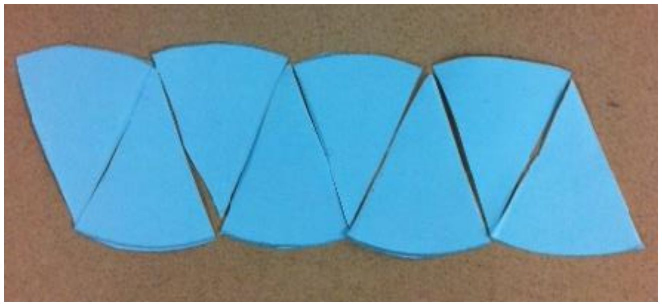
1. Cut out a Circle and Fold it then Cut it into at least 8 Congruent Sectors (or pizza slices).

***SOLUTION***



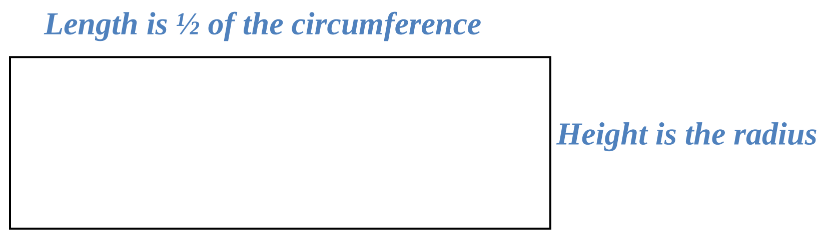
2. Lay the slices next to each other to create a rectangle like shape.

***SOLUTION***
***Teachers should encourage a few groups to cut their sectors in half again creating 16 sectors to show that the parallelogram becomes more rectangular as the number of rectangles increases.***



3. Think about the dimensions of your "rectangle" in terms of the original circle's Circumference and radius. Sketch and label it here.

***SOLUTION***




4. Since the Area of a Rectangle is found by multiplying the length of its base by its height, Find the Area of your "rectangle" by doing this calculation as well.

My Rectangle created from a circle has an approximate area of:

***SOLUTION***

$$\begin{aligned} Area &= base \bullet height \\ &= \frac{1}{2}C \bullet r \\ &= \frac{1}{2}(2\pi r) \bullet r \\ &= \pi r^2 \end{aligned}$$

5. How does your formula compare with the formula you know to be the Area of a Circle?

***SOLUTION***
***They are the same!***

## Investigating Arc Length and the Area of a Sector

***COMMENTS***
***This website has an interactive feature to help with the culminating discovery of the Arc Length Formula***: http://www.mathopenref.com/arclength.html

***SOLUTION***
***A sample solution is provided here:***

In this portion of the investigation you will look at the relationship between a central angle and its intercepted arc.

1. Decide how long you would like your radius to be. r = ***4cm*** (don't forget units!)

2. Using your compass, draw a circle with the above radius on a separate piece of paper. What is the circumference of your circle? Don't forget units! ***C=25.133cm***

3. Divide your circle, using a protractor, into four equal parts (hint: use two diameters). What is the measure of each central angle in the circle you constructed?

   ***Central Angle = 90º***

4. Write a fraction that compares the measure of the central angle to the total number of degrees in a circle. Then simplify this fraction.

   ***90/360 = .25***

5. Keeping in mind your answer above, what would be the length of the arc formed by one of the central angles?

   ***Arc Length should be ¼ of Circumferemce, so 6.28cm***

6. Write a fraction that compares the arc length computed above to the total circumference of the circle. Then simplify this fraction.

   ***6.28/25.133 = .25***

7. Make a conjecture about the proportion of the measure of a central angle and the proportion of its intercepted arc? (Hint: What do you notice about the fraction you found in question 4 and the fraction you found in question 6?) **Come up with a general formula for this conjecture.**

***SOLUTION***

$$\frac{90}{360} = \frac{6.28}{25.133}$$

$$\frac{\textit{Central Angle}^{\circ}}{360^{\circ}} = \frac{\textit{Arc Length}}{\textit{Circumference}}$$

8. Compare your findings and formula with your peers around you. Does your formula work with different sized circles?

   **SOLUTION**

***Yes. If students do not have the same formula, check to see if the reciprocal property of proportions or the Exchange Property of Proportions can be used to rearrange their proportion to look the same as the others. If not, make necessary corrections.***

Write your discovery as:

**Arc Length Proportion**

***SOLUTION***

$$\frac{Central\ Angle^{o}}{360^{o}} = \frac{Arc\ Length}{Circumference}$$

$$\frac{CA}{360} = \frac{Arc\ Length}{2\pi r}$$

**Arc Length Formula**

***SOLUTION***

$$2\pi r \bullet \frac{CA}{360} = Arc\ Length$$

In the next portion of this investigation we will look at the relationship between the central angle and the sector area it creates.

9. What is the area of your circle? Don't forget units! ***A=50.265 cm²*** 

10. Using your divided circle from the beginning of the investigation, what is the area of one of the sectors? ***Sector is ¼ of the circle so it has 12.566cm²***

11. Write a fraction that compares the sector area computed above to the total area of the circle. Then simplify this answer. ***12.566/50.265 = .25***

12. What do you notice about the simplified fraction of the sector area and the fraction you found in problem 4 of the central angle? ***Same ratio***

13. Make a conjecture about the proportion of the measure of a central angle and the proportion of its sector area? **Come up with a general formula for this conjecture.**

***SOLUTION***

$$\frac{90}{360} = \frac{12.566}{50.265}$$

$$\frac{Central\ Angle^{o}}{360^{o}} = \frac{Area\ of\ Sector}{Area\ of\ Circle}$$

14. Compare your findings and formula with your peers around you. Does your formula work with different sized circles?

***SOLUTION***

***Yes. If students do not have the same formula, check to see if the reciprocal property of proportions or the Exchange Property of Proportions can be used to rearrange their proportion to look the same as the others. If not, make necessary corrections.***

Write your discovery as:

**Area of Sector Proportion**

***SOLUTION***

$$\frac{Central\ Angle^{o}}{360^{o}} = \frac{Area\ of\ Sector}{Area\ of\ Circle}$$

$$\frac{CA}{360} = \frac{Area\ of\ Sector}{\pi r^2}$$

**Area of Sector Formula**

***SOLUTION***

$$\pi r^2 \bullet \frac{CA}{360} = Area\ of\ Sector$$

***COMMENTS***

***Some students will be able to memorize the proportion and use it correctly in problem solving, but there are students who will want a formula to substitute into and get an answer, so it is important to note the differences if the Proportion and the Formula and allow students to use whichever method works for them.***

## Arc Length and Area of a Sector

Define Arc Length.

Define Sector of a Circle.

**Part 1: Hands on Activity -** COOKIE LAB

Materials: Large Soft Cookie, String, Protractor, Ruler, Knife, Paper Towel

1. Find the circumference of the cookie in cm using the string and the ruler.

   Circumference = _______cm

2. Find the measure of the diameter in cm.

   Diameter = ________cm

3. What is the ratio of the Circumference to the Diameter?

   $\frac{C}{d} =$ ______

4. The formula for Area of a circle is $\pi r^2$; where r = radius of circle

5. Find the Area of the cookie. ____________$\text{cm}^2$

*Cut the cookie in half on the diameter. Then cut each half of the cookie into two **unequal** sectors. You will have 4 different pieces of cookie.*

6. Using the protractor, find the Angle Measure of each sector's central angle.

   Angle 1 = _______° Angle 2 = _______°

   Angle 3 = _______° Angle 4 = _______°

*You may now eat your cookie!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!*

7. The formula for the **length of an arc** in a circle is $Arc\ \ Length = \dfrac{2\pi r\theta}{360}$

where r = radius of circle and $\theta$= central angle

8. Using the Arc Length formula, find the measure of each sector's arc length.

Arc Length 1 = ________cm

Arc Length 2 = ________cm

Arc Length 3 = ________cm

Arc Length 4 = ________cm

9. What is the total length of the 4 arcs? _________cm

10. How does it compare to the circumference of the cookie?

11. The formula for the **Area of a Sector** is $Area\ \ of\ \ Sector = \dfrac{\pi r^2\theta}{360}$

where r = radius of circle and $\theta$ = central angle

12. Find the Area of each sector.

Area of sector 1 = _________$cm^2$

Area of sector 2 = _________ $cm^2$

Area of sector 3 = _________ $cm^2$

Area of sector 4 = _________ $cm^2$

13. What is the total area of the four sectors? _________ $cm^2$

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**Part 2: Understanding the Formulas**

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4. Since the Area of a Rectangle is found by multiplying the length of its base by its height, Find the Area of your "rectangle" by doing this calculation as well.

My Rectangle created from a circle has an approximate area of:

5. How does your formula compare with the formula you know to be the Area of a Circle?

## Investigating Arc Length and the Area of a Sector

In this portion of the investigation you will look at the relationship between a central angle and its intercepted arc.

1. Decide how long you would like your radius to be. r = __________ (don't forget units!)

2. Using your compass, draw a circle with the above radius on a separate piece of paper. What is the circumference of your circle? Don't forget units!

3. Divide your circle, using a protractor, into four equal parts (hint: use two diameters). What is the measure of each central angle in the circle you constructed?

4. Write a fraction that compares the measure of the central angle to the total number of degrees in a circle. Then simplify this fraction.

5. Keeping in mind your answer above, what would be the length of the arc formed by one of the central angles?

6. Write a fraction that compares the arc length computed above to the total circumference of the circle. Then simplify this fraction.

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8. Compare your findings and formula with your peers around you. Does your formula work with different sized circles?

Write your discovery as:

**Arc Length Proportion** **Arc Length Formula**

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9. What is the area of your circle? Don't forget units!

10. Using your divided circle from the beginning of the investigation, what is the area of one of the sectors?

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14. Compare your findings and formula with your peers around you. Does your formula work with different sized circles?

Write your discovery as:

**Area of Sector Proportion** **Area of Sector Formula**

## Formative Assessment Lesson: Sectors of Circles

**Back to Task Table**

*Source: Formative Assessment Lesson Materials from Mathematics Assessment Project*
http://map.mathshell.org/materials/download.php?fileid=1284

**ESSENTIAL QUESTIONS**:

- How do you compute perimeters, areas, and arc lengths of sectors using formulas?
- How do you find the relationships between arc lengths, and areas of sectors after scaling?

**TASK COMMENTS:**

Tasks and lessons from the Mathematics Assessment Project are specifically designed to help teachers effectively formatively assess their students. The way the tasks and lessons are designed gives the teacher a clear understanding of what the students are able to do and not do. Within the lesson, teachers will find suggestions and question prompts that will help guide students towards understanding. For more information access the MAP website:
http://www.map.mathshell.org/materials/background.php?subpage=formative

The task, *Sectors of Circles*, is a Formative Assessment Lesson (FAL) that can be found at the website: http://map.mathshell.org/materials/lessons.php?taskid=441&subpage=concept

The FAL document provides a clear lesson design, from the opening of the lesson to the closing of the lesson.

The PDF version of the task can be found at the link below:
http://map.mathshell.org/materials/download.php?fileid=1284

**STANDARDS ADDRESSED IN THIS TASK:**

**Find arc lengths and areas of sectors of circles**
**MGSE9-12.G.C.5** Derive using similarity the fact that the length of the arc intercepted by an angle is proportional to the radius, and define the radian measure of the angle as the constant of proportionality; derive the formula for the area of a sector.

**Standards for Mathematical Practice**
This lesson uses all of the practices with emphasis on:

**1. Make sense of problems and persevere in solving them** by requiring students to interpret and make meaning of a problem and find a logical starting point, and to monitor their progress and change their approach to solving the problem, if necessary.

**7. Look for and make use of structure** by expecting students to apply rules, look for patterns and analyze structure.

## Volumes of Cylinders, Cones, Pyramids, and Spheres

Back to Task Table

### Standards Addressed in this Task

**MGSE9-12.G.GMD.1** Give informal arguments for geometric formulas.

a. Give informal arguments for the formulas of the circumference of a circle and area of a circle using dissection arguments and informal limit arguments.
b. Give informal arguments for the formula of the volume of a cylinder, pyramid, and cone using Cavalieri's principle.

**MGSE9-12.G.GMD.2** Give an informal argument using Cavalieri's principle for the formulas for the volume of a sphere and other solid figures.

**MGSE9-12.G.GMD.3** Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems.

### Standards for Mathematical Practice

**1. Make sense of problems and persevere in solving them** by requiring students to interpret and make meaning of a problem and find a logical starting point, and to monitor their progress and change their approach to solving the problem, if necessary.

**2. Reason abstractly and quantitatively** by requiring students to make sense of quantities and their relationships to one another in problem situations.

**3. Construct viable arguments and critique the reasoning of others** by engaging students on discussion of why they agree or disagree with responses, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

**4. Model with mathematics** by expecting students to apply the mathematics concepts they know in order to solve problems arising in everyday situations, and reflect on whether the results are sensible for the given scenario.

**5. Use appropriate tools strategically** by expecting students to consider available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a compass, a calculator, software, etc.

### Common Student Misconceptions

1. An informal survey of students from elementary school through college showed the number pi to be the mathematical idea about which more students were curious than any other. There are at least three facets to this curiosity: the symbol $\pi$ itself, the number 3.14159..., and the formula for the area of a circle. All of these facets can be addressed here, at least briefly.
2. Many students want to think of infinity as a number. Avoid this by talking about a quantity that becomes larger and larger with no upper bound.
3. The inclusion of the coefficient 1/3 in the formulas for the volume of a pyramid or cone and 4/3 in the formula for the volume of a sphere remains a mystery for many students. In high school, students should attain a conceptual understanding of where these coefficients come from.

4. Concrete demonstrations, such as pouring water from one shape into another should be followed by more formal reasoning.

Bonaventura Francesco Cavalieri (1598 – November 30, 1647) was an Italian mathematician. We will use his works as the basis for this task. Cavalieri's Principle states:

> The volumes of two solids are equal if the areas of corresponding sections drawn parallel to some given plane are equal.

This can best be understood by looking at a stack of pennies. The volume of two stacks of the same number of pennies is the same, even if one stack is not vertically aligned.

***SOLUTION***

Let's apply this to derive the formula for the volume of a right cylinder. If you cut a super thin slice, or cross section, out of a cylinder what shape would result? ***a circle***

What is the Area Formula for that shape? $A=$ ***Area of a Circle Formula:*** $A=\pi r^2$

How many slices would you need to stack up to create your cylinder?

***Sample Responses might include: As many as it takes to get as high as the cylinder. It would depend upon the thickness of your slice. As your slice gets thinner the number of slices increases. The number needs to represent the height of the cylinder.***

Complete the Volume of a Cylinder Formula:

***SOLUTION***

$V = Area\ of\ the\ Base \bullet height$

$V = B \bullet h$

$V = \pi r^2 h$

Cavalieri principle. *Encyclopedia of Mathematics.* URL:http://www.encyclopediaofmath.org/index.php?title=Cavalieri_principle&oldid=17072

**Part 1: Understanding the Formulas**

***COMMENTS***

***If you have a set of solids this could be demonstrated using water and actually pouring the water from the cone into the cylinder. If you have multiple sets of solids, you could allow student groups to investigate the formulas at their own pace, using water, salt, rice, etc.***

***These demonstrations and derivations of the following 3 formulas are in direct keeping with the spirit of the standards to "give an informal argument".***

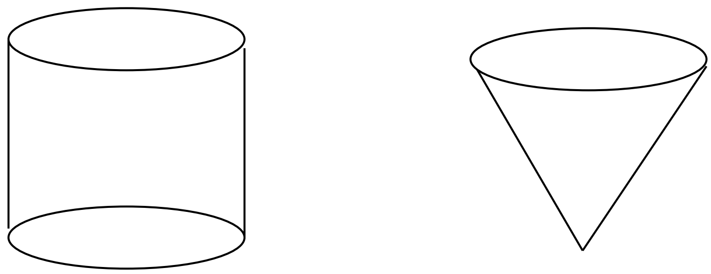
Let's investigate the relationship between a Cone and its corresponding Cylinder with the same height and radius. LABEL the height, h and radius, r on each diagram below.

1. If the cylinder was full of water and you poured it into the cone, how many times would it fill up the cone completely?

   ***SOLUTION***
   ***It would fill it exactly 3 times.***

   If the cone was full of water, how much of the cylinder would it fill up?

   ***It would fill up exactly 1/3 of the cylinder.***



2. Complete the Formulas below:
   **Volume of a Cylinder Formula vs. Volume of a Cone Formula**

***SOLUTION***

$V = Area\ of\ the\ Base \bullet height$

$V = B \bullet h$

$V = \pi r^2 h$

***SOLUTION***

$V = \frac{1}{3} volume\ of\ cylinder$

$= \frac{1}{3}\pi r^2 h$

***It is important to remind students that the height of a cone is not the slant height.***

3. Describe the relationship between the formulas and the amount the cone filled the cylinder.

***The Formula is really what happened!***

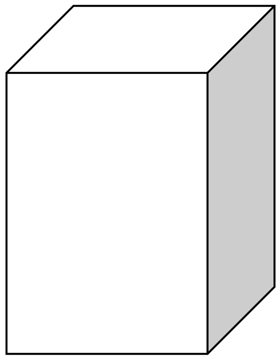
Let's investigate the relationship between a Pyramid and its corresponding Rectangular Prism with the same height, length, and width. LABEL the height, h, length, l, and width, w on each diagram below.

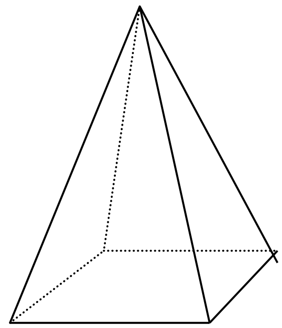
4. If the rectangular prism was full of water and you poured it into the pyramid, how many times would it fill up the pyramid completely?

***Again, exactly 3 times!***

If the pyramid was full of water, how much of the rectangular prism would it fill up?

***Again, it will fill up exactly 1/3.***





5. Complete the Formulas below:

**Right Rectangular Prism Volume Formula** **vs.** **Right Pyramid Volume Formula**

***SOLUTION***

$V = Area\ of\ the\ Base \bullet height$

$V = B \bullet h$

$V = lwh$

***SOLUTION***

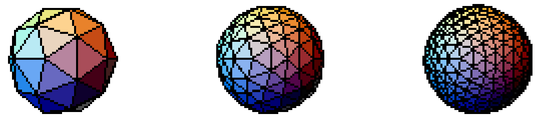
$V = \frac{1}{3} volume\ of\ prism$

$= \frac{1}{3} lwh$

***Remind students again that the height of a pyramid, like a cone, must be perpendicular to the base. The Slant Height is not the height.***

6. Describe the relationship between the formulas and the amount the pyramid filled the prism.

***The Formula uses the 1/3 just like the cone formula and just like what really happens!***



An image from
http://mathworld.wolfram.com/GeodesicDome.html

Let's investigate the Volume of a Sphere Formula.

What is the formula for the Surface Area of a Sphere?

***SOLUTION***

$SA = 4\pi r^2$

If that surface is divided up into triangles that are actually the bases of triangular pyramids that fill up the entire space inside of the sphere, then the Volume of the Sphere would equal the sum of the Volumes of all those pyramids.

Let's see if we can use this concept to derive the formula for the volume of a sphere:

## SOLUTION

$$V_{pyramid} = \frac{1}{3} \bullet B \bullet h$$

$$= \frac{1}{3} \bullet B \bullet r$$

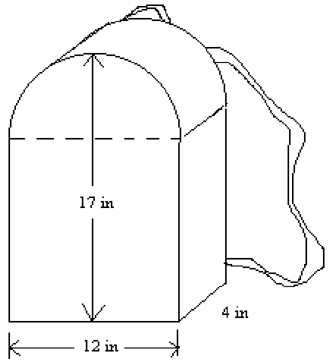
$$V_{all\ the\ pyramids} = \frac{1}{3} \bullet SA \bullet r$$

$$= \frac{1}{3} \bullet 4\pi r^2 \bullet r$$

$$V_{sphere} = \frac{4}{3}\pi r^3$$

## Part 2: Applications

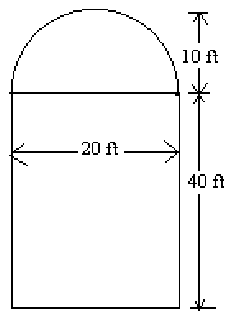
1. Approximate the Volume of the Backpack that is 17in x 12in x 4in.




***SOLUTION***

$$V = prism + \frac{1}{2}cylinder$$
$$= lwh + \frac{1}{2}(\pi r^2 h)$$
$$= \left((17-6)\bullet 12 \bullet 4\right) + \frac{1}{2}\left(\pi 6^2\right)$$
$$= 528 + 18\pi \, in^3 \approx 584.55 in^3$$

2. Find the Volume of the Grain Silo shown below that has a diameter of 20ft and a height of 50ft.




***SOLUTION***

$$V = \frac{1}{2}sphere + cylinder$$
$$= \frac{1}{2}(\frac{4}{3}\pi r^3) + (\pi r^2 h)$$
$$= \left(\frac{2}{3} \bullet \pi \bullet 10^3\right) + \left(\pi \bullet 10^2 \bullet 40\right)$$
$$= \frac{2000}{3}\pi + 4000\pi \, ft^3$$
$$= \frac{14000}{3}\pi \, ft^3 \approx 14660.77 \, ft^3$$

3. The diameter of a baseball is about 1.4 in.
How much leather is needed to cover the baseball?
How much rubber is needed to fill it?

***SOLUTION***

**Leather to cover is SA**

$$SA = 4\pi r^2$$
$$= 4\pi \bullet .7^2$$
$$= 1.96\pi\, in^2 \approx 6.16\, in^2$$

**Rubber to fill is V**

$$V = \frac{4}{3}\pi r^3$$
$$= \frac{4}{3}\pi \bullet .7^3$$
$$= \frac{343}{750}\pi\, in^3 \approx 1.44 in^3$$

4. The volume of a cylindrical watering can is 100cm$^3$. If the radius is doubled, then how much water can the new can hold?

***SOLUTION***

$$V = \pi r^2 h$$
$$V_{new} = \pi(2r)^2 h$$
$$V_{new} = 4\pi r^2 h$$
$$V_{new} = 4(old\ volume)$$
$$V_{new} = 4 \bullet 100 = 400 cm^3$$

***It is important to note that since only the radius is doubled and not the height, you cannot just multiply by $2^2$.***

## Volumes of Cylinders, Cones, Pyramids, and Spheres

Bonaventura Francesco Cavalieri (1598 – November 30, 1647) was an Italian mathematician. We will use his works as the basis for this task. Cavalieri's Principle states:

> The volumes of two solids are equal if the areas of corresponding sections drawn parallel to some given plane are equal.

This can best be understood by looking at a stack of pennies. The volume of two stacks of the same number of pennies is the same, even if one stack is not vertically aligned.

Let's apply this to derive the formula for the volume of a right cylinder. If you cut a super thin slice, or cross section, out of a cylinder what shape would result?

What is the Area Formula for that shape? $A =$

How many slices would you need to stack up to create your cylinder?

Complete the Volume of a Cylinder Formula:

$V = Area\ of\ the\ Base \bullet height$

$V = B \bullet h$

$V =$

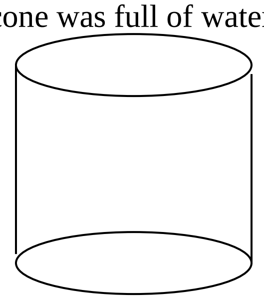
Cavalieri principle. *Encyclopedia of Mathematics.* URL: http://www.encyclopediaofmath.org/index.php?title=Cavalieri_principle&oldid=17072

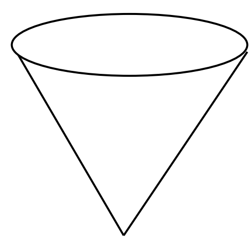
### Part 1: Understanding the Formulas

Let's investigate the relationship between a Cone and its corresponding Cylinder with the same height and radius. LABEL the height, h and radius, r on each diagram below.

1. If the cylinder was full of water and you poured it into the cone, how many times would it fill up the cone completely?

   If the cone was full of water, how much of the cylinder would it fill up?





2. Complete the Formulas below:

**Volume of a Cylinder Formula** **vs.** **Volume of a Cone Formula**

$V = Area\ of\ the\ Base \bullet height$

$V = B \bullet h$

$V =$

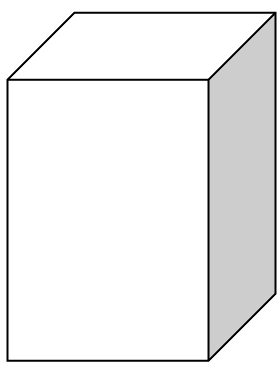
$V =$

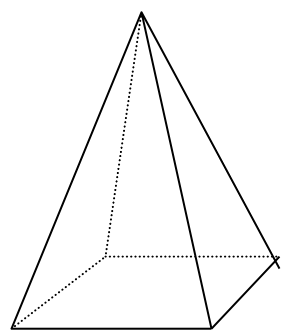
3. Describe the relationship between the formulas and the amount the cone filled the cylinder.

Let's investigate the relationship between a Pyramid and its corresponding Rectangular Prism with the same height, length, and width. LABEL the height, h, length, l, and width, w on each diagram below.

4. If the rectangular prism was full of water and you poured it into the pyramid, how many times would it fill up the pyramid completely?

   If the pyramid was full of water, how much of the rectangular prism would it fill up?





5. Complete the Formulas below:

**Right Rectangular Prism Volume Formula** **vs.** **Right Pyramid Volume Formula**

$V = Area\ of\ the\ Base \bullet height$

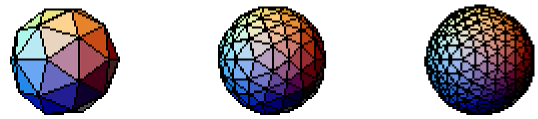
$V = B \bullet h$

$V =$

$V =$

6. Describe the relationship between the formulas and the amount the pyramid filled the prism.

Let's investigate the Volume of a Sphere Formula**.**



An image from
http://mathworld.wolfram.com/GeodesicDome.html

What is the formula for the Surface Area of a Sphere?

$SA =$

If that surface is divided up into triangles that are actually the bases of triangular pyramids that fill up the entire space inside of the sphere, then the Volume of the Sphere would equal the sum of the Volumes of all those pyramids.

Let's see if we can use this concept to derive the formula for the volume of a sphere:

$V_{pyramid} =$

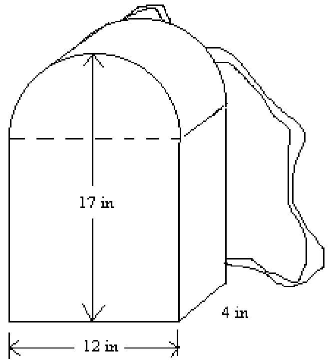
$V_{all\ the\ pyramids} =$

$=$

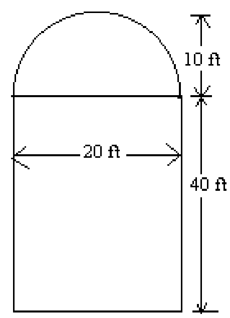
$V_{sphere} = \frac{4}{3}\pi r^3$

## Part 2: Applications

1. Approximate the Volume of the Backpack that is 17in x 12in x 4in.




2. Find the Volume of the Grain Silo shown below that has a diameter of 20ft and a height of 50ft.




3. The diameter of a baseball is about 1.4 in.
   How much leather is needed to cover the baseball?
   How much rubber is needed to fill it?

4. The volume of a cylindrical watering can is $100cm^3$. If the radius is doubled, then how much water can the new can hold?

# Math Lab

Spheres: Surface Area and Volume
Date ______________________

Name _________________________
Period ________________________

1. What is the equation for the following:
   a. Circumference: ________________________________________________
   b. Surface Area of a Circle: ________________________________________
   c. Surface Area of a Sphere: ________________________________________
   d. Volume of a Sphere: ____________________________________________

2. Item One ____________________________________
   a. What is the Circumference of your item? ______________________
   b. What is the Diameter of your item based on your circumference? ______
   c. What is the Radius of your item based on your circumference? _______
   d. What is the equation for Surface area of a Sphere? ___________________
   e. What is the Surface area of your item? _________________________
   f. What is the Volume of your item? _____________________

3. Item Two ____________________________________
   a. What is the Circumference of your item? ______________________
   b. What is the Diameter of your item based on your circumference? ______
   c. What is the Radius of your item based on your circumference? _______
   d. What is the equation for Surface area of a Sphere? ___________________
   e. What is the Surface area of your item? _________________________
   f. What is the Volume of your item? _____________________

4. Item Three ____________________________________
   a. What is the Circumference of your item? ______________________
   b. What is the Diameter of your item based on your circumference? ______
   c. What is the Radius of your item based on your circumference? _______
   d. What is the equation for Surface area of a Sphere? ___________________
   e. What is the Surface area of your item? _________________________
   f. What is the Volume of your item? _____________________

5. Item Four ____________________________________
   a. What is the Circumference of your item? ______________________
   b. What is the Diameter of your item based on your circumference? ______
   c. What is the Radius of your item based on your circumference? _______
   d. What is the equation for Surface area of a Sphere? ___________________
   e. What is the Surface area of your item? _________________________

f. What is the Volume of your item? ___________________

Conclusion Questions

6. How does the radius length relate to the surface area of a sphere? Explain and use examples from your data above.

7. If you were to triple the radius of a sphere, what would happen to the volume? Explain using examples.

8. If you were to triple the radius of a sphere, what would happen to the surface area? Explain using examples.

9. By rounding pi to 3.14 instead of using the actual pi button on your calculator, what would happen to your results? Why? Explain using examples.

## Formative Assessment Lesson: Evaluating Statements about Enlargements

**Back to Task Table**

*Source: Formative Assessment Lesson Materials from Mathematics Assessment Project*
http://map.mathshell.org/materials/download.php?fileid=678

**ESSENTIAL QUESTIONS:**

- How do you computing perimeters, areas and volumes using formulas?
- How do you find the relationships between perimeters, areas, and volumes of shapes after scaling?

**TASK COMMENTS:**

Tasks and lessons from the Mathematics Assessment Project are specifically designed to help teachers effectively formatively assess their students. The way the tasks and lessons are designed gives the teacher a clear understanding of what the students are able to do and not do. Within the lesson, teachers will find suggestions and question prompts that will help guide students towards understanding. For more information access the MAP website:
http://www.map.mathshell.org/materials/background.php?subpage=formative

The task, *Evaluating Statements about Enlargements*, is a Formative Assessment Lesson (FAL) that can be found at the website:
http://map.mathshell.org/materials/lessons.php?taskid=213&subpage=concept

The FAL document provides a clear lesson design, from the opening of the lesson to the closing of the lesson.

The PDF version of the task can be found at the link below:
http://map.mathshell.org/materials/download.php?fileid=678

**STANDARDS ADDRESSED IN THIS TASK:**

**Explain volume formulas and use them to solve problems**
**MGSE9-12.G.GMD.1** Give informal arguments for geometric formulas.

a. Give informal arguments for the formulas of the circumference of a circle and area of a circle using dissection arguments and informal limit arguments.
b. Give informal arguments for the formula of the volume of a cylinder, pyramid, and cone using Cavalieri's principle.

**MGSE9-12.G.GMD.2** Give an informal argument using Cavalieri's principle for the formulas for the volume of a sphere and other solid figures.

**MGSE9-12.G.GMD.3** Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems.

**Standards for Mathematical Practice**
This lesson uses all of the practices with emphasis on:

**3. Construct viable arguments and critique the reasoning of others** by engaging students on discussion of why they agree or disagree with responses, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

**7. Look for and make use of structure** by expecting students to apply rules, look for patterns and analyze structure.

# Formative Assessment Lesson: Calculating Volume of Compound Objects

**Back to Task Table**

*Source: Formative Assessment Lesson Materials from Mathematics Assessment Project*
http://map.mathshell.org/materials/download.php?fileid=684

## ESSENTIAL QUESTIONS:

- How do you compute measurements using formulas?
- How do you decompose compound shapes into simpler ones?
- How do you use right triangles and their properties to solve real-world problems?

## TASK COMMENTS:

Tasks and lessons from the Mathematics Assessment Project are specifically designed to help teachers effectively formatively assess their students. The way the tasks and lessons are designed gives the teacher a clear understanding of what the students are able to do and not do. Within the lesson, teachers will find suggestions and question prompts that will help guide students towards understanding. For more information access the MAP website:
http://www.map.mathshell.org/materials/background.php?subpage=formative

The task, *Calculating Volume of Compound Objects*, is a Formative Assessment Lesson (FAL) that can be found at the website:
http://map.mathshell.org/materials/lessons.php?taskid=216&subpage=concept

The FAL document provides a clear lesson design, from the opening of the lesson to the closing of the lesson.

The PDF version of the task can be found at the link below:
http://map.mathshell.org/materials/download.php?fileid=684

## STANDARDS ADDRESSED IN THIS TASK:

**Explain volume formulas and use them to solve problems**

**MGSE9-12.G.GMD.1** Give informal arguments for geometric formulas.

a. Give informal arguments for the formulas of the circumference of a circle and area of a circle using dissection arguments and informal limit arguments.
b. Give informal arguments for the formula of the volume of a cylinder, pyramid, and cone using Cavalieri's principle.

**MGSE9-12.G.GMD.2** Give an informal argument using Cavalieri's principle for the formulas for the volume of a sphere and other solid figures.

**MGSE9-12.G.GMD.3** Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems.

**Standards for Mathematical Practice**
This lesson uses all of the practices with emphasis on:

**1. Make sense of problems and persevere in solving them** by requiring students to interpret and make meaning of a problem and find a logical starting point, and to monitor their progress and change their approach to solving the problem, if necessary.

**6. Attend to precision** by requiring students to calculate efficiently and accurately; and to communicate precisely with others by using clear mathematical language to discuss their reasoning.

## A Golden Crown? (Short Cycle Task)

**Back to Task Table**

*Source: Balanced Assessment Materials from Mathematics Assessment Project*
http://www.map.mathshell.org/materials/download.php?fileid=816

### ESSENTIAL QUESTIONS:

- How do you analyze proportional relationships and use them to solve real world mathematical problems?

### TASK COMMENTS:

Tasks and lessons from the Mathematics Assessment Project are specifically designed to help teachers effectively formatively assess their students. The way the tasks and lessons are designed gives the teacher a clear understanding of what the students are able to do and not do. Within the lesson, teachers will find suggestions and question prompts that will help guide students towards understanding. For more information access the MAP website:
http://www.map.mathshell.org/materials/background.php?subpage=summative

The task, *A Golden Crown?*, is a Mathematics Assessment Project Assessment Task that can be found at the website:
http://www.map.mathshell.org/materials/tasks.php?taskid=282&subpage=expert

The PDF version of the task can be found at the link below:
http://www.map.mathshell.org/materials/download.php?fileid=816

The scoring rubric can be found at the following link:
http://www.map.mathshell.org/materials/download.php?fileid=817

### STANDARDS ADDRESSED IN THIS TASK:

**Explain volume formulas and use them to solve problems**

**MGSE9-12.G.GMD.1** Give informal arguments for geometric formulas.

a. Give informal arguments for the formulas of the circumference of a circle and area of a circle using dissection arguments and informal limit arguments.
b. Give informal arguments for the formula of the volume of a cylinder, pyramid, and cone using Cavalieri's principle.

**MGSE9-12.G.GMD.2** Give an informal argument using Cavalieri's principle for the formulas for the volume of a sphere and other solid figures.

**MGSE9-12.G.GMD.3** Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems.

**Standards for Mathematical Practice**
This task uses all of the practices with emphasis on:

**2. Reason abstractly and quantitatively** by requiring students to make sense of quantities and their relationships to one another in problem situations.

**3. Construct viable arguments and critique the reasoning of others** by engaging students on discussion of why they agree or disagree with responses, decide whether they make sense, and ask useful questions to clarify or improve the arguments.
**6. Attend to precision** by requiring students to calculate efficiently and accurately; and to communicate precisely with others by using clear mathematical language to discuss their reasoning.

**7. Look for and make use of structure** by expecting students to apply rules, look for patterns and analyze structure.

## Bestsize Cans (Short Cycle Task)

**Back to Task Table**

*Source: Balanced Assessment Materials from Mathematics Assessment Project*
http://www.map.mathshell.org/materials/download.php?fileid=820

### ESSENTIAL QUESTIONS:

- How do you explain volume formulas and use them to solve problems?

### TASK COMMENTS:

Tasks and lessons from the Mathematics Assessment Project are specifically designed to help teachers effectively formatively assess their students. The way the tasks and lessons are designed gives the teacher a clear understanding of what the students are able to do and not do. Within the lesson, teachers will find suggestions and question prompts that will help guide students towards understanding. For more information access the MAP website:
http://www.map.mathshell.org/materials/background.php?subpage=summative

The task, *Bestsize Cans*, is a Mathematics Assessment Project Assessment Task that can be found at the website:
http://www.map.mathshell.org/materials/tasks.php?taskid=284&subpage=expert

The PDF version of the task can be found at the link below:
http://www.map.mathshell.org/materials/download.php?fileid=820

The scoring rubric can be found at the following link:
http://www.map.mathshell.org/materials/download.php?fileid=821

### STANDARDS ADDRESSED IN THIS TASK:

**Explain volume formulas and use them to solve problems**

**MGSE9-12.G.GMD.1** Give informal arguments for geometric formulas.

a. Give informal arguments for the formulas of the circumference of a circle and area of a circle using dissection arguments and informal limit arguments.
b. Give informal arguments for the formula of the volume of a cylinder, pyramid, and cone using Cavalieri's principle.

**MGSE9-12.G.GMD.2** Give an informal argument using Cavalieri's principle for the formulas for the volume of a sphere and other solid figures.

**MGSE9-12.G.GMD.3** Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems.

**Standards for Mathematical Practice**

This task uses all of the practices with emphasis on:

**2. Reason abstractly and quantitatively** by requiring students to make sense of quantities and their relationships to one another in problem situations.

**3. Construct viable arguments and critique the reasoning of others** by engaging students on discussion of why they agree or disagree with responses, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

**6. Attend to precision** by requiring students to calculate efficiently and accurately; and to communicate precisely with others by using clear mathematical language to discuss their reasoning.

**7. Look for and make use of structure** by expecting students to apply rules, look for patterns and analyze structure.

## Funsize Cans (Short Cycle Task)

Back to Task Table

*Source: Balanced Assessment Materials from Mathematics Assessment Project*
http://www.map.mathshell.org/materials/download.php?fileid=756

**ESSENTIAL QUESTIONS:**

- How do you explain volume formulas and use them to solve problems?

**TASK COMMENTS:**

Tasks and lessons from the Mathematics Assessment Project are specifically designed to help teachers effectively formatively assess their students. The way the tasks and lessons are designed gives the teacher a clear understanding of what the students are able to do and not do. Within the lesson, teachers will find suggestions and question prompts that will help guide students towards understanding. For more information access the MAP website:
http://www.map.mathshell.org/materials/background.php?subpage=summative

The task, *Funsize Cans,* is a Mathematics Assessment Project Assessment Task that can be found at the website:
http://www.map.mathshell.org/materials/tasks.php?taskid=252&subpage=apprentice

The PDF version of the task can be found at the link below:
http://www.map.mathshell.org/materials/download.php?fileid=756

The scoring rubric can be found at the following link:
http://www.map.mathshell.org/materials/download.php?fileid=757

**STANDARDS ADDRESSED IN THIS TASK:**

**Explain volume formulas and use them to solve problems**
**MGSE9-12.G.GMD.1** Give informal arguments for geometric formulas.

a. Give informal arguments for the formulas of the circumference of a circle and area of a circle using dissection arguments and informal limit arguments.
b. Give informal arguments for the formula of the volume of a cylinder, pyramid, and cone using Cavalieri's principle.

**MGSE9-12.G.GMD.2** Give an informal argument using Cavalieri's principle for the formulas for the volume of a sphere and other solid figures.

**MGSE9-12.G.GMD.3** Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems.

**Standards for Mathematical Practice**
This task uses all of the practices with emphasis on:

**3. Construct viable arguments and critique the reasoning of others** by engaging students on discussion of why they agree or disagree with responses, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

**7. Look for and make use of structure** by expecting students to apply rules, look for patterns and analyze structure.

## Propane Tanks (Short Cycle Task)

**Back to Task Table**

*Source: Balanced Assessment Materials from Mathematics Assessment Project*
http://www.map.mathshell.org/materials/download.php?fileid=828

**ESSENTIAL QUESTIONS:**

- How do you explain volume formulas and use them to solve problems?
- How do you visualize relationships between two-dimensional and three-dimensional objects?

**TASK COMMENTS:**

Tasks and lessons from the Mathematics Assessment Project are specifically designed to help teachers effectively formatively assess their students. The way the tasks and lessons are designed gives the teacher a clear understanding of what the students are able to do and not do. Within the lesson, teachers will find suggestions and question prompts that will help guide students towards understanding. For more information access the MAP website:
http://www.map.mathshell.org/materials/background.php?subpage=summative

The task, *Propane Tanks*, is a Mathematics Assessment Project Assessment Task that can be found at the website:
http://www.map.mathshell.org/materials/tasks.php?taskid=288&subpage=expert

The PDF version of the task can be found at the link below:
http://www.map.mathshell.org/materials/download.php?fileid=828

The scoring rubric can be found at the following link:
http://www.map.mathshell.org/materials/download.php?fileid=829

**STANDARDS ADDRESSED IN THIS TASK:**

**Explain volume formulas and use them to solve problems**
**MGSE9-12.G.GMD.1** Give informal arguments for geometric formulas.

a. Give informal arguments for the formulas of the circumference of a circle and area of a circle using dissection arguments and informal limit arguments.
b. Give informal arguments for the formula of the volume of a cylinder, pyramid, and cone using Cavalieri's principle.

**MGSE9-12.G.GMD.2** Give an informal argument using Cavalieri's principle for the formulas for the volume of a sphere and other solid figures.

**MGSE9-12.G.GMD.3** Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems.

**Standards for Mathematical Practice**

This task uses all of the practices with emphasis on:
**2. Reason abstractly and quantitatively** by requiring students to make sense of quantities and their relationships to one another in problem situations.

**3. Construct viable arguments and critique the reasoning of others** by engaging students on discussion of why they agree or disagree with responses, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

**6. Attend to precision** by requiring students to calculate efficiently and accurately; and to communicate precisely with others by using clear mathematical language to discuss their reasoning.

**7. Look for and make use of structure** by expecting students to apply rules, look for patterns and analyze structure.

## Tennis Balls in a Can

[Back to Task Table](#)

*Developed by James Madden and the Louisiana Math and Science Teacher Institute On-Ramp.*
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### Common Core State Standards

**Visualize relationships between two-dimensional and three-dimensional objects.**

**MGSE9-12.G.GMD.4** Identify the shapes of two-dimensional cross-sections of three-dimensional objects, and identify three-dimensional objects generated by rotations of two-dimensional objects.

### Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

### Introduction

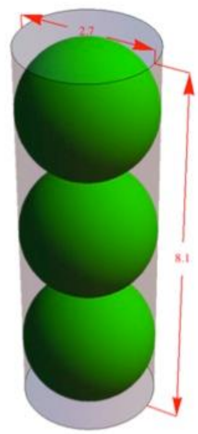
This task is inspired by the derivation of the volume formula for the sphere. If a sphere of radius 1 is enclosed in a cylinder of radius 1 and height 2, then the volume not occupied by the sphere is equal to the volume of a "double-napped cone" with vertex at the center of the sphere and bases equal to the bases of the cylinder. This can be seen by slicing the figure parallel to the base of the cylinder and noting the areas of the annular slices consisting of portions of the volume that are *inside* the cylinder but *outside* the sphere are the same as the areas of the slices of the double-napped cone (and applying Cavalieri's Principle). This almost magical fact about slices is a manifestation of Pythagorean Theorem. The visualization required here is used in calculus, in connection with procedures for calculating volumes by various slicing procedures.

### Materials

- Pencil
- Handout

## <u>Tennis Balls in a Can</u>

The official diameter of a tennis ball, as defined by the International Tennis Federation, is at least 2.575 inches and at most 2.700 inches. Tennis balls are sold in cylindrical containers that contain three balls each. To model the container and the balls in it, we will assume that the balls are 2.7 inches in diameter and that the container is a cylinder the interior of which measures 2.7 inches in diameter and $3 \times 2.7 = 8.1$ inches high.

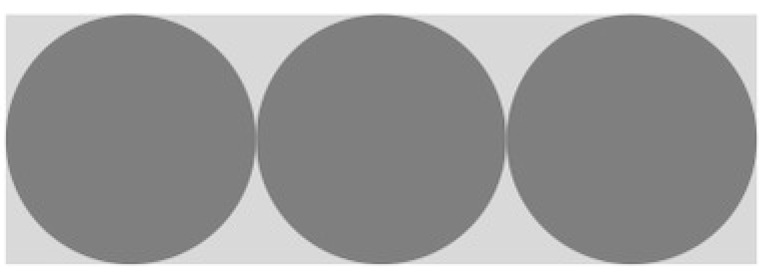


(a) Lying on its side, the container passes through an X-ray scanner in an airport. If the material of the container is opaque to X-rays, what outline will appear? With what dimensions?

***The shadow is a rectangle measuring 2.7 inches by 8.1 inches.***

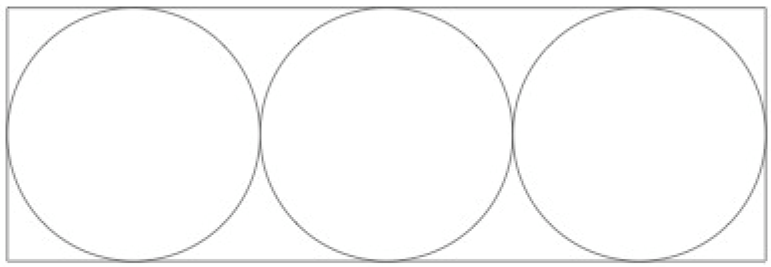
(b) If the material of the container is partially opaque to X-rays and the material of the balls is completely opaque to X-rays, what will the outline look like (still assuming the can is lying on its side)?

***The shadow is a light rectangle (2.7 × 8.1 inches) with three disks inside. It looks like a traffic light:***



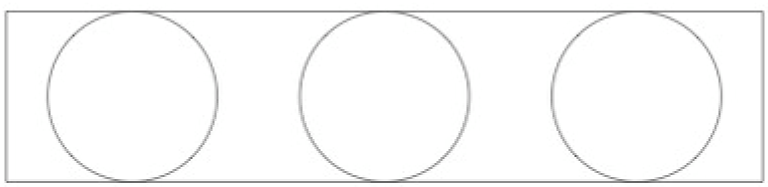
(c) The *central axis* of the container is a line that passes through the centers of the top and bottom. If one cuts the container and balls by a plane passing through the central axis, what does the intersection of the plane with the container and balls look like? (The intersection is also called a *cross section*. Imagine putting the cut surface on an ink pad and then stamping a piece of paper. The stamped image is a picture of the intersection.)

***The image is similar to the previous one, but now only the outlines are seen:***



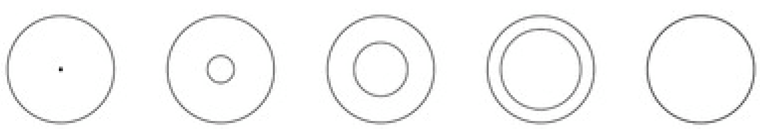
(d) If the can is cut by a plane parallel to the central axis, but at a distance of 1 inch from the axis, what will the intersection of this plane with the container and balls look like?

***The intersection with the container is a narrower rectangle. The intersections with the balls are smaller circles. Because each ball touches the container along its whole "equator," the circles must touch the long sides of the rectangle:***



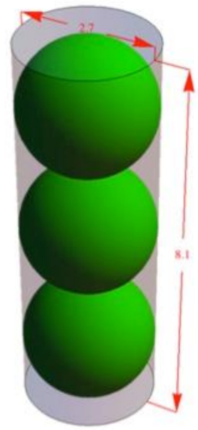
(e) If the can is cut by a plane parallel to one end of the can—a horizontal plane—what are the possible appearances of the intersections?

***The intersections are two concentric circles, except when $w = 0, 2.7, 5.4, 8.1$ and when $w = 1.35, 4.05, 6.75$. In the former case, we see a circle (from the container) and a point (where the plane touches a sphere). In the latter case, we see a single circle corresponding to a place where the equator of a ball touches the container.***



## **Tennis Balls in a Can**

The official diameter of a tennis ball, as defined by the International Tennis Federation, is at least 2.575 inches and at most 2.700 inches. Tennis balls are sold in cylindrical containers that contain three balls each. To model the container and the balls in it, we will assume that the balls are 2.7 inches in diameter and that the container is a cylinder the interior of which measures 2.7 inches in diameter and $3 \times 2.7 = 8.1$ inches high.




(a) Lying on its side, the container passes through an X-ray scanner in an airport. If the material of the container is opaque to X-rays, what outline will appear? What dimensions?

(b) If the material of the container is partially opaque to X-rays and the material of the balls is completely opaque to X-rays, what will the outline look like (still assuming the can is lying on its side)?

(c) The *central axis* of the container is a line that passes through the centers of the top and bottom. If one cuts the container and balls by a plane passing through the central axis, what does the intersection of the plane with the container and balls look like? (The intersection is also called a *cross section*. Imagine putting the cut surface on an ink pad and then stamping a piece of paper. The stamped image is a picture of the intersection.)

(d) If the can is cut by a plane parallel to the central axis, but at a distance of 1 inch from the axis, what will the intersection of this plane with the container and balls look like?

(e) If the can is cut by a plane parallel to one end of the can—a horizontal plane—what are the possible appearances of the intersections?

## Formative Assessment Lesson: 2D Representations of 3D Objects

**Back to Task Table**

*Source: Formative Assessment Lesson Materials from Mathematics Assessment Project*
http://map.mathshell.org/materials/download.php?fileid=1280

**ESSENTIAL QUESTIONS:**

- How do you visualize two-dimensional cross-sections of representations of three-dimensional objects?
- How do you recognize and draw two-dimensional cross-sections at different points along a plane of a representation of a three-dimensional object?

**TASK COMMENTS:**

Tasks and lessons from the Mathematics Assessment Project are specifically designed to help teachers effectively formatively assess their students. The way the tasks and lessons are designed gives the teacher a clear understanding of what the students are able to do and not do. Within the lesson, teachers will find suggestions and question prompts that will help guide students towards understanding. For more information access the MAP website:
http://www.map.mathshell.org/materials/background.php?subpage=formative

The task, *2D Representations of 3D Objects*, is a Formative Assessment Lesson (FAL) that can be found at the website:
http://map.mathshell.org/materials/lessons.php?taskid=439&subpage=concept

The FAL document provides a clear lesson design, from the opening of the lesson to the closing of the lesson.

The PDF version of the task can be found at the link below:
http://map.mathshell.org/materials/download.php?fileid=1280

**STANDARDS ADDRESSED IN THIS TASK:**

**MGSE9-12.G.GMD.4** Identify the shapes of two-dimensional cross-sections of three-dimensional objects, and identify three-dimensional objects generated by rotations of two-dimensional objects.

**Standards for Mathematical Practice**
This lesson uses all of the practices with emphasis on:

**3. Construct viable arguments and critique the reasoning of others** by engaging students on discussion of why they agree or disagree with responses, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

**5. Use appropriate tools strategically** by expecting students to consider available tools when solving a mathematical problem.  These tools might include pencil and paper, concrete models, a ruler, a protractor, a compass, a calculator, software, etc.

**6. Attend to precision** by requiring students to calculate efficiently and accurately; and to communicate precisely with others by using clear mathematical language to discuss their reasoning.

**7. Look for and make use of structure** by expecting students to apply rules, look for patterns and analyze structure.

## Culminating Task: Rate of Change Lab

Back to Task Table

**Standards Addressed in this Task**
**MGSE9-12.G.GMD.1** Give informal arguments for geometric formulas.

a. Give informal arguments for the formulas of the circumference of a circle and area of a circle using dissection arguments and informal limit arguments.
b. Give informal arguments for the formula of the volume of a cylinder, pyramid, and cone using Cavalieri's principle.

**MGSE9-12.G.GMD.2** Give an informal argument using Cavalieri's principle for the formulas for the volume of a sphere and other solid figures.

**MGSE9-12.G.GMD.3** Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems.

**Standards for Mathematical Practice**
**1. Make sense of problems and persevere in solving them** by requiring students to interpret and make meaning of a problem and find a logical starting point, and to monitor their progress and change their approach to solving the problem, if necessary.

**2. Reason abstractly and quantitatively** by requiring students to make sense of quantities and their relationships to one another in problem situations.

**3. Construct viable arguments and critique the reasoning of others** by engaging students on discussion of why they agree or disagree with responses, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

**4. Model with mathematics** by expecting students to apply the mathematics concepts they know in order to solve problems arising in everyday situations, and reflect on whether the results are sensible for the given scenario.

**5. Use appropriate tools strategically** by expecting students to consider available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a compass, a calculator, software, etc.

**Common Student Misconceptions**
1. An informal survey of students from elementary school through college showed the number pi to be the mathematical idea about which more students were curious than any other. There are at least three facets to this curiosity: the symbol $\pi$ itself, the number 3.14159..., and the formula for the area of a circle. All of these facets can be addressed here, at least briefly.
2. Many students want to think of infinity as a number. Avoid this by talking about a quantity that becomes larger and larger with no upper bound.
3. The inclusion of the coefficient 1/3 in the formulas for the volume of a pyramid or cone and 4/3 in the formula for the volume of a sphere remains a mystery for many students. In high school, students should attain a conceptual understanding of where these coefficients come from.

4. Concrete demonstrations, such as pouring water from one shape into another should be followed by more formal reasoning.

***COMMENTS***

***This lab is adapted from a College Board Pre-AP Workshop presentation. It is excellent for all students because it drills similar triangles, averaging, and measuring skills. It is especially important for students on track to take AP Calculus, or Calculus as a freshman in college.***

Materials: 2 paper snow cone cups per group, scissors, 1 cylindrical plastic cup, salt, 2 rulers with cm markings, lab sheet, dust buster for spills

Lab Sheet:

1. Measure the height of your cone. h = ________cm
2. Measure the diameter of your cone. d = ______cm r = _______cm
3. Snip off a small tip of your cone.
4. Place the second cone under the first, as a cap.
5. When the teacher instructs you, fill the cone with salt.
6. During 15 second intervals, you will be removing the overlay cone and letting the salt drip from the cone into your cup. You will then be recording the new height of your salt below. You will not be able to record the radius and volume at this time.
7. We will continue doing this until all of the groups have emptied their cones of salt.

8. Re-Record your original height and radius as Time Interval 1. This is the only radius, you know. You will have to calculate all the other radii and all of the volumes.

***Students will use similar triangles to determine each new radius. Changing this lab to measure the slant height would create more similar triangle work, as students would then need to calculate each new height as well.***

| Time Interval | Height | Radius | Volume |
|---|---|---|---|
| 1 | | | |
| 2 | | | |
| 3 | | | |
| 4 | | | |
| 5 | | | |
| 6 | | | |
| 7 | | | |
| 8 | | | |
| 9 | | | |
| 10 | | | |
| 11 | | | |
| 12 | | | |
| 13 | | | |
| 14 | | | |
| 15 | | | |

9. Calculate the missing boxes on your chart.

10. Calculate the Rate of Change of the Volume from one interval to the next and record below.

| **Interval** | **per 15 sec** | **per second** |
|---|---|---|
| **1 to 2** | | |
| **2 to 3** | | |
| **3 to 4** | | |
| **4 to 5** | | |
| **5 to 6** | | |
| **6 to 7** | | |
| **7 to 8** | | |
| **8 to 9** | | |
| **9 to 10** | | |
| **10 to 11** | | |
| **11 to 12** | | |
| **12 to 13** | | |
| **13 to 14** | | |
| **14 to 15** | | |

11. Calculate the Average Rate of Change of Volume.

12. Using your groups Average Rate of Change, how long would it take to fill your cylindrical cup, if the salt had to first go through your cone?

    Radius of Cup: __________cm

    Height of Cup: ___________cm

    Volume of Cup: ____________$cm^3$

    Length of time to fill cup: __________________seconds

13. How many times would you need to fill your cone?

# Culminating Task: Rate of Change Lab

Materials: 2 paper snow cone cups per group, scissors, 1 cylindrical plastic cup, salt, 2 rulers with cm markings, lab sheet, dust buster for spills

Lab Sheet:

1. Measure the height of your cone. h = ________cm

2. Measure the diameter of your cone. d = ______cm r = _______cm

3. Snip off a small tip of your cone.

4. Place the second cone under the first, as a cap.

5. When the teacher instructs you, fill the cone with salt.

6. During 15 second intervals, you will be removing the overlay cone and letting the salt drip from the cone into your cup. You will then be recording the new height of your salt below. You will not be able to record the radius and volume at this time.

7. We will continue doing this until all of the groups have emptied their cones of salt.

8. Re-Record your original height and radius as Time Interval 1. This is the only radius, you know. You will have to calculate all the other radii and all of the volumes.

| Time Interval | Height | Radius | Volume |
|---|---|---|---|
| **1** | | | |
| **2** | | | |
| **3** | | | |
| **4** | | | |
| **5** | | | |
| **6** | | | |
| **7** | | | |
| **8** | | | |
| **9** | | | |
| **10** | | | |
| **11** | | | |
| **12** | | | |
| **13** | | | |
| **14** | | | |
| **15** | | | |

9. Calculate the missing boxes on your chart.

10. Calculate the Rate of Change of the Volume from one interval to the next and record below.

| **Interval** | **per 15 sec** | **per second** |
|---|---|---|
| **1 to 2** | | |
| **2 to 3** | | |
| **3 to 4** | | |
| **4 to 5** | | |
| **5 to 6** | | |
| **6 to 7** | | |
| **7 to 8** | | |
| **8 to 9** | | |
| **9 to 10** | | |
| **10 to 11** | | |
| **11 to 12** | | |
| **12 to 13** | | |
| **13 to 14** | | |
| **14 to 15** | | |

11. Calculate the Average Rate of Change of the Volume.

12. Using your groups Average Rate of Change, how long would it take to fill your cylindrical cup, if the salt had to first go through your cone?

    Radius of Cup: ___________cm

    Height of Cup: ____________cm

    Volume of Cup: _____________$cm^3$

    Length of time to fill cup: ____________________seconds

13. How many times would you need to fill your cone?