Mathematics: Roots of Polynomials: An Introduction

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1. Multiplying two linear factors

Before you use this study guide, you should be familiar with factoring, solving, and graphing quadratics. You may want to refer to Britannica Study Guides on factoring, *Factoring by Finding a Common Factor* and *Factoring Trinomials*.

You are going to learn techniques that allow you to factor polynomials of higher degree. As you will discover, these techniques are intimately related to finding the roots of the polynomials. One of the main goals of these techniques is to "reduce" the problem to a point where you can use the techniques that work with quadratics. That is why it is important to know how to factor a quadratic function and to solve a quadratic equation. The techniques we will learn are referred to as the Factor Theorem and the Rational Root Theorem.

We begin with a review of multiplying two linear factors. Linear factors are expressions in the form, $(ax + b)$, where x is a variable and a and b are real numbers. They are called linear factors because if you were to graph $y = ax + b$, you would get a straight line. They are of degree 1, since the highest exponent of the variable x is 1. Two linear factors can be multiplied using the FOIL (**F**irst **O**uter **I**nner **L**ast) technique.

Examples
1. $(x + 2)(x - 3) = x^2 - x - 6$
2. $(x - 6)(x + 5) = x^2 - x - 30$
3. $(x - 5)(x + 5) = x^2 - 25$
4. $(2x - 1)(3x - 4) = 6x^2 - 11x + 4$

Click this icon to see an animated presentation of example 1, multiplied using FOIL.

Click this icon to see example 2 multiplied by using a cross-problem format.

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2. Multiplying three linear factors

When you multiply, you should generally work with two factors at a time. When presented with three factors, you can pick any two (since multiplication can be done in any order), and multiply them together using techniques from the previous screen.

Example: Multiply $(x - 1)(x + 2)(x - 3)$.

We choose to multiply the second and the third factor because we already did that multiplication in the last screen.

$(x + 2)(x - 3) = x^2 - x - 6$

We then need to multiply this result by the remaining linear factor.

$(x - 1)(x^2 - x - 6)$

Once you have begun multiplying polynomials that have more than two terms, you may find it useful to organize your work vertically, as shown by clicking on the icon.

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3. The purpose of factoring

We can use polynomial equations to describe real-life situations. Finding the roots (x-intercepts) and turning points (peaks and valleys) to these equations yields important and meaningful information. In the case of business or physics situations, the results might indicate at what time an object will hit the ground, or when profits will be maximized. Click the icon to see a sample graph.

The polynomial shown above has roots $x = -4, -1, +5$. Notice that roots are also called x-intercepts and zeros. In this guide, we will show that factoring into linear factors allows you to easily determine the roots. In calculus, you discover a simple method to find turning points for any equation.

The points on the graph called *roots* are (–4, 0), (–1, 0), and (5, 0). It is worth emphasizing that these are points on the graph, and they are characterized by an important common feature—they all have 0 for the y-coordinate.

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4. Solving quadratic equations

There are four methods commonly used to solve a quadratic equation. We can solve by guessing and checking, by using the quadratic formula, by factoring the quadratic expression, or by completing the square. We show an example of how some of these techniques relate to each other.

We begin with the function $f(x) = 2x^2 + 5x - 12$. To graph this quadratic equation, you might want to know the x-intercepts, the y-intercept, and the turning point. Since the x-intercepts occur when $f(x) = 0$, we can replace $f(x)$ with 0 and solve the equation $0 = 2x^2 + 5x - 12$. To solve this equation, guessing and checking is time-consuming; factoring is difficult when the coefficient of x^2 is different from 1; and completing the square is generally difficult. Therefore we use the quadratic formula to solve this equation. Practice using the quadratic formula (which you can see when you click the first icon), and then check your answer by clicking the second icon.

As you can see, the solutions to the equation are $x = 1.5$ and -4. This means that $f(1.5) = 0$, and $f(-4) = 0$, which means that the x-intercepts are at -4 and 1.5, and that (1.5, 0), and (–4, 0) are points on the graph. Substituting zero for x in the equation results in $f(0) = -12$. Thus the y-intercept is –12 and the point (0, –12) is on the graph. The graph of a quadratic equation is a special curve known as a parabola. All parabolas have the same general "U" shape, involving one turning point, known as its vertex. In analytic geometry you will discover that its vertex occurs when $x = -b/2a$, from which the corresponding value of y can be determined and the point added to the graph. Click the icon to see the graph.

Since the number inside the square root of the quadratic formula (the determinant $b^2 - 4ac$) was a perfect square (121), the quadratic is factorable.

When you factor $2x^2 + 5x - 12$, you get $(2x - 3)(x + 4)$. See the Britannica Study Guide *Factoring Trinomials* if you need to review this. This means that the equation above, which we solved by looking for x-intercepts, can be rewritten as follows:

$2x^2 + 5x - 12 = 0$
$(2x - 3)(x + 4) = 0$

Using the zero product property, we can say that this means that solving the equations

$2x - 3 = 0$ and $x + 4 = 0$

will give us solutions to the equation. The solutions to these are $x = 1.5$, and $x = -4$, which match the previous roots.

Factoring enables us to find the x-intercepts. Sometimes factoring is faster than using the quadratic formula, and sometimes it is slower. Unfortunately, something like the quadratic formula does not exist for all polynomial functions. Factoring, and guessing and checking, are the only methods available until you learn calculus. The following screens review techniques that can help you factor and find roots.

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5. Prime factors of polynomials: ambiguity in the word factor

In the previous screen, we realized that we could factor the quadratic $2x^2 + 5x - 12$ because the determinant was 121, which is a perfect square. Actually, we should have been more precise. A perfect square determinant means that you can factor the polynomial into factors that have integer coefficients. When a polynomial cannot be factored by factors whose coefficients are integers, then we say that it is a prime. $2x^2 + 5x - 12$ is not prime because we factored it into $(2x - 3)(x + 4)$, which are factors with integer coefficients. There are other possible factorizations, however.

Multiply each row below and confirm that these factors also result in $2x^2 + 5x - 12$.

$(x - 1.5)(2x + 8)$
$2(x - 1.5)(x + 4)$
$(4x - 6)(0.5x + 2)$

Each row multiplies to make $2x^2 + 5x - 12$. However, these are factorizations that have non-integer coefficients.

If you are trying to factor in the common sense of the term, you are looking for a factorization with integer coefficients. When you are asked to factor $2x^2 + 5x - 12$, you are expected to answer $(2x - 3)(x + 4)$. However, if you are trying to factor in order to find roots, then any of the above factorizations will do. When finding roots, you care about being able to use the zero product property. It is not critical to find the factors with integer coefficients.

To confirm this, make each of the factorizations above equal to zero, and apply the zero product property. You will notice that the solutions in each case are $x = 1.5$, and $x = -4$. Thus, any one of these factorizations of $2x^2 + 5x - 12$ would allow us to find the roots.

In other words, the meaning of the word "factor" is somewhat ambiguous. Sometimes it is a noun, and means part of a product, and sometimes it is a verb, and means "write an equivalent expression as a product using factors with integer coefficients." In the rest of this guide, some of our solutions will include integer coefficients, and some of them will not.

To stay within the spirit of mathematics convention, the verb *to factor* will continue to mean factor using expressions with integer coefficients. To avoid confusion, we will not say that we are factoring cubics; we will say that we are "writing the cubic as a product of factors." This allows us to work with factors with any kind of coefficients, even imaginary ones. A factor of polynomial $P(x)$ will be defined as an expression that divides into $P(x)$ leaving a remainder of zero, as shown in the next screen.

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6. Finding one factor from another: polynomial division

Let's pretend for a second that you do not know how to factor 141, but people tell you that they know for sure that 3 is a factor of 141. Perhaps they have added the digits of the number, 1 + 4 + 1. Since the sum is 6, and 6 is divisible by 3, 141 is divisible by 3. In any case, once you know one factor, you can divide 141 by 3 to get another factor. In this case, 141 ÷ 3 = 47, so 3 and 47 are factors of 141.

This will also work for polynomials. Let's pretend that you had to write $2x^2 + 5x - 12$ as a product of factors, and you had no idea what to do. Let's also pretend that someone tells you that $(x - 1.5)$ is a factor of $2x^2 + 5x - 12$. With this information, you can divide $2x^2 + 5x - 12$ by $(x - 1.5)$, to get another factor. Click the icon to see this long division.

As you can see above the division bar on the graphic, when you divide by $(x - 1.5)$, you get a remainder of zero, and the other factor of $2x^2 + 5x - 12$ is $(2x + 8)$. We can write that $2x^2 + 5x - 12 = (x - 1.5)(2x + 8)$. Again, if we are looking for factors with integer coefficients, then it is difficult to see how to get there from here. But if we are just looking for the roots of $y = 2x^2 + 5x - 12$, then we have all that we need. Using the zero product property, we get $x - 1.5 = 0$ and $2x + 8 = 0$, which gives us $x = 1.5, -4$. Notice that when we divide a quadratic by a linear factor, we get a polynomial of degree 1 as a result.

We use a very similar procedure to factor cubics. We work very hard to find one factor, and then perform the long division. As in the example above, this always results in another factor whose degree is one less than the original polynomial. Since a cubic is of degree 3, dividing by a linear factor results in a polynomial of degree 2, or a quadratic. Factoring a quadratic is generally easy. Even if the quadratic is prime, you can always find roots of a quadratic using the quadratic formula. So, the real question becomes how to find that first factor.

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7. Finding a factor of a polynomial

To find a factor of a polynomial, you could guess and check, performing long division on many factors, waiting until you found one with remainder zero. Obviously, this could take a long time. Tools have been invented to make the job easier.

You may have noticed by now that every time we create a factor in the form ($x - a$), then a is a root. This should be obvious. Applying the zero product property to the factor ($x - a$), we get the equation $x - a = 0$. The solution to this equation will always be $x = a$. The factor theorem formalizes this relationship.

Factor Theorem
For a polynomial, $P(x)$, if $P(r) = 0$, then r is a root of $P(x)$, and ($x - r$) is a factor of $P(x)$.

This is a useful result because it eliminates the need for you to perform long division over and over again to test for a factor. Instead, you can just find a number, r, that makes the polynomial become zero. Then you can conclude that ($x - r$) is a factor of the polynomial.

Guessing and checking to find a value for x that makes a polynomial be zero can take a long time if you have to choose from all possible real numbers. Thankfully, the rational root theorem usually reduces the set of possible numbers to a much smaller set.

Rational Root Theorem
The rational number n/d is a root of $P(x)$ if and only if n is a factor of the constant term in $P(x)$, and d is a factor of the leading coefficient.

Let's understand this theorem using our familiar quadratic polynomial example.

Example: Use the rational root theorem to write the set of possible rational roots for $P(x) = 2x^2 + 5x - 12$.

The theorem states that any rational root of this polynomial has to be a rational number whose numerator is a factor of 12, and whose denominator is a factor of 2. First, we make a list of the integer factors of 12 and 2.

Factors of 12 are 12, –12, 6, –6, 4, –4, 3, –3, 2, –2, 1, –1.

Factors of 2 are 2, –2, 1, –1.

Then, we make a list of all possible fractions (the rational number n/d) with factors of 12 on top, and factors of 2 on the bottom. They are:

12, –12, 6, –6, 4, –4, 3, –3, 3/2, –3/2, 2, –2, 1, –1, 1/2, –1/2

The rational root theorem says that these are the only numbers that are possible candidates for rational roots. On the next screen, we discuss the strategy for testing these numbers.

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8. Strategies for testing possible rational roots

In the last screen, we determined that

12, –12, 6, –6, 4, –4, 3, –3, 3/2, –3/2, 2, –2, 1, –1, 1/2, –1/2

is the list of possible rational roots for the polynomial $P(x) = 2x^2 + 5x - 12$.

It is usually a good strategy to begin testing using small positive integers, as these are the easiest to evaluate. In this case, find $P(1)$, $P(2)$, and so on until one of these is zero. We try 1 first. Since $P(1) = 2(1)^2 + 5(1) - 12 = -5$, 1 is not a root. Since $P(2) = 2(2)^2 + 5(2) - 12 = 6$, 2 is not a root.

However, something interesting just occurred that can help us solve our problem.

We are looking for a number that makes the polynomial become 0. We just saw that $P(1) = -5$, and $P(2) = 6$. $P(1)$ was below zero, and $P(2)$ was above zero. Perhaps there is a number between 1 and 2 that will solve our problem. Remember that each time you evaluate the polynomial for a number in place of x, you are finding a point on the graph of the polynomial. Click the icon to see a table of points and their plots on a graph.

As you can see, it is true that there must be a root in between $x = 1$ and $x = 2$. Keep in mind that we are able to say this because all polynomials are continuous. Looking at our list of possibilities, we see that there is one that lies between 1 and 2, namely 3/2, or 1.5. We test it and discover that $P(1.5)$ does equal 0. Therefore $(x - 1.5)$ is a factor of the polynomial $P(x) = 2x^2 + 5x - 12$, and 1.5 is a root.

Make a table of point coordinates as you evaluate your polynomial for different values of x. If you find the y-coordinates of your points are moving quickly away from zero in absolute value, then you are probably headed in the wrong direction. If, on the other hand, you ever see the y-coordinate change signs, pause and consider that there must be a root in between those two x values. However, remember that we are only finding rational roots in this way. The graph may cross the x-axis on an irrational number, like *pi* or the square root of 3. Click the icon to see a table of these two points for the possible rational roots of $2x^2 + 5x - 12$.

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9. Irrational roots

Let's find the roots of the polynomial $P(x) = 2x^3 + 4x^2 – 6x – 12$.

As always, one should start by seeing if any common factor exists in order to simplify the problem. In this case, a common factor of 2 exists. Thus, $P(x) = 2(x^3 + 2x^2 – 3x – 6)$ and the 2 can be ignored in finding the roots since $2 \cdot 0 = 0$.

By the rational root theorem, the possible roots for the factored polynomial are 6, –6, 3, –3, 2, –2, 1, and –1. Again, we evaluate $P(1)$, $P(2)$, and so on. Again, we see that the value of the polynomial changes sign between $x = 1$, and $x = 2$. $P(1) = 2((1)^3 + 2(1)^2 – 3(1) – 6)) = 2(–6) = –12$, and $P(2) = 2((2)^3 + 2(2)^2 – 3(2) – 6)) = 2(4) = 8$. Click the icon to see a graph of these points.

However, no possible candidate exists on our list between 1 and 2. Remember that it is called the rational root theorem. It only talks about the roots that are rational. The root between 1 and 2 must be irrational.

Since this is a cubic, we hope that there is at least one rational root so that we can divide by a factor and get to the quadratic that will allow us to use the quadratic formula. Continuing to evaluate positive integers, we get $P(3) = 60$. Since the values are getting larger, we try going in the other direction on our list of possible roots. We get $P(–1) = –4$, and $P(–2) = 0$. Luckily, there is a rational root. We can now divide the polynomial $2(x^3 + 2x^2 – 3x – 6)$ by the linear $(x + 2)$. Try this on your own and then click the icon to check your answer.

We can now write $2x^3 + 4x^2 – 6x – 12$ as $2(x + 2)(x^2 – 3)$. Using the zero product property, we get the equations $x + 2 = 0$, and $x^2 – 3 = 0$. Solve the quadratic and then click the icon to see our solution and the roots of this polynomial.

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Glossary

coefficient
numerical part of a term. For example, the 4 in $4x$ is a coefficient.

continuous
a graph of a curve or line is continuous if you can trace it with your pencil without lifting your pencil from the paper (and without adding additional marks to the graph in the process!)

cubic
an expression that can be expressed as a polynomial of degree 3

determinant
for a quadratic $ax^2 + bx + c$, the expression $b^2 - 4ac$. It is useful for determining the nature of the roots.

factor
noun: parts of a multiplication expression. 4 and $(x - 2)$ are the factors of the expression $4(x - 2)$.

irrational
a real number that cannot be written in the form n/m where n and m are integers

linear factor
a factor that can be written in the form $(x - a)$, which is a polynomial of degree 1

polynomial
a function in which the expression is a sum of monomials in one variable with non-negative exponents

prime
can only be factored into the factors 1 and itself

quadratic
an expression that can be expressed as a polynomial of degree 2

quadratic factor
a factor that can be written in the form $(ax^2 + bx + c)$, which is a polynomial of degree 2

rational
a real number that can be expressed as a quotient n/m where n and m are integers

root
the value of a variable x that makes the value of a polynomial of x become zero

trinomial
a polynomial with three terms

turning point
a point on the graph of a function where the graph has a high point (a point higher than any near it), or a low point (a point lower than any near it)

variable

algebraic terms that can take more than one value. Letters such as x, y, and t are often used as variables.

zero product property
if $a \cdot b = 0$, then $a = 0$ or $b = 0$, or both a and $b = 0$

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Teacher's Notes

This material is generally taught in a course following advanced algebra, typically in the fourth year of high school math. Before starting, students should be familiar with factoring quadratic expressions and solving quadratic equations.

Finding roots of higher-degree polynomials is so entwined with solving quadratic equations that we review those techniques in the context of the new material. Although we focus on cubic functions of degree 3, we finish by showing how the techniques can be used to find the roots of higher-degree polynomials.

Polynomial long division is presented in this guide, although we leave synthetic division and the remainder theorem to a later guide. Instead, students focus on testing for roots by using the factor theorem. By substituting values in for x, they will perform a skill with which they are more familiar, and they will more naturally understand that they are finding coordinates for points on the graph that they can subsequently plot. The continuity of polynomials is mentioned, and the significance of a sign change in the value of the polynomial is discussed. We keep our lessons simple, but we try to lay the groundwork for more complex concepts, such as the Descartes Rule of Signs and the upper bound theorem.

We also point out the distinction between "factor" as a verb, which means to "write an expression as a product of factors whose coefficients are integers," and "factor" as a noun, which means "a part of a product." A factor does not have to have integer coefficients; it can just be part of a term. We point out that if the goal is to find roots, then any factoring will do. If the goal is to factor, then the factors should have integer coefficients.

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Glossary

Click on underlined words in the text to go to the Glossary for a definition. To see the entire glossary for this Study Guide, click on the Glossary link located at the end of the contents list.

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