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# geometry

This ancient branch of mathematics deals with points, lines, surfaces, and solids-and their relationships. In particular, **geometry** may be thought of as offering (1) precise definitions of many different figures; (2) construction methods for drawing figures; (3) a wealth of facts about the figures; and, most important, (4) ways to prove the facts.

### EUCLIDEAN GEOMETRY

**Geometry** was thoroughly organized in about 300 BC, when the Greek mathematician Euclid gathered what was known at the time, added original work of his own, and arranged 465 propositions into 13 books, collectively called 'Elements'. The books covered not only plane and solid **geometry** but also much of what is now known as algebra, trigonometry, and advanced arithmetic. (See also Euclid.)

Down through the ages, the propositions have been rearranged, and many of the proofs are different, but the basic idea presented in the 'Elements' has not changed. In the work facts are not just cataloged but are developed in an orderly way.

Even in 300 BC, **geometry** was recognized to be not just for mathematicians. Anyone can benefit from the basic teachings of **geometry**, which are how to follow lines of reasoning, how to say precisely what is intended, and especially how to prove basic concepts by following these lines of reasoning. Taking a course in **geometry** is beneficial for all students, who will find that learning to reason and prove convincingly is necessary for every profession. It is true that not everyone must prove things, but everyone is exposed to proof. Politicians, advertisers, and many other people try to offer convincing arguments. Anyone who cannot tell a good proof from a bad one may easily be persuaded in the wrong direction. **Geometry** provides a simplified universe, where points and lines obey believable rules and where conclusions are easily verified. By first studying how to reason in this simplified universe, people can eventually, through practice and experience, learn how to reason in a complicated world.

**Geometry** in ancient times was recognized as part of everyone's education. Early Greek philosophers asked that no one come to their schools who had not learned the 'Elements' of Euclid. There were, and still are, many who resisted this kind of education. It is said that Ptolemy I asked Euclid for an easier way to learn the material. Euclid told him there was no 'royal road' to **geometry**. The same message applies to readers of this article. They will not learn what **geometry** is all about. What they will learn is the basic shapes of some of the figures dealt with in **geometry** and a few facts about them. It takes a **geometry** course, with textbook and teacher, to show the complete and orderly arrangement of the facts and how each is proved.

### Point, Lines, and Angles

Geometric figures are based on points and lines. Euclid began his 'Elements' with a series of definitions, starting with these two:

A point is that which has no part.

A line is breadthless length.

Because he had not explained the terms part, breadth, or length, he was not defining these figures in terms of simpler things. The fact is that points and lines are so simple that they cannot be defined perfectly, but other things can be defined in terms of them. Definitions in this article will not be stated in full detail, as they would be in a **geometry** textbook, but they will be described briefly to show how they are linked, leading from points and lines to very special figures.

A line is usually drawn with arrowheads to show that it extends without end in both directions. A ray is half of a line-with just one end point. Two rays with the same end point form an angle. If the rays are the two halves of the same line, the angle is a straight angle. For measuring purposes, a straight angle may be thought to be like a book opened flat on a desk. An angle opened half that far is a right angle; its sides are perpendicular to one another.

For smaller angular measurements, the right angle is divided into 90 equal parts, each part being one degree of arc; therefore, half a right angle is 45 degrees, written as 45°. Four right angles measure 360°.

**Some different geometric figures are (from top ...**

Some different geometric figures are (from top to bottom): a line, a ray, an angle, a straight angle, a straight angle shown as an open book, and a right angle.

Any angle smaller than a right angle is acute; those larger than a right angle but smaller than a straight angle are obtuse. Any angle larger than a straight angle is said to be reflex. It has the same sides as an angle that is not reflex, but the angle is measured the long way around; there is certainly a difference between a book opened 340° and one opened 20°.

**Triangles**

The sides of an angle are unending rays, but the rays can be cut off into segments without changing the opening between the rays, which is the means by which angles are measured. Segment lengths are important in the next chain description, which leads from lines to the most special kinds of triangles.

**Two different depictions of a reflex angle.**

Two different depictions of a reflex angle.

A line segment is part of a line, with two end points. A broken line is made up of line segments joined end to end; if the ends of the broken line meet, it is a closed broken-line, or polygon. A polygon with three sides is a triangle. If none of the angles of a triangle are right angles, it is an oblique triangle. If all three angles are acute, it is an acute triangle. If an acute triangle has two equal sides it is an acute-isosceles triangle. (Any isosceles triangle has two equal sides.) If all three sides of a triangle are equal, it is an equilateral triangle. There are other interesting kinds of triangles not described here, but every kind will fit into one of the six categories shown on the chart. A triangle with one right angle is a right triangle. An oblique triangle may be either acute or obtuse (having one obtuse angle). All obtuse triangles are oblique, but so are acute triangles. A scalene triangle has no equal sides. An equilateral triangle has three, but this does not disqualify it from being isosceles. Equilateral triangles form a subset of acute-isosceles triangles.

An excellent way of representing subsets and the like is the use of Venn diagrams. Each set or subset is represented by a circle or a blob of some other shape, as shown in the diagram. One circle shown inside another means that one set is contained in the other, as the set of acute triangles is shown within the set of oblique triangles. The area where two circles overlap represents the
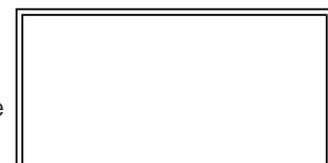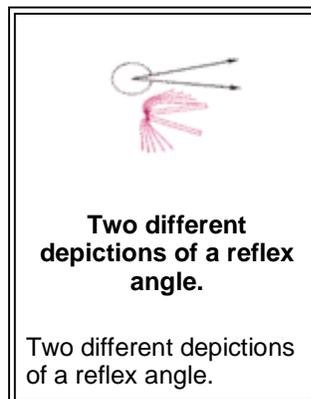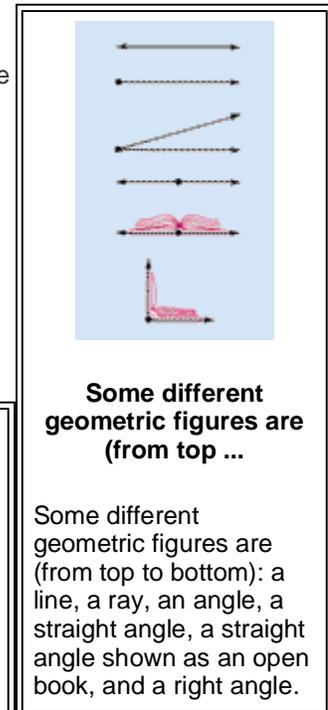
intersection of sets, as that for acute-isosceles triangles, represented by the purple region.



**Triangles**

Triangles

Every kind of triangle fits into this diagram somewhere. It is, in fact, the same diagram as the previous one, as can be seen by comparing the colors. Right triangles, though not named on this Venn diagram, are represented by the yellow and green regions.
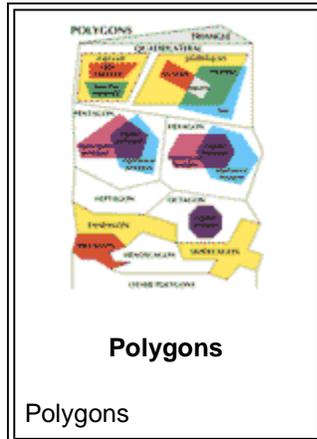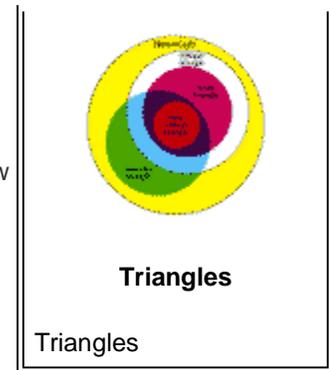


**Polygons**

Polygons

The regions on a Venn diagram represent sets, often with unlimited numbers of elements, as in the case of triangles. The regions are not supposed to look like the members of the set themselves, but for polygons that have more sides than triangles, it is possible to have the shapes of the regions illustrate the figures themselves as well as to show how the sets are related. This is done on the diagram, which represents part of the vast set of polygons.
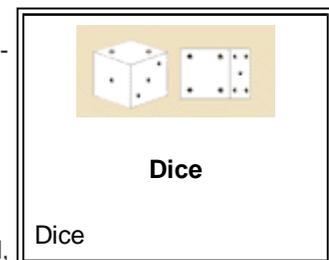
**Quadrilaterals**

A polygon with four sides is a quadrilateral. Quadrilaterals exist in many more shapes than do triangles, but not all have special names. That is why the diagram shows some space inside 'quadrilateral' in addition to that filled with specific kinds.

The kite is a shape not often mentioned in **geometry** books, but it is a familiar and interesting shape, with two pairs of adjacent equal sides. A parallelogram has both pairs of opposite sides parallel. If the same figure is both a kite and a parallelogram, it is an equilateral quadrilateral, or a rhombus. Another special kind of parallelogram is the rectangle, with four right angles. A square, with four right angles and all sides equal, is a special kind of rectangle, rhombus, kite, and parallelogram.

Another family of quadrilaterals is the trapezoids, which have one pair of parallel sides. It might be thought that a parallelogram is a special example of a trapezoid, just as an equilateral triangle is a special example of an isosceles triangle. But in this case it works out better to specify that a trapezoid has only two parallel sides. If the other two sides are equal, it is an isosceles trapezoid; if it has right angles at one end, it is a right trapezoid. But there can be no right-isosceles trapezoids; if a trapezoid were both right and isosceles it would be a rectangle. It would then have another pair of parallel sides and be completely disqualified from being a trapezoid.



**Perspective view of a cubical room**

Perspective view of a cubical room
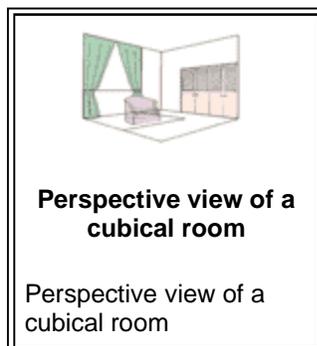
People see the many kinds of quadrilaterals more often than they realize. Squares that appear in man-made forms, however, are not usually seen as squares. The diagram, for example, shows a perspective view of a cubical room, containing a square doorway, a square window, and a square rug. The view is a realistic one, such as a camera would make, and it shows the doorway as an isosceles trapezoid, the window as a right trapezoid, and the floor as a kite. The walls appear as trapezoids that are neither right nor isosceles, and the rug is no special kind of quadrilateral at all. The picture of the dice in the diagram is another kind of view of squares, not quite so realistic as the view of the room but still quite close to what is actually seen. The faces that appear include two rectangles, a rhombus, and two parallelograms but no squares, even though the faces of dice are square.



**Dice**

Dice

**Other Polygons**

Pentagons, with five corners, and hexagons, with six, can take on even more shapes than quadrilaterals;

named kinds of pentagons and hexagons are even more scarce than named kinds of quadrilaterals. Three common adjectives applied to polygons of more than four sides are equilateral, having all sides equal; equiangular, having all angles equal; and regular, both equilateral and equiangular. These are shown in the polygon chart for pentagons and hexagons.

The names equilateral, equiangular, and regular are not usually used with quadrilateral. In the case of triangles, every equilateral triangle is also equiangular and, therefore, regular.

The heptagon has seven corners, and the octagon has eight. The enneagon, nine corners, is sometimes called a nonagon. The other prefixes used are the Greek names for numbers, but nona- is Latin. Other Latin names, however, are quadrilateral, which means 'four sides,' and triangle, which means 'three angles.' A more consistent name for quadrilaterals that is also used is tetragon. A three-sided polygon can be called a trigon as part of the name trigonometry, which means 'triangle measurement.' Deca- (10), hendeca- (11), and dodeca- (12) are also Greek prefixes. These Greek prefixes are used to form the names decagon, hendecagon, and dodecagon.

### Curves

**Geometry** includes the study of many kinds of curves. The one that gets the most attention is the circle. It is a set of points that are all a certain distance, the radius, from a certain point, the center. An interesting variation is to use one point and one line and do such things as finding points twice as far from the line as from the point or, what is the same thing, half as far from the point as they are from the line.

This technique is important enough so that special names are used for the line, the point, and the relation between distances. The line is the directrix (plural directrixes, also directrices). The point is the focus (plural focuses or foci). The ratio of distance-from-focus to distance-from-directrix is the eccentricity, or e.

If points are half as far from the focus as from the directrix, e=12. Distances from the directrix are measured perpendicularly, so that the curve produced is not the same as the one using two points. For e=12, an oval is produced that is about 15% longer than it is wide-definitely not a circle. This kind of oval is an ellipse. It is the same shape at both ends, and may be thought of as having two foci and two directrices, one at each end. A smaller eccentricity gives a fatter ellipse; a larger eccentricity, a thinner one. The size of the curve will vary with the distances between the focus and the directrix, but the shape is always the same for a certain eccentricity.

Every eccentricity between 0 and 1 produces an ellipse. If e=1, a different curve is formed-a parabola-and every point on a parabola is equidistant from the focus and the directrix of the parabola. This curve does not close at all, but its 'arms' extend indefinitely, with the curve growing wider all the time. It includes, for example, two points that are 100 feet from the focus and 100 feet from the directrix, and if the focus is 1 12 inches from the directrix these two points are just under ten feet from each other.
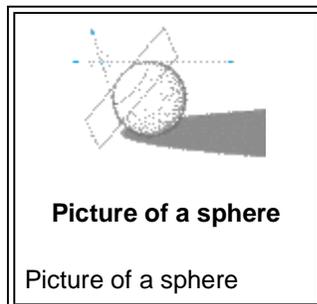
Curves with eccentricities of more than 1 are even more eccentric. They include points on both sides of the directrix, which could not happen for smaller eccentricities because points on one side of the directrix are necessarily farther from the focus than they are from the directrix. These two-part curves are hyperbolas. Each part, or branch, is a mirror image of the other; like the ellipse, the hyperbola may be thought of as having two foci and two directrices.

The hyperbola opens out in a very different way from the parabola. To show this, imagine that the previous picture is here reduced to about 130 of its size. The hyperbolas stick out in four directions each. They look rather like a flat letter X because the curves get so close to two intersecting lines, called the asymptotes. The parabola may look a little like a V, but it does not straighten out the way the hyperbolas do. It does not have asymptotes but continues to curve, seeming more and more in appearance like two parallel lines. The parabola is really in a class by itself, and yet it is much like an ellipse with an eccentricity of 0.999 or a hyperbola with an eccentricity of 1.001. These latter two curves would have their other foci a thousand times as far away as the distance between the original focus and directrix. The parabola is sometimes thought of as having 'another focus' at an infinite distance-in either direction.

The ellipse, parabola, and hyperbola-and sometimes the circle-are called conic sections because they are exactly the shapes formed by the intersection of a plane with a conical surface.

Conic sections can also be illustrated by a picture of a sphere touching a plane, such as a ball sitting on a floor, illuminated by a tiny bright light. For experiments similar to this illustration, it is best to use a flashlight with its reflector either removed or covered with a black paper cone.

The sphere casts a shadow that is actually a solid cone of darkness. This cone cannot be seen, but the place where it hits the floor is what is usually taken as the shadow of the ball. In the picture, the shadow is an unending parabola. If the light were any higher, it would close into an ellipse. If the light were lower, it would open out into one branch of a hyperbola. The other branch can be visualized by imagining the light cone extended right through its vertex-the light-to form what is called the other nappe of the cone. For either an ellipse or a parabola, the other nappe never touches the floor, but for the hyperbola its outline forms another branch that is the same shape as the shadow.

**The kind of conic section produced by the inter...**

The kind of conic section produced by the intersection of a plane and conical surface is determined by the angle at which the plane intersects the surface.

**Picture of a sphere**

Picture of a sphere

This drawing is related directly to the diagram showing the conic sections with a directrix and focus. The point where the sphere touches the plane is the focus. To find the directrix, notice on the sphere itself the border between the lighted and shaded parts. It is a circle, and the plane that contains the circle extends to meet the plane of the floor in a line that is the directrix. With this in mind, one can visualize all sorts of sphere shadows. Even the circle can be formed by putting the light directly above the sphere. For this curve the directrix disappears because the plane that is supposed to form it by intersecting the floor is parallel to the floor-so there is no intersection. Each curve can be formed using many different spheres, each of which forms different cones. The smaller the sphere, the closer to it the light must be to make the proper-size shadow.

**Solids**

Solid **geometry** deals with three-dimensional figures, such as spheres and cones. These figures have curved surfaces. A solid with only plane surfaces, however, is a polyhedron (plural polyhedrons or polyhedra). Like polygons, polyhedra can be named by using Greek numeral prefixes. A tetrahedron has four triangular faces.

A pentahedron, with five faces, can also be a pyramid, but with any kind of quadrilateral as its base. There are several other kinds of pentahedra. Each pentahedron has two triangular faces at opposite ends, with three quadrilaterals in between. Each pentahedron, therefore, has six vertices (points) and nine edges (line segments). If these are counted as variations of the same basic shape, there are only two kinds of pentahedra.

**Hexahedra**
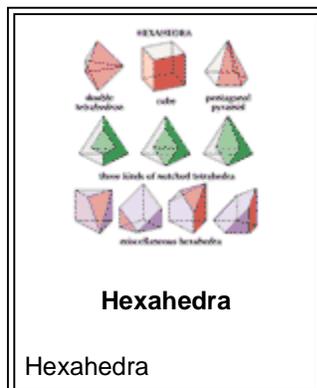
Hexahedra

Even with such restrictions, there are ten kinds of hexahedra, with six faces. Some of them have the same numbers of vertices and edges but in different arrangements. Each hexahedron can be distorted into variations that look different from one another. If, for example, one of the corners is cut off either of the pentahedra, making a new face there, it may appear that a new kind of hexahedron has been discovered that is not among the ten, even though it is. Every hexahedron has four more edges than it has vertices.

The great variety among the hexahedra shows that it is not enough to name a solid by its number of faces alone. Figures are classified according to other properties, as suggested by the names of the pentahedra. Two important sets of polyhedra are prisms and pyramids, either of which may use any kind of polygons as bases. A pyramid with its top cut off parallel to the base becomes a
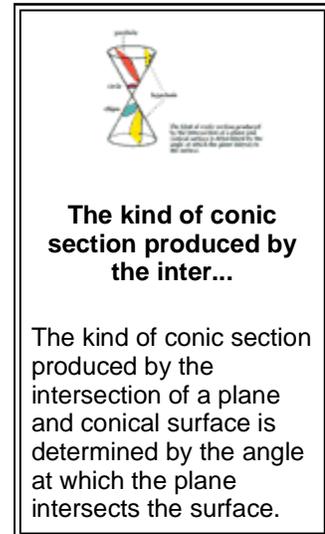
frustum of a pyramid. Either a prism or a pyramid can have its top cut off not parallel to the base, forming new kinds of polyhedra-truncated prisms or truncated pyramids.



**Regular polyhedra**
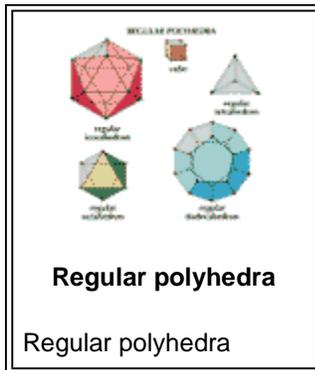
Regular polyhedra

Only five kinds of polyhedra are worthy of the name regular. Three of them have equilateral triangles as their faces; one has squares; and the other, regular pentagons. The double tetrahedron-one tetrahedron on top of another-shown among the hexahedra above, almost qualifies, but it has two different kinds of corners. The five regular polyhedra are known as Platonic bodies, because the Greek philosopher Plato investigated them about a hundred years before Euclid's time. The Greek mathematician Archimedes, who lived about the same time as Euclid, extended the investigation to solids that are almost regular and found them closely related to the regular ones. For two examples, consider the cube



**Cuboctahedron and rhombic dodecahedron**

Cuboctahedron and rhombic dodecahedron
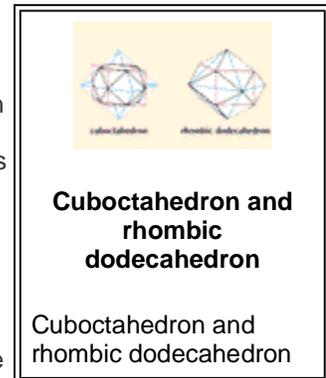
and the regular octahedron. One can be put inside the other so that all 12 edges of each solid touch the edges of the other exactly at their midpoints. The region that is included inside both polyhedra is a 14-faced solid with 12 vertices, a cuboctahedron. The same two solids form a framework for building a polyhedron that encloses both of them; it has 12 faces and 14 vertices. Each face is a rhombus, so the solid is a rhombic dodecahedron. Solids like these are sometimes called semiregular.

**Non-Euclidean Geometry**

In the 19th century, many mathematicians began questioning one of Euclid's main premises: that, simply stated, two lines are parallel if, no matter how far they are extended in either direction, they never intersect, but always remain the same distance apart from each other. The German mathematician Bernhard Riemann, by extending Euclid's basically two-dimensional **geometry** into three or more dimensions, showed among other things that there are no lines parallel to the given line. This idea eventually became essential to Einstein's development of the theory of relativity. (See also Einstein.)

Spherical **geometry**, another field developed in the 19th century, is concerned with circles, globes, and spheres and their properties. Spheres have no corners of any kind but only one smooth surface. No straight lines can be drawn on a sphere, but any two points on it can be connected by an arc which, if extended, would go all the way around the sphere, dividing it into hemispheres (see Hemisphere). This is the largest kind of circle that can be drawn on a sphere, and it is called a great circle. Great circles can form angles and triangles and other polygons. They can, in fact, do anything that lines do on a plane except be parallel. Any two great circles meet each other at two diametrically opposite points. Small circles (for example, the lines of latitude on Earth above and below the equator) can be parallel, but they do not have the other properties of straight lines. (See also Latitude and Longitude.)



**Spherical triangle**

Spherical triangle
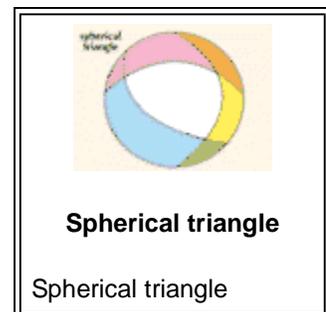
One of the better-known facts of plane **geometry**, another 19th-century development, is that the angles of a triangle add up to one straight angle, or 180°. This may appear to have nothing to do with parallel lines, but the relationship cannot be proved without using parallel lines. It should therefore be expected that the sum of the angles in a spherical triangle would not be the same as that in a plane triangle. The angles of a spherical triangle always add up to more than 180°. The larger the triangle, the greater the amount by which the sum exceeds 180°.
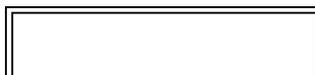
'Saddle' **geometry**, or hyperbolic **geometry**, whose development is generally attributed to German mathematician Carl Friedrich Gauss, is based on a surface
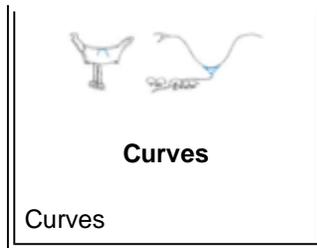
**Curves**

Curves

that curves in two directions at once, like a saddle or certain mountain passes. Such a surface cannot be extended indefinitely, like a plane, nor can it meet itself in a shape as tidy as a sphere. The angles of a triangle (shown in blue in the diagram) drawn on the surface of a saddlelike mountain pass add up to less than 180°. The figure shows a natural rock formation that is much like a triangle in saddle **geometry**. While spherical **geometry** has no parallels, in saddle **geometry** many lines can be drawn through the same point, all parallel to the same line. (See also Gauss.)

In 1975 Polish mathematician Benoit Mandelbrot introduced fractal **geometry** as a way to describe irregularly shaped objects or natural phenomena-such as coastlines, snowflakes, and tree branches-that could not be described by Euclidean **geometry**. Mandelbrot coined the word fractal to signify certain complex geometric shapes. The word is derived from the Latin fractus, meaning 'fragmented' or 'broken' and refers to the fact that these objects are self-similar-that is, their component parts resemble the whole. He stated that natural forms have the tendency to repeat themselves on an ever smaller scale, so that if each component is magnified it will look basically like the object as a whole. This **geometry** has been applied to the fields of physiology, chemistry, and mechanics.

**Construction**

In the illustrations for this article, points and lines have been shown in color, which has been helpful in showing relationships. These representations, however, are definitely not realistic. Lines do not even have thickness, much less color. Here is a picture that comes closer to a true representation:

In the diagram, the border line between the red and white areas is neither red nor white, and it has no thickness; it is much like a true line. The place where this line runs into the black region is much like a true point, with no dimensions at all. It is much more practical for everyday calculations to represent points by dots and to represent lines by paths of black or color. Nonetheless, the reasoning applied to the figures can be exact even if the diagrams are not.

**Representation of lines**

Representation of lines
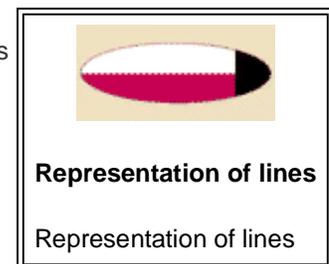
**Drafter's tools.**

A drafter, whose job is to draw plans and sketches of machinery or other technical structures, must use many geometrical facts and specialized equipment to work out these facts while on the job (see Mechanical Drawing). Euclid showed that just about every geometric shape can be drawn by using combinations of two basic procedures described in the following rules: (1) Through two given points, one and only one straight line can be drawn, and (2) with a given point as center and a given line segment as radius, one and only one circle can be drawn. These suggest two instruments that are thought of as the tools of the draftsman's trade: an unmarked straightedge-for ruling lines, but not for measuring them-and a simple compass-for drawing circles of various sizes. These instruments are enough to do everything from copying a line segment to constructing a square with the same area as any given polygon.

**The impossible problems.**

**Impossible problems**

Impossible problems
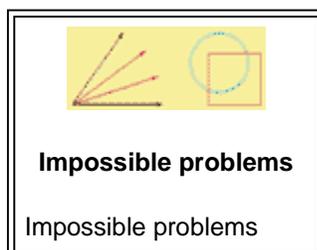
There were three classical problems that gave geometers a lot of trouble: trisecting an angle, squaring a circle, and doubling a cube. It is easy to construct an angle three times as big as another, but to start with the big angle and cut it into three equal pieces proved difficult. Another problem called for constructing a square and a circle with the same area. No matter which figure was given, there was no way to make the other exactly the same size. Nor could anyone find a construction to make one cube with twice the volume of another. If the edge of a cube is doubled, its volume becomes eight times as large-not twice.

For centuries these three problems bothered the mathematical world. In looking for solutions, many useful investigations of many parts of mathematics were made, especially in the area of nonrational numbers (see

[Numeration Systems and Numbers](#)). Finally, in the late 1800s, mathematicians settled once and for all exactly what a compass and straightedge could and could not do, and they proved that these three problems are not within the capabilities of the two instruments. However, the fact that these constructions cannot be done with the traditional tools of **geometry** does not mean that they are impossible.

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